Title: Confined Contextuality and Weak Measurement

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Abstract:

Confined Contextuality and Weak Measurement

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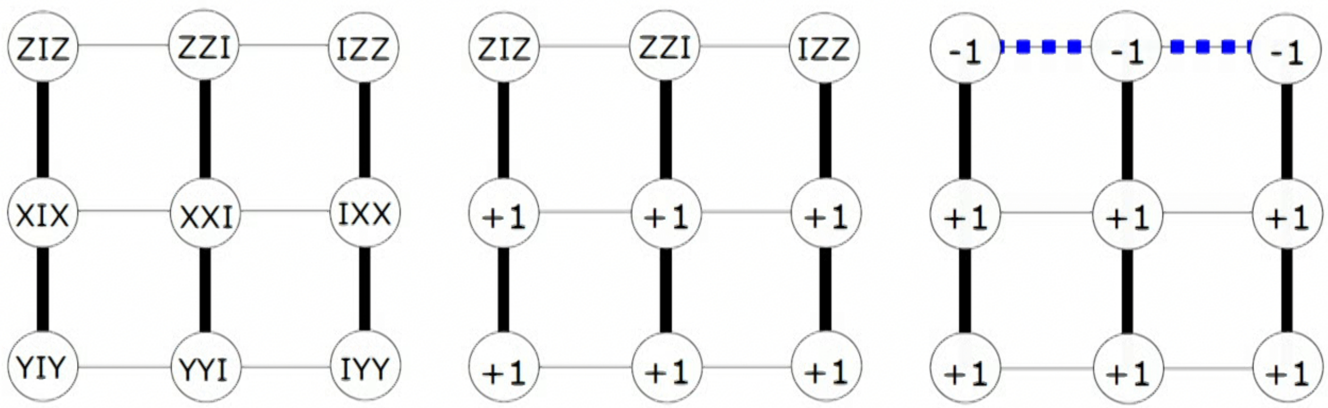
Confining Contextuality (1)

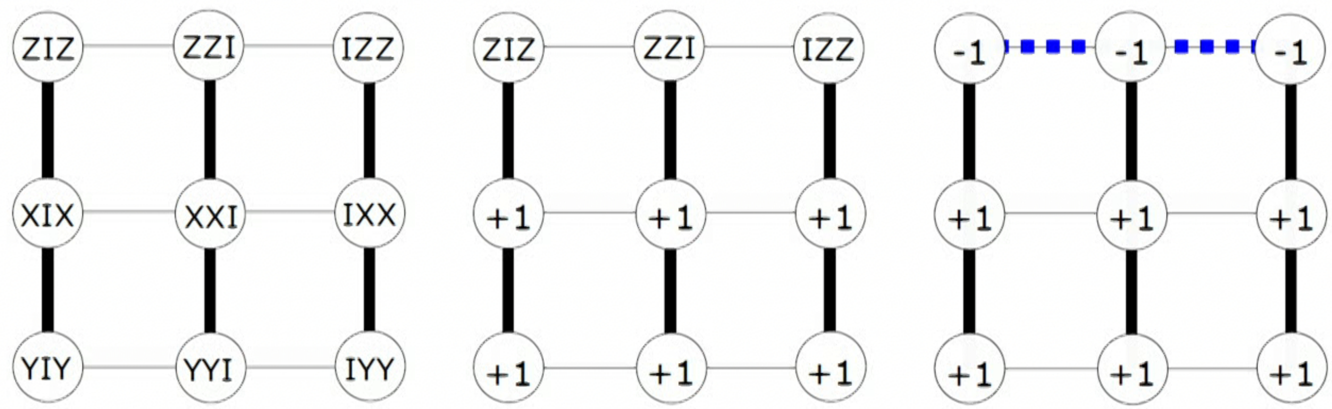
Using both pre- and post-selection (PPS), one can confine the explicit conflict between the predictions of quantum theory and any noncontextual hidden variable theory (NCHVT) to a particular measurement context. For example, consider the following BKS set, with the product state pre-selection $|\sigma_x = +1\rangle^{\otimes 3}$ and post-selection $|\sigma_y = +1\rangle^{\otimes 3}$. For a particular preparation, an NCHVT should predict the outcomes of any measurement, and thus this PPS forces certain value assignments.



(a) BKS 3-spin Square (b) Pre- / postselection (c) Paradoxical values

Confining Contextuality (2)

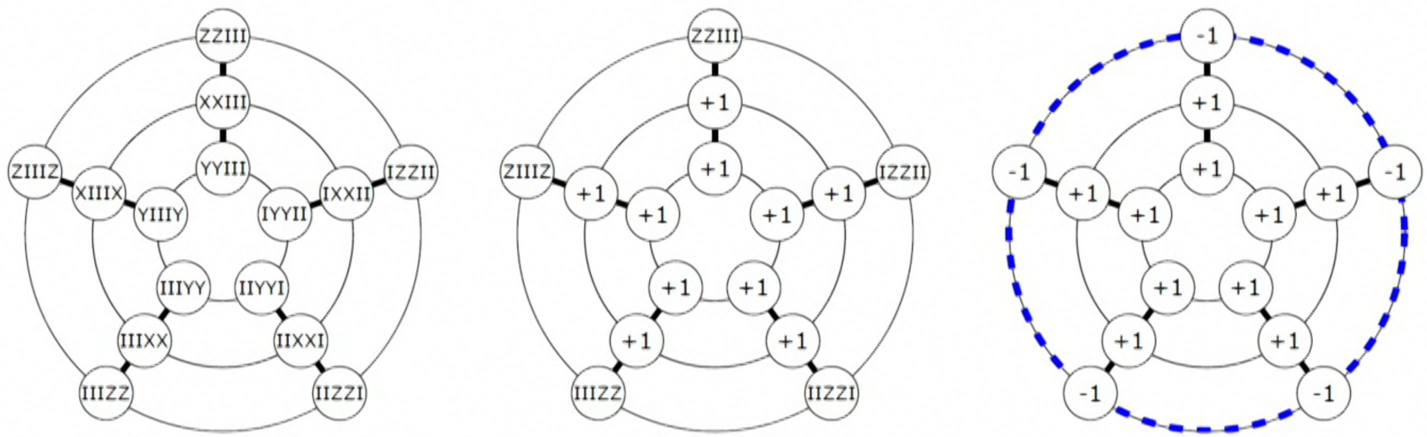
Moving beyond the PPS, the NCHVT can be extended in any context where all but one value has already been assigned, since this leaves only one viable prediction for the NCHVT. Another way to say this, as shown by Leifer and Pusey, is that any projector that is assigned a probability of 1 by the Aharonov-Bergmann-Lebowitz formula must also be assigned a 1 in any NCHVT. A PPS-Paradox is any case where this causes the NCHVT to violate an explicit prediction of quantum theory.



(a) BKS 3-spin Square (b) Pre- / postselection (c) Paradoxical values

Confining Contextuality (3)

The 3-qubit square generalizes to the family of *N*-qubit Wheel BKS sets, as does the pigeonhole paradox.



(a) 5-qubit Wheel (b) Pre- / Post-Selection (c) Paradoxical Values

The Weak Value

We call an ensemble of experiments in which a given state $|\psi\rangle$ is prepared in each run, and the outcome $|\phi\rangle$ is obtained by a final measurement, a PPS ensemble. For such an ensemble, the weak value of every observable A of a system, with spectral decomposition $A = \sum_i \lambda_i \Pi^i$, is defined as,

$$A_w \equiv \frac{\langle\phi|A|\psi\rangle}{\langle\phi|\psi\rangle} = \sum_i \lambda_i \frac{\langle\phi|\Pi^i|\psi\rangle}{\langle\phi|\psi\rangle} = \sum_i \lambda_i \Pi^i_w. \quad (1)$$

If a pointer is weakly coupled to the system during the time between pre- and post-selection, such that the interaction introduces almost no disturbance, the ensemble average shift of the pointer is given by the weak value. Note that weak values are defined by the PPS, even if no weak measurement takes place.

The ABL Rule and Weak Values

For a given PPS, the ABL rule gives the probability to find a given outcome by a projective measurement made during the intervening time. It can be written in terms of the weak values of the projectors in the basis B of the intermediate measurement as,

$$P_{ABL}(\Pi_i = 1 | B, |\psi\rangle, |\phi\rangle) = \frac{|\Pi_w^i|^2}{\sum_{j\in B} |\Pi_w^j|^2}, \quad (2)$$

which is particularly useful for analyzing PPS-Paradoxes. For any complete basis B, $\sum_{i\in B} \Pi^i = I$, and thus $\sum_{i\in B} \Pi_w^i = 1$. Using this fact, it follows that for any basis with cardinality two, $\Pi_w^i = 1$ implies,

$$P_{ABL}(\Pi_i = 1) = \frac{|1|^2}{|1|^2 + |0|^2} = 1, \quad (3)$$

and thus any NCHVT must assign a 1 to these projectors. It also follows for any basis that $\Pi_w^i = 0$ implies an NCHVT assignment of 0.

The 3-box Paradox

Consider a pre-selection $|\psi\rangle = (|1\rangle + |2\rangle + |3\rangle)/\sqrt{3}$ and post-selection $|\psi\rangle = (|1\rangle + |2\rangle - |3\rangle)/\sqrt{3}$, such that a particle begins and ends with equal probability 1/3 to be found in each of three boxes. The weak values of the projectors onto the three boxes are, $|1\rangle\langle 1|_w = 1$, $|2\rangle\langle 2|_w = 1$, and $|3\rangle\langle 3|_w = -1$.

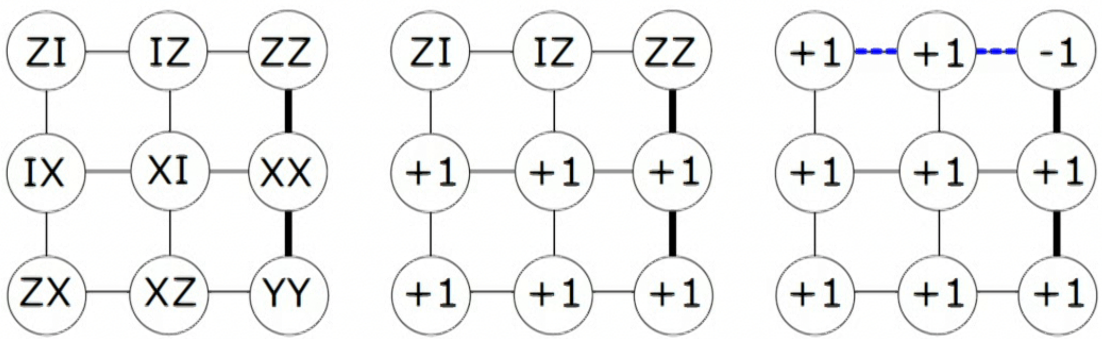
Using the ABL rule for two different coarse-grained bases, we can find some values that must be assigned by an NCHVT. The basis $B_1 = (|1\rangle\langle 1| + |3\rangle\langle 3|, |2\rangle\langle 2|)$, and the basis $B_2 = (|2\rangle\langle 2| + |3\rangle\langle 3|, |1\rangle\langle 1|)$. The ABL formula gives probabilities,

$$P_{ABL}(|2\rangle\langle 2| = 1|B_1) = P_{ABL}(|1\rangle\langle 1| = 1|B_2) = \frac{|1|^2}{|1-1|^2 + |1|^2} = 1, \qquad (4)$$

which forces both projectors $|1\rangle\langle 1|$ and $|2\rangle\langle 2|$ to be assigned a 1 in the NCHVT, and thus violates the quantum prediction for the fine-grained basis $B_3 = (|1\rangle\langle 1|, |2\rangle\langle 2|, |3\rangle\langle 3|)$.

The Quantum Cheshire Cat

We can also consider confined contextuality in the Peres-Mermin Square, a different proof of the BKS theorem for 2 qubits, by pre-selecting the Bell state, $|\psi\rangle = (|00\rangle + |01\rangle + |10\rangle - |11\rangle)/2$ and post-selecting the product state $|\phi\rangle = (|00\rangle + |01\rangle + |10\rangle + |11\rangle)/2$, we force six value assignments. Applying the ABL rule then causes the NCHVT to violate the quantum prediction for the top row.



