Title: The contextual fraction as a measure of contextuality

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Abstract:

The contextual fraction as a measure of contextuality

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Perimeter Institute,
24th July 2017

Overview

- Contextuality is a fundamental non-classical phenomenon
- Studied as a resource for quantum computing and information processing: e.g. for nonlocal games, MBQC, MSD
- **Contextual fraction** is a quantitative measure with many desirable properties
  1. General: applies in any measurement scenario
  2. Precise relationship to violations of Bell inequalities
  3. Computable using linear programming
  4. Monotone wrt operations that don't introduce contextuality

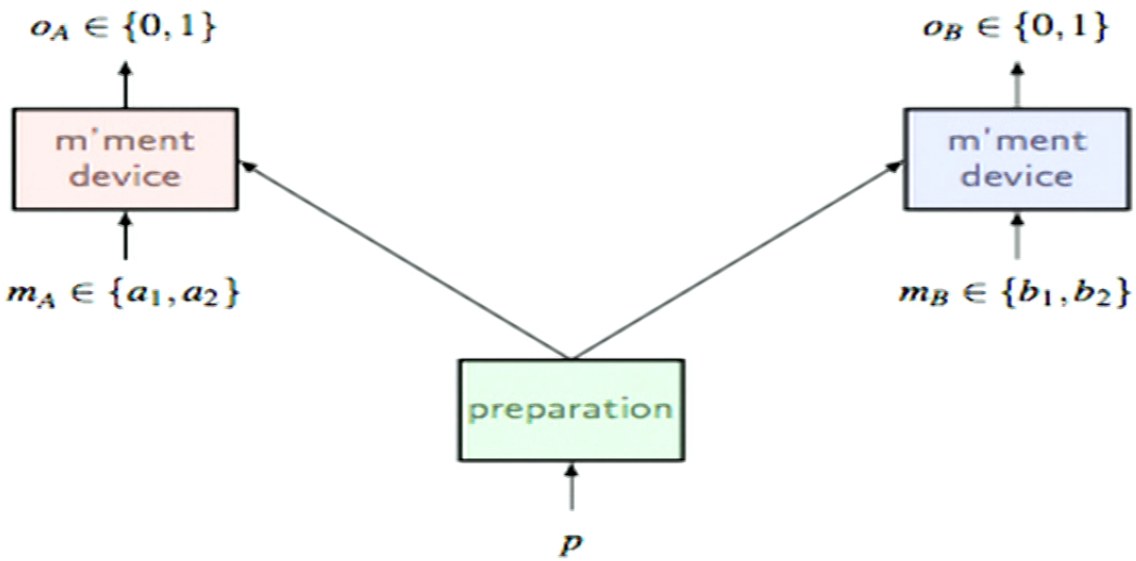
  $\rightsquigarrow$ **resource theory**
- Relates to quantifiable advantages in QC and QIP tasks

Plan

- Formalism: contextuality as a feature of empirical data (Abramsky & Brandenburger)
- Contextual fraction as a quantitative measure
- Computational explorations
- Towards resource theory for contextuality
- Contextual fraction and quantum advantages

Empirical data (e.g. CHSH experiment)

A	B	(0,0)	(0,1)	(1,0)	(1,1)
a_1	b_1	$1/2$	0	0	$1/2$
a_1	b_2	$3/8$	$1/8$	$1/8$	$3/8$
a_2	b_1	$3/8$	$1/8$	$1/8$	$3/8$
a_2	b_2	$1/8$	$3/8$	$3/8$	$1/8$



Measurement scenario $\langle X, \mathscr{M}, O \rangle$

A	B	(0,0)	(0,1)	(1,0)	(1,1)
a_1	b_1				
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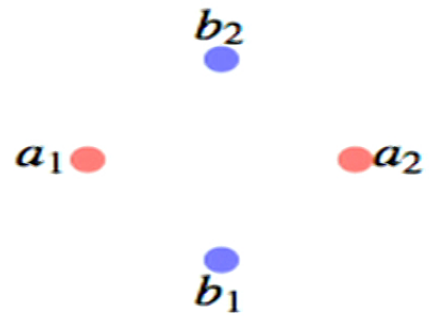

X a finite set of measurements — e.g.

$$X = \{a, a', b, b'\}$$

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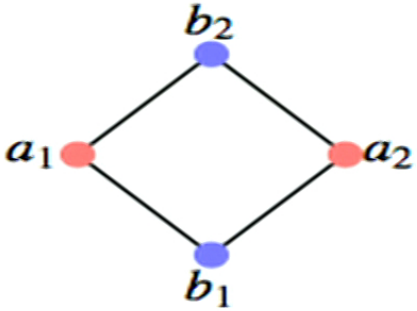

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$\mathscr{M}$ the (maximal) contexts — e.g.

$$\mathscr{M} = \{\{a, b\}, \quad \{a, b'\}, \quad \{a', b\}, \quad \{a', b'\}\}$$

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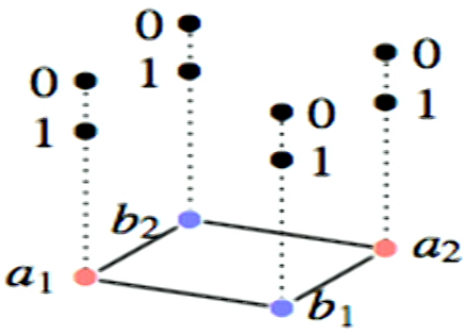

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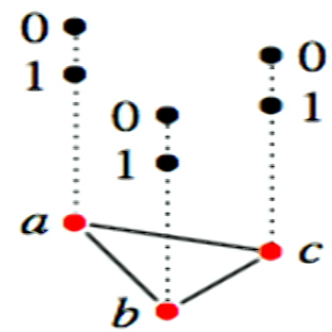
O a finite set of outcomes — e.g.

