Noncontextuality: how we should define it, why it is natural, and what to do about its failure

Robert Spekkens

Contextuality:
Conceptual Issues,
Operational Signature, and applications
July 24, 2017
What we want in a notion of nonclassicality

- Subject to direct experimental test
- Constitutes a resource
- Applicable to a broad range of physical scenarios
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- Subject to direct experimental test
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Failure to admit a locally causal model: ✔ ✔ ✗
What is needed to witness the failure of local causality
What we want in a notion of nonclassicality

- Subject to direct experimental test
- Constitutes a resource
- Applicable to a broad range of physical scenarios

Failure to admit a locally causal model: ✓ ✓ x
Failure to admit a noncontextual model: ✓
What is needed to witness the failure of local causality

What is needed to witness the failure of noncontextuality
What we want in a notion of nonclassicality

<table>
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What we want in a notion of nonclassicality

Subject to direct experimental test
Constitutes a resource
Applicable to a broad range of physical scenarios

Failure to admit a locally causal model ✓ ✓ ✗
Failure to admit a noncontextual model ? ✓ ✓
What we want in a notion of nonclassicality

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Defining the principle of noncontextuality
Operational theory

\[ p(X|M, P) \]
Operational theory

\[ p(X|M, P) \]

Ontological model of an operational theory

\[ \lambda \in \Lambda \quad \text{Ontic state space} \]

causally mediates between P and M
Operational theory

\[ p(X|M, P) \]

Ontological model of an operational theory

\[ \lambda \in \Lambda \quad \text{Ontic state space} \]

\[ P \leftrightarrow \mu(\lambda|P) \quad M \leftrightarrow \xi(X|M, \lambda) \]

\[ p(X|M, P) = \sum_{\lambda} \xi(X|M, \lambda) \mu(\lambda|P) \]

causally mediates between P and M
An ontological model of an operational theory is **noncontextual** if

Operational equivalence of two experimental procedures \[\rightarrow\] Equivalent representations in the ontological model

Preparation
Noncontextuality
Operational equivalence classes of preparations

\[ P \sim P' \]

\[ \forall M : p(X|P, M) = p(X|P', M) \]
Difference of Equivalence class
Example from quantum theory

Different density op's

\[ \rho \quad \rho' \]
Example from quantum theory

\[ \frac{1}{2} I = \frac{1}{2} |0 \rangle \langle 0| + \frac{1}{2} |1 \rangle \langle 1| \]

\[ \frac{1}{2} I = \frac{1}{2} |+ \rangle \langle +| + \frac{1}{2} |- \rangle \langle -| \]
Preparation noncontextual model

\[ \mu(\lambda) \]

\[ \lambda \]
Preparation **contextual model**

\[ \mu(\lambda | P) \]

\[ \mu(\lambda | P') \]

\[ \lambda \]

\[ P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9 \]

\[ M_1, M_2, M_3, M_4, M_5, M_6, M_7, M_8, M_9, M_{10} \]
Measurement
Noncontextuality
Operational equivalence classes of measurements

\[ M \simeq M' \]

\[ \forall P : p(X|P, M) = p(X|P, M') \]
Example from quantum theory
Example from quantum theory

\[
\{ |\psi_1\rangle\langle \psi_1|, I - |\psi_1\rangle\langle \psi_1| \} = \{ |\psi_1\rangle\langle \psi_1|, I - |\psi_1\rangle\langle \psi_1| \}
\]

\[
I - |\psi_1\rangle\langle \psi_1| = |\psi_2\rangle\langle \psi_2| + |\psi_3\rangle\langle \psi_3| = |\psi_2\rangle\langle \psi_2| + |\psi_3\rangle\langle \psi_3|
\]
Measurement

noncontextual model
Measurement contextual model
Measurement
noncontextual model
\[ P \simeq P' \]
\[ \forall M : p(X|P, M) = p(X|P', M) \]

\[ M \simeq M' \]
\[ \forall P : p(X|P, M) = p(X|P, M') \]

Preparation noncontextuality:
\[ \mu(\lambda|P) = \mu(\lambda|P') \]

Measurement noncontextuality:
\[ \xi(X|\lambda, M) = \xi(X|\lambda, M') \]

The best explanation of context-independence at the operational level is context-independence at the ontological level.
Transformation noncontextuality is defined similarly

