Title: Towards a mathematical theory of contextuality

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Abstract:

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Department of Computer Science, University of Oxford

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There are several current approaches, some represented at this meeting.

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Foundational significance: leading us to a deeper understanding of contextuality and its consequences.

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A **measurement scenario** is $(X, \mathcal{M}, O)$: measurement labels, sets of compatible measurements, outcomes.

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$$e = \{e_C\}_{C \in \mathcal{M}}$$

of probability distributions $e_C \in \mathrm{Prob}(O^C)$, on the joint outcomes of performing the measurements in each context.

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Non-contextuality is existence of a "global section": joint distribution $d \in \mathrm{Prob}(O^X)$ such that

$$e_C = d|_C \qquad \text{for all } C \in \mathcal{M}$$

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Bundle Pictures[2]

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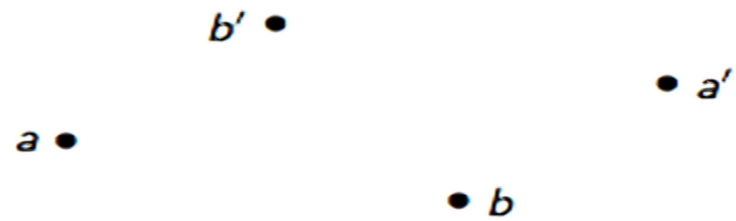

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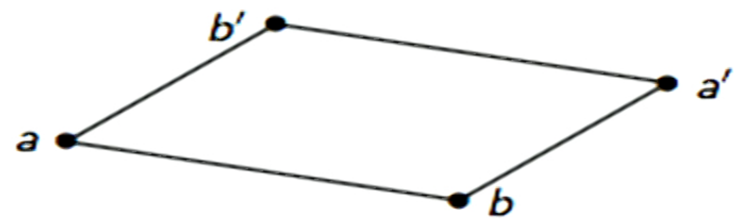

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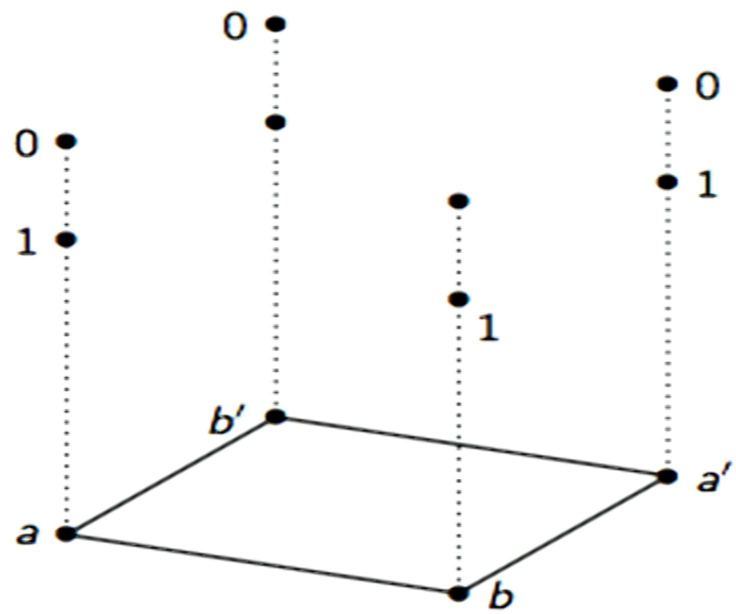

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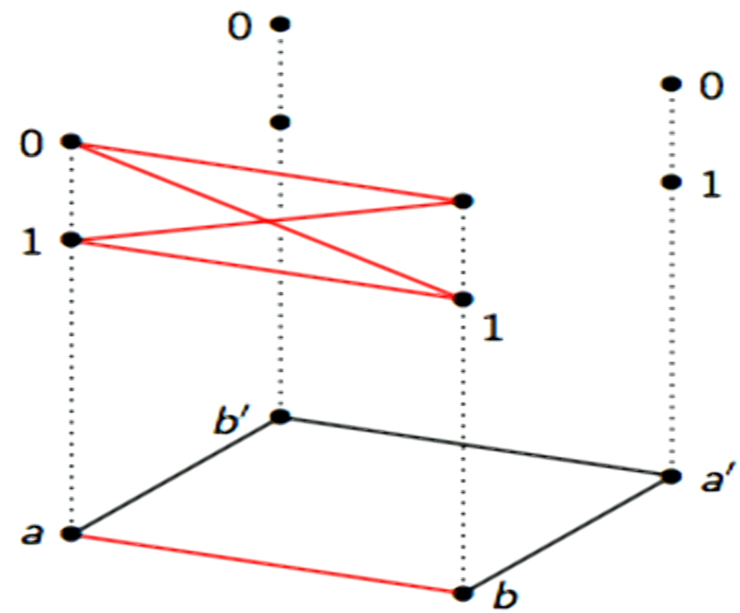

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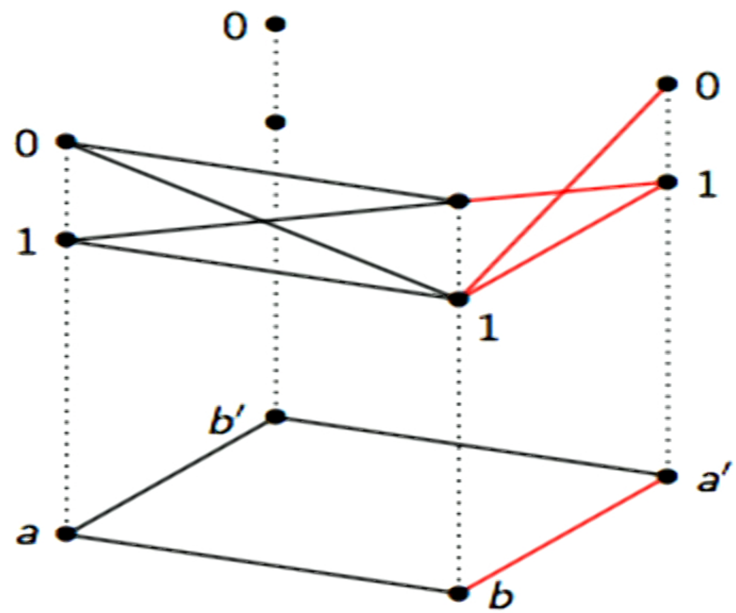