$$O = \{0, 1\}$$

Measurement scenario: 'Triangle'

		(0,0)	(0,1)	(1,0)	(1,1)
a	b				
b	c				
c	a				



Measurements:

$$X = \{a, b, c\}$$

Contexts:

$$\mathscr{M} = \{\{a, b\}, \quad \{b, c\}, \quad \{c, a\}\}$$

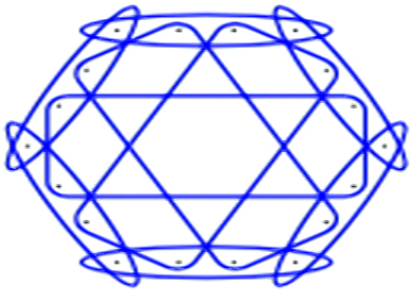
Outcomes:

$$O = \{0, 1\}$$

Measurement scenario: 18-vector KS (Cabello)

- A set of 18 variables: $X = \{A, \ldots, O\}$
- A set of outcomes: $O = \{0, 1\}$
- A measurement cover: $\mathscr{M} = \{C_1, \ldots, C_9\}$
  Contexts given by columns in the following table / hyperedges in hypergraph

C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
A	A	H	H	B	I	P	P	Q
B	E	I	K	E	K	Q	R	R
C	F	C	G	M	N	D	F	M
D	G	J	L	N	O	J	L	O



(Graphics by Ravi)

Empirical models

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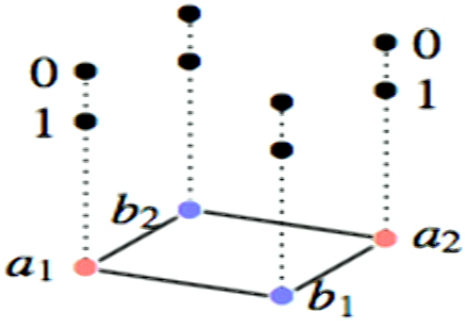

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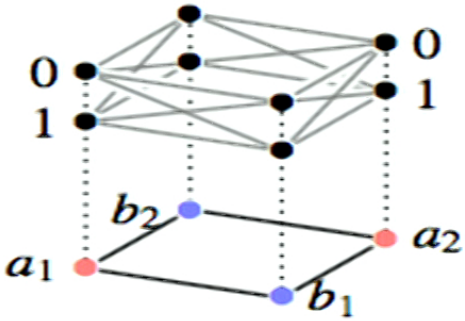

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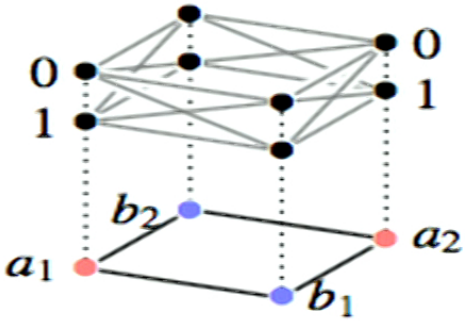

- **Empirical model**: family $\{e_C\}_{C \in \mathscr{M}}$ where each $e_C \in \mathrm{Prob}(O^C)$

Empirical models

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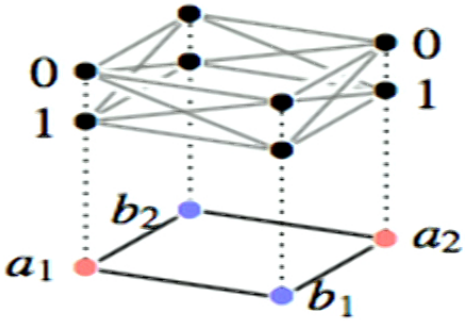

- **Empirical model**: family $\{e_C\}_{C \in \mathscr{M}}$ where each $e_C \in \mathrm{Prob}(O^C)$
- Distribution for each context:

$$e_{\{a,b\}} = \mathrm{prob}(o_1, o_2 | a, b), \quad \ldots, \quad e_{\{a',b'\}} = \mathrm{prob}(o_1, o_2 | a', b')$$

- 'Local' consistency:

$$\mathrm{prob}(o_1 | a, b) = \mathrm{prob}(o_1 | a, b') = \mathrm{prob}(o_1 | a), \text{ etc.}$$

no-signalling

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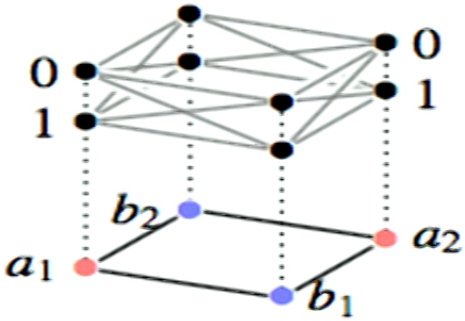

