

Title: Kochen-Specker contextuality: a hypergraph approach with operational equivalences

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Abstract:

Kochen-Specker contextuality: a hypergraph approach with operational equivalences

Ana Belén Sainz,
A. Acín , T. Fritz, A. Leverrier

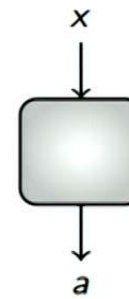
Comm. in Mat. Phys. vol. 334, pp 533-628 (2015)

Outline

- Hypergraph framework
- Bell scenarios
- Compatible-observables scenarios
- Graph theory advantage
- Quantum from principles

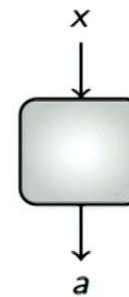
Contextuality scenarios

- Set of measurements
- Set of outcomes



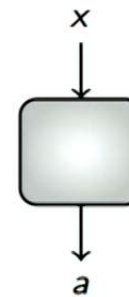
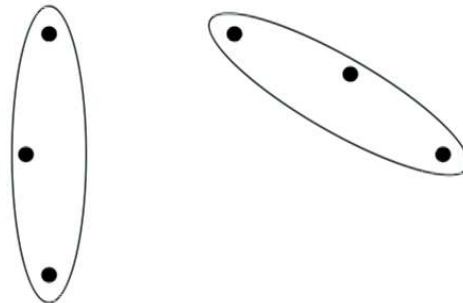
Contextuality scenarios

- Set of measurements
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- Operational equivalences → identify outcomes of different measurements: same probability



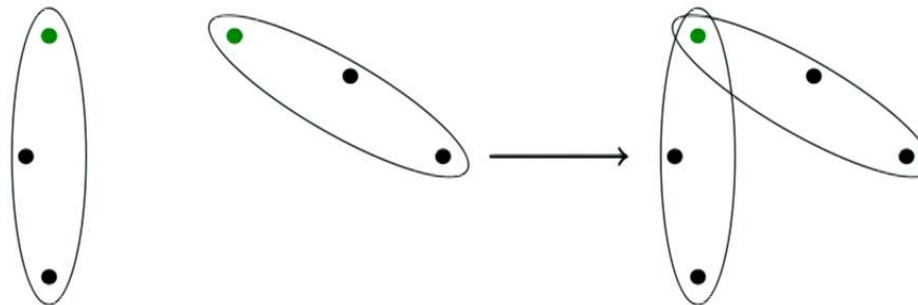
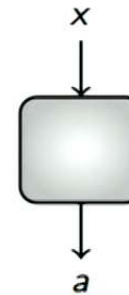
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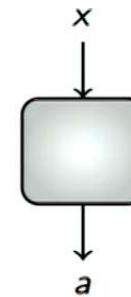
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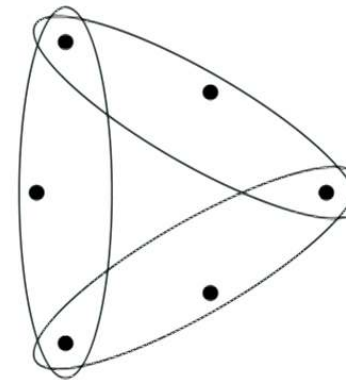
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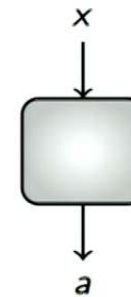
Hypergraph:

- Vertices \rightarrow events – measurement outcome
- Hyperedges \rightarrow complete measurements – set of outcomes



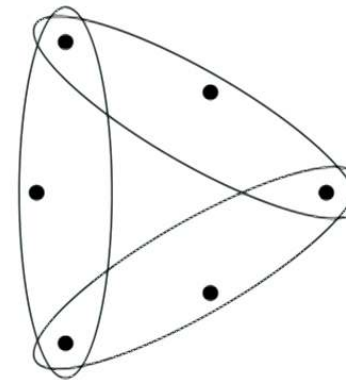
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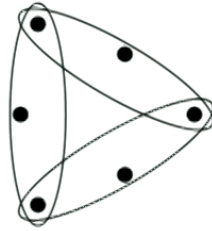