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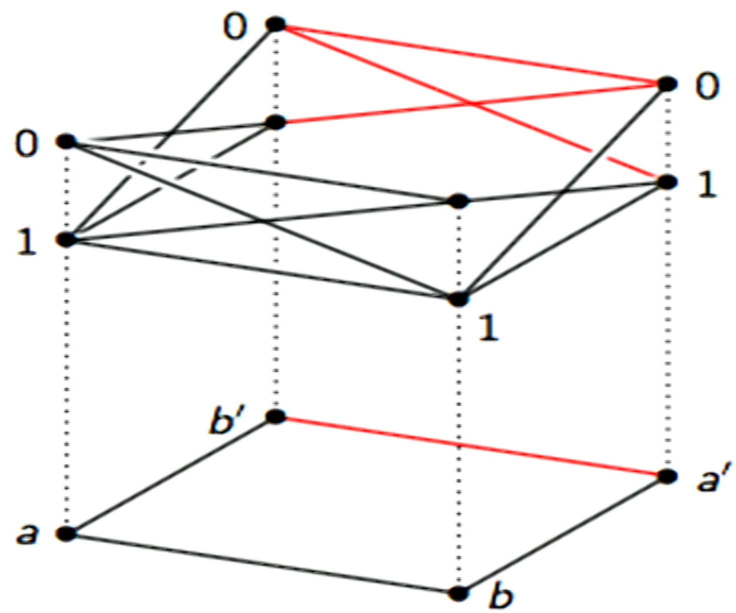

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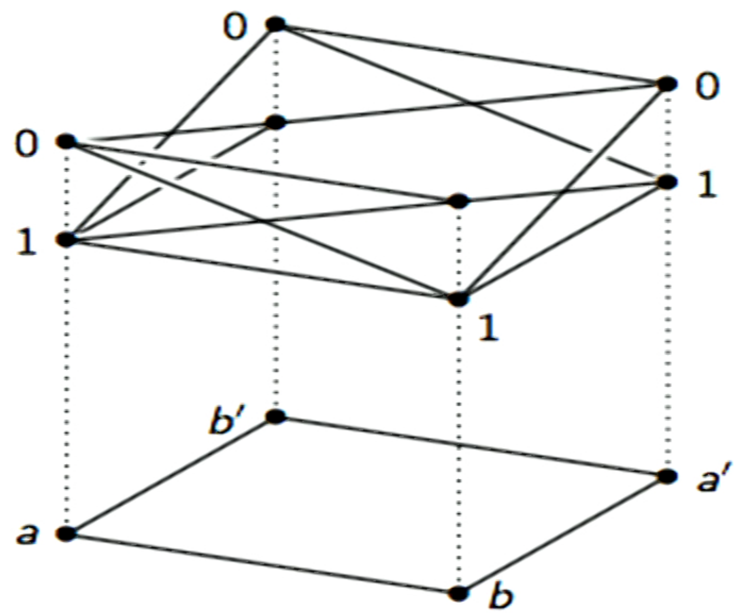

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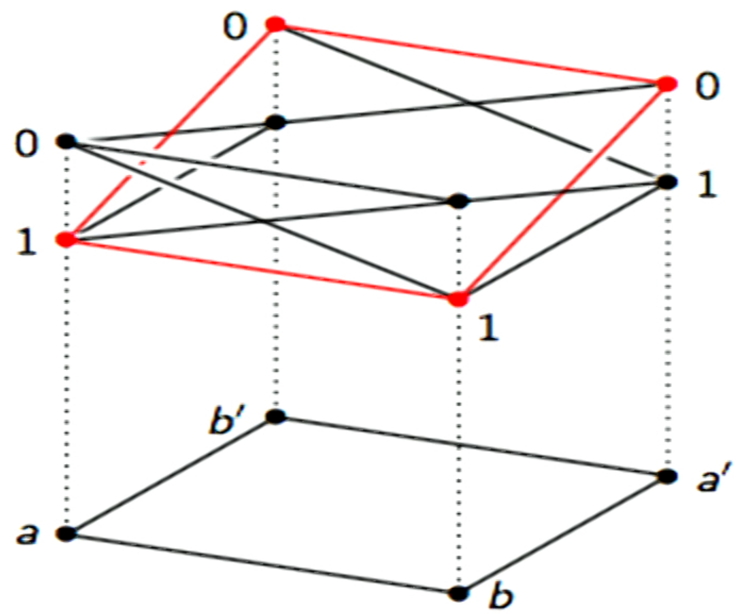

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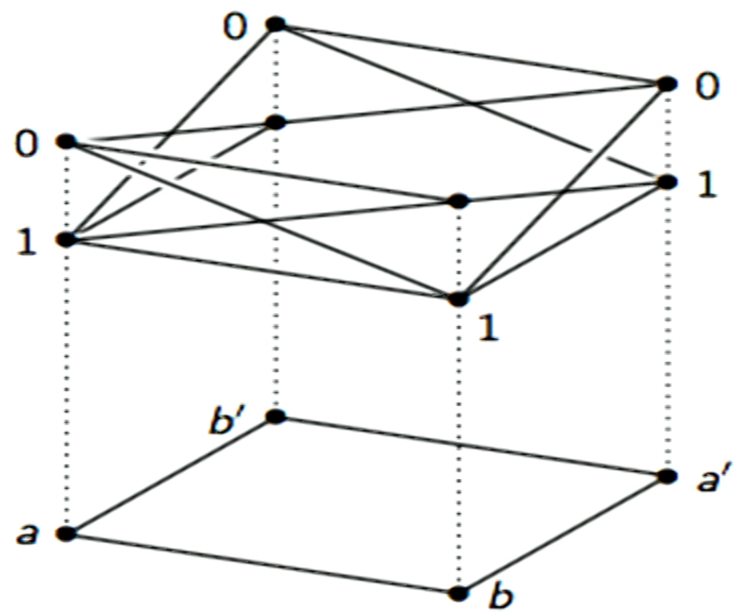

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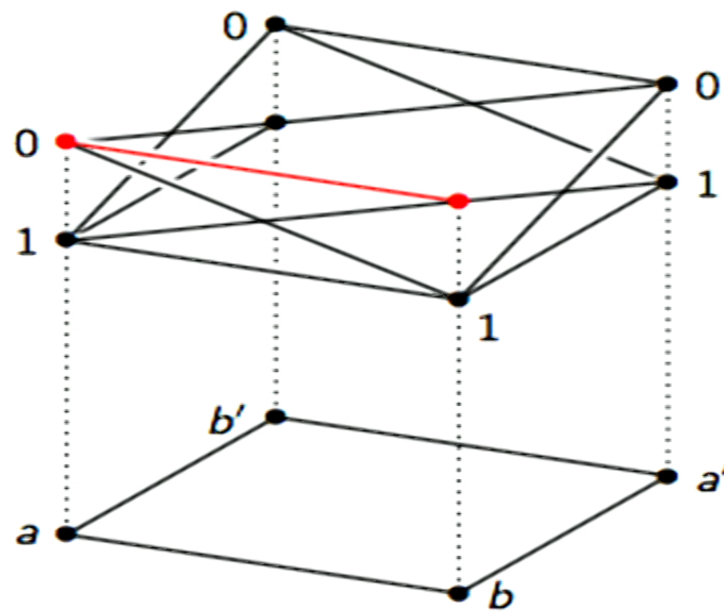