(d) Peres-Mermin Square

(e) Pre- / Post-Selection

(f) Paradoxical Values

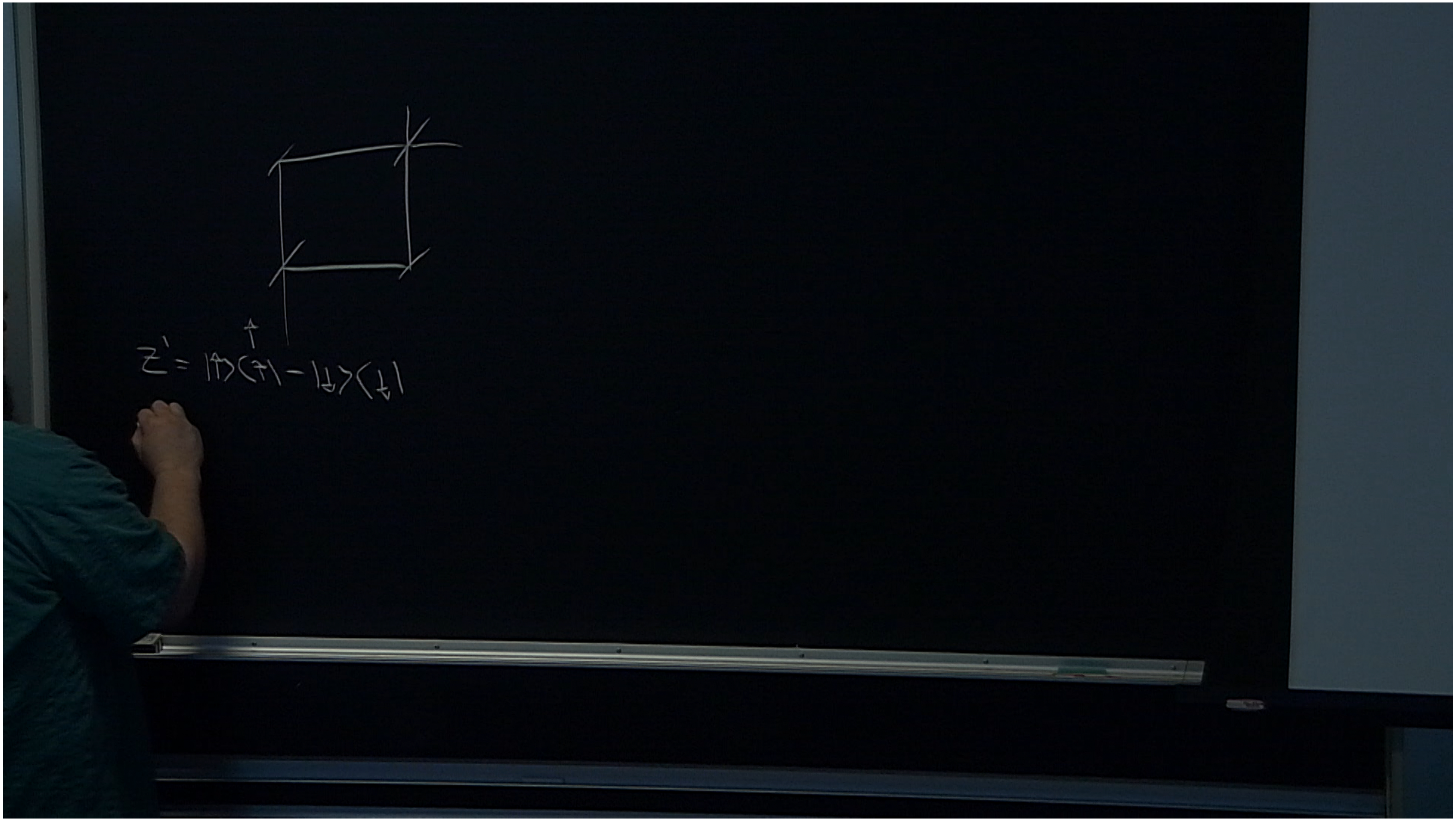
The 4-box Paradox

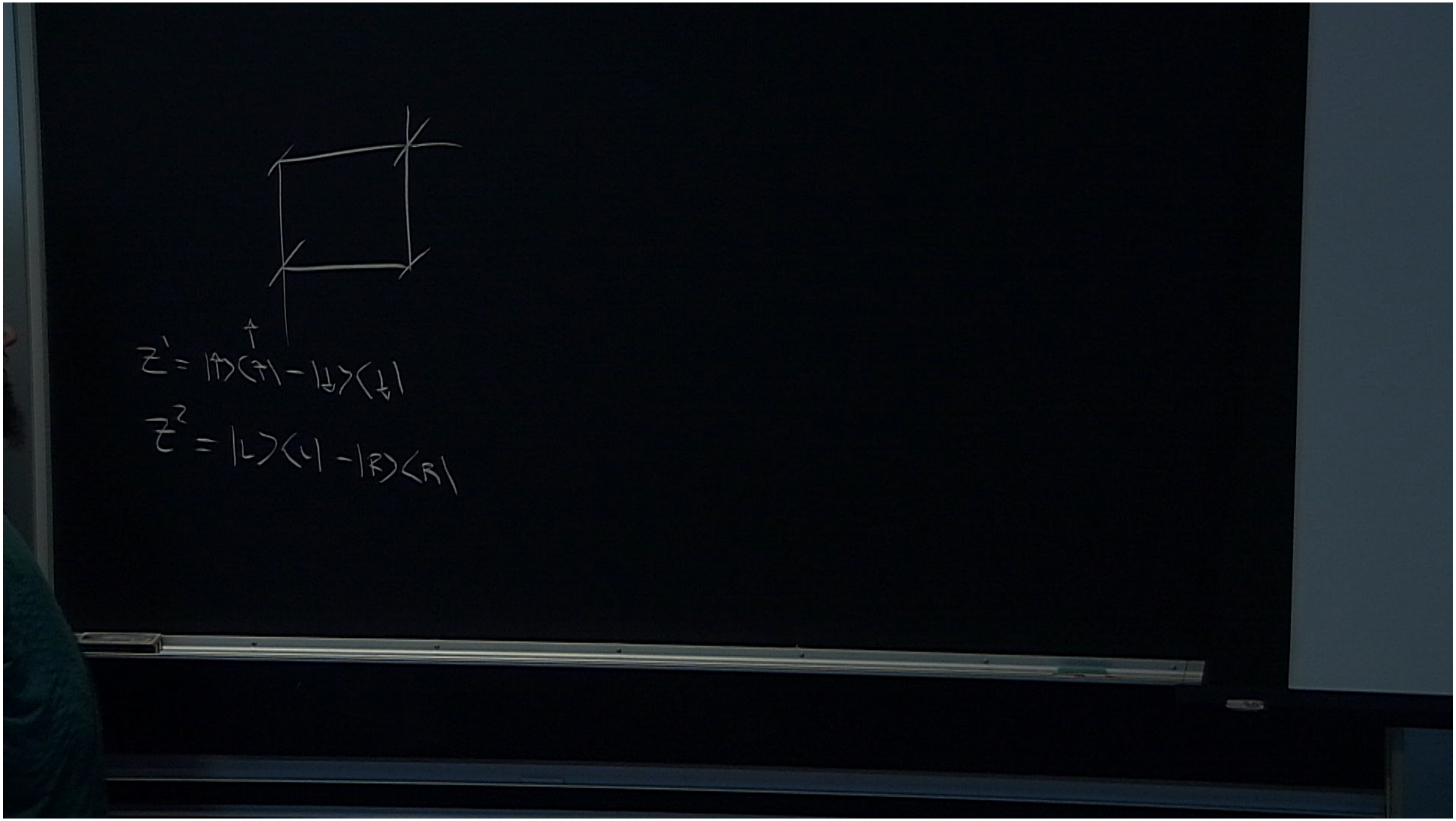
Both the Quantum Pigeonhole effect and the Quantum Cheshire Cat can be interpreted as 4-box paradoxes. Letting the projectors in the context with violation represent four boxes, we have the following weak values: $|1\rangle\langle 1|_w = 1/2$, $|2\rangle\langle 2|_w = 1/2$, $|3\rangle\langle 3|_w = 1/2$, and $|4\rangle\langle 4|_w = -1/2$. Considering the three bases $B_1 = (|1\rangle\langle 1| + |4\rangle\langle 4|, |2\rangle\langle 2| + |3\rangle\langle 3|)$, $B_2 = (|2\rangle\langle 2| + |4\rangle\langle 4|, |1\rangle\langle 1| + |3\rangle\langle 3|)$, $B_3 = (|3\rangle\langle 3| + |4\rangle\langle 4|, |1\rangle\langle 1| + |2\rangle\langle 2|)$, we obtain,

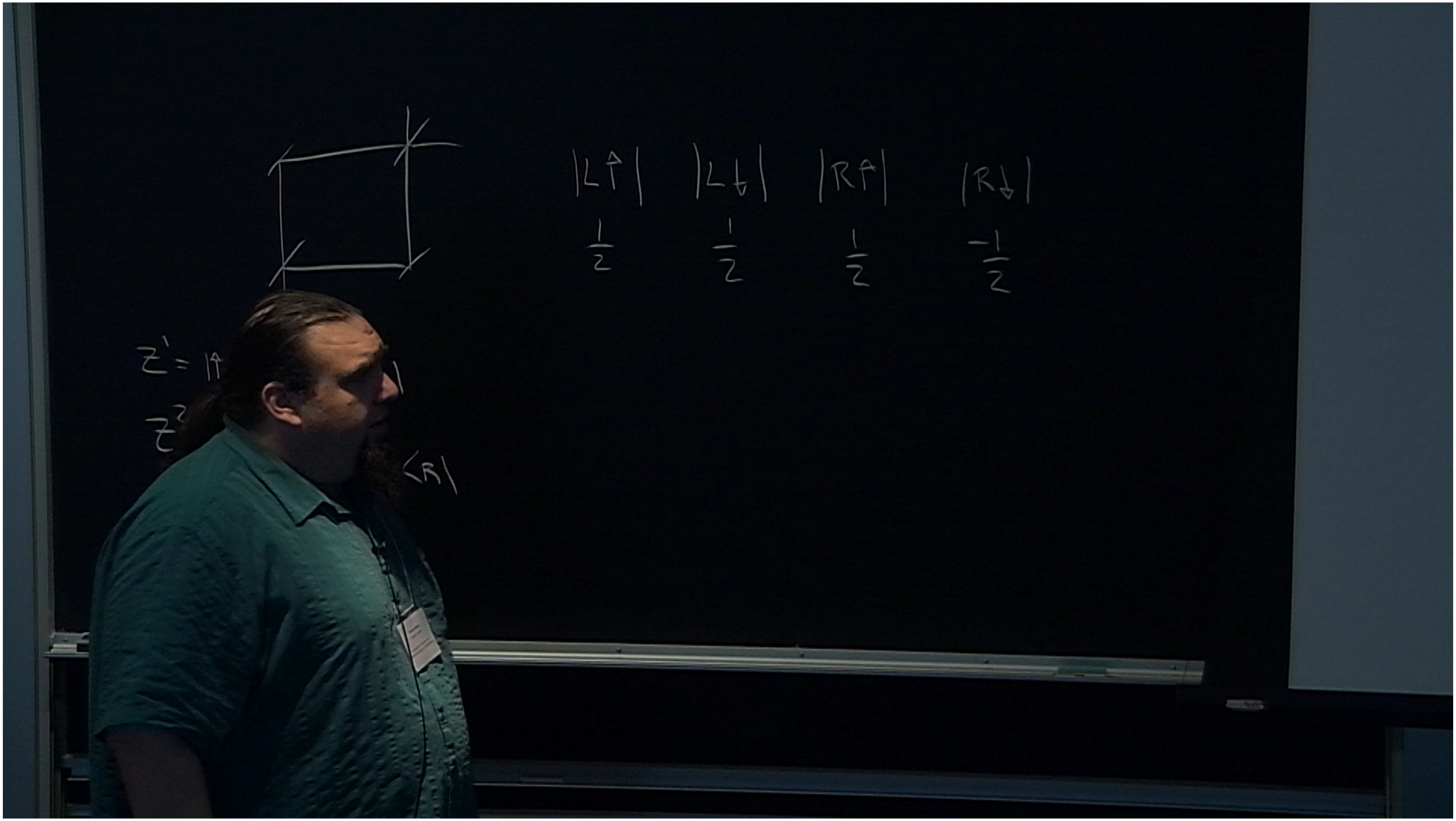
$$P_{ABL}(|2\rangle\langle 2| + |3\rangle\langle 3| = 1|B_1) = P_{ABL}(|1\rangle\langle 1| + |3\rangle\langle 3| = 1|B_2) \quad (5)$$