The only natural assumption is universal noncontextuality
Comparison with other approaches
Other approaches agree that to assume noncontextuality is to assume an inference of the form:

Some equivalence relation holding between two experimental procedures $\rightarrow$ Equivalent representations in the ontological model

But
- focus on measurements
- disagree on the nature of the equivalence relation
- disagree on the nature of the ontological representation
Disagreement #1:
The nature of the equivalence relation between measurement procedures
The competing proposal:

Suppose B and C are two incompatible measurements that are each compatible with A

“The experimental apparatus used for measuring, e.g., [A] must be the same when [it] is measured together with [B], and when it is measured together with [C].”

Sameness of apparatus not a necessary condition:
Sameness of apparatus not a necessary condition:
Sameness of apparatus not a sufficient condition:

![Diagram showing two setups with M2 and M3 disconnected from M1](image-url)
Operational equivalence says: the relevant notion of sameness for two measurement procedures on a system is sameness of statistics for all preparation procedures on that system.
Operational equivalence says: the relevant notion of sameness for two measurement procedures on a system is **sameness of statistics for all preparation procedures on that system.**

It is the grounds by which we justify modeling two measurements equivalently in QT.
Disagreement #2:
The nature of the ontological representation of measurements
Measurement
noncontextual model
measurement noncontextuality and outcome determinism = KS-noncontextuality

But, in face of a violation of KS-noncontextuality, we could give up outcome determinism
Theorem: A universally noncontextual ontological model assigns outcomes to a quantum measurement deterministically if and only if the measurement is sharp


All real-world measurements have some degree of unsharpness
\[ M_S \simeq M'_S \]

Thus

\[ \xi(X|\lambda_S, M_S) = \xi(X|\lambda_S, M'_S) \]
No evidence for $M_{sa} \simeq M_{sa}'$

No justification for $\xi(X|\lambda_s, \lambda_a, M_{sa}) = \xi(X|\lambda_s, \lambda_a, M_{sa}')$
From $M_S \simeq M'_S$ all we can infer is

$$\int d\lambda_a \xi(X|\lambda_s, \lambda_a, M_{sa}) \mu(\lambda_a|P_a) = \int d\lambda_a \xi(X|\lambda_s, \lambda_a, M'_{sa}) \mu(\lambda_a|P'_a)$$
Challenges to direct experimental testability
How do we derive constraints on experimental statistics from noncontextuality when the preparations and measurements are noisy/unsharp?

The problem of contending with unsharpness
How do we experimentally establish operational equivalence?

**Problem of inexact operational equivalences**

\[ P \simeq P' \]
\[ \forall M : p(X|P, M) = p(X|P', M) \]

Noncontextuality

\[ M \simeq M' \]
\[ \forall P : p(X|P, M) = p(X|P, M') \]

Noncontextuality

\[ \mu(\lambda|P) = \mu(\lambda|P') \]

Preparation

\[ \xi(X|\lambda, M) = \xi(X|\lambda, M') \]

Measurement
How do we experimentally establish operational equivalence?

The problem of the universal quantifier

\[
P \simeq P' \\
\forall M : p(X|P, M) = p(X|P', M)
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Preparation noncontextuality

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\mu(\lambda|P) = \mu(\lambda|P')
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\[
M \simeq M' \\
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\]

Measurement noncontextuality

\[
\xi(X|\lambda, M) = \xi(X|\lambda, M')
\]
The problem of contending with unsharpness
Kunjwal and RWS, “From the Kochen-Specker theorem to noncontextuality inequalities without assuming determinism”, PRL 115, 110403 (2015)


Kunjwal and RWS, “Translating proofs of the Kochen-Specker theorem without KS-uncolourability into noise-robust noncontextuality inequalities”, forthcoming

(See talk on Friday by Ravi Kunjwal)