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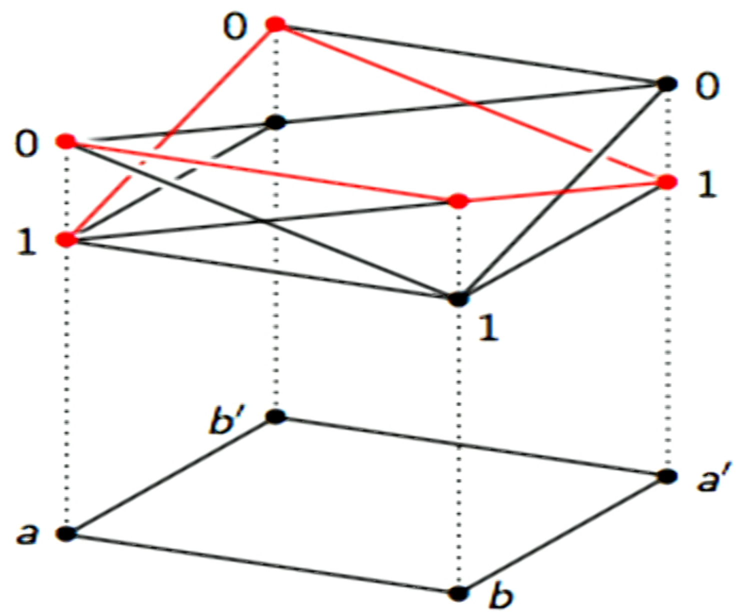

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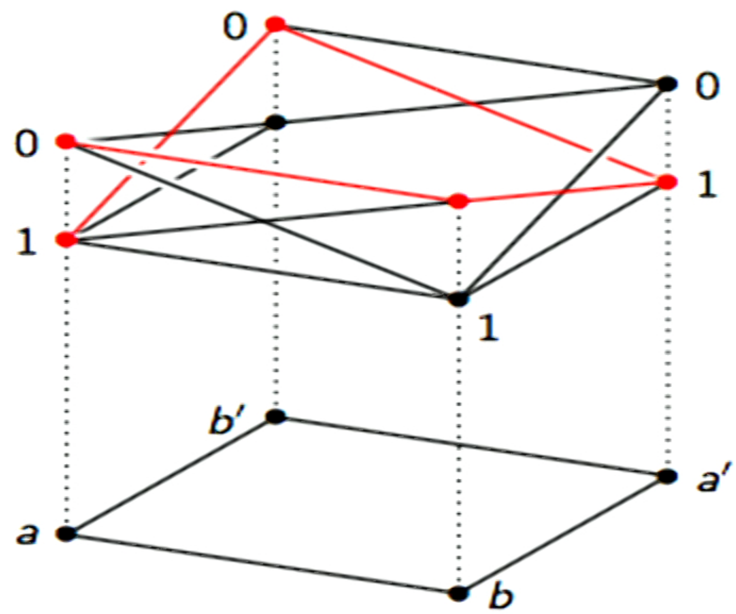

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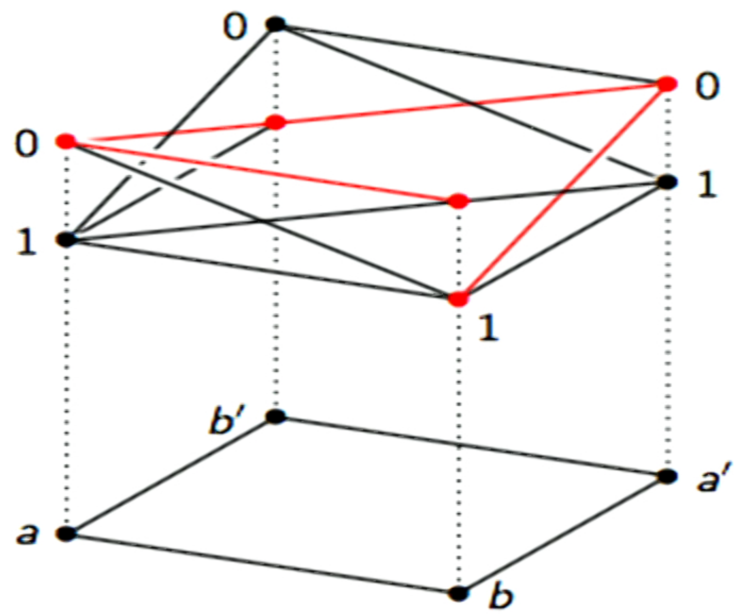

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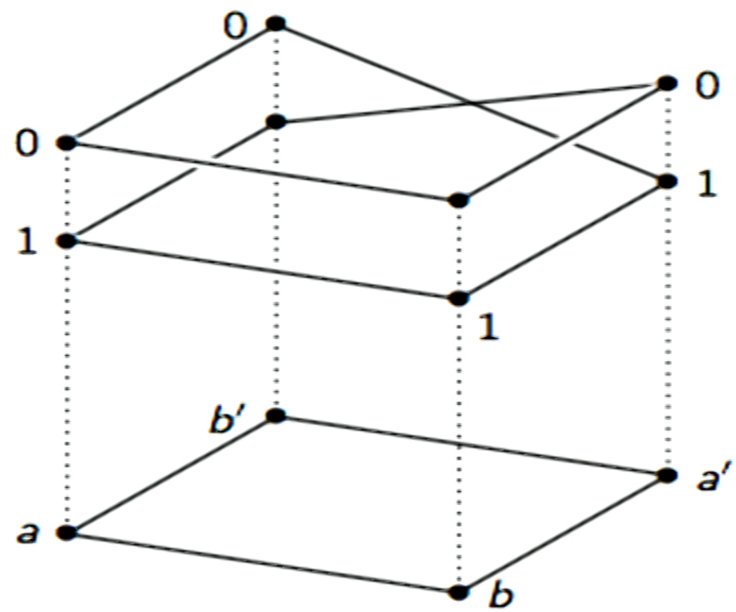