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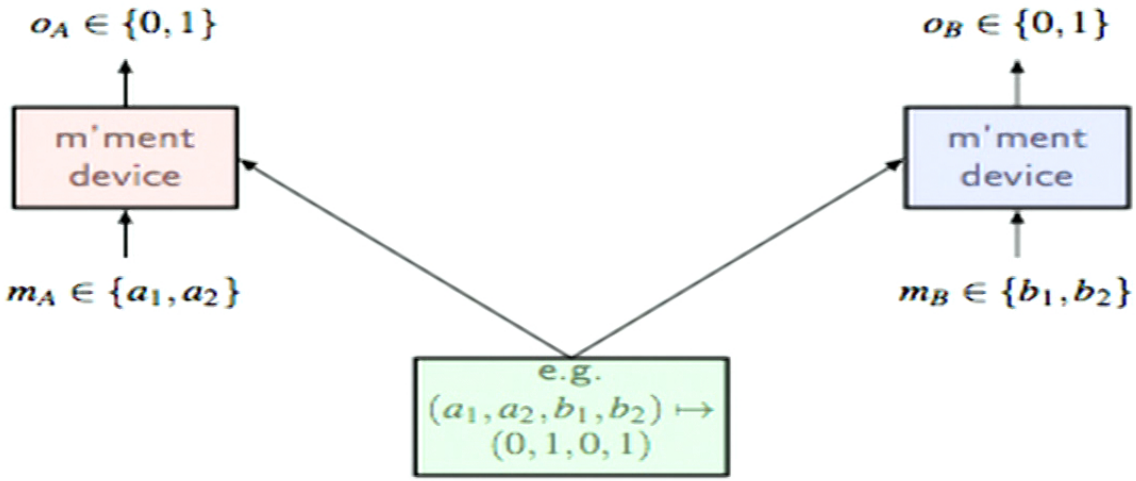
Classical data



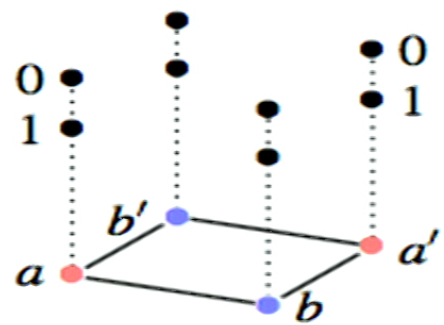
- A preparation should correspond to some probability distribution over global assignments (i.e. instruction sets)
- Existence of such a distribution is equivalent to the existence of factorisable hidden variables

Classical v contextual data

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	$1/2$	0	0	$1/2$
(a,b')	$1/2$	0	0	$1/2$
(a',b)	$1/2$	0	0	$1/2$
(a',b')	0	$1/2$	$1/2$	0

Classical data arises as a convex combination of **global assignments**:

$$(a,a',b,b') \mapsto (0,0,0,0),$$
$$(a,a',b,b') \mapsto (0,0,0,1),$$
$$\dots \quad ,$$
$$(a,a',b,b') \mapsto (1,1,1,1)$$



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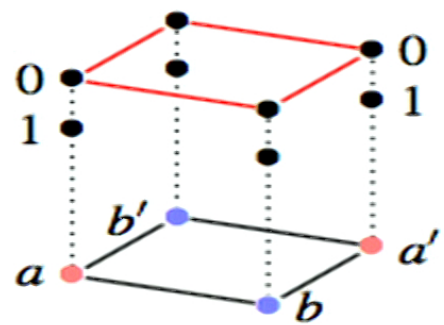

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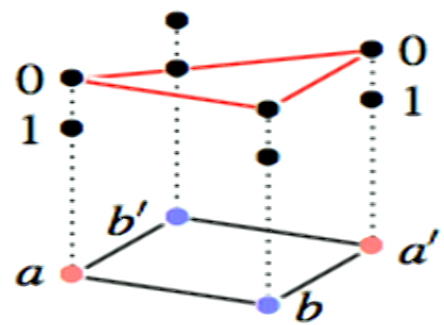

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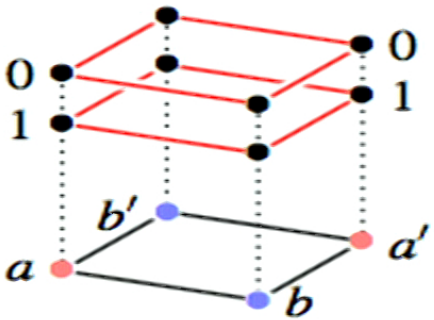

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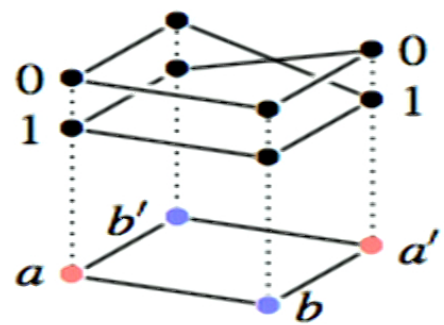
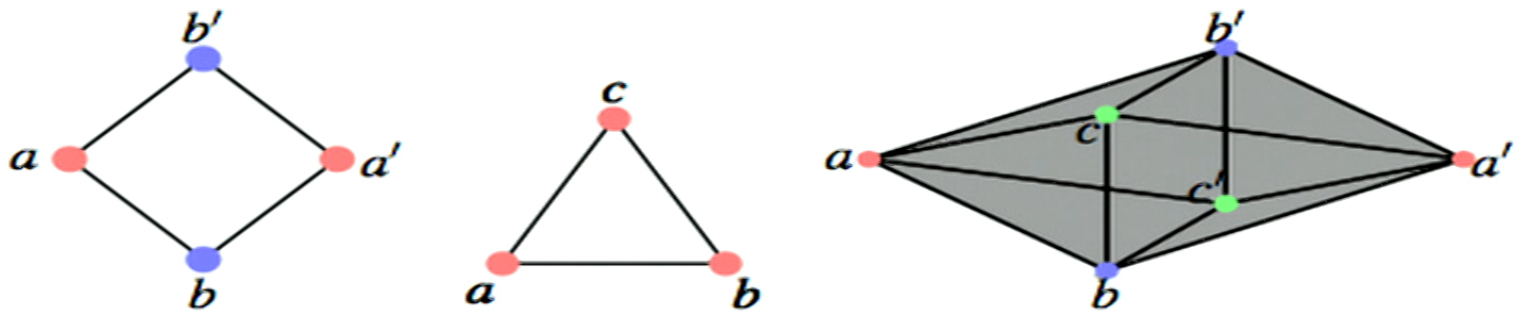


