Title: Aharonov vs. Spekkens round II: Contextuality in Pre- and Post-Selection Paradoxes

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Abstract:

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Aharonov vs. Spekkens Round II: Contextuality in Pre- and Post-Selection Paradoxes

Matthew Leifer
Chapman University

July 25, 2017

KS at 50 – Canada at 150

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Aharonov vs. Spekkens Round II: Contextuality in Pre- and Post-Selection Paradoxes

Matthew Leifer
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- "Progress through paradox"[a]:
  - Three box paradox
  - Quantum pigeonhole principle
  - Quantum Cheshire cats
  - Anomalous weak values

[a] Y. Aharonov and D. Rohrlich, "Quantum Paradoxes" (Wiley, 2005).

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Spekkens

- A vast array of seemingly puzzling quantum phenomena occur in classical models with a restriction on how much you can know about the system[1].
- Those that do not, seem to fall under the rubric of (Spekkens) contextuality[2].

[1] R. Spekkens, *Phys. Rev. A* 75:032110 (2007).
[2] R. Spekkens, *Phys. Rev. A* 71:052108 (2005).

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Outline

- We will consider two classes of pre- and post-selection paradoxes:
  - Paradoxes with strong measurements:
    - Logical Pre- and Post-Selection paradoxes are related to statistical proofs of the Kochen-Specker theorem[3] and are proofs of Spekkens contextuality[4].
    - Includes: Three-box paradox, Hardy's paradox, Quantum pigeonhole paradox, Vaidman's "failure of the product rule".
    - Other pre- and post-selection paradoxes are not, e.g. Quantum Cheshire cat.
  - Paradoxes with weak measurements:
    - Any anomalous weak value is a proof of Spekkens contextuality[5].
    - Includes things that are noncontextual with strong mesurements, e.g. quantum Cheshire cat.

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[3] ML and R. Spekkens, *Phys. Rev. Lett.* 95 200405 (2005).
[4] M. Pusey and ML, *Proc. QPL 2015*, arXiv:1506.07850 (2015).
[5] M. Pusey, *Phys. Rev. Lett.* 113 200401 (2014).

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The Three Box Paradox

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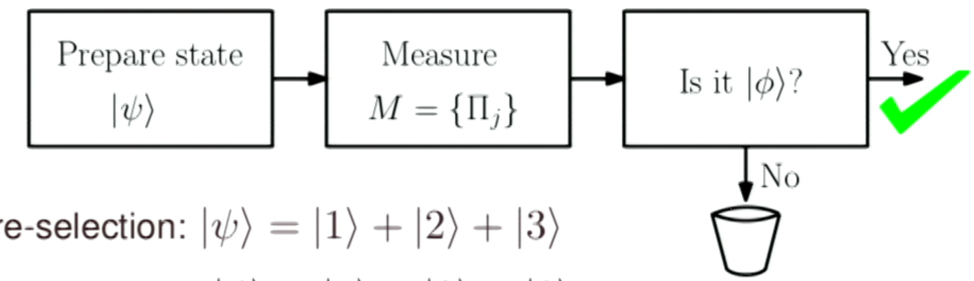
Three box paradox



- Pre-selection: $|\psi\rangle = |1\rangle + |2\rangle + |3\rangle$
- Post-selection: $|\phi\rangle = |1\rangle + |2\rangle - |3\rangle$
- Two possible intermediate measurements:
  - M_1: Is ball in box 1? $\Pi_1 = |1\rangle\langle 1|, \quad \Pi_{2\vee 3} = |2\rangle\langle 2| + |3\rangle\langle 3|$

$$\mathbb{P}(\Pi_1|\psi, M_1, \phi) = 1$$

  - M_2: Is ball in box 2? $\Pi_2 = |2\rangle\langle 2|, \quad \Pi_{1\vee 3} = |1\rangle\langle 1| + |3\rangle\langle 3|$

$$\mathbb{P}(\Pi_2|\psi, M_2, \phi) = 1$$

Y. Aharonov and L. Vaidman, *J. Phys. A* 24 pp. 2315–2328 (1991).

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Kochen-Specker Contextuality

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Kochen-Specker (KS) Noncontextuality

- *Outcome determinism*: At any given time, the system has a definite value for every observable.
  - For every orthonormal basis $\{|\psi_j\rangle\}$, precisely one of them is asigned the value 1, the rest 0.
- *Noncontextuality*: The outcome assigned to an observable does not depend on which other (commuting) observables it is measured with.
  - The value assigned to a basis vector does not depend on which basis it occurs in, e.g.

$$|1\rangle\,,|2\rangle\,,|3\rangle$$

vs.

