Title: Thermal stability and fault-tolerance in symmetry protected topological phases

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Abstract: <p>A recent breakthrough in the condensed matter community is the identification and characterization of a rich set of ordered states, known as symmetry protected topological (SPT) phases. These phases are not only fascinating from the perspective of fundamental physics but have also found powerful applications in quantum computation. Very little is known about the thermal stability of SPT ordered systems, or whether their associated computational properties may survive at non-zero temperature. In this talk I will give a brief introduction to SPT phases of matter, and address the question of whether these phases can survive at non-zero temperature. In particular, I will focus on understanding the thermal stability and fault-tolerant properties of models studied in the theory of quantum computation.</p>

Thermal stability and fault-tolerance in symmetry protected topological phases

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Aleksander Kubica, Stephen Bartlett

based on arXiv:1611.05450

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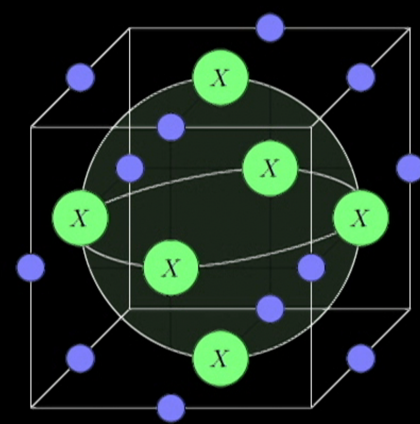
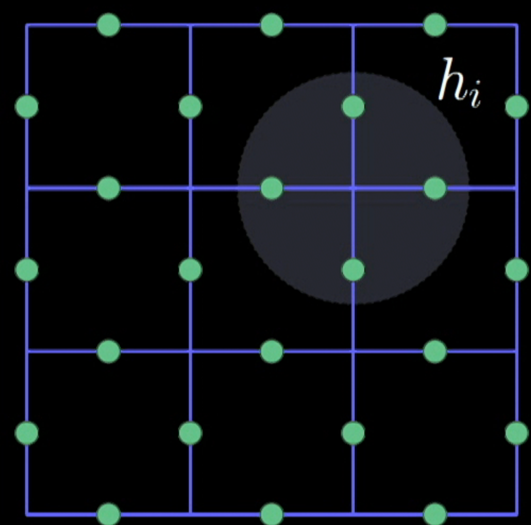
ARC CENTRE OF EXCELLENCE FOR ENGINEERED QUANTUM SYSTEMS

This talk

- Focussing on symmetry protected topological (SPT) order and its stability at nonzero temperature
  - SPT phases of matter are fascinating in their own right
  - Have broad applications in quantum information and fault-tolerance



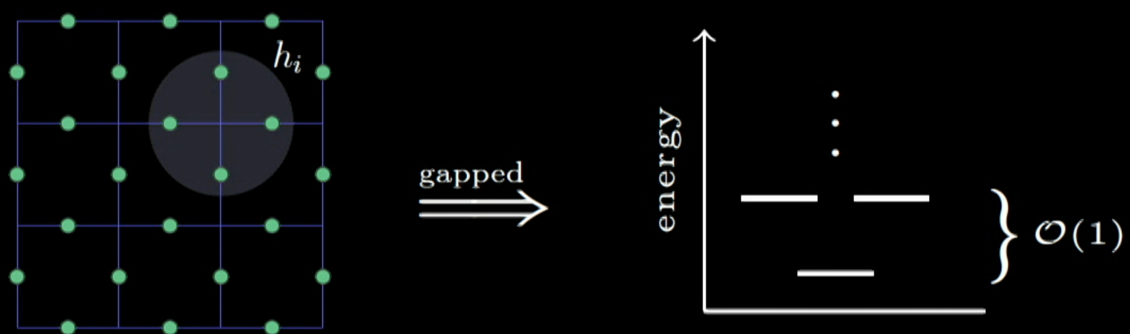
Many-body phases of matter



- Hilbert space $\mathcal{H} = \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d \otimes \cdots$
- Local Hamiltonian $H = \sum_i h_i$