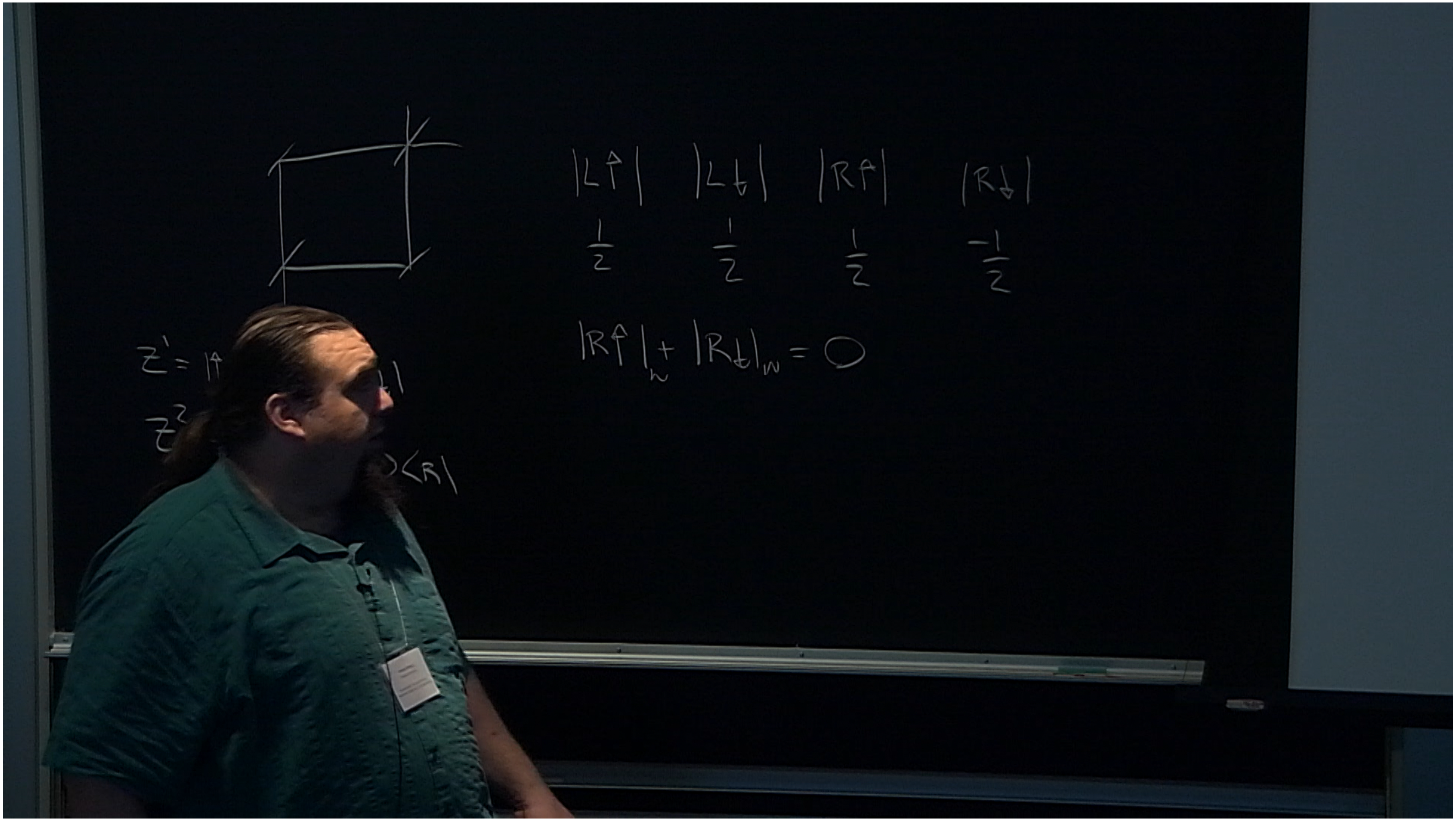
$$= P_{ABL}(|1\rangle\langle 1| + |2\rangle\langle 2| = 1|B_3) = 1,$$

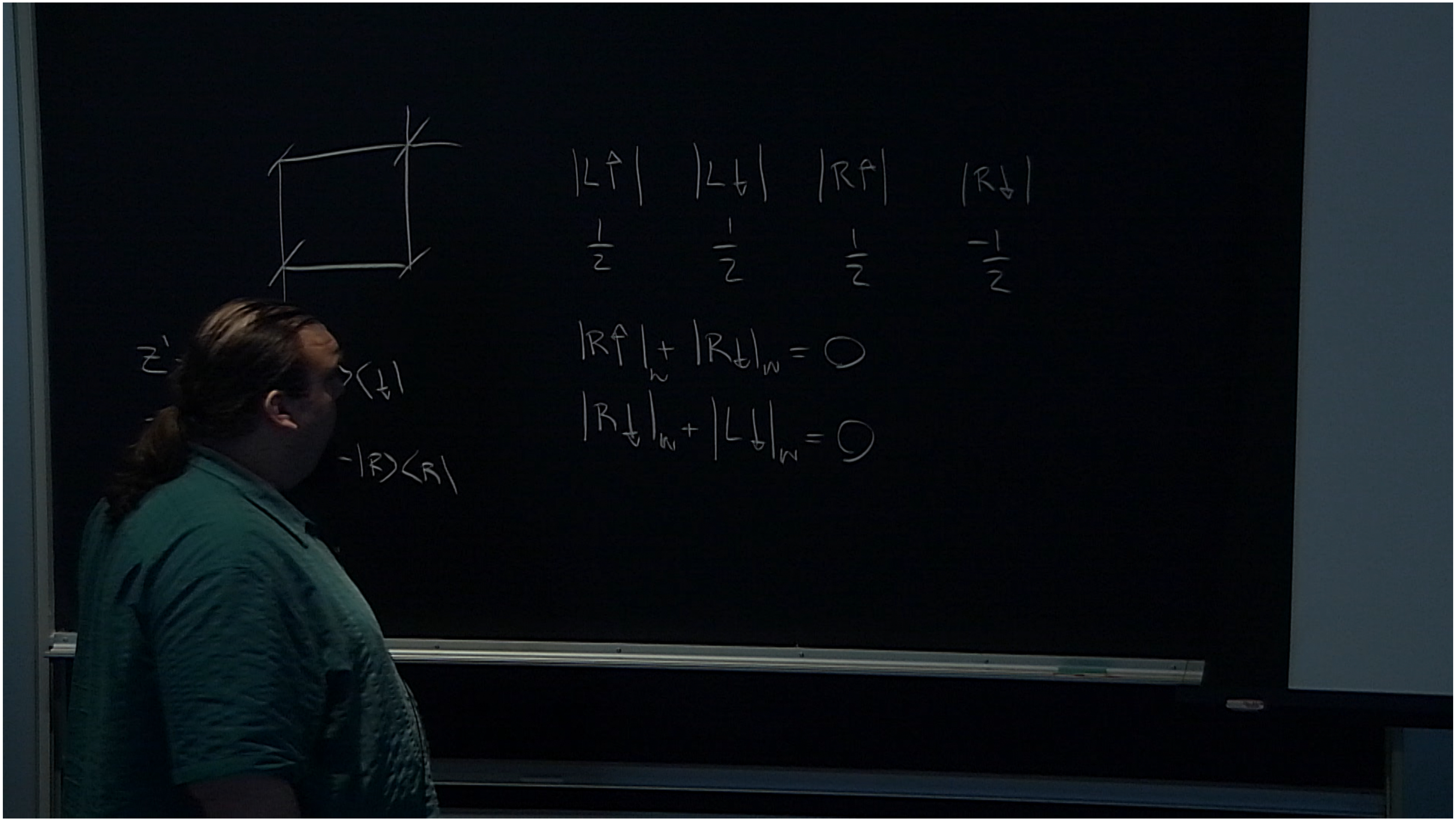
This leads to NCHVT assignments of 0 to all four of these projectors in their fine-grained basis, which again violates the quantum prediction.

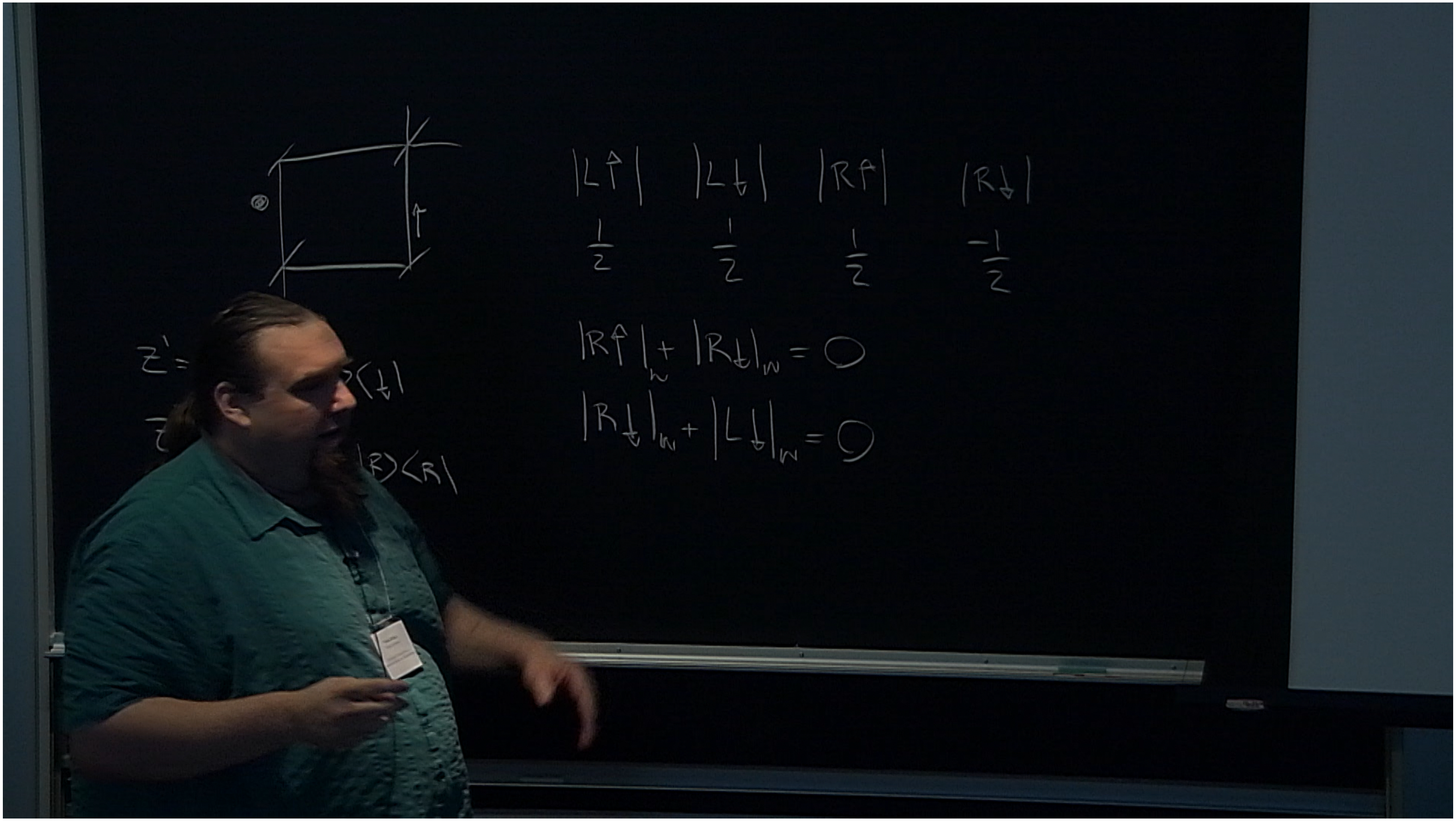



Z^1 = |↑⟩⟨↑| − |↓⟩⟨↓|
Z^2 = |L⟩⟨L| − |R⟩⟨R|


|L↑|
|L↓|
|R↑|
|R↓|
1/2
1/2
1/2
-1/2


|L↑| |L↓| |R↑| |R↓|
1/2 1/2 1/2 −1/2
|R↑|_w + |R↓|_w = 0


|L↑| |L↓| |R↑| |R↓|
1/2   1/2   1/2   -1/2
|R↑|_w + |R↓|_w = 0
|R↓|_w + |L↓|_w = 0
-|R⟩⟨R|