$$|1\rangle\,,|2\rangle+|3\rangle\,,|2\rangle-|3\rangle\,.$$

S. Kochen and E. Specker, *J. Math. Mech.* 1 pp. 59–87 (1967).

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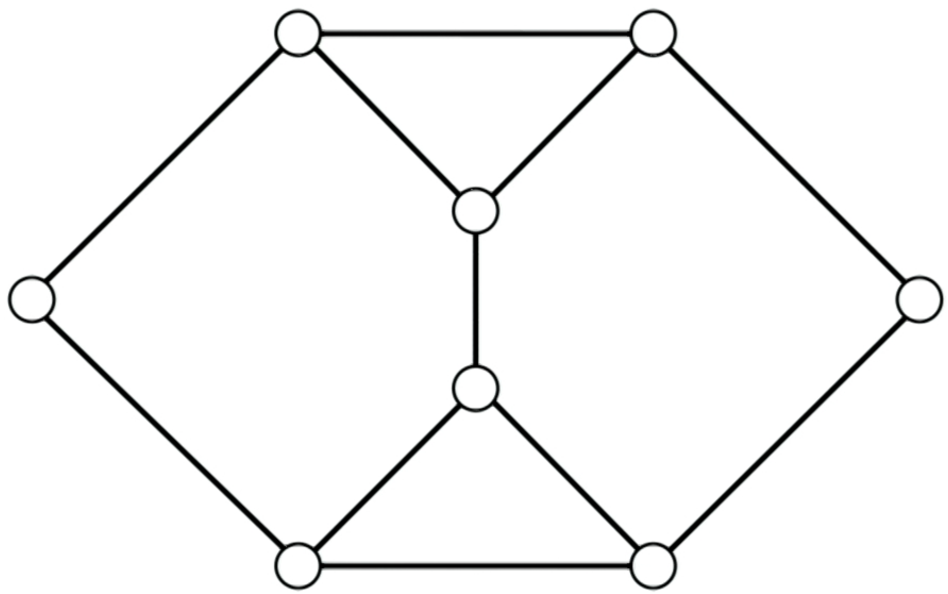
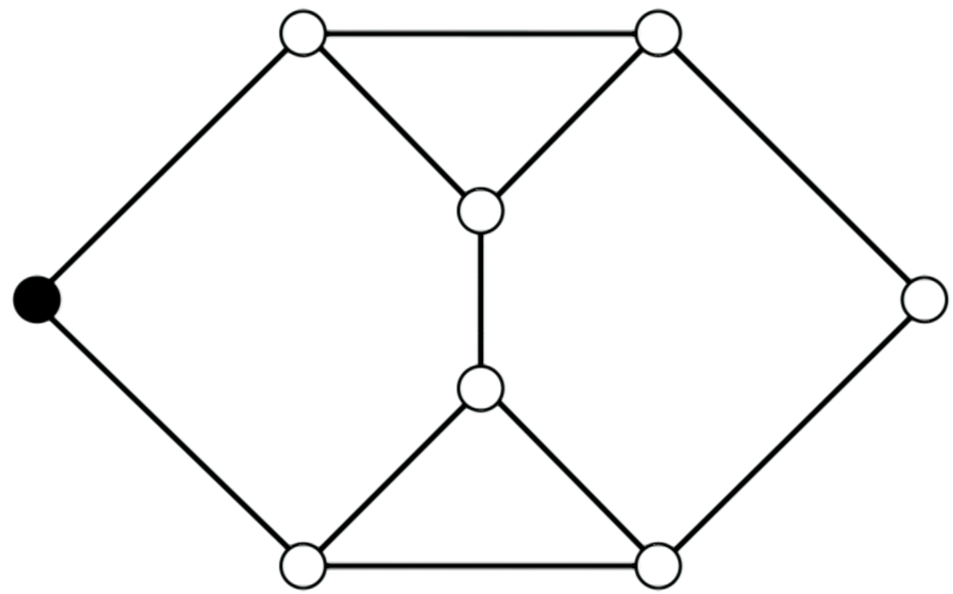
Clifton's contextuality proof



R. Clifton, *Am. J. Phys.* 61 443 (1993).

Clifton's contextuality proof



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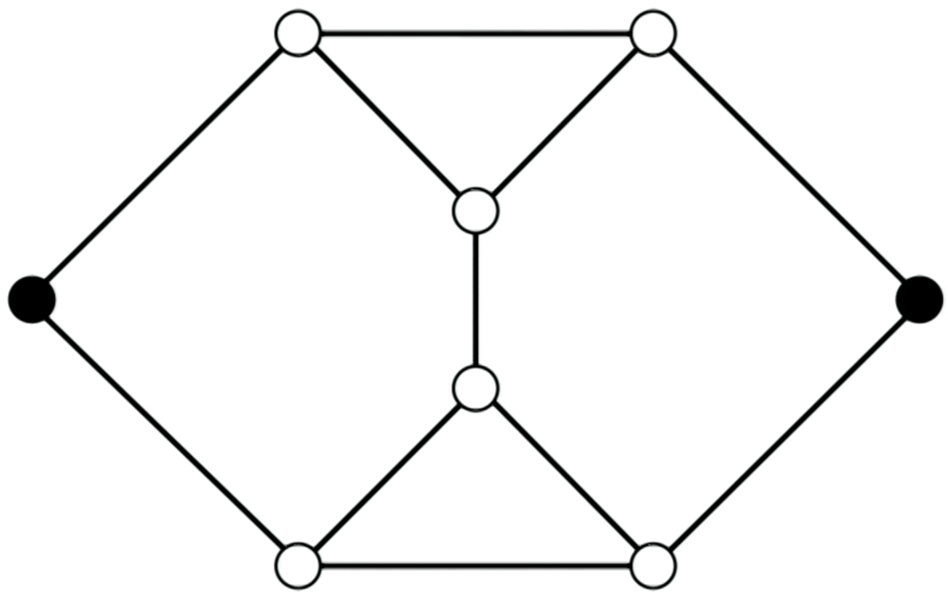
Clifton's contextuality proof