Gapped many-body phases

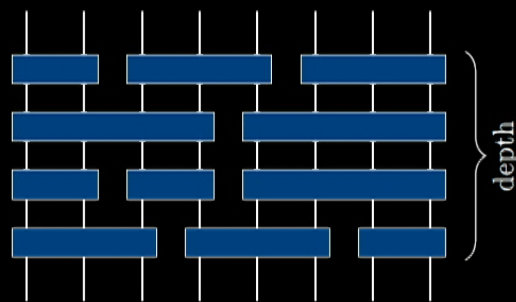
- Interested in systems with a gap



- Phase: Gapped ground states with similar properties

Gapped many-body phases

Phase: Gapped ground states $|\psi_1\rangle \sim |\psi_2\rangle \iff$ they are related by a constant depth unitary circuit



Note: most of the time $\log(L) \equiv$ const.

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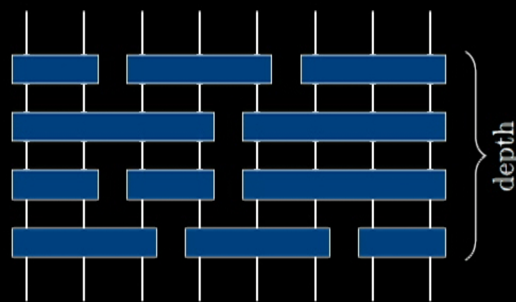

- Trivial phase: $\sim$ product state
- Topologically ordered: $\nsim$ product state
- Much richer with symmetries!

Note: most of the time $\log(L) \equiv$ const.

Symmetry protected topological order

- Gapped Hamiltonian H with symmetries S_1, S_2, S_3, $\cdots$

$$[H, S_1] = [H, S_2] = [H, S_3] = \cdots = 0$$

Symmetry protected topological order

- Gapped Hamiltonian H with symmetries $S_1, S_2, S_3, \cdots$

$$[H, S_1] = [H, S_2] = [H, S_3] = \cdots = 0$$

- A ground state $|\psi\rangle$ is SPT ordered if:

1. There exists a constant depth circuit U such that

$$U|\psi\rangle = |\phi_{\text{prod}}\rangle \sim |000\ldots\rangle$$

2. No symmetric constant depth circuit can map $|\psi\rangle$ to a product state

Example: SPT order in 1D

- Easiest example: 1D cluster state global *onsite* symmetry.

$$H = -\sum_j Z_{j-1} X_j Z_{j+1}$$

Chen *et al.* arXiv:1106.4772

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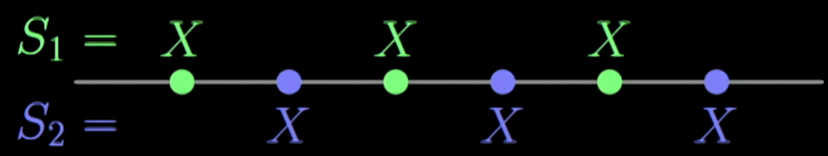
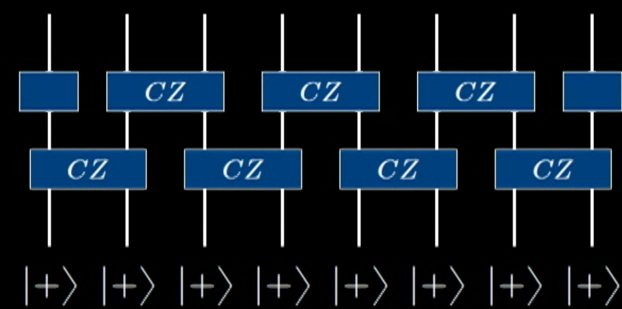




- Has a symmetry $[S_1, H] = [S_2, H] = 0$

- Each individual CZ gate breaks the symmetry
- Any sublinear preparation must break the symmetry
- Higher dimensional generalizations: group cohomology models