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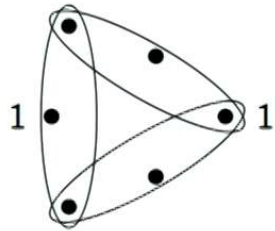
Contextuality Scenarios



- **Probabilistic Model:** $\mathcal{G}(H)$

$p : V \rightarrow [0, 1]$, such that $\sum_{v \in e} p(v) = 1$, for each $e \in E$.

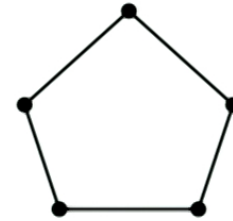
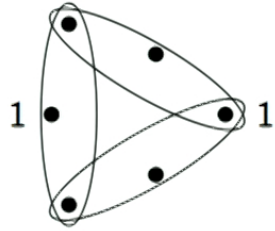
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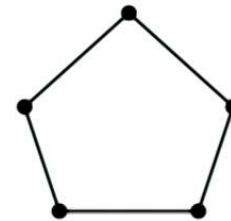
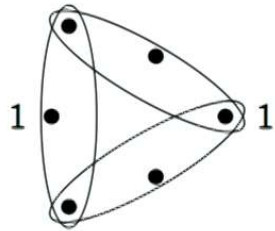
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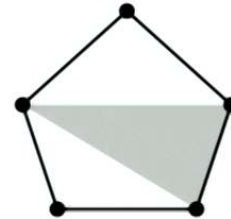
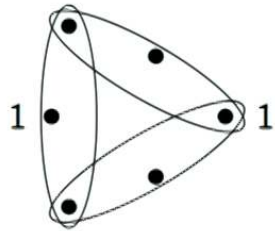
Deterministic model: $p(v) \in \{0, 1\}$

- **Classical models:** $\mathcal{C}(H)$

$$p(v) = \sum_{\lambda} q_{\lambda} p_{\lambda}(v)$$

$p_{\lambda}(v)$: deterministic models on H , $q_{\lambda} \geq 0$

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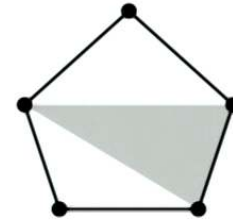
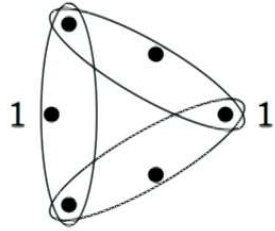
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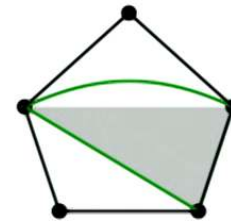
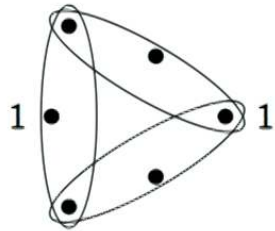
$p_{\lambda}(v)$: deterministic models on H , $q_{\lambda} \geq 0$

- **Quantum models:** $\mathcal{Q}(H)$

$\exists \mathcal{H}, \rho, \{P_v : v \in V\}, \sum_{v \in e} P_v = \mathbb{1}_{\mathcal{H}} \quad \forall e \in E$