The Peres-Mermin square

\[
\begin{array}{ccc}
X \otimes I & I \otimes X & X \otimes X \\
I \otimes Z & Z \otimes I & Z \otimes Z \\
X \otimes Z & Z \otimes X & Y \otimes Y \\
\end{array}
\]

\[
(X \otimes I)(I \otimes X)(X \otimes X) = I \otimes I,
\]

\[
(I \otimes Z)(Z \otimes I)(Z \otimes Z) = I \otimes I,
\]

\[
(X \otimes Z)(Z \otimes X)(Y \otimes Y) = I \otimes I,
\]

\[
(X \otimes I)(I \otimes Z)(X \otimes Z) = I \otimes I,
\]

\[
(I \otimes X)(Z \otimes I)(Z \otimes X) = I \otimes I,
\]

\[
(X \otimes X)(Z \otimes Z)(Y \otimes Y) = -I \otimes I.
\]
The Peres-Mermin square

\[
\begin{array}{ccc}
X \otimes I & I \otimes X & X \otimes X \\
I \otimes Z & Z \otimes I & Z \otimes Z \\
X \otimes Z & Z \otimes X & Y \otimes Y \\
\end{array}
\]

\[
\begin{align*}
[X \otimes I]_\lambda [I \otimes X]_\lambda [X \otimes X]_\lambda &= +1, \\
[I \otimes Z]_\lambda [Z \otimes I]_\lambda [Z \otimes Z]_\lambda &= +1, \\
[X \otimes Z]_\lambda [Z \otimes X]_\lambda [Y \otimes Y]_\lambda &= +1, \\
[X \otimes I]_\lambda [I \otimes Z]_\lambda [X \otimes Z]_\lambda &= +1, \\
[I \otimes X]_\lambda [Z \otimes I]_\lambda [Z \otimes X]_\lambda &= +1, \\
[X \otimes X]_\lambda [Z \otimes Z]_\lambda [Y \otimes Y]_\lambda &= -1.
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    [I \otimes X]_\lambda[Z \otimes I]_\lambda[Z \otimes X]_\lambda &= +1, \\
    [X \otimes X]_\lambda[Z \otimes Z]_\lambda[Y \otimes Y]_\lambda &= -1.
\end{align*}
\]
Distn’s have support only on values such that

\[ m_{11}m_{12}m_{13} = 1 \]
\[ m_{21}m_{22}m_{23} = 1 \]
\[ m_{31}m_{32}m_{33} = 1 \]
\[ m_{11}m_{21}m_{31} = 1 \]
\[ m_{12}m_{22}m_{32} = 1 \]
\[ m_{13}m_{23}m_{33} = -1 \]
An indeterministic noncontextual assignment

\[ m_{11} \] [+]
\[ m_{12} \] [+]
\[ m_{13} \] [+]

\[ m_{21} \] [+]
\[ m_{22} \] (\( \frac{1}{2} [++]+ \frac{1}{2} [-] \))
\[ m_{23} \] (\( \frac{1}{2} [++] + \frac{1}{2} [-] \))

\[ m_{31} \] [+]
\[ m_{32} \] (\( \frac{1}{2} [++] + \frac{1}{2} [-] \))
\[ m_{33} \] (\( \frac{1}{2} [+] + \frac{1}{2} [-] \))
An indeterministic noncontextual assignment

\[
\begin{array}{ccc}
  m_{11} & +1 & \color{green}\checkmark \\
  m_{21} & +1 & \color{brown}\checkmark \\
  m_{31} & +1 & \color{blue}\checkmark \\
  m_{12} & +1 & \color{green}\checkmark \\
  m_{22} & 0 & \color{orange}\checkmark \\
  m_{32} & 0 & \color{cyan}\checkmark \\
  m_{13} & +1 & \color{yellow}\checkmark \\
  m_{23} & 0 & \color{pink}\checkmark \\
  m_{33} & 0 & \color{teal}\checkmark \\
\end{array}
\]
An indeterministic noncontextual assignment

\[ m_{11} +1 \quad m_{12} +1 \quad m_{13} +1 \]

\[ m_{21} +1 \quad m_{22} 0 \quad m_{23} 0 \]

\[ m_{31} +1 \quad m_{32} 0 \quad m_{33} 0 \]
Sources

\[ S \]

\[ S = 1 \]

\[ S = 2 \]

\[ P_1 \]

\[ P_2 \]
\[ P^\text{ave}_S \quad = \quad \sum_s \mu(\lambda, s | S) = \mu(\lambda | S) \]
\[ \mu(\lambda|P^\text{ave}_s) = \sum_s \mu(\lambda, s|S) = \mu(\lambda|S) \]

\[ P^\text{ave}_S \simeq P^\text{ave}_{S'} \]