Chen *et al.* arXiv:1106.4772

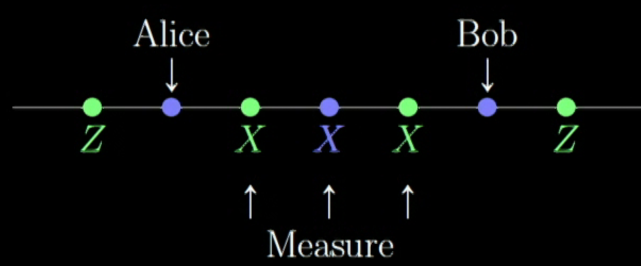
Application 1: MBQC and computational phases of matter

- SPTs are robust resources for measurement based quantum computation

$$\text{SPT order} \longrightarrow \text{MBQC}$$

$$\text{Topo order} \longrightarrow \text{Quantum code}$$

- Example: generating EPR pairs using any state in the cluster phase:



$$|\psi_{\text{out}}\rangle \sim |00\rangle + |11\rangle$$

Else *et al.* arXiv:1201.4877, arXiv:1207.4805,
Miller *et al.* arXiv:1409.6242, Raussendorf *et al.* arXiv:1609.07549

Application 2: Fault-tolerant gates in topological codes

- SPTs have found application in the classification of transversal gates in topological codes

Yoshida arXiv:1503.07208, arXiv:1509.03626

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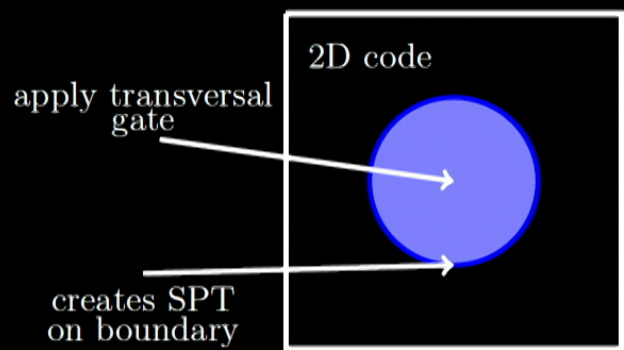
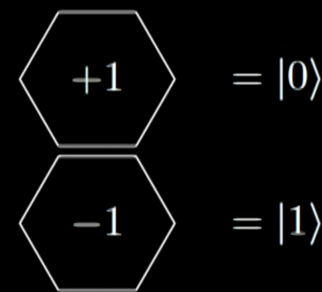
- SPTs have found application in the classification of transversal gates in topological codes
- In the 2D color code, transversal $P = \sqrt{Z}$ creates the cluster state:



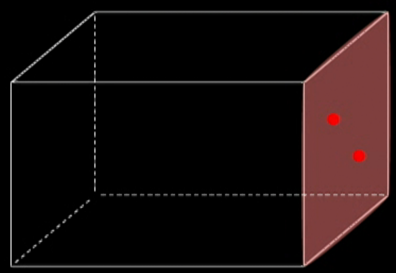


- Protecting symmetry comes from parity constraints on anyons
- Framework to systematically construct fault-tolerant gates in d-dimensional topological code using $(d-1)$-dimensional SPT order

Yoshida arXiv:1503.07208, arXiv:1509.03626

Application 3: boundary theories

- When a d-dim SPT is placed on a lattice with a boundary, the $(d-1)$-dim boundary usually (always?) has 'protected modes':



- The boundary theory must either
  - break the symmetry
  - be gapless
  - have topological order
- New *anomalous* phases can appear on the boundary of a 3d theory which cannot be realised in a 2d system
- New exotic SET phases obtained by gauging the symmetry

Chen *et al.* arXiv:1106.4752, Wang *et al.* arXiv:1512.09111

The questions

- Two seemingly unrelated questions:
  1. Does SPT order survive at nonzero temperature? Do the associated computational structures persist at nonzero temperature?
  2. What is the underlying order giving rise to the high-threshold in the topological formulation of MBQC?
- Goal of the talk:
  1. We rule out thermal stability of a large class of SPT models.
  2. Provide evidence of a correspondence:

$$\text{SPT at } T>0 \rightarrow \text{Fault-tolerant MBQC}$$

Defining SPT order at nonzero temperature

- Interested in the equilibrium Gibbs ensemble of H at $\beta = T^{-1}$

$$\rho(\beta) = \frac{e^{-\beta H}}{\mathrm{Tr}\, e^{-\beta H}}$$

- What is the circuit depth required to prepare it, from a classical ensemble (e.g. $H_{\mathrm{cl}} = -\sum_v X_v$) (approximately, with ancillas)