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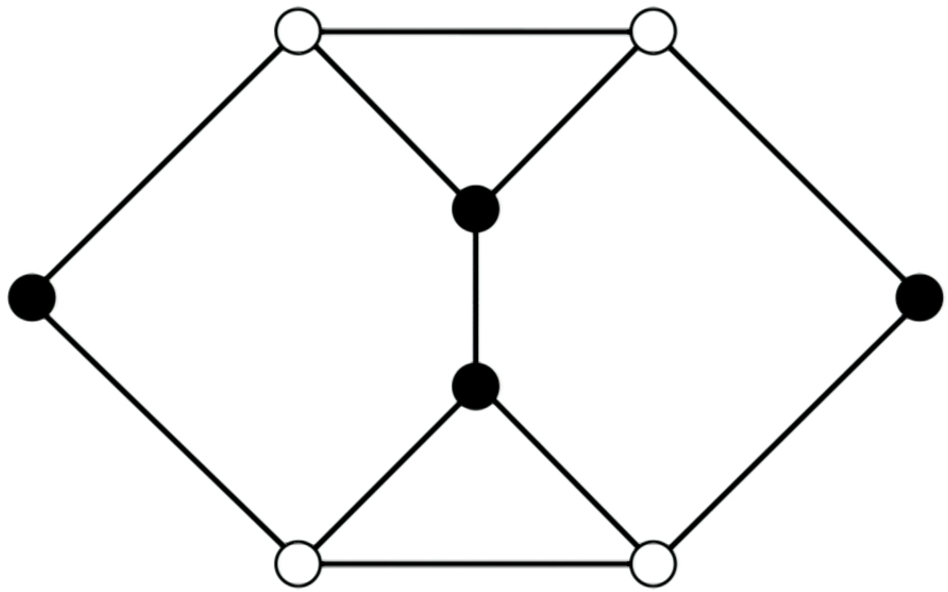
Clifton's contextuality proof



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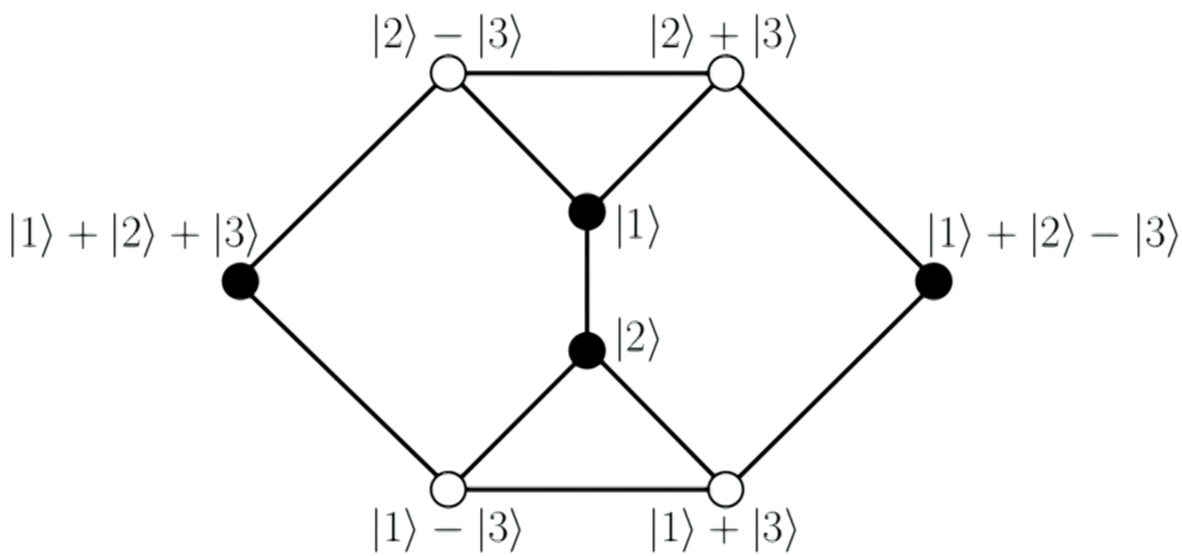
Clifton's contextuality proof



- All logical pre- and post-selection paradoxes are related to a proof of (KS) contextuality in the same way[6].

R. Clifton, *Am. J. Phys.* 61 443 (1993).

[6] ML and R. Spekkens, *Phys. Rev. Lett.* 95 200405 (2005).

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A KS noncontextual model

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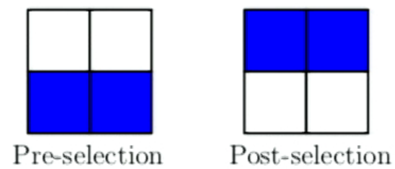
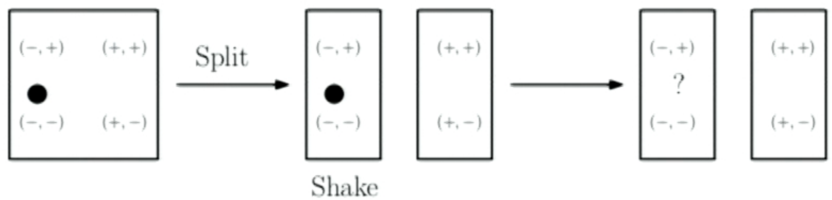
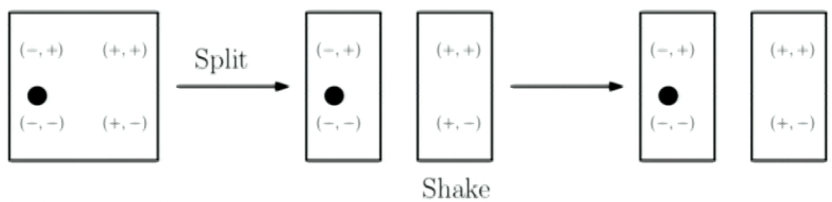
The partitioned box paradox



- “Left”-measurement:



- “Right”-measurement:



ML and R. Spekkens, Int. J. Theor. Phys. 44 pp. 1977–1987 (2005).