|L↑| |L↓| |R↑| |R↓|
1/2 1/2 1/2 -1/2
|R↑|_w + |R↓|_w = 0
|R↓|_w + |L↓|_w = 0

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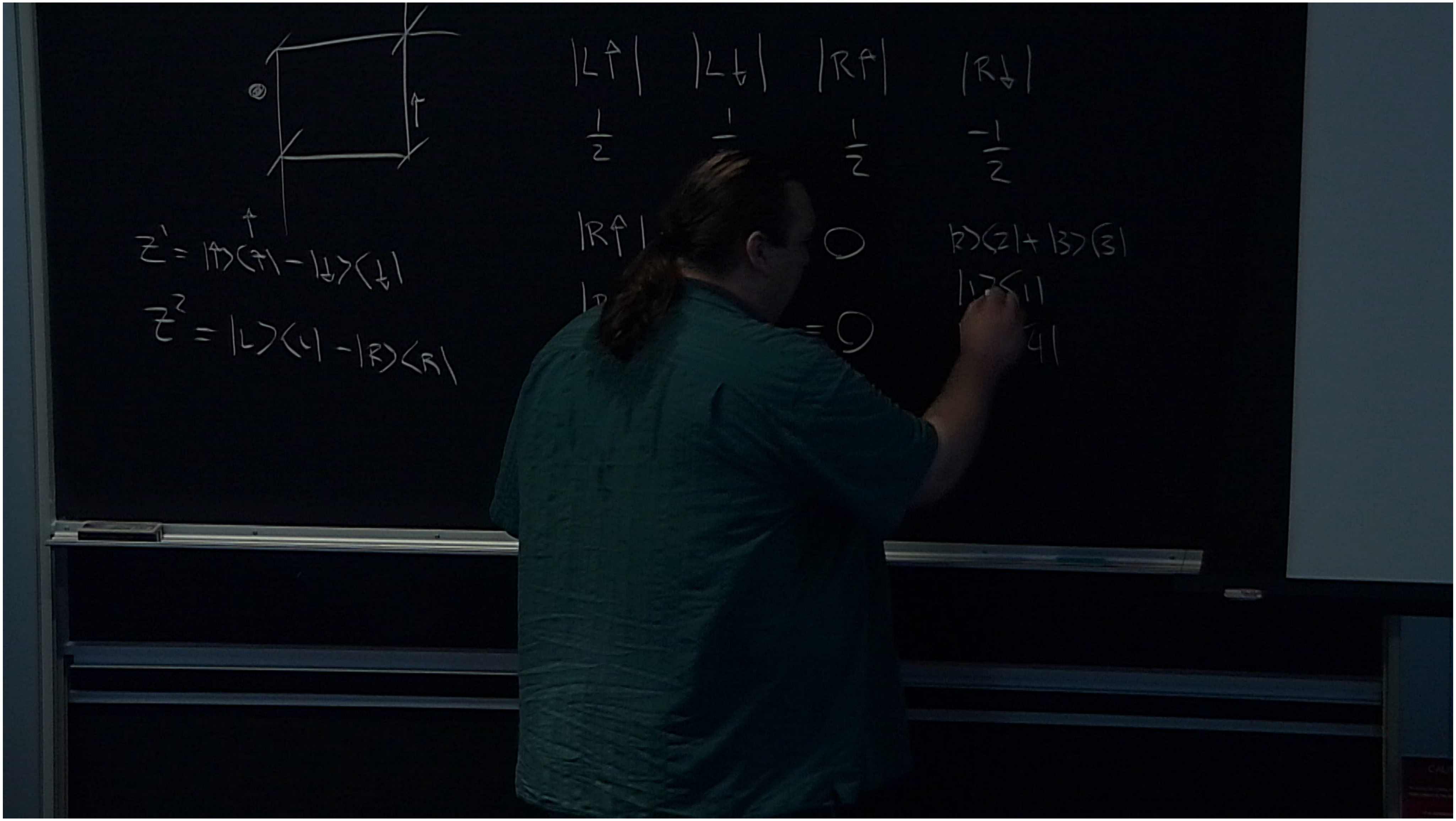
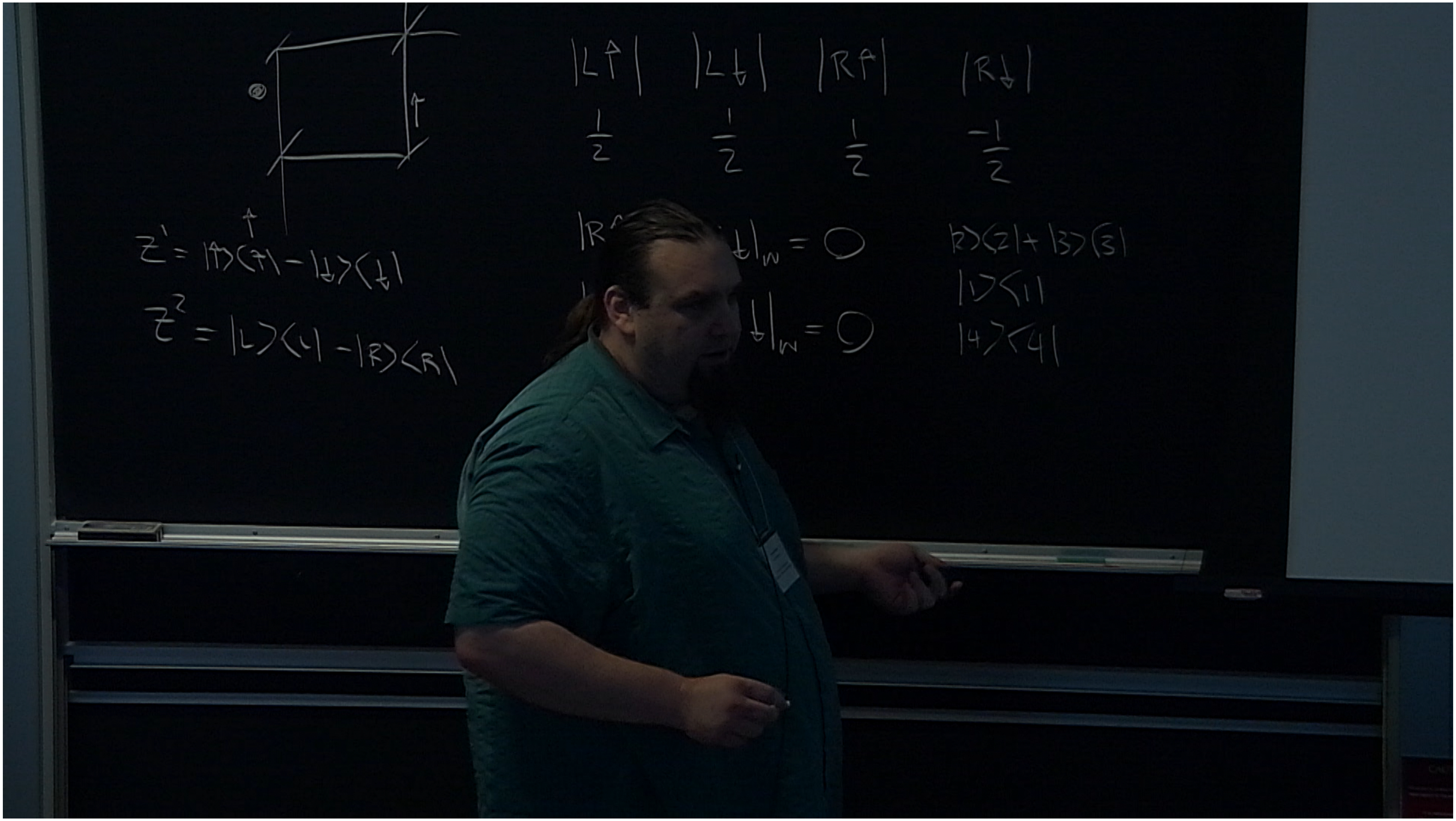

|L↑| |L↓| |R↑| |R↓|
Z¹ = |↑⟩⟨↑| − |↓⟩⟨↓|
Z² = |L⟩⟨L| − |R⟩⟨R|



|L↑| |L↓| |R↑| |R↓|
1/2 1/2 1/2 -1/2
Z¹ = |↑⟩⟨↑| − |↓⟩⟨↓|
Z² = |L⟩⟨L| − |R⟩⟨R|
|2⟩⟨2| + |3⟩⟨3|
|1⟩⟨1|
|4⟩⟨4|

Interpreting PPS-Paradoxes (1)

- In the language of contextuality, the violation of a quantum prediction gives rise to a No-Go theorem, from which the conclusion is simply that quantum mechanics does not admit a NCHVT — and the paradoxical values are physically meaningless.
- Aharonov and others take the alternate approach of attempting to give a tangible physical interpretation to the paradoxical NCHVT values themselves, leading to the conclusions like: The object is in none of the boxes in an N-box paradox, which gives rise to the pigeonhole effect, or; The mass and spin of a neutron have become spatially separated, as in the Cheshire Cat.

Interpreting PPS-Paradoxes (2)

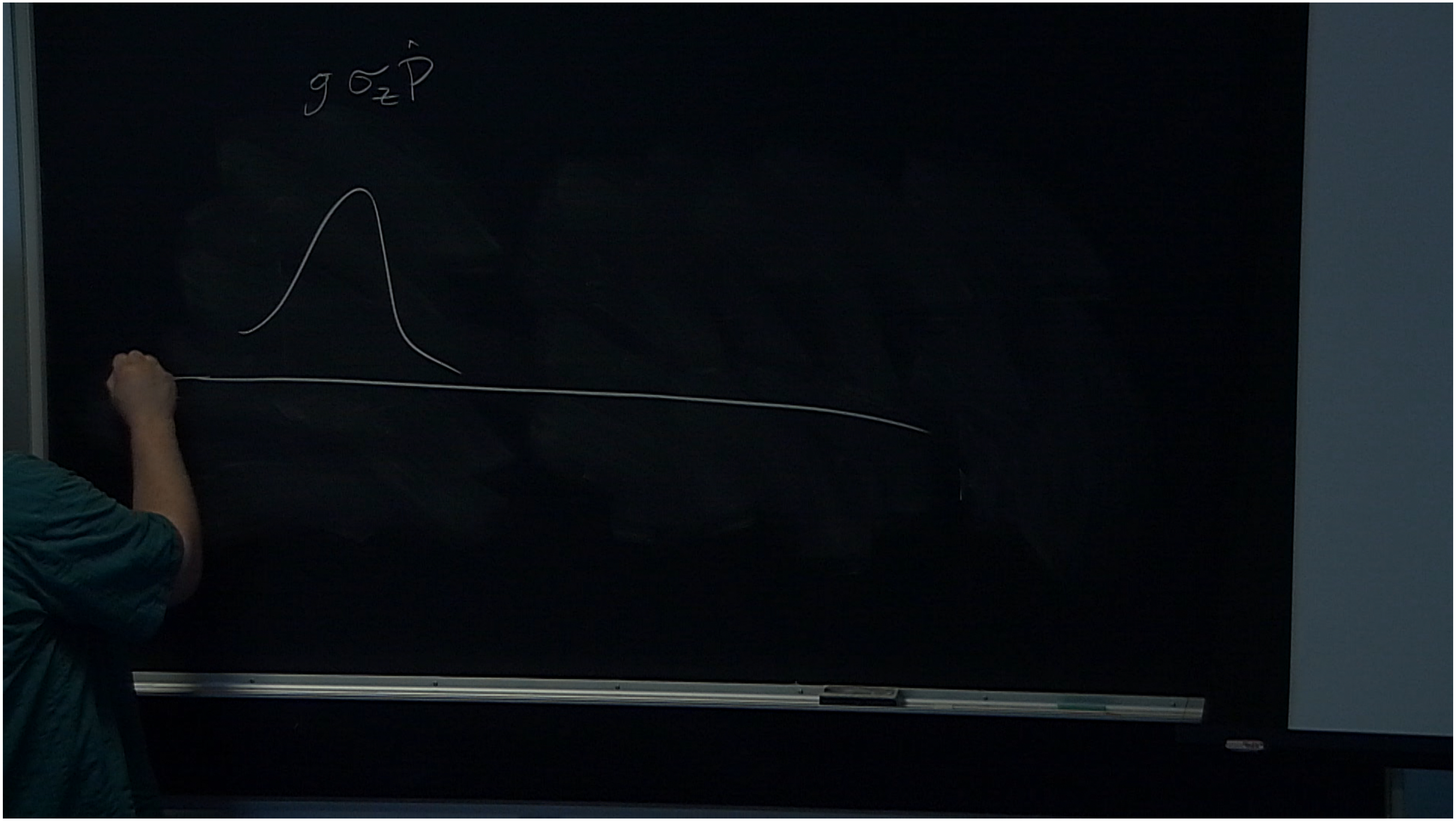
- I tend to favor the former viewpoint, since the latter is prone to internal contradictions of logic, and there is no experiment that can validate all of the NCHVT assignments.
- In fact, if one weakly measures the projectors in the contradictory context, introducing an arbitrarily small disturbance to the state, the physical values that appear are the weak values — which manifestly disagree with some predictions of the NCHVT.
- In short, the weak values are physical, the NCHVT assignments are not, and while the weak values may by exotic, they are also internally consistent.