$$p(v) = \text{tr}(\rho P_v)$$

Contextuality Scenarios



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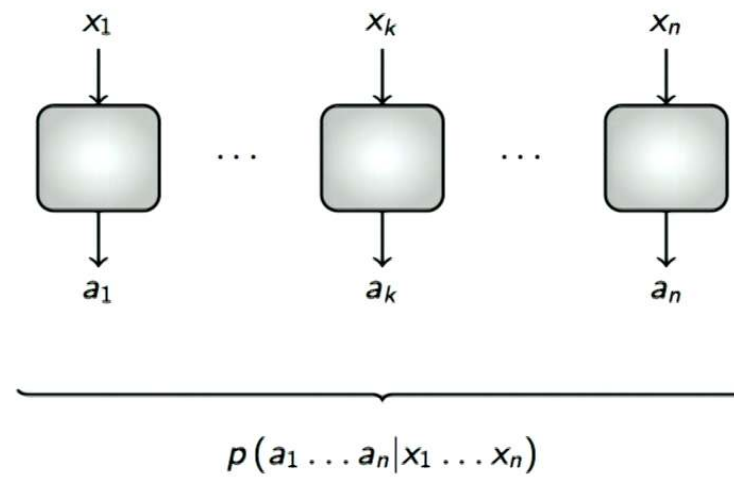
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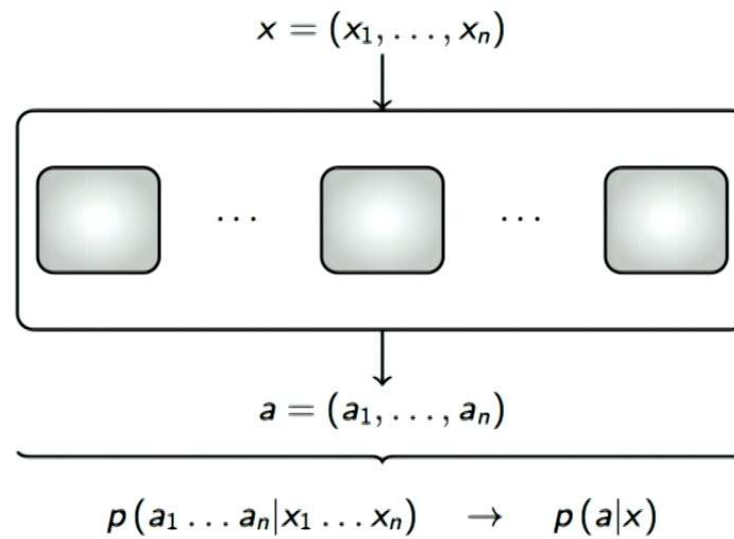
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Bell scenarios



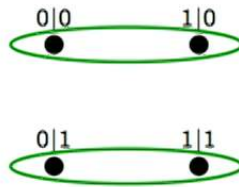
Bell scenarios



Bell scenarios

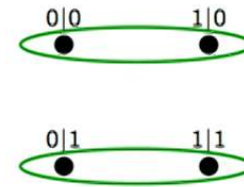
$$(n, m, d) \rightarrow \mathcal{B}_{n,m,d} ?$$