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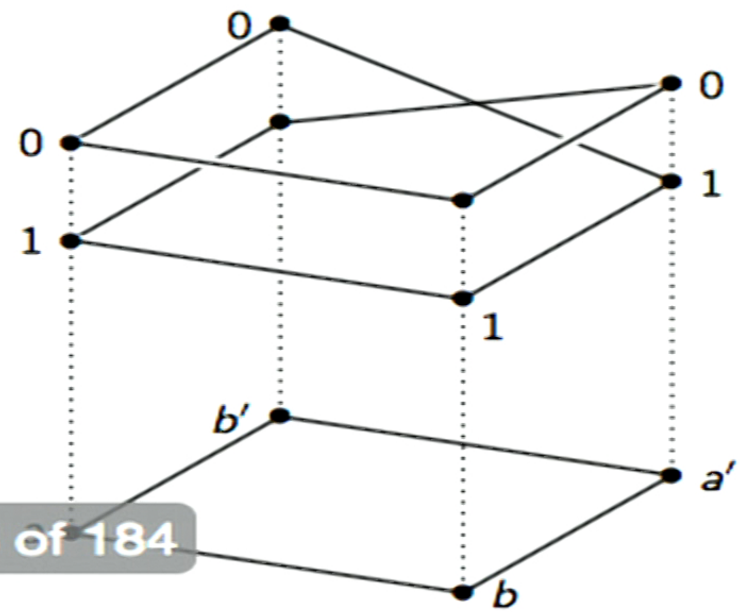

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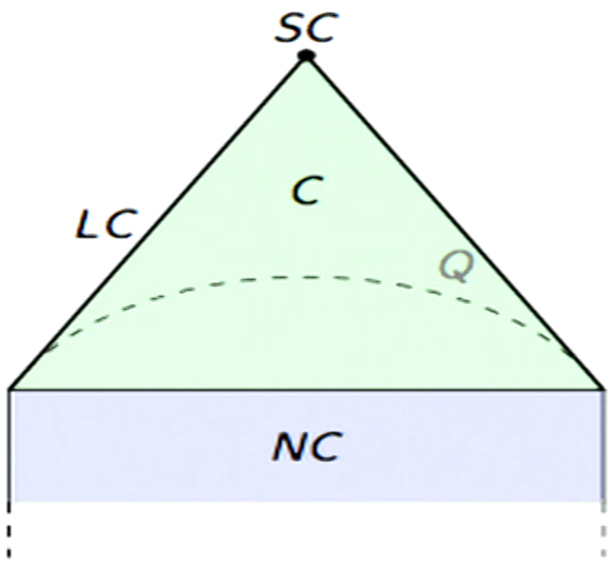

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For any given measurement scenario:

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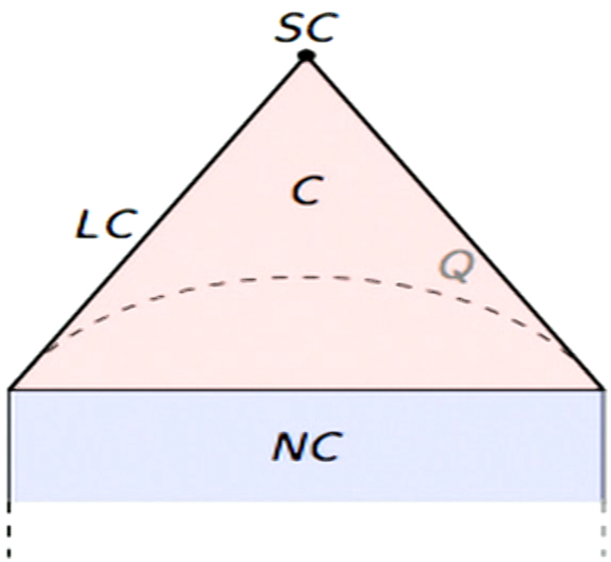

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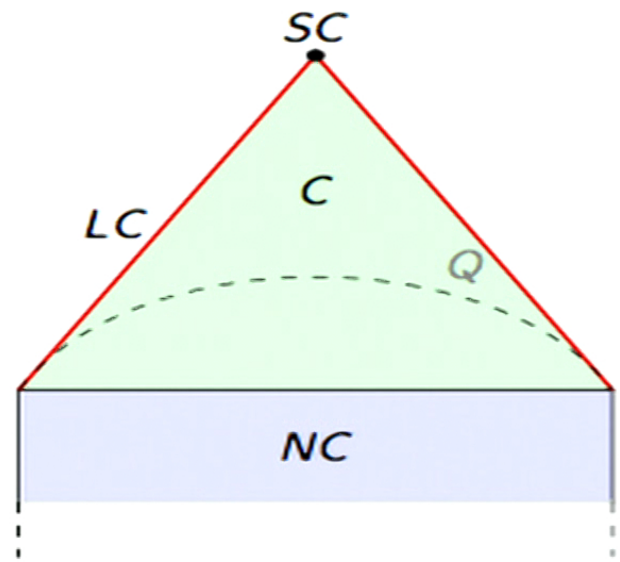

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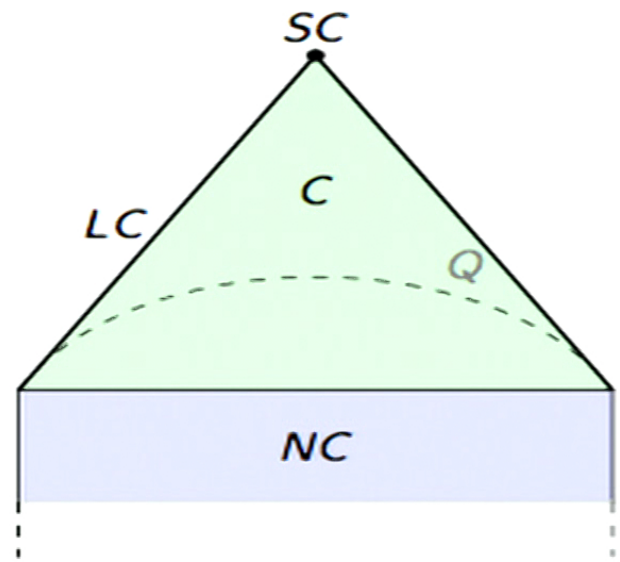

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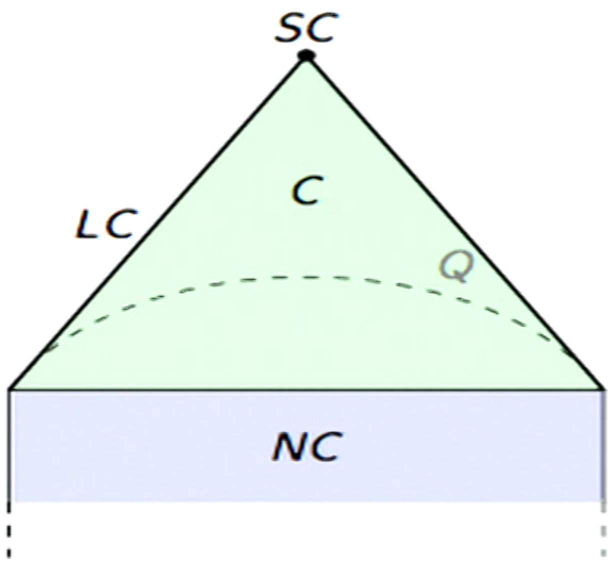

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- The quantum monad.
- Quantitative measures and resource theory for contextuality: Shane's talk.