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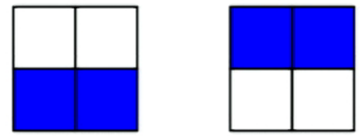
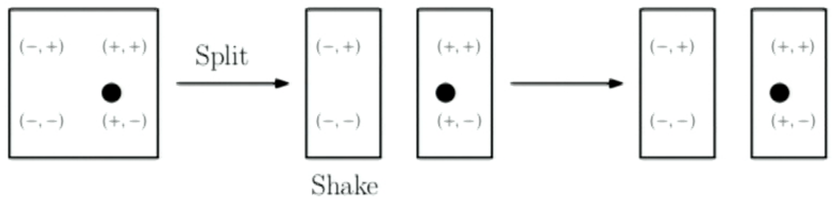
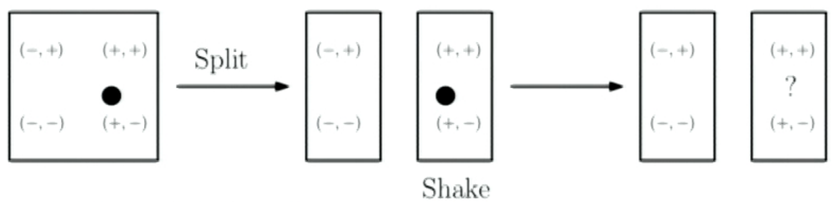


Pre-selection Post-selection

- "Left"-measurement:



- "Right"-measurement:



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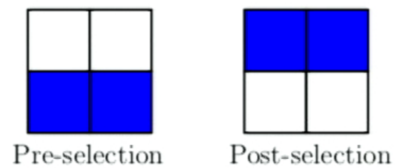
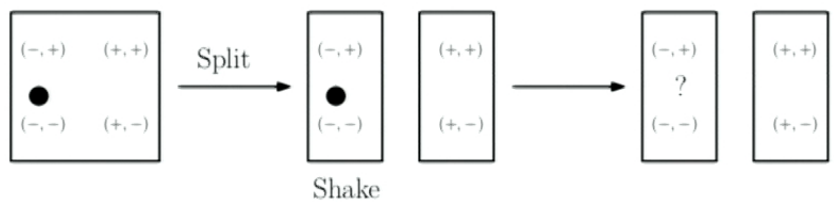
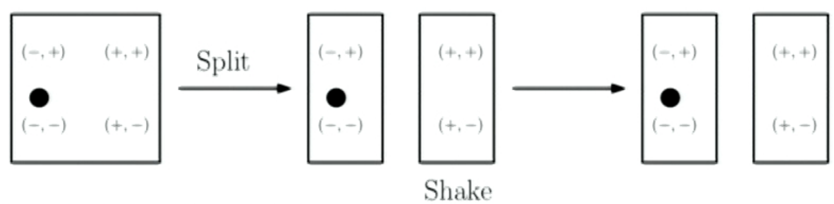


- "Left"-measurement:



- "Right"-measurement:



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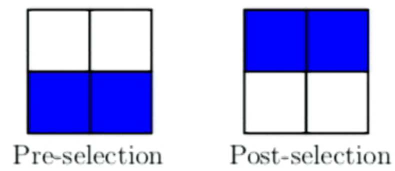

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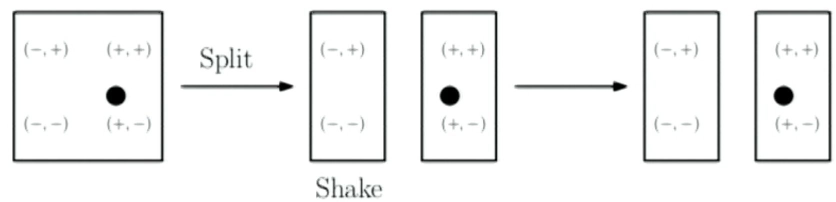

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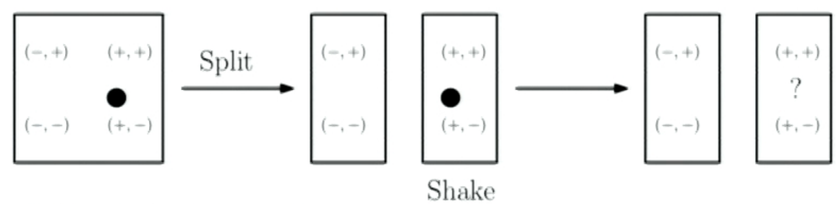

ML and R. Spekkens, Int. J. Theor. Phys. 44 pp. 1977–1987 (2005).