Alice



H_A

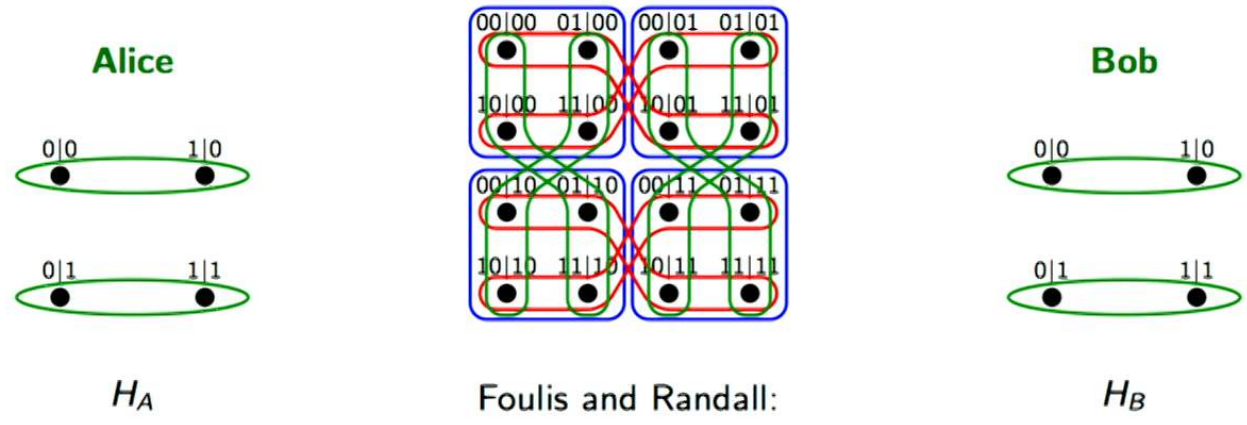
Bob



H_B

Bell scenarios

$$(n, m, d) \rightarrow \mathcal{B}_{n,m,d} ?$$



H_A

Foulis and Randall:

H_B

$$H_{AB} = H_A \otimes_{FR} H_B$$

Bell Scenarios

Product Scenarios

Alice: H_A , Bob: H_B \longrightarrow Joint scenario: $H_A \otimes H_B$

Foulis-Randall product

$$\mathcal{B}_{n,m,d} = \otimes_{k=1}^n H_k = \mathcal{B}_{1,m,d}^{\otimes n}$$

- $\mathcal{G}(\mathcal{B}_{n,m,d}) = \mathcal{NS}(n, m, d)$
- $\mathcal{C}(\mathcal{B}_{n,m,d}) = \mathcal{C}(n, m, d)$
- $\mathcal{Q}(\mathcal{B}_{n,m,d}) = \mathcal{Q}(n, m, d) \longrightarrow P_{a_1 \dots a_n | x_1 \dots x_n} = \prod_j P_{a_j | x_j}$

Compatible-observables scenarios¹

Scenario : (X, O, \mathcal{M})

- X : observables
- O : outcomes
- \mathcal{M} : measurement cover \rightarrow sets of compatible measurements
 $C \in \mathcal{M}$ measurement context

¹Abramsky Brandenburger, NJP 13 (2011), no. 11, 113036.

Compatible-observables scenarios¹

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Specker's triangle:

$$X = \{A, B, C\}$$

$$\mathcal{M} = \{\{A, B\}, \{B, C\}, \{A, C\}\}$$

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Compatible-observables scenarios

$$(X, O, \mathcal{M}) \longrightarrow H[X]$$

- Vertices:

- Hyperedges:

Compatible-observables scenarios

$$(X, O, \mathcal{M}) \longrightarrow H[X]$$

- Vertices: $(C, s) : C \in \mathcal{M}, s \in O^C$

Specker's triangle: $(ab|AB), (bc|BC), (ac|AC)$

- Hyperedges:

Compatible-observables scenarios

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- Hyperedges: Measurement protocols

- Choose and measure an observable (A) .
- Depending on the outcome, choose a compatible observable (A') .
- Measure (A') , ...

Compatible-observables scenarios

$$(X, O, \mathcal{M}) \rightarrow H[X]$$

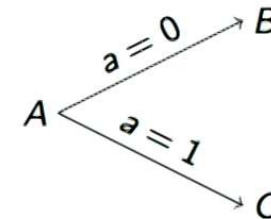
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$\{(00|AB), (01|AB), (10|AC), (11|AC)\}$

Compatible-observables scenarios

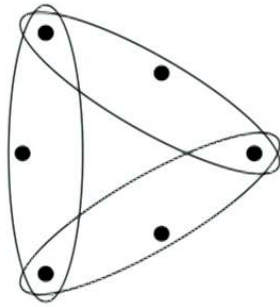
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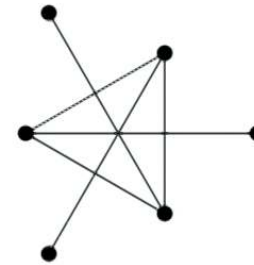
$$\text{Empirical model } P_C(s) \Leftrightarrow \text{Probabilistic model } p(C, s)$$