Cohomology of contextuality[3]

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The coboundary maps are defined by

$$\delta^0((r_i))_{ij} := r_i|_{C_{ij}} - r_j|_{C_{ij}}, \qquad \delta^1((r_{ij})) := r_{ij}|_{C_{ijk}} - r_{ik}|_{C_{ijk}} + r_{jk}|_{C_{ijk}}$$

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Now the homological algebra machine can run. We can define the cohomology group in dimension i:

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If we want to fix attention on a particular local section $s_1 : C_1 \rightarrow O$, we can use **relative cohomology** to pick out those cocycles with $r_1 = s_1$.

Defining the obstruction

Now we use the cohomology machinery. From the connecting homomorphism of the long exact sequence, we can define a map γ which for each local assignment $s : C_1 \to O$ assigns an element $\gamma(s) \in H^1(\Sigma, C_1)$ in the first relative cohomology group.

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For all s, $\gamma(s) = \mathbf{0}$ iff s can be extended to a compatible family (r_i) with $r_1 = s$, i.e. to a global section. Thus if $\gamma(s) \neq \mathbf{0}$, the empirical model is logically contextual at s. If this holds for all s, the model is strongly contextual.

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Thus cohomology gives us a **computable invariant**, which provides a **sufficient condition**, and a **witness**, for contextuality.

N.B. The condition is not in general necessary: there are false positives.

For the expert

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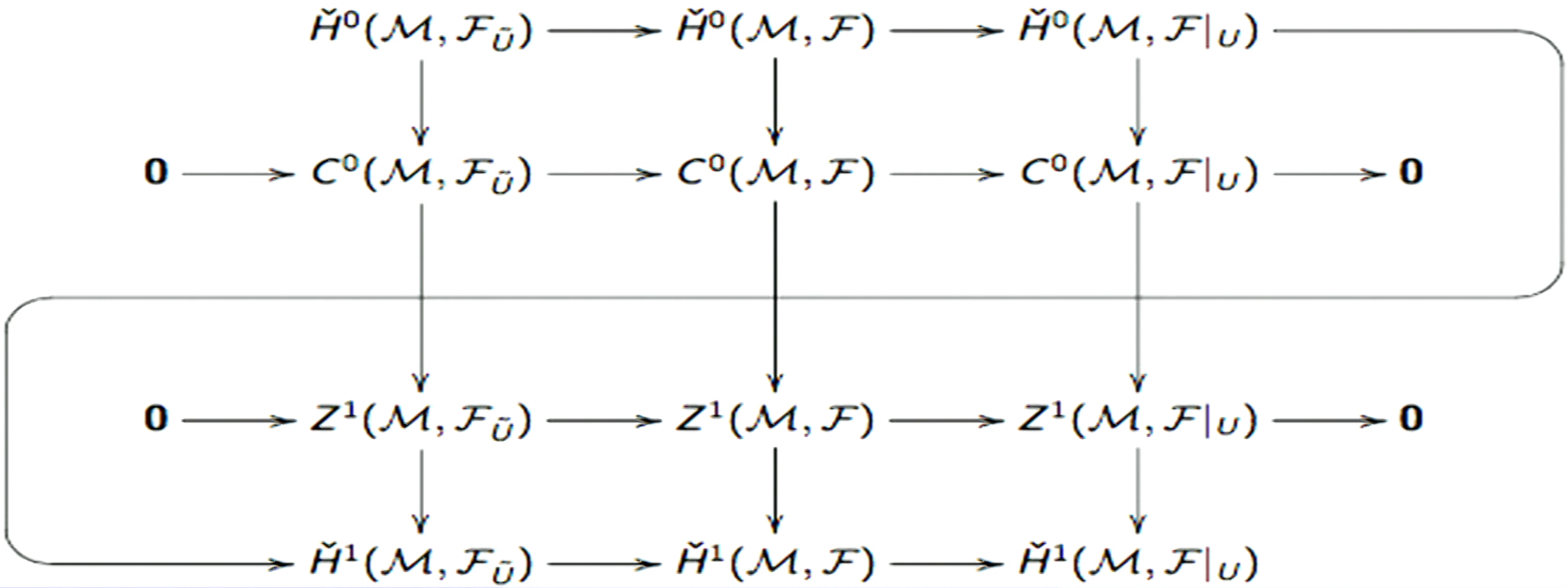
The no-signalling (compatibility) property of the model allows us to use the Snake Lemma of homological algebra to construct the connecting homomorphism:



Applications

We have computed cohomological witnesses for contextuality for many examples, including: GHZ, Kochen-Specker constructions, Peres-Mermin magic square, . . .

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We have also shown that cohomology detects contextuality for a large class of examples, including All-versus-Nothing arguments in the sense of Mermin.

$$\mathsf{AvN}_R(\Sigma) \;\Rightarrow\; \mathsf{SC}(\mathsf{Aff}\,\Sigma) \;\Rightarrow\; \mathsf{CSC}_R(\Sigma) \;\Rightarrow\; \mathsf{CSC}_{\mathbb{Z}}(\Sigma) \;\Rightarrow\; \mathsf{SC}(\Sigma)\,.$$

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- A group in Hannover are using our approach to study multipartite entanglement monogamies, with applications to the ground state problem for complex many-body quantum systems.

Characterization of state contextuality

A quantum realization of an empirical model $e : (X, \mathcal{M}, O)$ is given by a state, together with measurements corresponding to the labels in X. These jointly determine the probabilities, via the Born rule.