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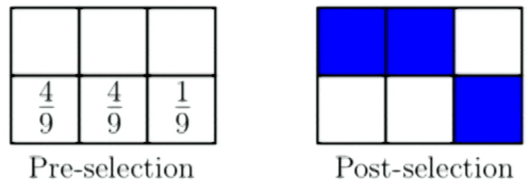
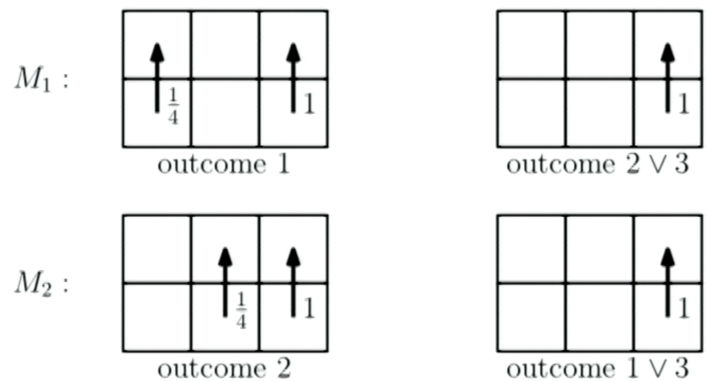
The partitioned box paradox

- We can reproduce the predictions of the three-box paradox exactly by adding more states and changing the update rules.
  - New pre- and post-selection:



  - New state-update rules:



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Spekkens Contextuality

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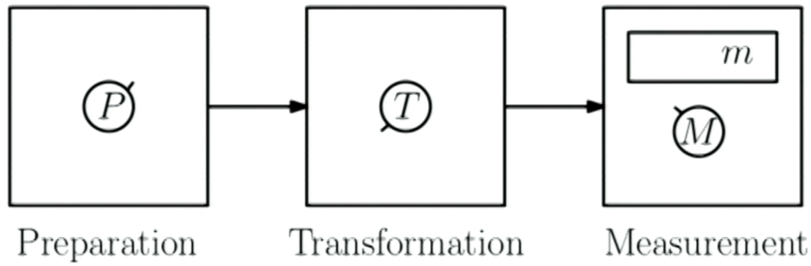
Spekkens Noncontextuality

- Operational theory:



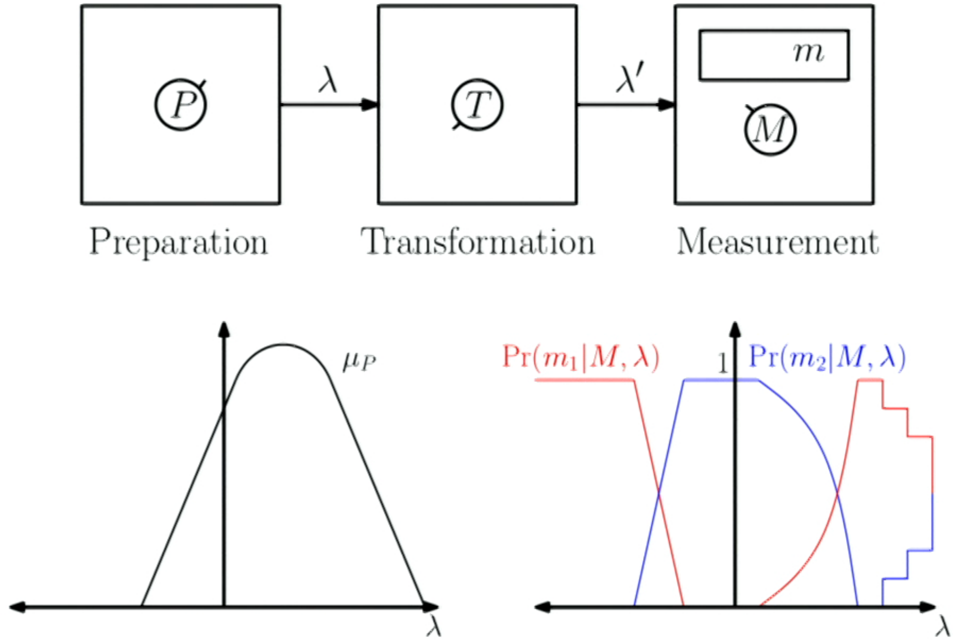
$$\mathbb{P}(m|P, M, T)$$

- In quantum theory:

$$\mathbb{P}(m|P, M, T) = \mathrm{Tr}\left(E_m^M \mathcal{E}_T(\rho_P)\right)$$

R. Spekkens, *Phys. Rev. A* 71:052108 (2005).

Ontological models



$$\mathbb{P}(m|P,M,T) = \int_{\Lambda'}\int_{\Lambda} \Pr(m|M,\lambda')d\Gamma_T(\lambda'|\lambda)d\mu_P(\lambda)$$

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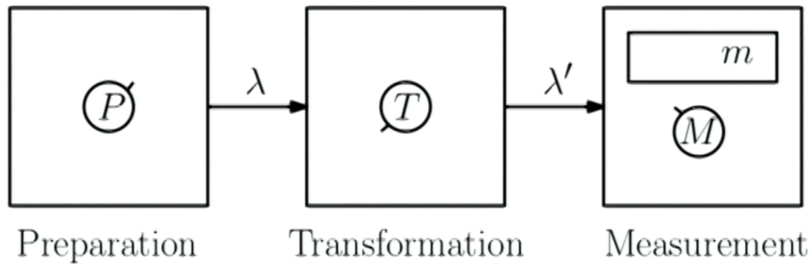
Transformation noncontextuality



Definition. An ontological model is *transformation noncontextual* if, whenever

$$\mathbb{P}(m|P, M, T) = \mathbb{P}(m|P, M, S)$$

for all P, M, m, we have

$$\Gamma_T = \Gamma_S.$$

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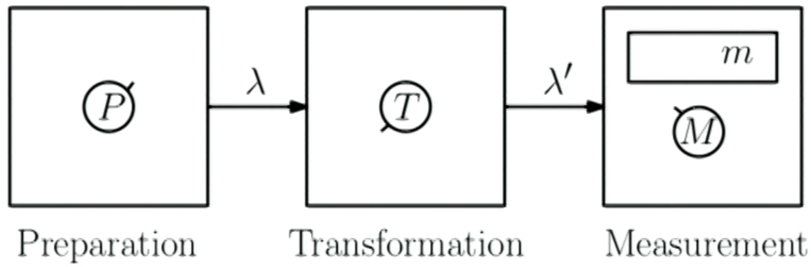
Transformation noncontextuality



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- In quantum theory, Γ_T only depends on $\mathcal{E}_T$.