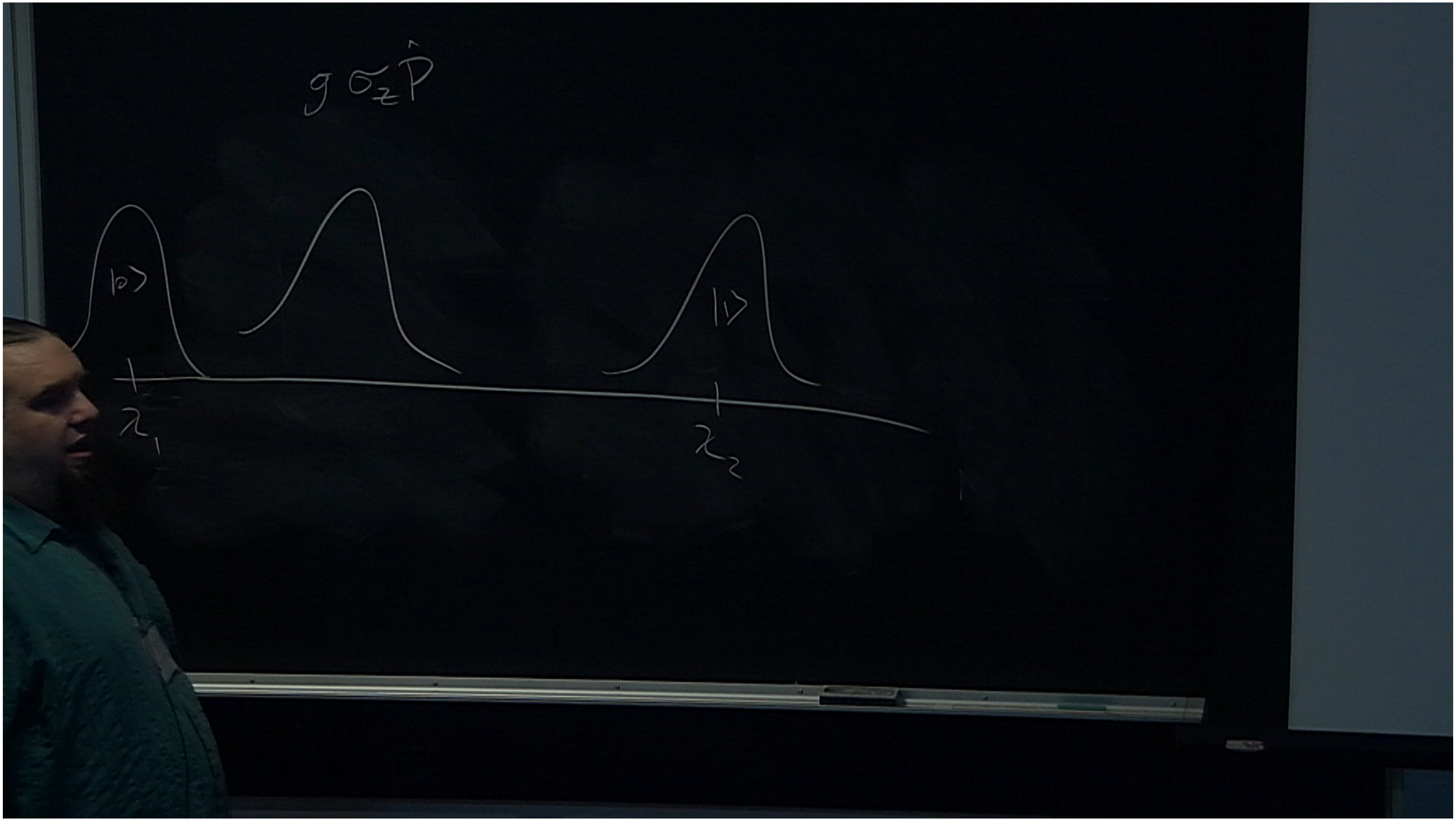
Paradoxes and Anomalous Weak Values

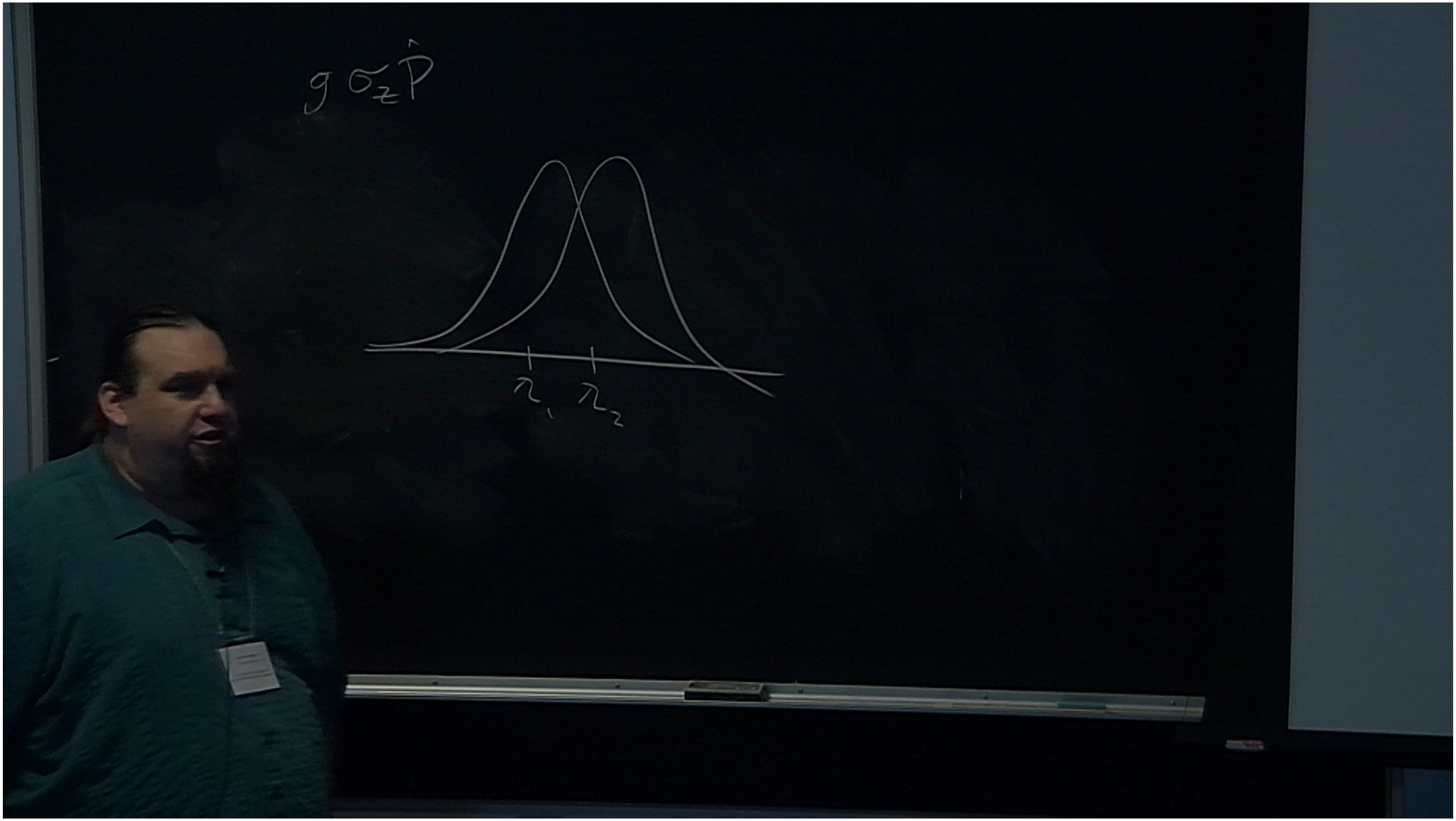
- For a projector, an *anomalous* weak value is one whose real part falls outside the range $[0, 1]$ or one with a nonzero imaginary part. No NCHVT assignment, or stochastic average of such assignments, can produce an anomalous value, and thus the presence of anomalous weak values rules out NCHVTs.
- Note that without projectors with negative weak values, it would also be impossible to show confined contextuality, or to construct any PPS-Paradox.
- For confined contextuality within BKS sets, the negative projector weak value is a necessary consequence of the algebraic structure of the BKS set itself.

Weak vs. Strong Measurement

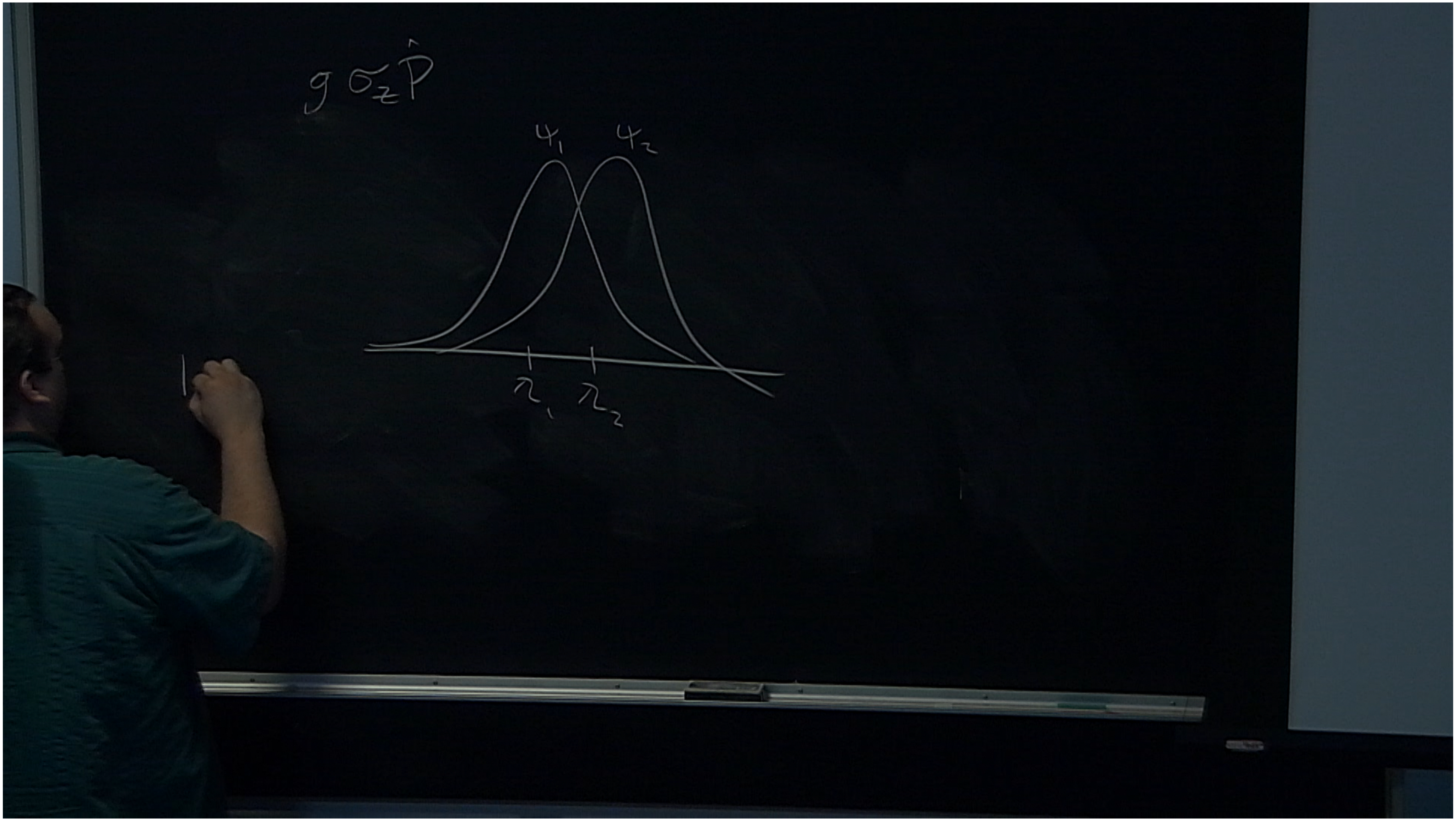
- In a strong measurement using a Gaussian pointer state $1/\sqrt{2\pi\sigma^2}e^{x^2/2\sigma^2}$, an impulsive coupling between the system and the pointer moves the pointer to one of several well-separated locations, each corresponding to an eigenvalue of the measured observable. The measurement is strong because the translation d induced by the coupling is much larger then the width of the Gaussian $d \gg \sigma$.
- In a weak measurement using the same Gaussian pointer, the impulsive coupling translates the pointer to one of several locations, but the translation d is much smaller than the width of the Gaussian $d \ll \sigma$, which results in interference between the different terms.