Preparation noncontextuality

\[ \mu(\lambda|S) = \mu(\lambda|S') \]
Source version of the Peres-Mermin square

\[
\begin{align*}
X \otimes I & \quad I \otimes X & \quad X \otimes X \\
I \otimes Z & \quad Z \otimes I & \quad Z \otimes Z \\
X \otimes Z & \quad Z \otimes X & \quad Y \otimes Y
\end{align*}
\]

9 binary-outcome sources defining the same average preparation

\[
\begin{align*}
(X \otimes I)(I \otimes X)(X \otimes X) &= I \otimes I, \\
(I \otimes Z)(Z \otimes I)(Z \otimes Z) &= I \otimes I, \\
(X \otimes Z)(Z \otimes X)(Y \otimes Y) &= I \otimes I, \\
(X \otimes I)(I \otimes Z)(X \otimes Z) &= I \otimes I, \\
(I \otimes X)(Z \otimes I)(Z \otimes X) &= I \otimes I, \\
(X \otimes X)(Z \otimes Z)(Y \otimes Y) &= -I \otimes I.
\end{align*}
\]
By preparation noncontextuality

\[ \forall i, i' : P^{\text{ave}}_{S_i} \simeq P^{\text{ave}}_{S_{i'}} \iff \forall i, i' : \mu(\lambda|S_i) = \mu(\lambda|S_{i'}) \equiv \nu(\lambda) \]
By preparation noncontextuality

\[ \forall i, i' : P_{S_i}^{\text{ave}} \simeq P_{S_{i'}}^{\text{ave}} \iff \forall i, i' : \mu(\lambda|S_i) = \mu(\lambda|S_{i'}) \equiv \nu(\lambda) \]

\[
\text{pr}(m_j, s_i|M_j, S_i) = \sum_{\lambda \in \Lambda} \xi(m_j|M_j, \lambda) \mu(s_i, \lambda|S_i) \\
= \sum_{\lambda \in \Lambda} \xi(m_j|M_j, \lambda) \mu(s_i|\lambda, S_i) \mu(\lambda|S_i) \\
= \sum_{\lambda \in \Lambda} \xi(m_j|M_j, \lambda) \mu(s_i|\lambda, S_i) \nu(\lambda)
\]
\[ \text{pr}(m_j, s_i | M_j, S_i) = \sum_{\lambda \in \Lambda} \xi(m_j | M_j, \lambda) \mu(s_i | \lambda, S_i) \nu(\lambda) \]
\[
\Pr(m_j, s_i | M_j, S_i) = \sum_{\lambda \in \Lambda} \xi(m_j | M_j, \lambda) \mu(s_i | \lambda, S_i) \nu(\lambda)
\]

\[
\lambda 
\]

\[
\text{Corr} = \frac{1}{9} \sum_{i=1}^{9} \langle s_i m_i \rangle S_i, M_i
\]
\[ \text{pr}(m_j, s_i | M_j, S_i) = \sum_{\lambda \in \Lambda} \xi(m_j | M_j, \lambda) \mu(s_i | \lambda, S_i) \nu(\lambda) \]

\[ \text{Corr} = \frac{1}{9} \sum_{i=1}^{9} \langle s_i m_i \rangle S_i, M_i \]

\[ = \sum_{\Lambda} \text{Corr}(\lambda) \nu(\lambda) \]

\[ \text{Corr}(\lambda) = \frac{1}{9} \sum_{i=1}^{9} \langle s_i \rangle \lambda \langle m_i \rangle \lambda \]
\[
\text{pr}(m_j, s_i | M_j, S_i) = \sum_{\lambda \in \Lambda} \xi(m_j | M_j, \lambda) \mu(s_i | \lambda, S_i) \nu(\lambda)
\]

\[
\text{Corr} = \frac{1}{9} \sum_{i=1}^{9} \langle s_i m_i \rangle_{S_i, M_i} = \sum_{\lambda} \text{Corr}(\lambda) \nu(\lambda)
\]

\[
\text{Corr}(\lambda) = \frac{1}{9} \sum_{i=1}^{9} \langle s_i \lambda | m_i \rangle_{\lambda}
\]

\[
\text{Corr} \leq \frac{5}{9}
\]

Noncontextuality inequality
The problem of inexact operational equivalences
How do we experimentally establish operational equivalence?  
**Problem of inexact operational equivalences**

\[
P \simeq P' \\
\forall M : p(X|P, M) = p(X|P', M) \\
\uparrow \\
M \simeq M' \\
\forall P : p(X|P, M) = p(X|P, M') \\
\uparrow \\
\]

Preparation noncontextuality  
\[\mu(\lambda|P) = \mu(\lambda|P')\]

Measurement noncontextuality  
\[\xi(X|\lambda, M) = \xi(X|\lambda, M')\]

A version of the finite precision loophole

The problem of the universal quantifier
The problem of the universal quantifier
How do we experimentally establish operational equivalence? 