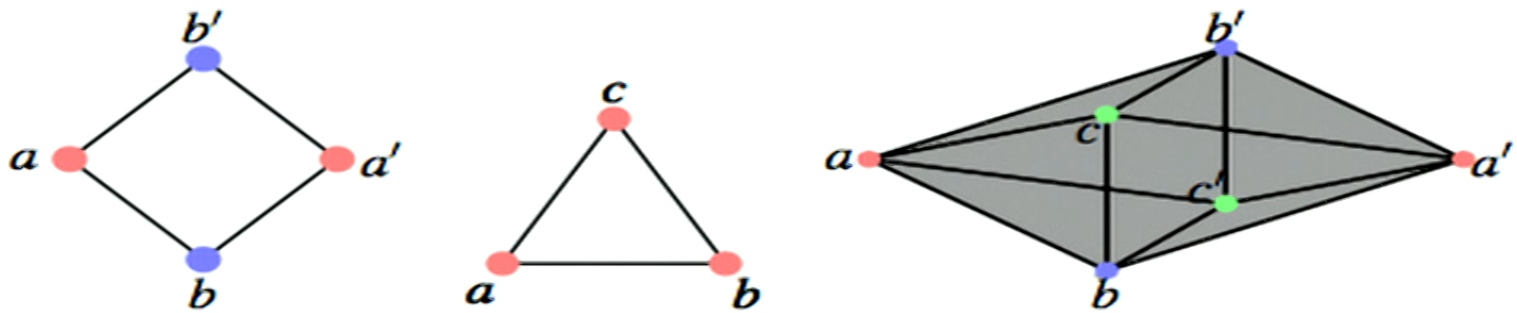
However, the above correlations **cannot** be obtained as a convex combination of global assignments!

Visual aid



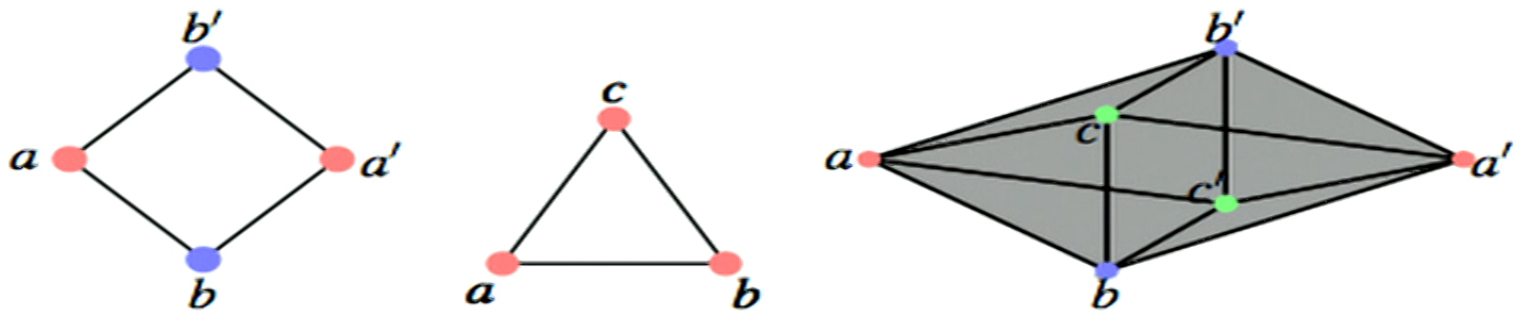
- Measurement scenario $\longrightarrow$ simplicial complex $(X, \mathscr{M})$

Visual aid



- Measurement scenario $\longrightarrow$ simplicial complex $(X, \mathscr{M})$
- Empirical model $\{e_C\}_{C \in \mathscr{M}}$:
  probability distribution (over functions into O) at each face
- Well-defined marginals $\longleftrightarrow$ (generalised) no-signalling

Visual aid



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$$\boxed{\text{Global distribution} \Longleftrightarrow \text{Classical}}$$

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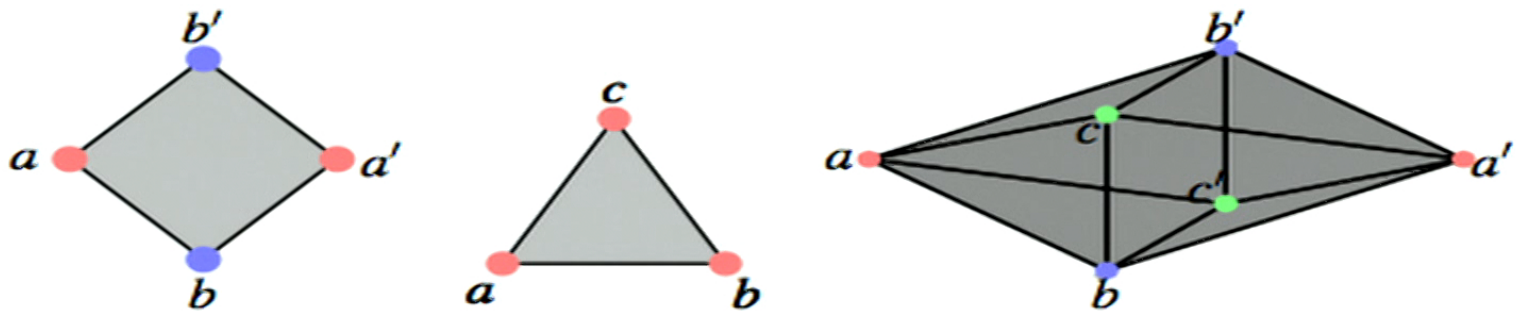