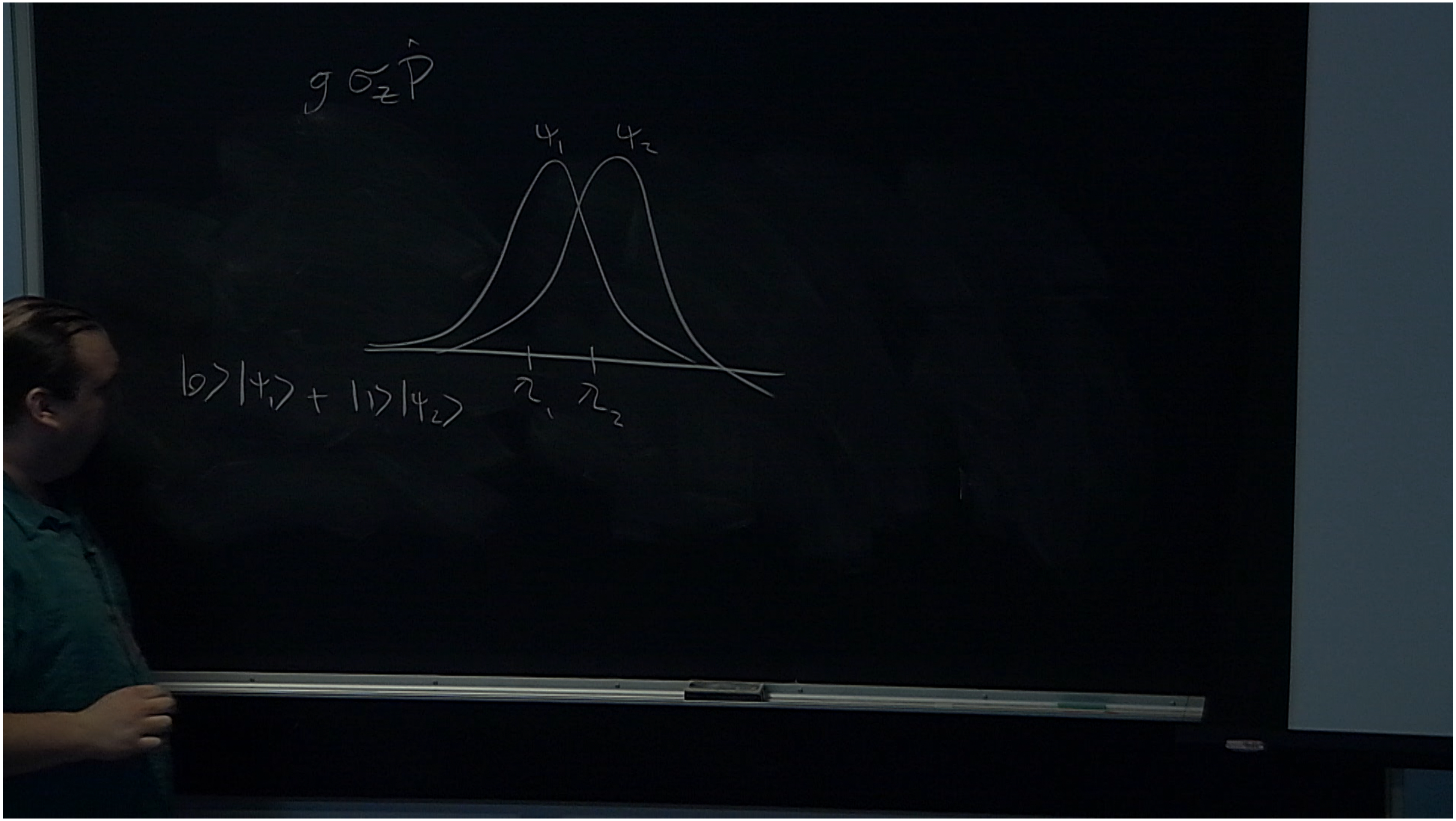

$g \sigma_z \hat{P}$


$g\sigma_z\hat{P}$
$|0\rangle$
$|1\rangle$
x_1
x_2



$g\,\sigma_z \hat{P}$
x_1
x_2

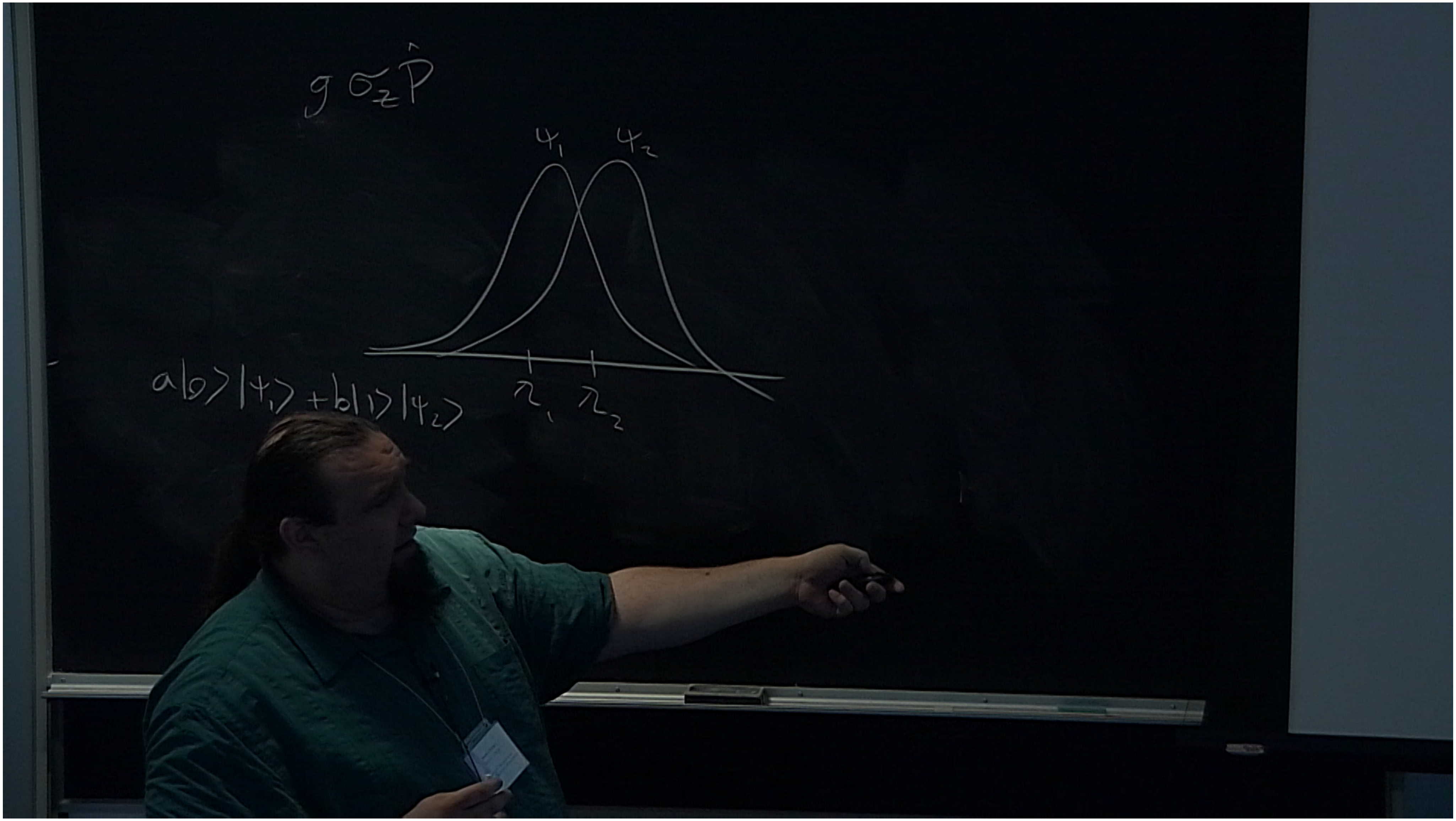

$g \sigma_z \hat{P}$
ψ_1
ψ_2
x_1
x_2

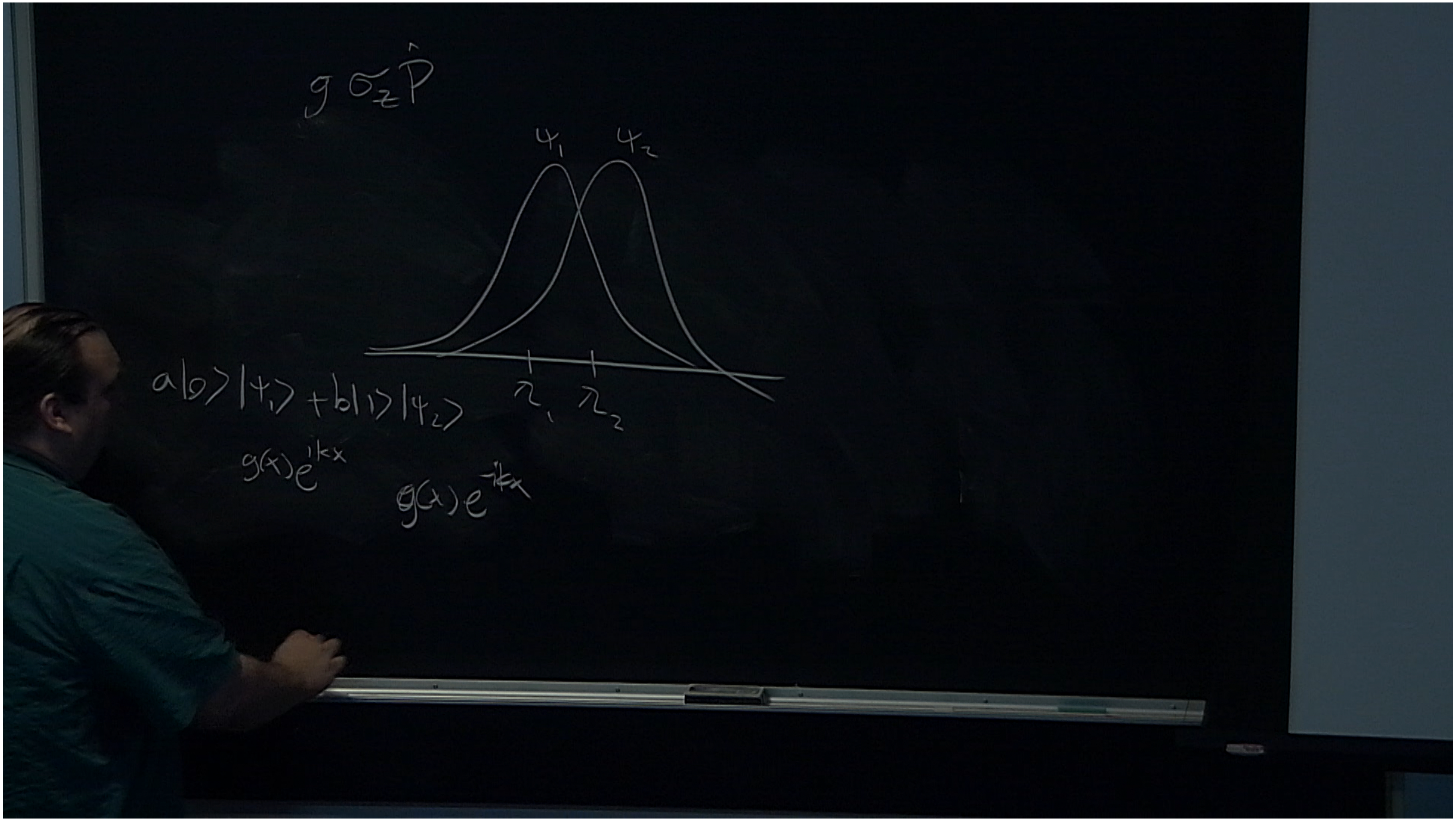


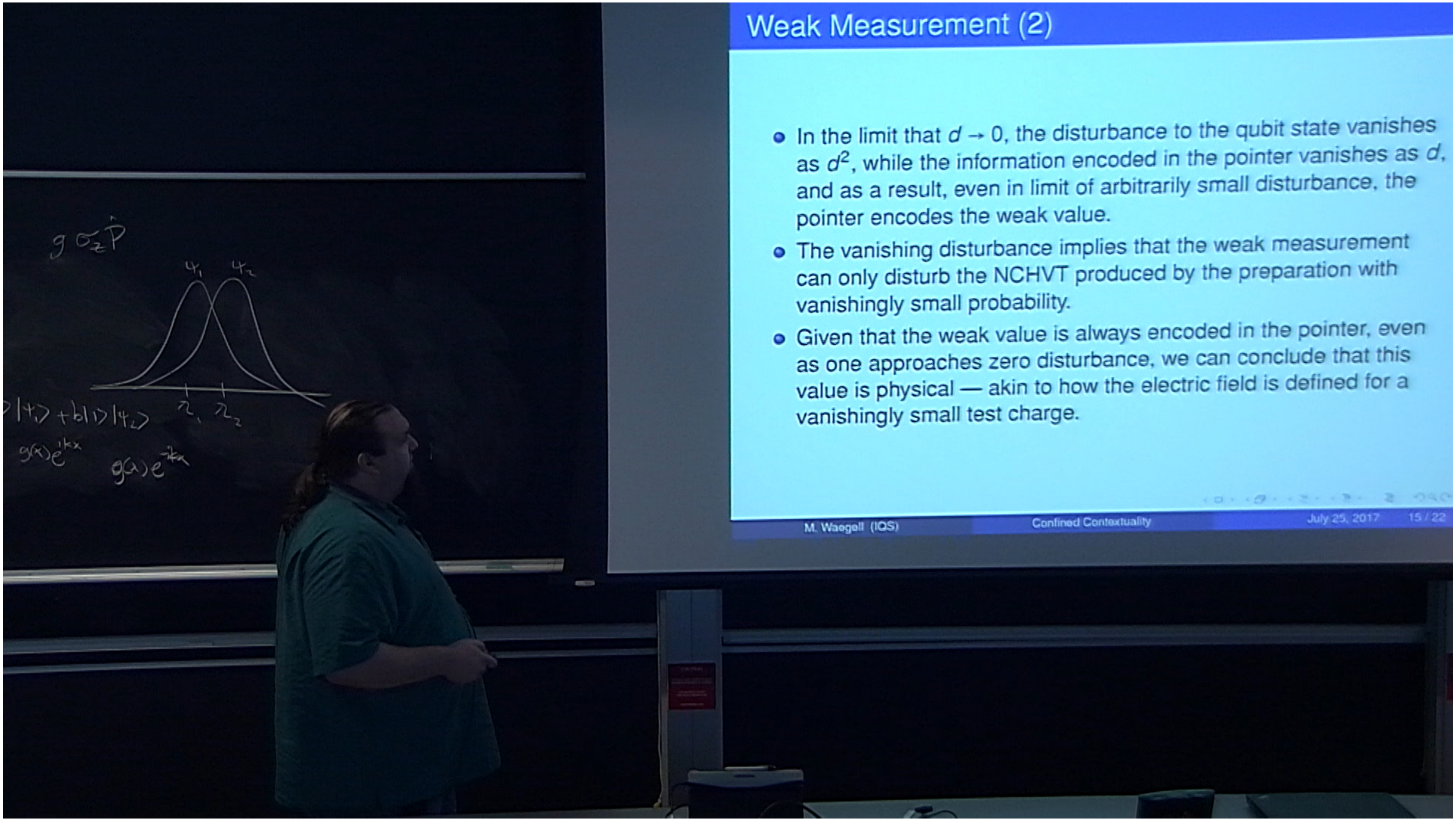
$g \sigma_z \hat{P}$
ψ_1 ψ_2
$|0\rangle|\psi_1\rangle + |1\rangle|\psi_2\rangle$
x_1 x_2

Weak Measurement (1)

- Consider an example where the spin σ_z of a qubit in the state $a|0\rangle + b|1\rangle$ is measured, resulting in the entangled state, $1/\sqrt{2\pi\sigma^2}(ae^{(x-d)^2/2\sigma^2}|0\rangle + be^{(x+d)^2/2\sigma^2}|1\rangle)$. The interference between the two Guassians is determined by the state onto which the qubit is projected after this coupling.
- Suppose the qubit is projected onto the state $f|0\rangle + g|1\rangle$. The resulting interference gives rise to $1/\sqrt{2\pi\sigma^2}(af^* e^{(x-d)^2/2\sigma^2} + bg^* e^{(x+d)^2/2\sigma^2}) \approx 1/\sqrt{2\pi\sigma^2}e^{(x-(\sigma_z)_w)^2/2\sigma^2}$, assuming $d \ll \sigma$.
- The weak value is, $(\sigma_z)_w = (af^* - bg^*)/(af^* + bg^*)$, which is fully determined by the PPS of the qubit.


$g\,\sigma_z \hat{P}$
ψ_1 ψ_2
$a|0\rangle|\psi_1\rangle + b|1\rangle|\psi_2\rangle$
x_1 x_2


g σ_z P̂
ψ_1 ψ_2
a|0>|ψ_1> + b|1>|ψ_2>
x_1 x_2
g(x)e^{ikx}
g(x)e^{-ikx}



Weak Measurement (2)