Def We say ρ is (r, ϵ) SPT-trivial if

$$\left\| \rho - \mathrm{Tr}_{\mathcal{H}'} \left(U \rho_{cl} U^\dagger \right) \right\|_1 < \epsilon$$

- ρ_{cl} is the Gibbs state of a classical Hamiltonian on an enlarged space
- U is a symmetric circuit of depth r
- $\mathcal{H}'$ is the ancillary space

First result: instability of onsite SPT phases

Informal statment: SPT models protected by global *onsite* symmetries are not thermally robust

- Recall onsite symmetries:

$$S_{\text{onsite}} = \prod_{\text{sites } i} u_i$$

- SPT models protected by onsite symmetry are generically classified by group cohomology (completely classified in $d = 1, 2$)

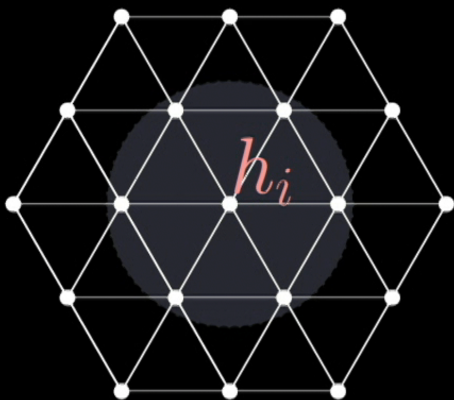
Result 1: Theorem: For any $T>0$, SPT models described by group cohomology are (r, ϵ) SPT-trivial for

- $r = \mathcal{O}(\log(L) \log\log(L))$
- $\epsilon = \text{poly}^{-1}(L)$

where L is linear size of a d dimensional lattice.

Onsite SPTs are trivial: Sketch in 2D

- Sketch of proof in 2D for nontrivial $\mathbb{Z}_2$ SPT phase



- Qubits on vertices of triangular lattice
- $H = \sum_v h_v$, commuting terms
- $\mathbb{Z}_2$ symmetry given by $S = \otimes_v X_v$
- Unique, gapped ground state

- Belongs to a nontrivial SPT phase: cannot find a local symmetric unitary U such that

$$UHU^\dagger = -\sum_v X_v$$

- Idea: introduce 'holes' to circumvent symmetry and give a low-depth preparation of the Gibbs state

Onsite SPTs are trivial: Sketch in 2D

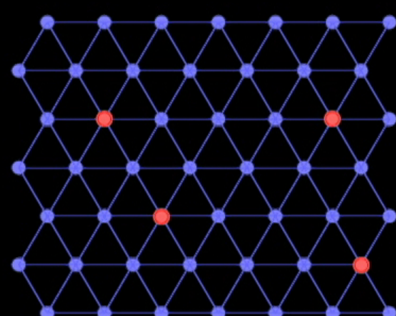
- First technical tool - approximation by 'imperfect Hamiltonian'

Remove terms with probability p_β

Hamiltonian $H = \sum_v h_v$ → Imperfect Hamiltonian $H(p_\beta)$

$$p_\beta = \frac{2}{1+e^{2\beta}}$$

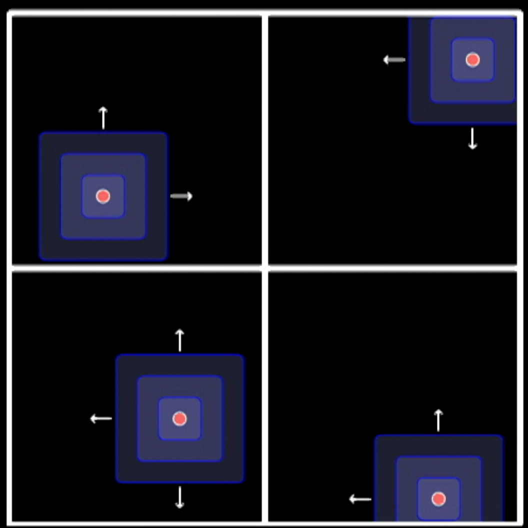
- Ground space of $H(p_\beta)$ approximates the Gibbs state of H up to $\text{poly}^{-1}(L)$ error