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Implications for state-update rules

Theorem. *Let* $\{\Pi_j\}$ *be a projective measurement and let* $\mathcal{E}$ *be the nonselective state-update rule*

$$\mathcal{E}(\rho) = \sum_j \Pi_j \rho \Pi_j .$$

Then,

$$\mathcal{E}(\rho) = p\rho + (1-p)\mathcal{C}(\rho),$$

where $\mathcal{C}$ *is a completely-positive, trace-preserving map and* $0 < p \leq 1$.

- Proof for special case $\{\Pi_1, \Pi_2\}$:

$$U_1 = \Pi_1 + \Pi_2 = I \qquad U_2 = \Pi_1 - \Pi_2$$

$$\mathcal{E}(\rho) = \frac{1}{2}U_1\rho U_1^\dagger + \frac{1}{2}U_2\rho U_2^\dagger = \frac{1}{2}\rho + \frac{1}{2}U_2\rho U_2^\dagger .$$

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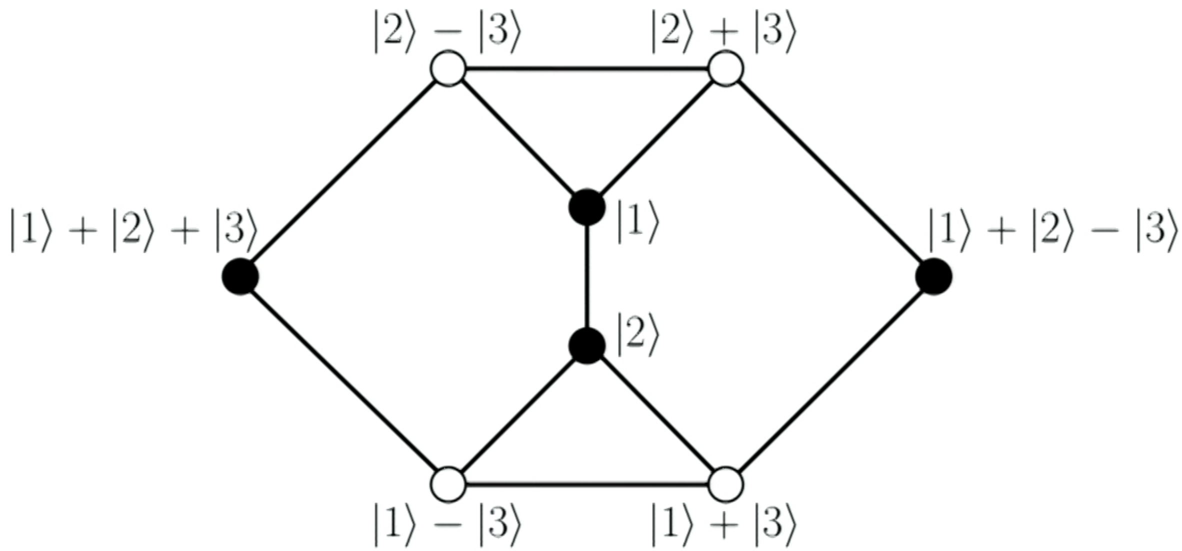
Proof of contextuality



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General Result

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Partial Boolean Algebras

Definition. A *partial boolean algebra* of projectors is a set $\mathcal{P}$ of projectors such that

- If $\Pi \in \mathcal{P}$ then $I - \Pi \in \mathcal{P}$.
- If $\Pi_1, \Pi_2 \in \mathcal{P}$ and $\Pi_1\Pi_2 = \Pi_2\Pi_1$ then $\Pi_1\Pi_2 \in \mathcal{P}$.

Definition. A function $f : \mathcal{P} \to \mathbb{R}$ satisfies the *algebraic conditions* if

- For all $\Pi \in \mathcal{P},\ 0 \leq f(\Pi) \leq 1$.
- $f(I) = 1$.
- $f(I - \Pi) = 1 - f(\Pi)$.
- If $\Pi_1, \Pi_2 \in \mathcal{P}$ and $\Pi_1\Pi_2 = \Pi_2\Pi_1$ then

$$f(\Pi_1 + \Pi_2 - \Pi_1\Pi_2) = f(\Pi_1) + f(\Pi_2) - f(\Pi_1\Pi_2).$$

Boolean Complexes

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Definition. A *boolean complex* of projectors is a set $\mathcal{P}$ of projectors such that

- If $\Pi \in \mathcal{P}$ then $I - \Pi \in \mathcal{P}$.
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Definition of a Logical Pre- and Post-Selection Paradox