Graph Theory advantage

Contextuality scenario: H

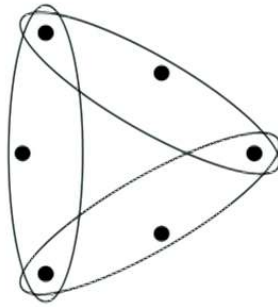


Non-orthogonality graph: $NO(H)$

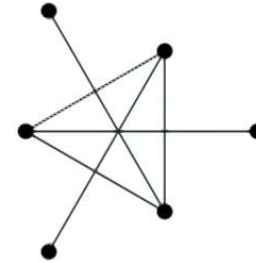


Graph Theory advantage

Contextuality scenario: H



Non-orthogonality graph: $\text{NO}(H)$



$$p \in \mathcal{C}(H) \text{ iff } \alpha^*(\text{NO}(H), p) = 1$$

Quantum models

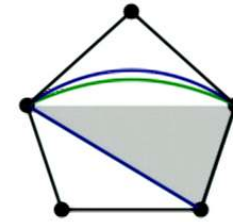
Quantum models cannot be characterised by the properties of $(\text{NO}(H), p)$

Example: $\exists H, H', p$ st:

- $p \in \mathcal{Q}(H)$
- $p \in \mathcal{Q}_1(H') \setminus \mathcal{Q}(H')$
- $\text{NO}(H) = \text{NO}(H')$

Almost quantum models

“Almost” quantum: Q^{1+AB} ⁴, Q_1 ⁵, \tilde{Q} ⁶



Attempts to characterise quantum \longrightarrow Almost quantum

Q_1

$$\exists \mathcal{H}, \rho, \{P_v : v \in V\}$$

$$\sum_{v \in e} P_v \leq \mathbb{1}_{\mathcal{H}} \quad \forall e \in E$$

$$p(v) = \text{tr}(\rho P_v)$$

$Q_1(H)$

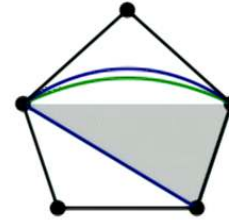
⁴ M. Navascués, S. Pironio and A. Acín, NJP 10 (7), 073013 (2008)

⁵ A. Acín, T. Fritz, A. Leverrier and ABS, CMP 334(2), 533-628 (2015)

⁶ M. Navascués, Y. Guryanova, M. Hoban and A. Acín, Nat Comm 6, 6288 (2015)

Almost quantum models

"Almost" quantum: \mathcal{Q}^{1+AB} , \mathcal{Q}_1 , $\tilde{\mathcal{Q}}$

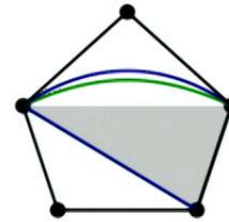


Attempts to characterise quantum \longrightarrow Almost quantum

$$p \in \mathcal{Q}_1(H) \text{ if and only if } \vartheta(\text{NO}(H), p) = 1$$

Almost quantum models

“Almost” quantum: \mathcal{Q}^{1+AB} , \mathcal{Q}_1 , $\tilde{\mathcal{Q}}$



Attempts to characterise quantum \longrightarrow Almost quantum

$$p \in \mathcal{Q}_1(H) \text{ if and only if } \vartheta(\text{NO}(H), p) = 1$$

Quantum from principles: Consistent Exclusivity

CE: Contextuality²

Exclusiveness : if $\exists e \in E$ with $\{u, v\} \subset e \Rightarrow u \perp v$

Constraint: $\sum_{v \in S} p(v) \leq 1$, S set of pairwise exclusive events



$$p(a|x) \in \text{CE}^1$$

²A. Cabello, S. Severini and A. Winter, arXiv:1010.2163 (2010) – A. Acín, T. Fritz, A. Leverrier, ABS, CMP 334, 533-628 (2015)– Joe Henson, arXiv:1210.5978 (2012)