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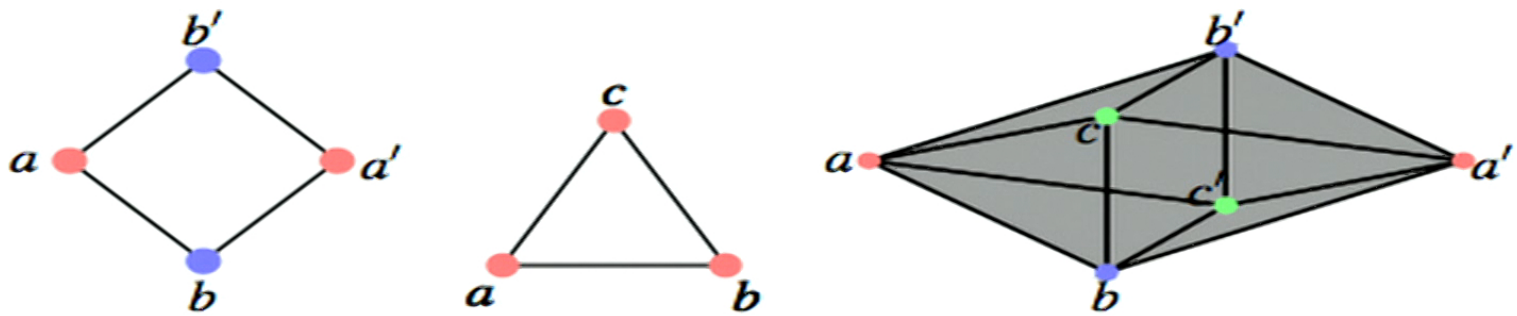

- Measurement scenario $\longrightarrow$ simplicial complex $(X, \mathscr{M})$
- Empirical model $\{e_C\}_{C \in \mathscr{M}}$:
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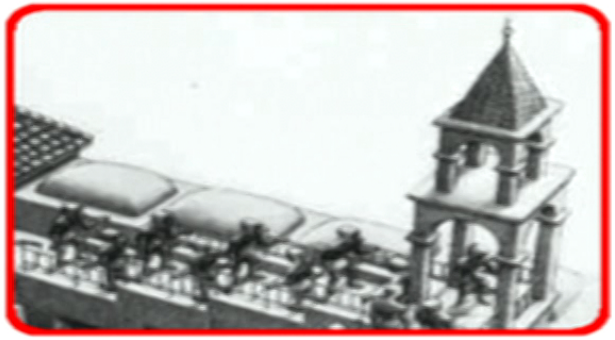
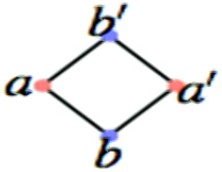
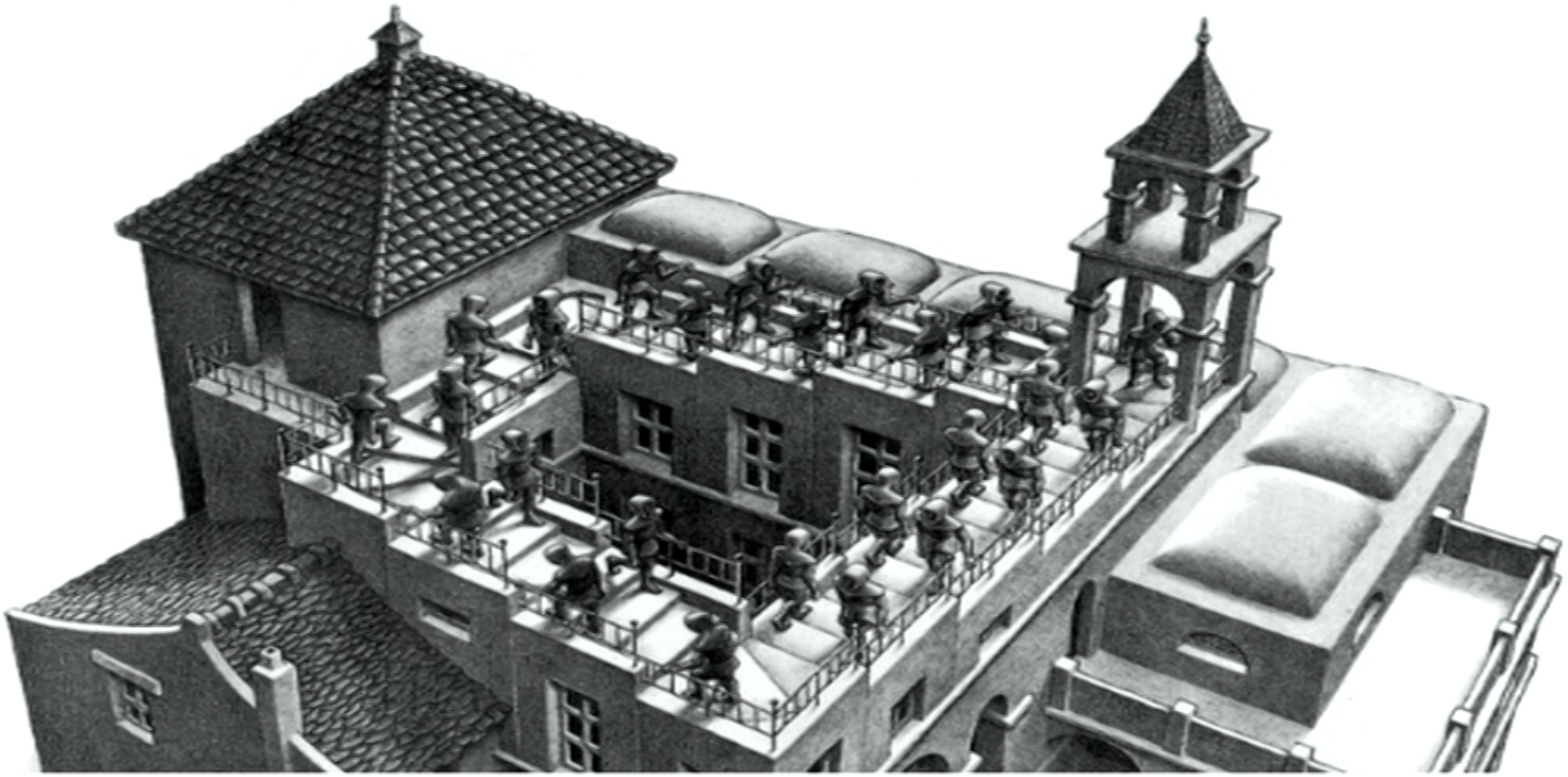
$$\boxed{\textbf{No}\ \text{global distribution} \iff \text{Contextual}}$$