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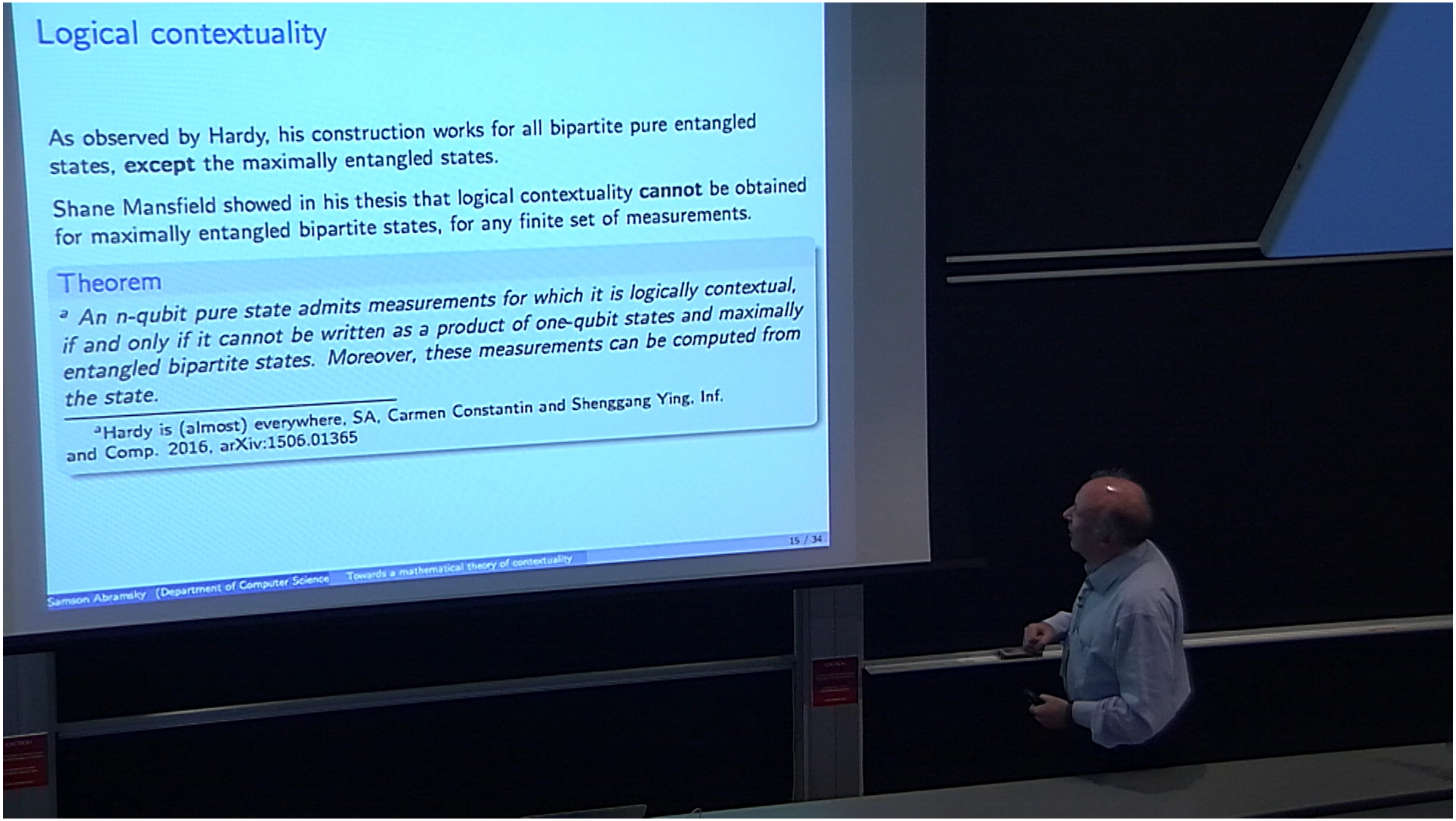
We shall briefly summarize the answers to these questions which have been obtained so far.

Logical contextuality

As observed by Hardy, his construction works for all bipartite pure entangled states, **except** the maximally entangled states.

Shane Mansfield showed in his thesis that logical contextuality **cannot** be obtained for maximally entangled bipartite states, for any finite set of measurements.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Samson Abramsky (Department of Computer Science
Towards a mathematical theory of contextuality

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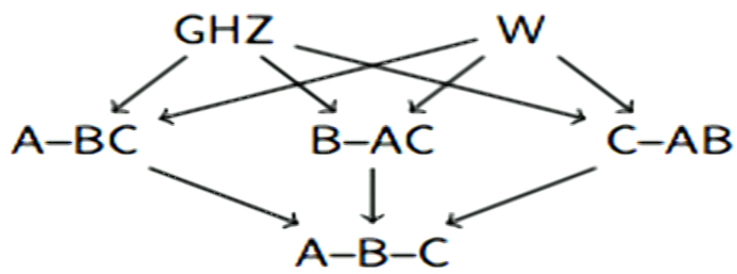
For three-qubit states, we use the Dur-Vidal-Cirac characterization of SLOCC-classes:



Theorem

[a] *Only states in the GHZ* SLOCC*-class can achieve strong contextuality with any finite set of measurements. Moreover, these states must be of a constrained form ("balanced"), and only equatorial measurements need be considered.*

[a] Minimal quantum resources for strong non-locality, SA, R. Barbosa, G. Carù, N. de Silva, K. Kishida and S. Mansfield, TQC 2017, arXiv:1705.09312

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Attenuating entanglement

We define a family $\{|\psi_n\rangle\}$ of tripartite states in distinct LU-classes within the GHZ SLOC-class. Two of the qubits are maximally entangled, and the entanglement with the third decreases with n.

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where

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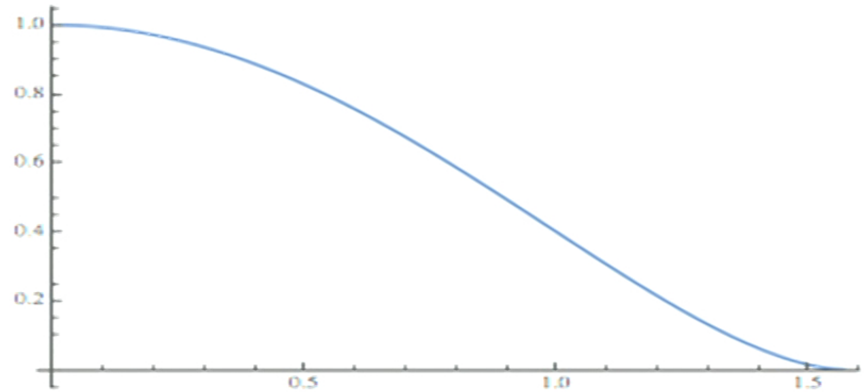
The von Neumann entanglement entropy between the first two qubits and the third as a function of λ:



Strong contextuality with weak entanglement

We define $|\psi_n\rangle := |\psi_{\lambda_n}\rangle$, where $\lambda_n := \frac{\pi}{2} - \frac{\pi}{n}$.

Theorem

[a] *For each $|\psi_n\rangle$, we can find n measurements for each qubit in the entangled pair, and a single measurement for the third qubit, such that the resulting empirical model is strongly contextual.*

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A construction by Barrett, Kent and Pironio uses essentially the same measurements on the maximally entangled bipartite state to achieve strong contextuality **in the limit**.

Our family of states, which require only weak entanglement with the third qubit to achieve full strong contextuality with a finite scenario, may be advantageous in experiments to find higher values for the **contextual fraction**.

Quantum witnesses for strong contextuality

A quantum witness for strong contextuality of an empirical model $e : (X, \mathcal{M}, O)$ is given by a state ψ, and a PVM $P_x = \{P_{x,o}\}_{o \in O}$ for each $x \in X$, such that $[P_{x,o}, P_{x',o'}] = \mathbf{0}$ whenever x and x' both occur in some $C \in \mathcal{M}$.