- Consider pre- and post-selection states $|\psi\rangle$ and $|\phi\rangle$.
- Consider a set $\mathcal{P}$ of projection operators that is closed under complements, i.e. if $\Pi \in \mathcal{P}$ then $I - \Pi \in \mathcal{P}$.
- Let
$$\mathbb{P}(\Pi|\psi, \phi) := \mathbb{P}(\Pi|\psi, \{\Pi, I - \Pi\}, \phi)$$
and suppose that $\mathbb{P}(\Pi|\psi, \phi) \in \{0, 1\}$ for all $\Pi \in \mathcal{P}$.
- Let $\mathcal{P}'$ be the smallest boolean complex containing $\mathcal{P}$.
- Then, a *Logical Pre- and Post-Selection (LPPS) paradox* occurs if $f(\Pi) = \mathbb{P}(\Pi|\psi, \phi)$ cannot be extended to a function on $\mathcal{P}'$ that satisfies the algebraic conditions.
- Example: three-box paradox
$$f(\Pi_1) = 1, f(\Pi_2) = 1 \qquad \Rightarrow \qquad f(\Pi_1 + \Pi_2) = 2.$$

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Definition of a Logical Pre- and Post-Selection Paradox

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- Example: three-box paradox
$$f(\Pi_1) = 1, f(\Pi_2) = 1 \qquad \Rightarrow \qquad f(\Pi_1 + \Pi_2) = 2.$$

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General result

Theorem. *All LPPS paradoxes are proofs of Spekkens contextuality.*

- A number of features of LPPS paradoxes are crucial for the proof:
  - That the pre- and post-selection states are non orthogonal.

$$|\langle\phi|\psi\rangle| > 0$$

  - That the intermediate measurement has Lüders rule (projection postulate) state update rule.

$$|\psi\rangle \to \frac{\Pi|\psi\rangle}{\langle\psi|\Pi|\psi\rangle}$$

  - That the probabilities $\mathbb{P}(\Pi|\psi, \phi)$ are all 0 or 1.

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The Quantum Cheshire Cat

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The Quantum Cheshire Cat

- Let $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$.
- Pre-selection: $|\psi\rangle = |+\rangle \otimes |0\rangle$.
- Post-selection: $|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$.
- Intermediate measurements:

$$\Pi_1 = |1\rangle\langle 1| \otimes I : \qquad \mathbb{P}(\Pi_1|\psi,\phi) = 0.$$

$$\Pi_{1+} = |1\rangle\langle 1| \otimes |+\rangle\langle +| : \qquad \mathbb{P}(\Pi_{1+}|\psi,\phi) = \frac{1}{4}.$$

$$\Pi_{1-} = |1\rangle\langle 1| \otimes |-\rangle\langle -| : \qquad \mathbb{P}(\Pi_{1-}|\psi,\phi) = \frac{1}{4}.$$

- Should have $f(\Pi_1) = f(\Pi_{1+}) + f(\Pi_{1-})$, but $0 \neq \frac{1}{4} + \frac{1}{4}$.

Y. Aharonov, S. Popescu, D. Rohrlich and P. Skrzypczyk, *N. J. Phys* 15 113015 (2013).

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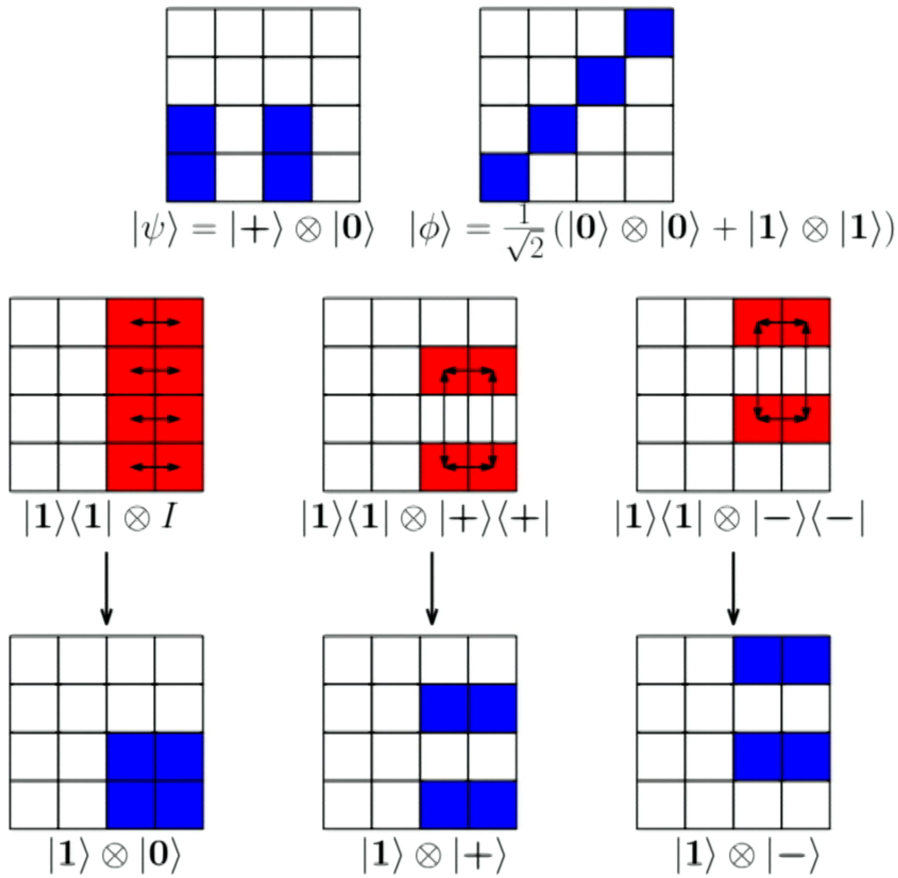
A Spekkens Noncontextual Model