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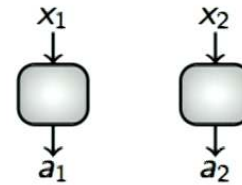
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$$p(a|x) \in \text{CE}^1$$



$$p(a_1 a_2 | x_1 x_2) := p(a_1 | x_1) p(a_2 | x_2) \in \text{CE}^1$$

$$p(a|x) \in \text{CE}^2$$

²A. Cabello, S. Severini and A. Winter, arXiv:1010.2163 (2010) – A. Acín, T. Fritz, A. Leverrier, ABS, CMP 334, 533-628 (2015)– Joe Henson, arXiv:1210.5978 (2012)

Quantum from principles: Consistent Exclusivity

Consistent Exclusivity: $\sum_{v \in S} p(v) \leq 1$

S independent set on $\text{NO}(H)$

CE¹ $p \in \mathcal{CE}^1(H)$ if and only if $\alpha(\text{NO}(H), p) = 1$

CE^k $p \in \mathcal{CE}^k(H)$ if and only if $\alpha(\text{NO}(H)^{\boxtimes k}, p^{\otimes k}) = 1$

CE[∞] $p \in \mathcal{CE}^\infty(H)$ if and only if $\Theta(\text{NO}(H), p) = 1$

Quantum from principles: Consistent Exclusivity

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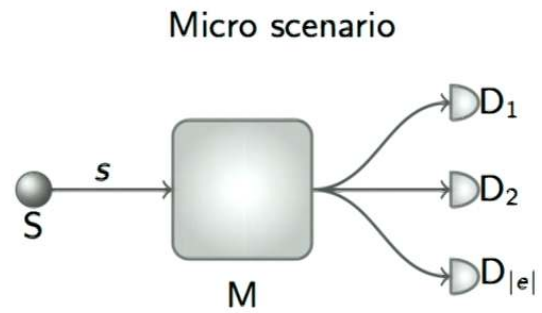
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$$\alpha \leq \Theta \leq \vartheta \Rightarrow \mathcal{Q}_1(H) \subseteq \mathcal{CE}^\infty(H)$$