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These must then satisfy, for all $C \in \mathcal{M}$ and $s \in O^C$:

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Example: the GHZ state, with X and Y measurements for each party.

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Quantum witnesses for strong contextuality

A quantum witness for strong contextuality of an empirical model $e : (X, \mathcal{M}, O)$ is given by a state ψ, and a PVM $P_x = \{P_{x,o}\}_{o \in O}$ for each $x \in X$, such that $[P_{x,o}, P_{x',o'}] = \mathbf{0}$ whenever x and x' both occur in some $C \in \mathcal{M}$.

These must then satisfy, for all $C \in \mathcal{M}$ and $s \in O^C$:

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Note that in the state-independent case, we have the condition:

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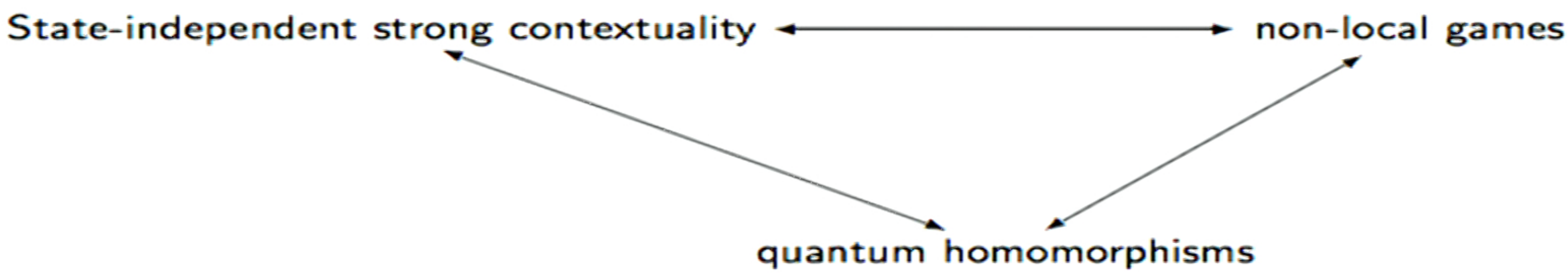
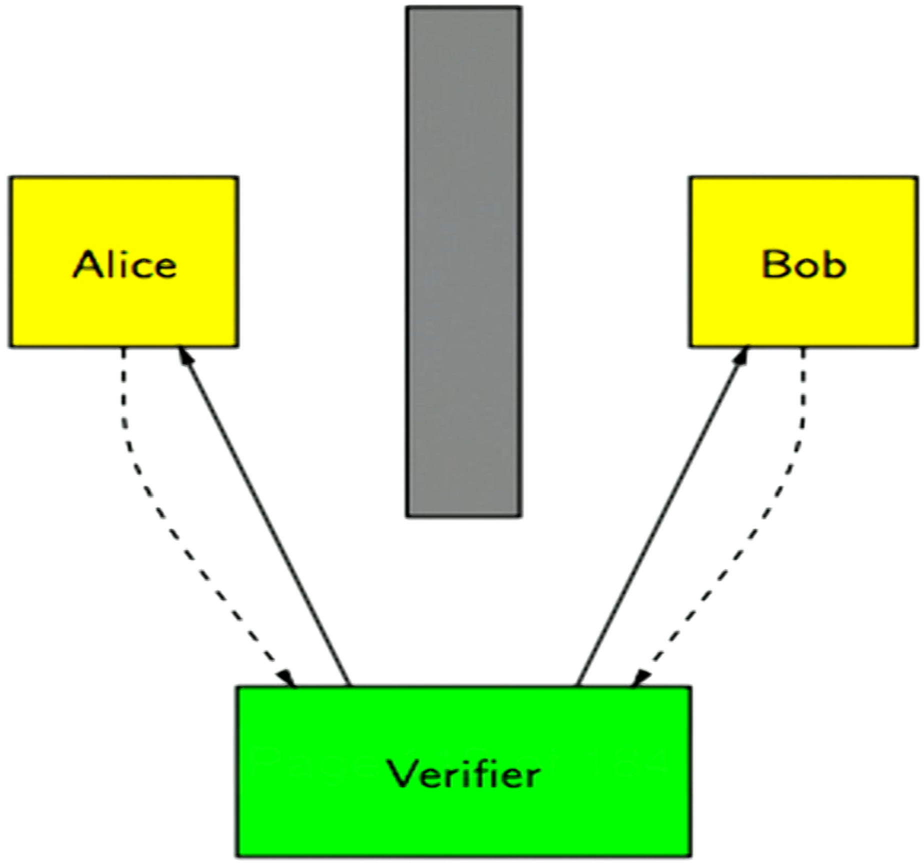
The quantum monad[5]

Exact three-way correspondence:



[5] The quantum monad on relational structures, SA, Rui Soares Barbosa, Nadish de Silva and Octavio Zapata, arXiv:1705.07310

Non-local games



The Mermin Magic Square

A	B	C
D	E	F
G	H	I

The values we can observe for these variables are 0 or 1.

We require that each row and the first two columns have even parity, and the final column has odd parity.

This translates into 6 linear equations over $\mathbb{Z}_2$:

$$A \oplus B \oplus C = 0 \qquad A \oplus D \oplus G = 0$$
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Of course, the equations are not satisfiable in $\mathbb{Z}_2$!

Alice-Bob games for binary constraint systems

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Mermin's construction shows that there is a quantum perfect strategy for the magic square.

Recent results

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Corollaries:

- There are finite systems of boolean equations which have quantum perfect strategies in infinite-dimensional Hilbert space, but not in any finite dimension.
- The question:

Given a binary constraint system, does a quantum perfect strategy exist?

  is undecidable.

Homomorphisms of relational structures

A relational vocabulary σ is a family of relation symbols R_i, each of arity $k_i \in \mathbb{N}$.

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What could it mean to quantize these fundamental structures?

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If only classical resources are allowed, the existence of a perfect strategy is equivalent to the existence of a homomorphism from $\mathcal{A}$ to $\mathcal{B}$.

Playing the homomorphism game with quantum resources

- There are finite-dimensional Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, and a pure state ψ on $\mathcal{H} \otimes \mathcal{K}$. Alice can only perform operations on $\mathcal{H}$, while Bob can only perform operations on $\mathcal{K}$.

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These resources are used as follows:

- Given a and $\mathbf{x}$, Alice measures $\mathcal{E}_{\mathbf{x}}^a$ on her part of ψ.
- Given x, Bob measures $\mathcal{F}_x$ on his part of ψ.
- They obtain the joint outcome $(\mathbf{y}, y)$ with probability $\psi^*(\mathcal{E}_{\mathbf{x},\mathbf{y}}^a \otimes \mathcal{F}_{x,y})\psi$.