Hastings arXiv:1106.6026, Siva *et al.* arXiv:1603.07805

Onsite SPTs are trivial: Sketch in 2D

- Second technical tool: local disentangler
- High probability of a missing term in each $\log^{\frac{1}{2}}(L) \times \log^{\frac{1}{2}}(L)$ region



- Can construct a symmetric disentangler near each missing term, e.g. for qubits

$$\mathcal{D}_v : h_v \mapsto X_v$$

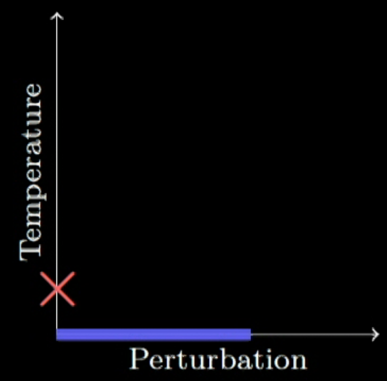
...and continue:

$$\mathcal{D} : h_{v'} \mapsto X_{v'}$$

- $\mathcal{O}(\log(L))$ spins to disentangle with gates of range $\leqslant \mathcal{O}(\log^{\frac{1}{2}}(L))$
- This gives a low-depth preparation of the Gibbs ensemble

Instability of SPT phases

- Works generally in d-dims using renormalization via Pachner moves
- For SPTs protected by onsite symmetry:



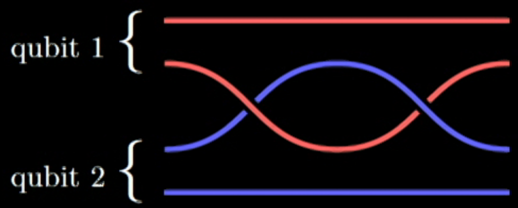
- Does there exist thermally stable SPT order?
  Thermal resources for MBQC?
  $\Longrightarrow$ Beyond group cohomology?

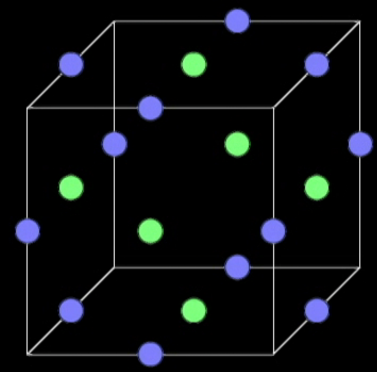
> Next result: The existence of a low temperature, SPT ordered phase: the 3D cluster model protected by a higher form symmetry

- Connection between higher form symmetries and error correction

Switching gears: The topological MBQC scheme

- Fault-tolerant MBQC using the 3d cluster state
- Computation proceeds by 'simulated' braiding of holes in the toric code (pair of *e* or *m* holes encoded a qubit)





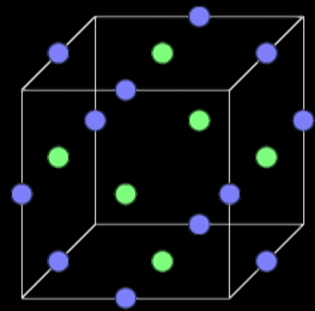
- Very high threshold scheme at ~ 1 %
- What underpins the thermal stability/high threshold of computation?
  - No thermal phase transition!
  - Lets explore in the context of SPT phases!