In the limit that d → 0, the disturbance to the qubit state vanishes as d2, while the information encoded in the pointer vanishes as d, and as a result, even in limit of arbitrarily small disturbance, the pointer encodes the weak value.
The vanishing disturbance implies that the weak measurement can only disturb the NCHVT produced by the preparation with vanishingly small probability.
Given that the weak value is always encoded in the pointer, even as one approaches zero disturbance, we can conclude that this value is physical — akin to how the electric field is defined for a vanishingly small test charge.
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July 25, 2017
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Weak Measurement (2)

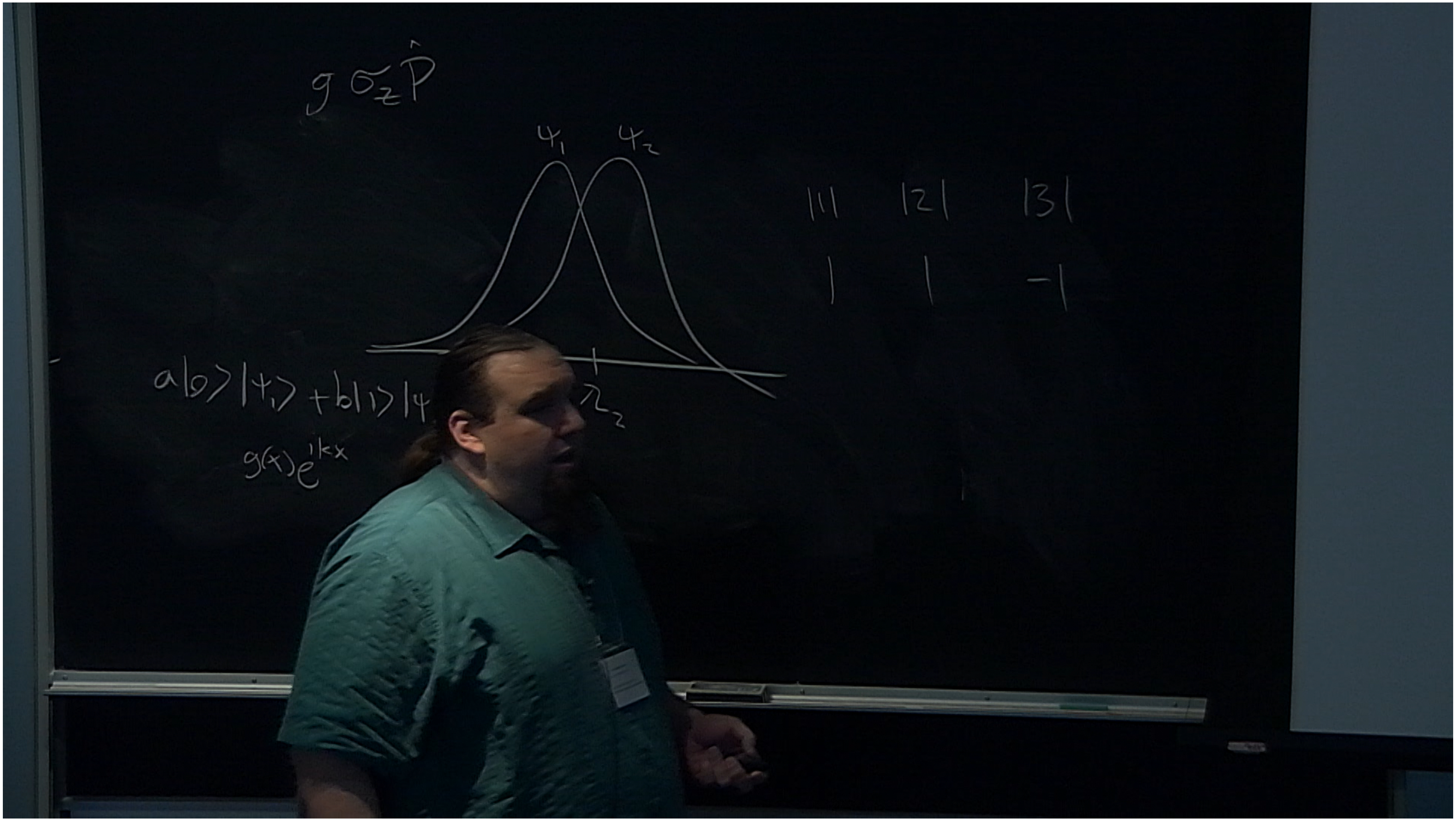
- In the limit that $d \to 0$, the disturbance to the qubit state vanishes as d^2, while the information encoded in the pointer vanishes as d, and as a result, even in limit of arbitrarily small disturbance, the pointer encodes the weak value.
- The vanishing disturbance implies that the weak measurement can only disturb the NCHVT produced by the preparation with vanishingly small probability.
- Given that the weak value is always encoded in the pointer, even as one approaches zero disturbance, we can conclude that this value is physical — akin to how the electric field is defined for a vanishingly small test charge.