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We can write the winning conditions explicitly in terms of the quantum operations:

$$
\begin{array}{lll}
\text{(QS1)} & \psi^*(\mathcal{E}_{\mathbf{x},\mathbf{y}}^a \otimes \mathcal{F}_{x,y})\psi = 0 & \text{if } x = \mathbf{x}_i \text{ and } y \neq \mathbf{y}_i \\
\text{(QS2)} & \psi^*(\mathcal{E}_{\mathbf{x},\mathbf{y}}^a \otimes I)\psi = 0 & \text{if } \mathbf{y} \notin R_a^{\mathcal{B}}.
\end{array}
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From quantum strategies to quantum homomorphisms[6]

Theorem

The existence of a quantum perfect strategy implies the existence of a strategy $(\psi, \{\mathcal{E}_{\mathbf{x}}\}, \{\mathcal{F}_x\})$ *with the following properties:*

- *The POVM's* $\mathcal{E}^i_{\mathbf{x}}$ *and* $\mathcal{F}_x$ *are projective.*

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- *If* $\mathbf{x} \in R^{\mathcal{A}}$ *and* $\mathbf{y} \notin R^{\mathcal{B}}$, *then* $\mathcal{E}_{\mathbf{x},\mathbf{y}} = \mathbf{0}$.

N.B. In passing to this special form, the dimension is **reduced;** the process by which we obtain projective measurements is not at all akin to dilation.

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This theorem shows that all the information determining the strategy is in Alice's operators. Moreover, Alice's operators must be chosen non-contextually, so that $\mathcal{E}^i_{\mathbf{x},y}$ is independent of the context $\mathbf{x}$. This means that we can define projectors $P_{x,y} := \mathcal{E}^i_{\mathbf{x},y}$ whenever $x = \mathbf{x}_i$.

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Quantum homomorphisms

A quantum homomorphism between relational structures $\mathcal{A}$ and $\mathcal{B}$ is a family of projectors $\{P_{x,y}\}_{x \in A, y \in B}$ in Proj(d) for some d, satisfying the following conditions:

- (QH1) For all $x \in A$, $\sum_{y \in B} P_{x,y} = I$.
- (QH2) For all $\mathbf{x} \in R^{\mathcal{A}}$, $x = \mathbf{x}_i$, $x' = \mathbf{x}_j$, and $y, y' \in B$, $[P_{x,y}, P_{x',y'}] = \mathbf{0}$. Thus we can define a projective measurement $P_{\mathbf{x}} = \{P_{\mathbf{x},\mathbf{y}}\}_{\mathbf{y}}$, where $P_{\mathbf{x},\mathbf{y}} := P_{x_1,y_1} \cdots P_{x_k,y_k}$.
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We write $\mathcal{A} \xrightarrow{q} \mathcal{B}$ for the existence of a quantum homomorphism from $\mathcal{A}$ to $\mathcal{B}$.

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For finite structures $\mathcal{A}$, $\mathcal{B}$, the following are equivalent:

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- (QH2) For all $\mathbf{x} \in R^{\mathcal{A}}$, $x = \mathbf{x}_i$, $x' = \mathbf{x}_j$, and $y, y' \in B$, $[P_{x,y}, P_{x',y'}] = \mathbf{0}$. Thus we can define a projective measurement $P_{\mathbf{x}} = \{P_{\mathbf{x},\mathbf{y}}\}_{\mathbf{y}}$, where $P_{\mathbf{x},\mathbf{y}} := P_{x_1,y_1} \cdots P_{x_k,y_k}$.
- (QH3) If $\mathbf{x} \in R^{\mathcal{A}}$ and $\mathbf{y} \notin R^{\mathcal{B}}$, then $P_{\mathbf{x},\mathbf{y}} = \mathbf{0}$.

$P_{x,y}P_{x,y'} = \mathbf{0}$ whenever $y \neq y'$.
We write $\mathcal{A} \xrightarrow{q} \mathcal{B}$ for the existence of a quantum homomorphism from $\mathcal{A}$ to $\mathcal{B}$.

Theorem

For finite structures $\mathcal{A}$, $\mathcal{B}$, the following are equivalent:

1. *There is a quantum perfect strategy for the homomorphism game from $\mathcal{A}$ to $\mathcal{B}$.*
2. $\mathcal{A} \xrightarrow{q} \mathcal{B}$.

An aside: the quantum monad

For each $d \in \mathbb{N}$ and σ-structure $\mathcal{A}$, we can define a structure $\mathcal{Q}_d\mathcal{A}$ such that there is a one-to-one correspondence

$$\mathcal{A} \xrightarrow{q}_d \mathcal{B} \quad \longleftrightarrow \quad \mathcal{A} \to \mathcal{Q}_d\mathcal{B}$$

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Example: quantum versions of graph invariants.

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Monads play a major role in programming language theory as encapsulating **computational effects**. E.g. their use in the Haskell programming language.

State-independent contextuality and quantum homomorphisms

Given an empirical model $e : (X, \mathcal{M}, O)$ we fix an order on X, and introduce a relation R_C for each context $C \in \mathcal{M}$.

We define a structure $\mathcal{A}_e$ with universe X, and for each C the relation $\{(x_1, \ldots, x_k)\}$, where $C = \{x_1 < \cdots < x_k\}$.

We define another structure $\mathcal{B}_e$ with universe O, and for each C the relation $\{\mathbf{o} \mid e_C(\mathbf{x} \mapsto \mathbf{o}) > 0\}$.

Theorem

There is a one-to-one correspondence between state-independent quantum witnesses for e, and quantum homomorphisms $\mathcal{A}_e \xrightarrow{q} \mathcal{B}_e$.

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Theorem

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Thus we see that there is a tight connection between state-independent strong contextuality *a la* Kochen-Specker and non-local games, via quantum homomorphisms. This underwrites and gives a physical warranty for the assumption of compatibility.

Final Remarks

Quantum solutions of binary constraint systems are subsumed as special cases of quantum homomorphisms.

There is also an infinite-dimensional version.

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Some further questions:

- What about **state-dependent** contextuality?
- Quantifying bounds on success probabilities in the non-perfect case – cf. the contextual fraction.

Envoi

Contextuality in physics raises deep questions about the nature of reality. But it is also a new kind of resource, which yields new possibilities in information processing tasks.
The challenge is to find methods to harness this resource, and understand its structure.

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Under the rubric of "local consistency, global inconsistency" contextuality is a pervasive notion, arising e.g. in constraint satisfaction, databases, distributed computation and elsewhere in classical computation.

Using the quantum monad and related constructions provides a systematic means of quantizing classical computational tasks in a structural way.

The Penrose Tribar