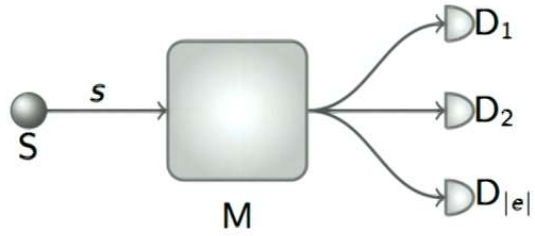
Macroscopic Non-Contextuality



$p(v)$

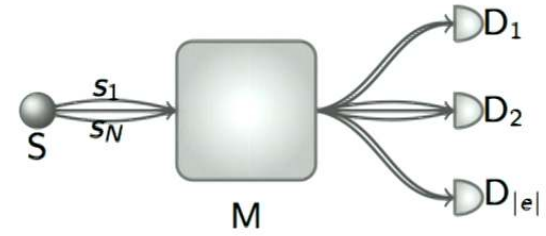
Macroscopic Non-Contextuality

Micro scenario



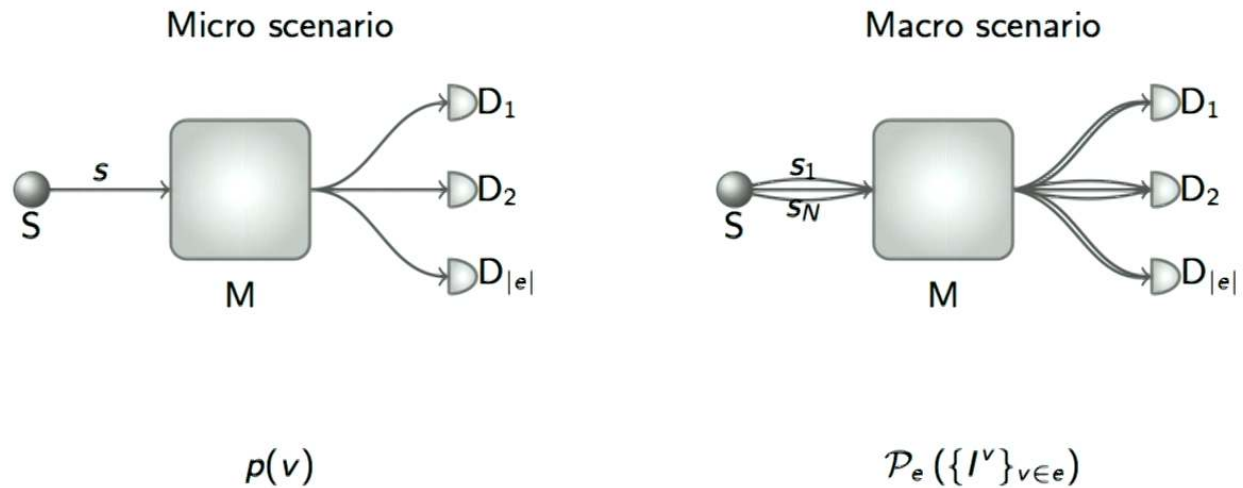
$$p(v)$$

Macro scenario



$$\mathcal{P}_e(\{I^v\}_{v \in e})$$

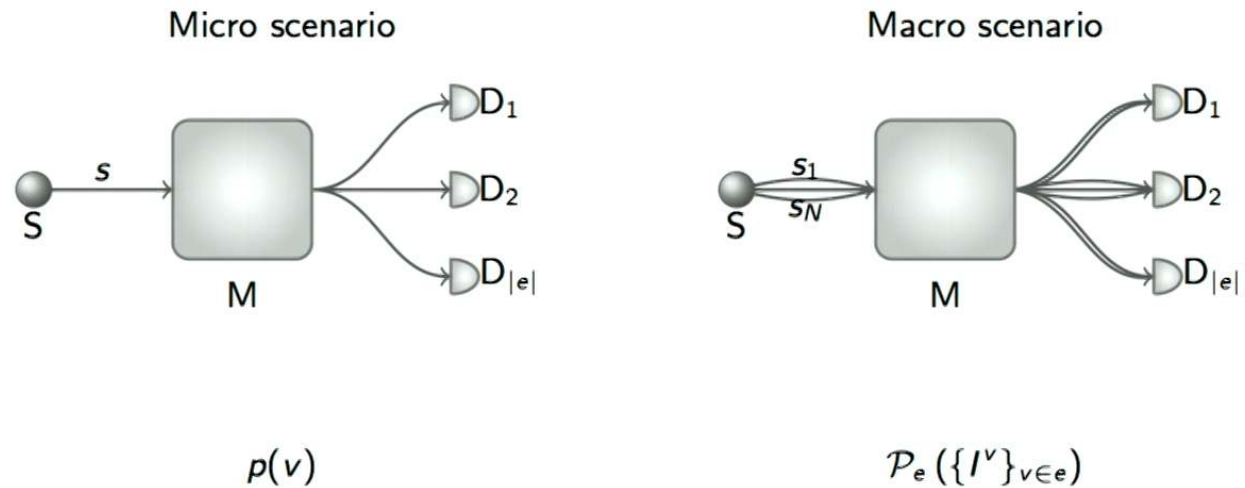
Macroscopic Non-Contextuality



Macroscopic Non-Contextuality:

$p(v)$ satisfies MNC if $(N \rightarrow \infty)$ $\mathcal{P}_e(\{I^v\}_{v \in e})$ is noncontextual

Macroscopic Non-Contextuality



Macroscopic Non-Contextuality:

$p(v)$ satisfies MNC if $(N \rightarrow \infty)$ $\mathcal{P}_e(\{I^v\}_{v \in e})$ is noncontextual

$p(v)$ satisfies MNC iff $p \in \mathcal{Q}_1$

Conclusions and open problems

- Framework to unify Nonlocality and Contextuality
- Useful connection to Graph Theory
- Temporal scenarios – sequences of measurements
- New principles to constrain probabilistic models
 - Distinguish quantum from almost quantum
- Applications

Thanks!

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T. Fritz, A. Leverrier and ABS, *EPTCS* 171, pp. 63-70 (2014).