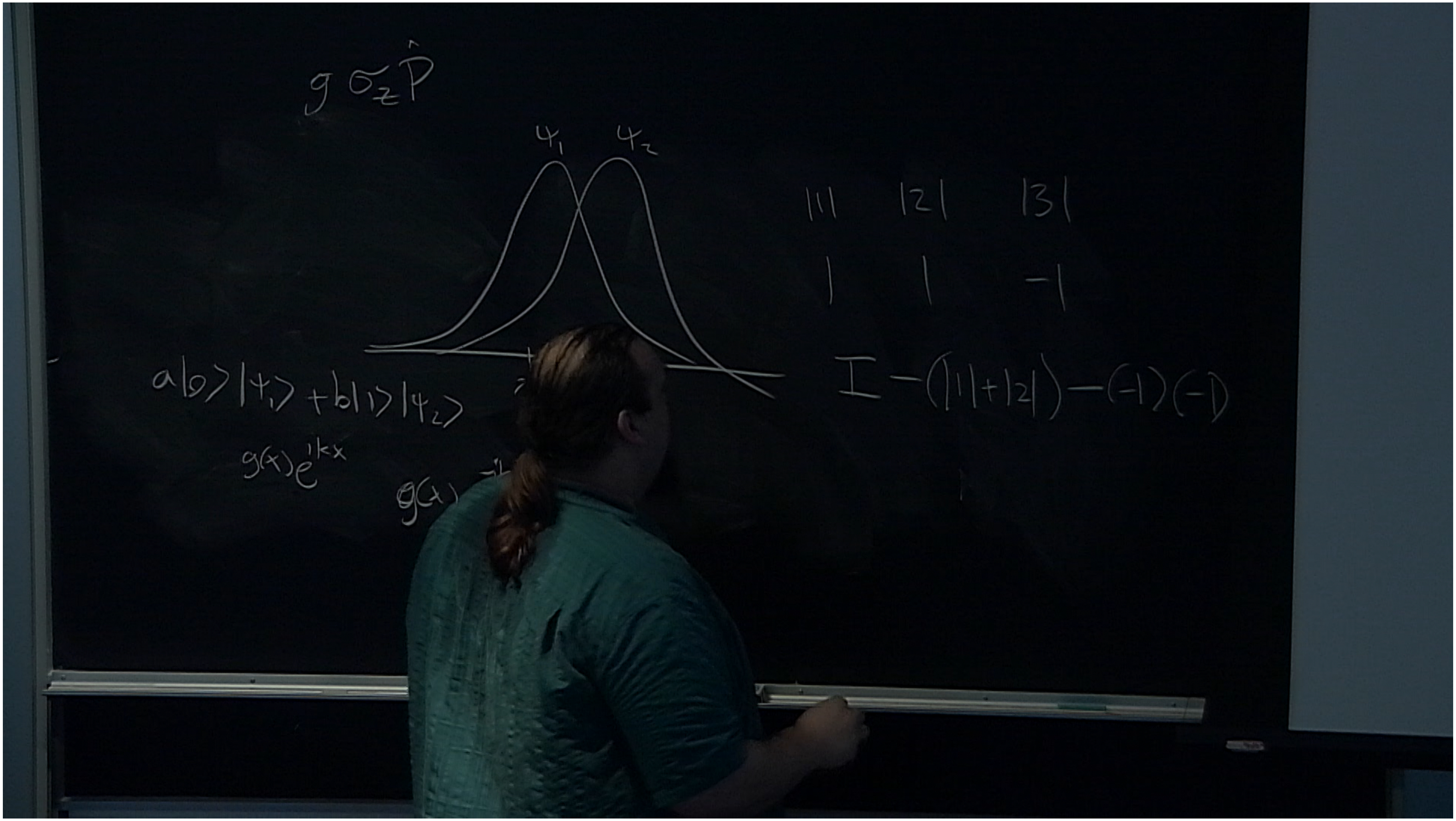
Contextuality Witness

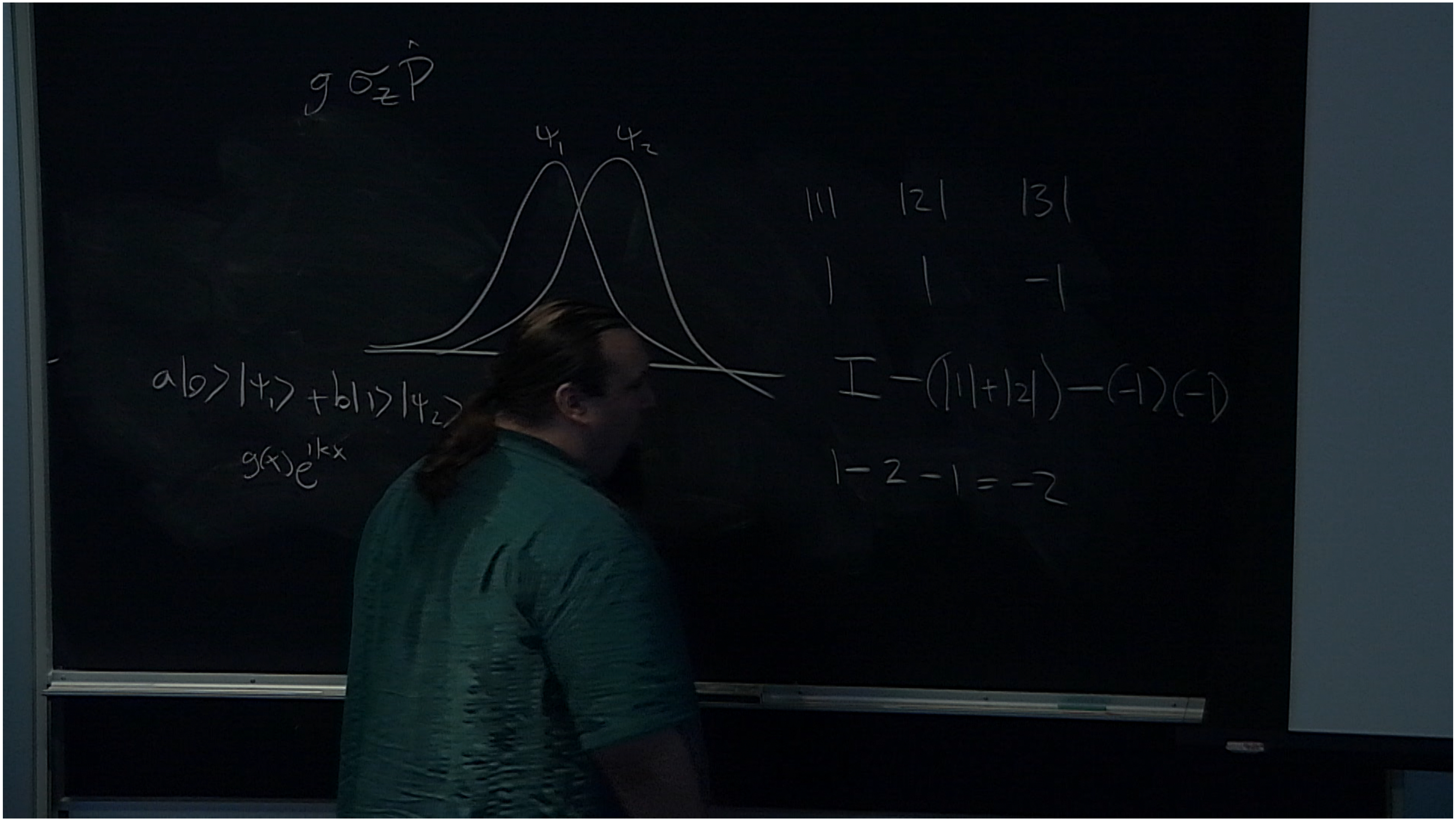
For each of the N-qubit Wheels we obtain a contradictory basis $B_C^N = \{\Pi^i\}$ containing 2^{N-1} rank-2 projectors. We construct the general contextuality observable $\hat{\mathcal{C}}^N$ for any specific choice of basis $\{\Pi^i\}$ and pre- and post-selection as,

$$\hat{\mathcal{C}}^N = I - \sum_{i=1}^{2^{N-1}} s_i \Pi^i, \tag{6}$$

with $s_i = \mathrm{sign}[\mathrm{Re}(\Pi^i_w)]$, using the theoretically predicted value of Π^i_w. Regardless of the signs s_i, if all $0 \leq \mathrm{Re}(\Pi^i_w) \leq 1$ (noncontextual), then $\mathrm{Re}(\hat{\mathcal{C}}^N_w) \geq 0$. Therefore, any $\mathrm{Re}(\hat{\mathcal{C}}^N_w) < 0$ is a proof of contextuality.

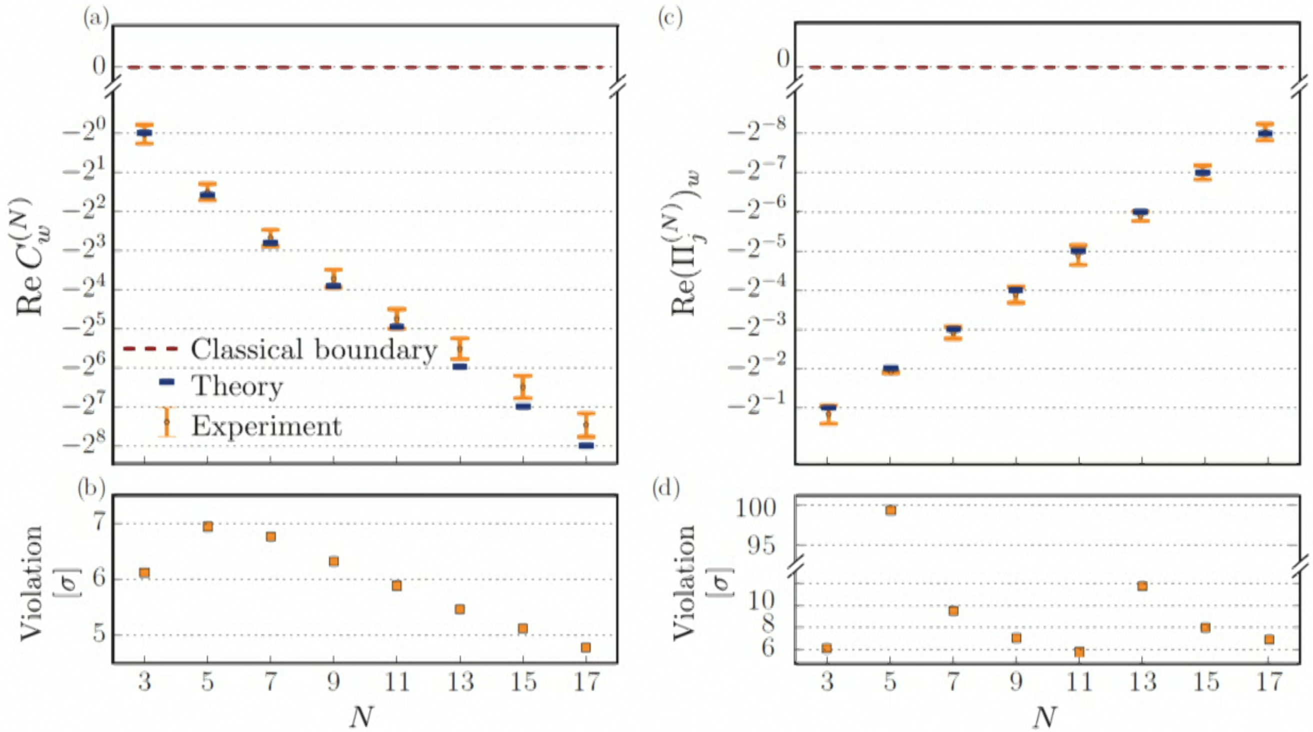

g σz P̂
ψ1 ψ2
|1| |2| |3|
1 1 −1
a|0>|ψ1> + b|1>|ψ
g(x)e^{ikx}





$g\,\sigma_z\hat{P}$
ψ_1 ψ_2
$|1|$ $|2|$ $|3|$
1 1 -1
$a|0\rangle|\psi_1\rangle + b|1\rangle|\psi_2\rangle$
$I - (|1|+|2|) - (-1)(-1)$
$g(x)e^{ikx}$
$1-2-1=-2$

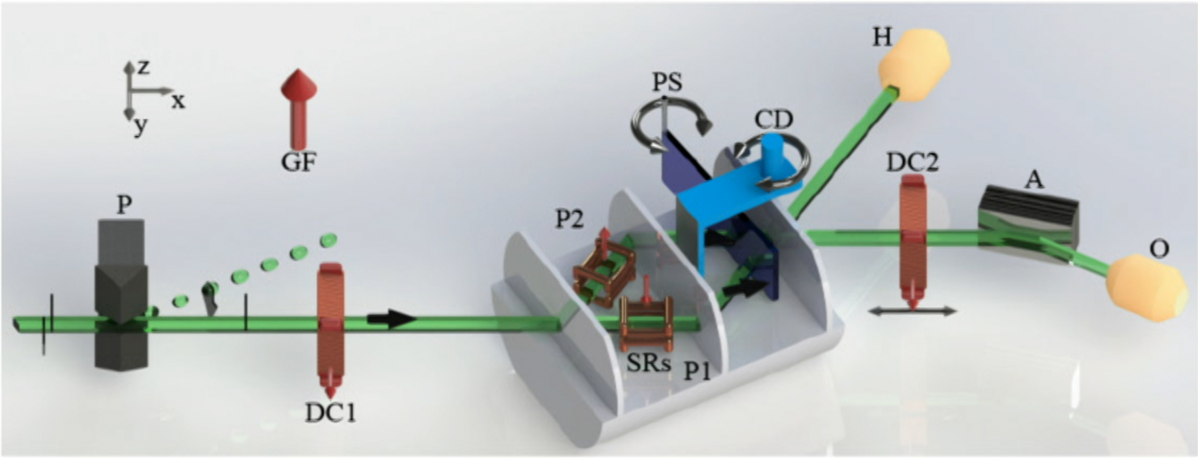
Results



(a)
(b)
(c)
(d)
Re $C_w^{(N)}$
Re$(\Pi_j^{(N)})_w$
Violation [σ]
Classical boundary
Theory
Experiment
N

Neutron Interferometer Setup

The experiment was conducted at the Institut Laue-Langevin in Grenoble France using neutron interferometry techniques. The quantum system in the experiment was the neutron spin degree of freedom, and the weak measurements were performed by coupling this to the path DOF in the interferometer.



Generalized Contextuality Witness

The confined contextuality proof used in our experiment can be generalized to a test that uses all pre- and post-selections that occur in the experimental setup, such that the complete ensemble is considered in every experiment. Let the pre-selection $|\psi_j\rangle$ be any of the $d = 2^N$ possible outcomes of an initial measurement of $\prod_j^{\otimes N} X_j$ on N qubits, and also let the post-selection $|\phi_k\rangle$ be any of the d possible outcomes of a final measurement of $\prod_k^{\otimes N} Y_k$. For each case, the Contextuality Witness can be written as,

$$\hat{\mathcal{C}}_{jk}^N = I - \sum_{i=1}^{2^{N-1}} s_{ijk} \Pi^{ijk}. \tag{7}$$

Combining the data from all runs of the experiment, we arrive at the quantity,

$$W = \sum_j^d \sum_k^d \frac{\langle \phi_k | \hat{\mathcal{C}}_{jk}^N | \psi_j \rangle}{\langle \phi_k | \psi_j \rangle} = d^2 I - \sum_{i,j,k} s_{ijk} \Pi_w^{ijk}, \tag{8}$$

and any $W < 0$ is a proof of contextuality.

Separable Weak Values

- For many examples of anomalous weak values for N qubits — including those given by our N-qubit Wheel family of BKS sets and PPS — the weak value can be factored into the tensor product of N single-qubit weak values (e.g. $(ZZ)_w = (Z)_w(Z)_w$).
- This happens whenever both the pre-selection and post-selection are product states, and the weakly-measured observable is itself factorable — which is always the case for observables in the N-qubit Pauli group, even if the contradictory context has entangled eigenstates.
- As a result, confined contextuality in the N-qubit Wheel can be witnessed using N distinct ensembles, with a single-qubit weak value extracted from each and then multiplied by the others.
- This may seem odd, but remember that quantum theory predicts the same weak value for separable PPS and observables, regardless of how the coupling is performed.

Conclusions

- Confined contextuality can be witnessed by weakly measuring anomalous weak values.
- PPS-Paradoxes should be resolved by considering the weak values, which are physical, rather than NCHVT assignments, which are not. They and are ultimately just examples of confined contextuality.
- In many cases, the anomalous weak values are separable, which greatly simplifies experimental procedures to test them.
- This is a powerful method for ruling out NCHVTs in large composite systems.

Thank You!

Thank You!