Contextuality analogy: Local consistency





Contextuality analogy: Global inconsistency



Acending and Descending, M.C. Escher

Contextuality analogy:
Local consistency v global inconsistency

Impossible figures in constructive geometry: e.g. the tribar

Strong contextuality

Strong contextuality:
no global assignment of outcomes to measurements is consistent with the data

E.g. K–S models, GHZ, the **PR box**, all AvN models:

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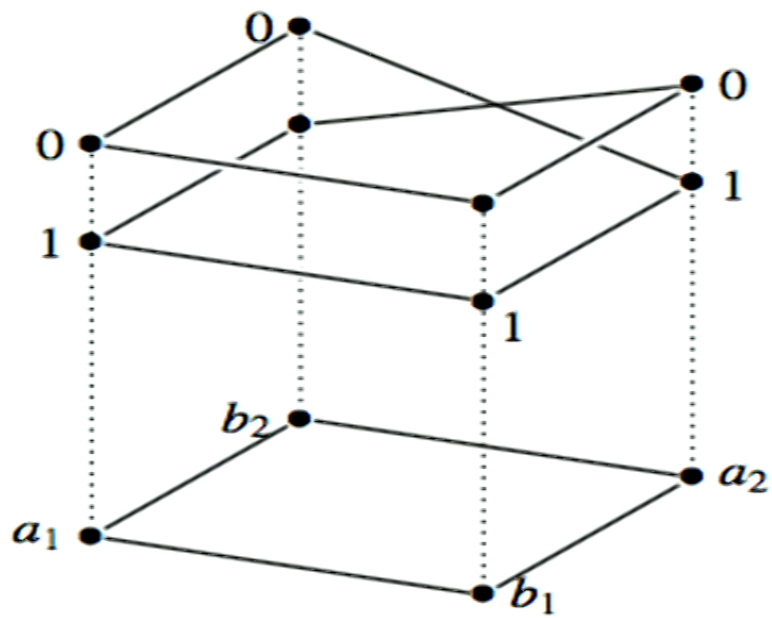

The contextual fraction

Contextuality as a linear system

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a,b)	$1/2$	0	0	$1/2$
(a,b')	$3/8$	$1/8$	$1/8$	$3/8$
(a',b)	$3/8$	$1/8$	$1/8$	$3/8$
(a',b')	$1/8$	$3/8$	$3/8$	$1/8$

- Flatten e to a vector $\mathbf{v}_e \in \mathbb{R}^m$, e.g.

$$\mathbf{v}_e = \{1/2, 0, 0, 1/2, \quad 3/8, 1/8, 1/8, 3/8, \quad 3/8, 1/8, 1/8, 3/8, \quad 1/8, 3/8, 3/8, 1/8\}$$

- Similarly for global assignments, e.g.

$$\mathbf{g}_1 = \{1,0,0,0, \quad 1,0,0,0, \quad 1,0,0,0, \quad 1,0,0,0\}$$

- Define $\mathbf{M} := [\mathbf{g}_1, \ldots, \mathbf{g}_n]$ with global assignments as columns
- e is **non-contextual** iff there exists a solution $\mathbf{d} \in \mathbb{R}^n$ with $\mathbf{d} \geq 0$ to

$$\mathbf{M}\mathbf{d} = \mathbf{v}_e$$

The Contextual fraction

Q: How much of the data can be explained classically?

The non-contextual fraction is the maximum weight λ over all convex decompositions

$$e = \lambda e^{NC} + (1 - \lambda) e'$$

where e^{NC} is a non-contextual model.

The Contextual fraction

Q: How much of the data can be explained classically?

The non-contextual fraction is the maximum weight λ over all convex decompositions

$$e = \lambda_{\max} e^{NC} + (1 - \lambda_{\max}) e^{SC}$$

where e^{NC} is a non-contextual model. e^{SC} is strongly contextual!

$$\mathrm{NCF}(e) = \lambda_{\max} \qquad \mathrm{CF}(e) = 1 - \lambda_{\max}$$

The Contextual fraction

Q: How much of the data can be explained classically?