**The problem of the universal quantifier**

\[ P \simeq P' \]
\[ \forall M : p(X|P, M) = p(X|P', M) \]

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\[ \forall P : p(X|P, M) = p(X|P, M') \]

Measurement noncontextuality

\[ \xi(X|\lambda, M) = \xi(X|\lambda, M') \]

Sufficient to consider a tomographically complete set
State tomography for a single qubit
Operational equivalence classes of preparations

\[ P \sim P' \]

\[ \forall M : p(X|P, M) = p(X|P', M) \]
Operational equivalence classes of preparations

\[ P \sim P' \]

\[ \forall M : p(X|P, M) = p(X|P', M) \]
Operational equivalence classes of preparations

\[ P \sim P' \]

\[ \forall M : p(X|P, M) = p(X|P', M) \]
Operational equivalence classes of measurements

\[ M \simeq M' \]
\[ \forall P : p(X|P, M) = p(X|P, M') \]
Mazurek, Pusey, Resch and RWS, “Deviations from quantum theory in the landscape of generalized probabilistic theories: direct constraints from experimental data”, forthcoming

See talk on friday by Mike Mazurek
Mazurek, Pusey, Resch and RWS, “Deviations from quantum theory in the landscape of generalized probabilistic theories: direct constraints from experimental data”, forthcoming

See talk on friday by Mike Mazurek

Note: In principle, one can never verify the claim that a given set of procedures is tomographically complete, but one can accumulate more and more evidence for it by trying one’s best to falsify the claim and failing to do so.
Why this notion of noncontextuality is natural
Leibniz’s principle of the identity of indiscernibles

If a physical model posits two scenarios that are empirically indistinguishable in principle but nonetheless are represented as ontologically distinct, this model should be rejected and replaced with one that makes these two scenarios ontologically the same.
The credentials of Leibniz’s principle
Fields before hole transformation

THE HOLE

Fields after hole transformation

THE HOLE

John Norton, SEP
The credentials of Leibniz’s principle

No fine-tuning
No overfitting
Nonclassicality of operational theory  =  Failure to admit of a noncontextual ontological model
Nonclassicality of operational theory = Failure to admit of a Leibnizian ontological model
What to do about the fact that quantum theory and nature fail to admit of a Leibnizian ontological model?
Abandon Leibniz’s principle? No.
Abandon Leibniz’s principle? No.

Give up on realism? No.
Abandon Leibniz’s principle? No.

Give up on realism? No.

Devise a notion of realism that goes beyond the ontological models framework and salvages Leibniz’s principle (and therefore the spirit of locality and noncontextuality)
No measurement of $A$ reveals any info about $S$

\[
P(Y|T\lambda_A) \quad P(X\lambda_A|S)
\]

where

\[
P(\lambda_A|S) = P(\lambda_A)
\]

\[
P(XY|ST) = \sum_{\lambda_A} P(Y|T\lambda_A)P(X\lambda_A|S)
\]

Satisfy noncontextuality inequalities
No measurement of A reveals any info about S

\[ P(Y|T\lambda_A) \]
\[ P(X\lambda_A|S) \]
\[ P(\lambda_A|S) = P(\lambda_A) \]
where
\[ \rho_A|S = \rho_A \]

\[ P(XY|ST) = \sum_{\lambda_A} P(Y|T\lambda_A)P(X\lambda_A|S) \]
\[ = \text{Tr}_A(\rho_{Y|TA}\rho_XA|S) \]

Satisfy noncontextuality inequalities

See: Leifer and RWS, PRA 88, 052130 (2013)
\[ M(P_1) \]
\[ M(P_2) \]
\[ M(P) = w_1 M(P_1) + (1-w_1) M(P_2) \]