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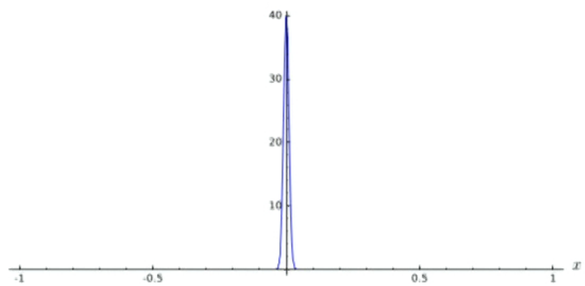
Anomalous Weak Values

A Brief Introduction to Weak Values

- To implement a strong projective measurement of observable $A = \sum_a a\Pi_a$:
  - Prepare a continuous variable pointer in a narrow Gaussian state with $\sigma \ll 1$.
  - Couple system to pointer with interaction Hamiltonian $H_I = A \otimes p$ for time $t = 1$ s.
  - Position density will evolve to sum of Gaussians centered at eigenvalues a with relative height $\langle\psi|\,\Pi_A\,|\psi\rangle$.
  - Measure pointer in position basis.



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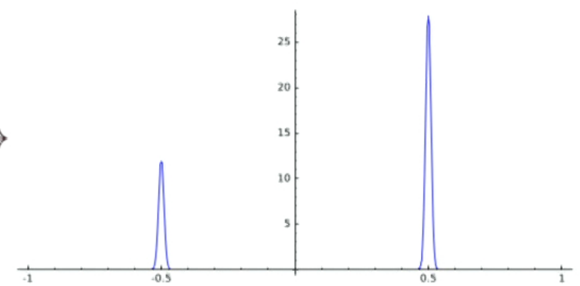

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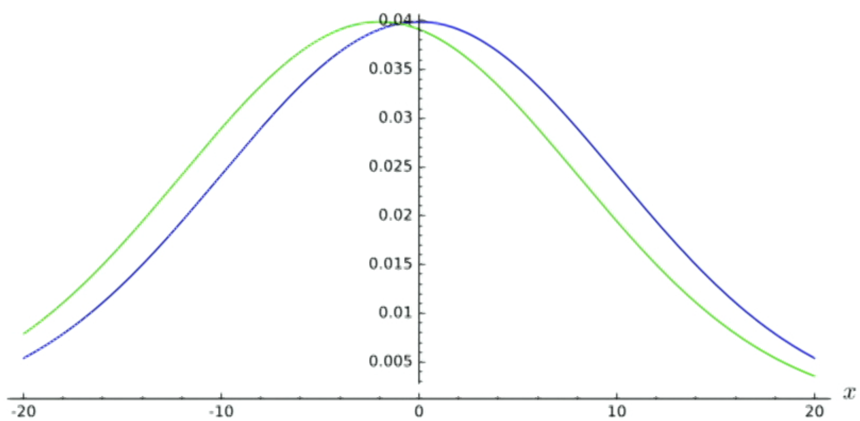
A Brief Introduction to Weak Values

- To implement a weak measurement of A, we choose a pointer state with $\sigma \gg 1$.
- Under pre- and post-selection, to first order, Gaussian just shifts by an amount

$$A_w = \text{Re}\left(\frac{\langle\phi|\,A\,|\psi\rangle}{\langle\phi|\psi\rangle}\right).$$



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Anomalous Weak Values

- A weak value is called *anomalous* if it lies outside the spectrum of A, i.e. for a projector, $\Pi_w < 0$ or $\Pi_w > 1$.
- Since A_w is a linear function of operators, it obeys all the algebraic conditions, except $0 \leq \Pi_w \leq 1$.
- It is also possible to prove that

$$\mathbb{P}(\Pi|\psi,\phi) = 0 \Rightarrow \Pi_w = 0 \qquad \mathbb{P}(\Pi|\psi,\phi) = 1 \Rightarrow \Pi_w = 1.$$

- Therefore, a violation of the algebraic conditions for strong measurements implies an anomalous weak value.
- Examples:
  - Three box paradox: $(\Pi_1)_w = 1$, $(\Pi_2)_w = 1$, $(\Pi_3)_w = -1$.
  - Cheshire cat: $(\Pi_1 \otimes I)_w = 0$, $(\Pi_{1+}) = \frac{1}{2}$, $(\Pi_{1-}) = -\frac{1}{2}$.

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Weak Values and Contextuality

Theorem. [7] *Negative weak values of projectors are proofs of Spekkens contextuality.*

- Can show that if the intermediate measurement does not disturb the ontic state then the median pointer value is always within the eigenspectrum.
- To obtain a negative value for a projector, need the ontic state to preferentially switch into a state that passes the post-selection when the pointer position is negative, and preferentially out of such states when the pointer position is positive.
- But the CPT-map describing the measurement disturbance is of the form

$$\mathcal{E}(\rho) = p\rho + (1-p)\mathcal{C}(\rho),$$

where $\mathcal{C}$ is CPT.

- In a transformation noncontextual value, p is too large to account for the observed negative bias of the pointer in the weak measurement limit.

[7] M. Pusey, *Phys. Rev. Lett.* 113 200401 (2014).

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Conclusions

Three Types of Pre- and Post-Selection Paradoxes

	Examples	Strong Version Proves Contextuality	Weak Version Proves Contextuality
Logical	Three-box, Pigeonhole Hardy, Failure of product rule	Yes	Yes
Non-logical, but violates algebraic conditions	Cheshire cat	No	Yes
No violation of algebraic conditions	Anomalous weak values for a qubit	No	Yes

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