The non-contextual fraction is the maximum weight λ over all convex decompositions

$$e = \lambda_{\max} e^{NC} + (1 - \lambda_{\max}) e^{SC}$$

where e^{NC} is a non-contextual model. e^{SC} is strongly contextual!

$$\mathrm{NCF}(e) = \lambda_{\max} \qquad \mathrm{CF}(e) = 1 - \lambda_{\max}$$

Contextual fraction

- $\mathrm{CF}(e) \in [0, 1]$
- e is **non-contextual** iff $\mathrm{CF}(e) = 0$
- e is **strongly contextual** iff $\mathrm{CF}(e) = 1$

(Non-)contextual fraction via linear programming

Checking non-contextuality of e corresponds to solving

$$\begin{array}{ll} \texttt{Find} & \mathbf{d} \in \mathbb{R}^n \\ \texttt{such that} & \mathbf{M}\mathbf{d} = \mathbf{v}_e \\ \texttt{and} & \mathbf{d} \geq \mathbf{0} \end{array}$$

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Computing the non-contextual fraction corresponds to solving the following **linear program**:

$$\begin{array}{ll} \texttt{Find} & \mathbf{c} \in \mathbb{R}^n \\ \texttt{maximising} & \mathbf{1} \cdot \mathbf{c} \\ \texttt{subject to} & \mathbf{M}\mathbf{c} \leq \mathbf{v}_e \\ \texttt{and} & \mathbf{c} \geq \mathbf{0} \end{array}$$

Computation explorations and example

Computational tools to calculate:

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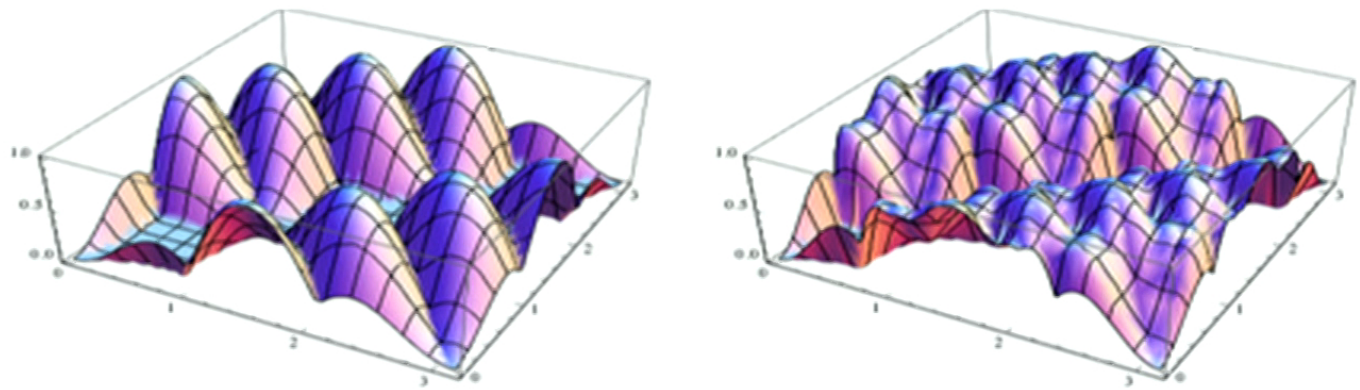

(a) (b)

Figure: CF(e) for equatorial measurements at ϕ_1 and ϕ_2 on each qubit of $\left|\psi_{\mathrm{GHZ}(n)}\right\rangle$ with: (a) $n = 3$; (b) $n = 4$.

Bell Inequality violations

Generalised Bell Inequalities

An **inequality** for a scenario $\langle X, \mathscr{M}, O \rangle$ is given by:

- A vector $\boldsymbol{\alpha} \in \mathbb{R}^m$
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- It is a **Bell inequality** if it is satisfied by every NC model i.e. it bounds $\boldsymbol{\alpha} \cdot \mathbf{v}_e$ for NC models
- A Bell inequality is **tight** if it is saturated by some NC model

Violation of a Bell Inequality

Normalisation:

- Generally, $\boldsymbol{\alpha} \cdot \mathbf{v}_e$ is limited only by

$$\|\alpha\| := \sum_{C \in \mathscr{M}} \max\left\{\alpha(C,s) \mid s \in O^C\right\}$$

- The **normalised violation** of a Bell inequality $\langle \alpha, R \rangle$ by e is

$$\frac{\max\{0, \boldsymbol{\alpha} \cdot \mathbf{v}_e - R\}}{\|\alpha\| - R} \quad \in [0,1]$$

Contextual fraction and Bell violations

Proposition

Let e be an empirical model

- Its violation of any Bell inequality is at most $\mathrm{CF}(e)$
- There exists a Bell inequality for which this is attained
- This Bell inequality is tight at "the" non-contextual model e^{NC}

$$e = \mathrm{NC}(e)\, e^{NC} + \mathrm{CF}(e)\, e^{SC}$$

Bell inequality violation and the contextual fraction

Quantifying Contextuality LP:

$$\begin{array}{ll} \texttt{Find} & \mathbf{c} \in \mathbb{R}^n \\ \texttt{maximising} & \mathbf{1} \cdot \mathbf{c} \\ \texttt{subject to} & \mathbf{M}\mathbf{c} \leq \mathbf{v}_e \\ \texttt{and} & \mathbf{c} \geq \mathbf{0} \end{array} .$$

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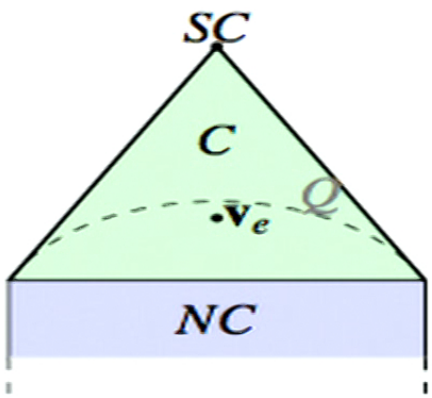

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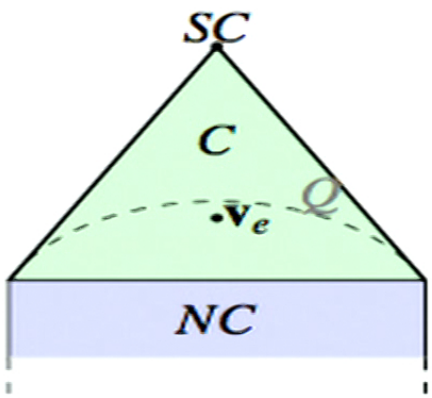

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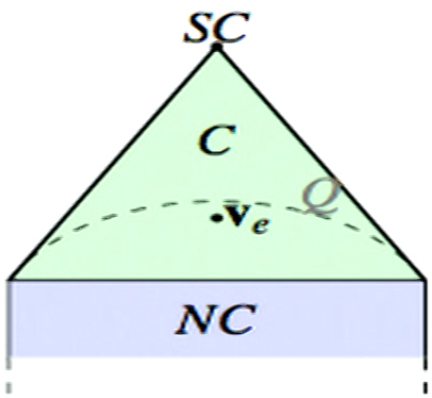

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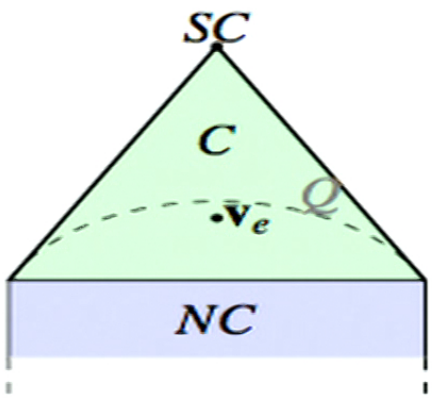

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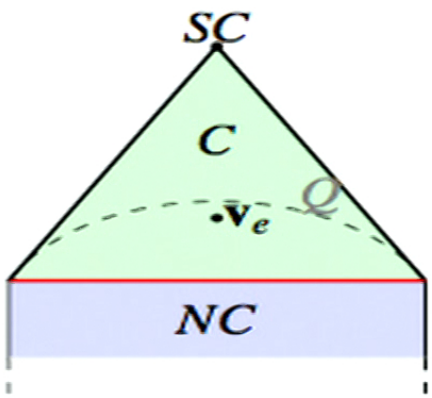

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computes tight Bell inequality (separating hyperplane)

Towards a resource theory of contextuality

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Q: What properties should a good measure satisfy?

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2. Contextuality is associated with quantum advantage in QC and QIP tasks

   Measure of contextuality $\rightsquigarrow$ quantify such advantages

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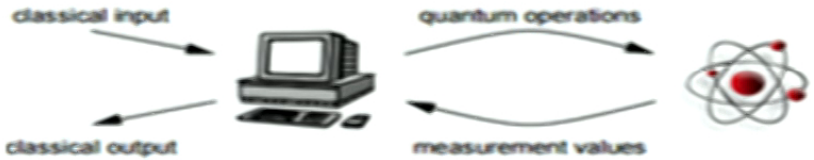
Contextuality and MBQC

Example: $\ell 2$-MBQC
measurement-based quantum computing scheme
(m input bits, l output bits)



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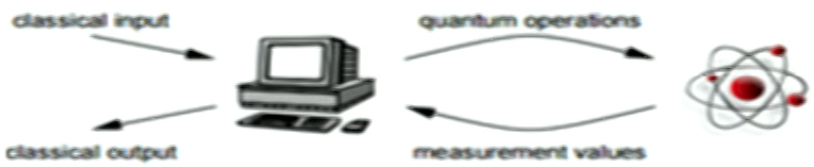

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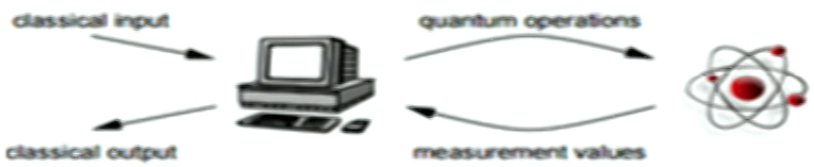

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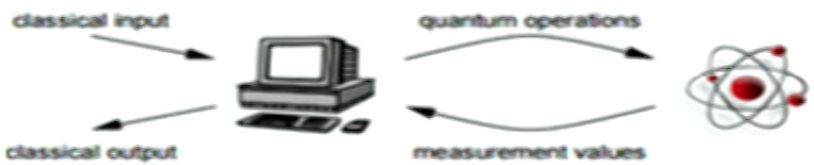

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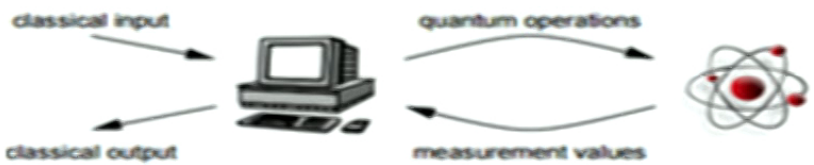

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where for Boolean functions f and g, $d(f,g) := 2^{-m}|\{\mathbf{i} \in 2^m \mid f(\mathbf{i}) \neq g(\mathbf{i})\}$

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- **Average probability of success** computing f (over all 2^m possible inputs): $\bar{p}_S$

$$\boxed{1 - \bar{p}_S \;\geq\; \mathrm{NCF}(e)\,\nu(f)}$$

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  - Negative probabilities, entropy measures, multipartiteness width, ...

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  - Connections with Contextuality-by-Default (Dzhafarov, Kujala, Larsson, . . .)

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  - What (else) is this resource useful for?

Contextual Fraction (Recap)

1. General: applicable in any measurement scenario
2. Normalised: allowing comparison across scenarios
3. 0 for non-contextuality . . . 1 for strong contextuality
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What else?

- Computational tools
- **Resource Theory:**
  1. Monotone wrt operations that don't introduce contextuality
  2. Measures quantifiable advantages in informatic tasks

Questions...

?

"The contextual fraction as a measure of contextuality"
Samson Abramsky, Rui Soares Barbosa, Shane Mansfield
arXiv:1705.07918[quant-ph]