Raussendorf *et al.* arXiv:0703143, arXiv:0510135

The 3D cluster model

- Cubic lattice with qubits on edges and faces first introduced by Raussendorf, Bravyi, Harrington 05

$$H_{\mathcal{C}} = -\sum_u K_u$$



$$K = \begin{matrix} & Z & \\ Z & X & Z \\ & Z & \end{matrix}$$

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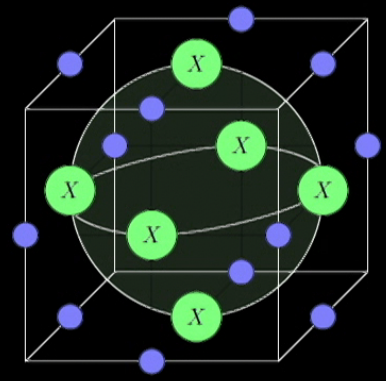
- Unique ground state: $K_u \left|\psi_{\mathcal{C}}\right\rangle = \left|\psi_{\mathcal{C}}\right\rangle$
- Constant depth preparation: $\left|\psi_{\mathcal{C}}\right\rangle = \prod_{\langle u,w \rangle} CZ_{u,w} \left|+\right\rangle^N$

Generalized (higher form) symmetries of the cluster model

- Generalized symmetry: $\mathbb{Z}_2 \times \mathbb{Z}_2$ 1-form symmetry.

$$S_{\mathcal{M}}(g) = \prod_{u \in \mathcal{M}} X_u, \qquad \mathcal{M} \text{ a 2-dim closed surface}$$



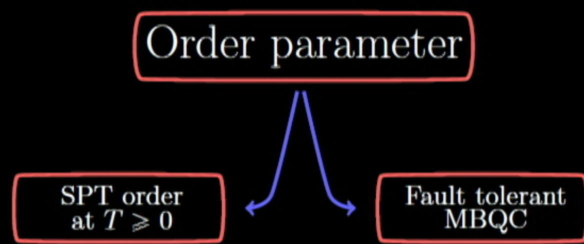
$$K = \begin{matrix} & Z & \\ Z & X & Z \\ & Z & \end{matrix}$$

$$[H, S_{\mathcal{M}}(g)] = 0$$

- A symmetry for each color
- Operators naturally arise in error correction in topological MBQC

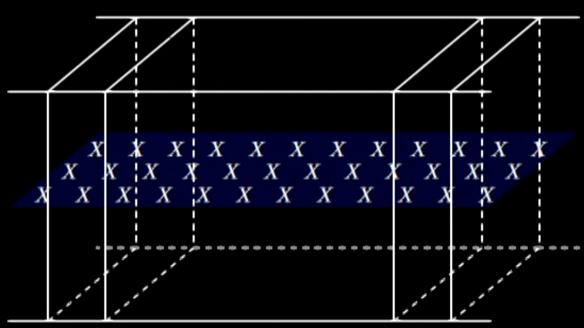
SPT order at $T \geqslant 0$ and topological MBQC

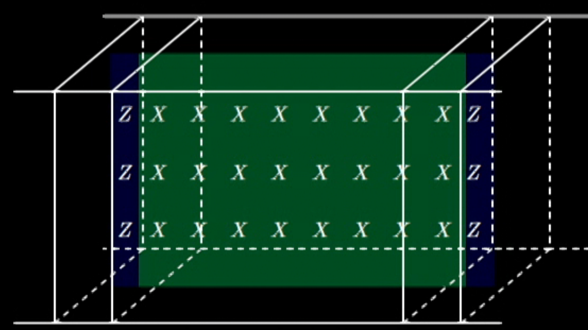
- Claim: correspondence between fault-tolerant MBQC and thermal SPT order
  - Manifests through nonlocal order parameters



Sheet order parameter

- Order parameters specified by choosing a 2D submanifold (with boundary)
- Take two operators M_1 and M_2 on $\Gamma \cong S^1 \times I$:





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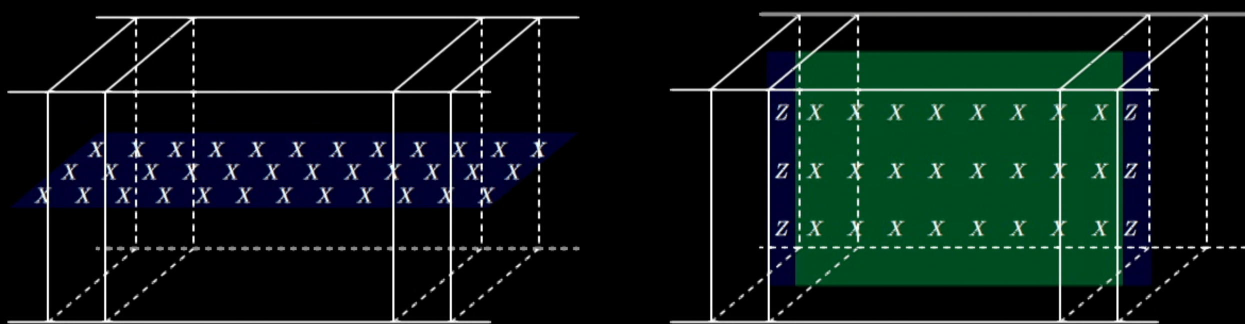
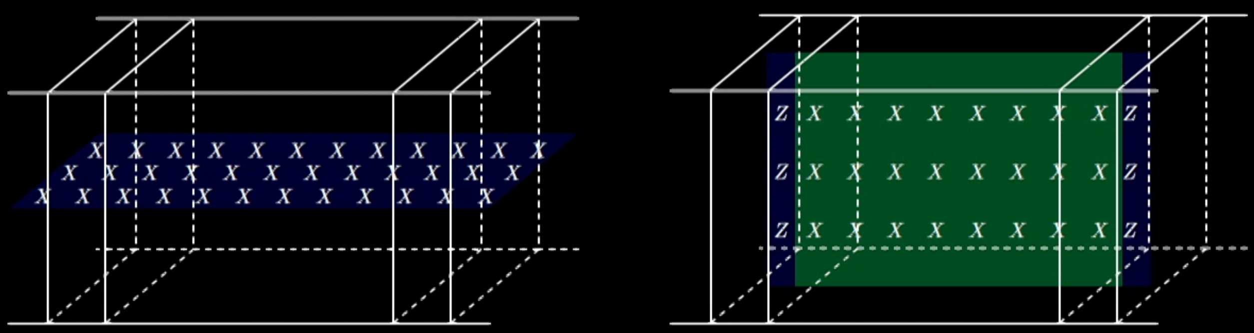


- We allow error corrected expectation values
- Φ_{ec} consists of measurement of 1-form operators and a recovery

$$O_\Gamma = \frac{1}{2}\,\text{Tr}\left(\Phi_{\text{ec}}(\rho)(M_1 + M_2)\right)$$

Sheet order parameter distinguishes SPT phases

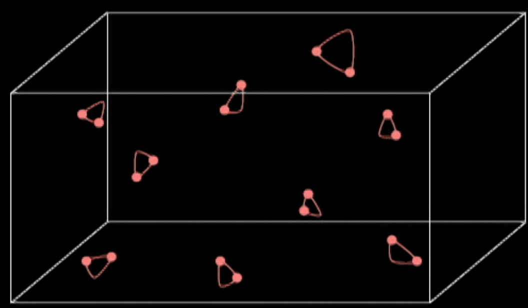
- SPT-trivial phase has low values of O_Γ
  - While $[M_1, M_2] = 0$...
    they anticommute in a neighbourhood of their boundary



- An (r, ϵ) SPT-trivial state needs $r \sim L$ to be a joint eigenstate of the anticommuting boundaries
- Error correction cannot improve this

Sheet order parameter distinguishes SPT phases

- Cluster phase has near maximal values of O_Γ
- Perform error correction to return to $S_{\mathcal{M}}(g) = +1$



1. Excitations are string like objects
2. Syndrome = boundaries of strings
3. Apply correction map to return to $+1$-eigenspace of 1-form operators

- Correction succeeds if no homologically nontrivial excitations
- This protocol succeeds below T_c (via mapping to 3D random plaquette Ising gauge model) like 3D toric code

Result 2: There exists a temperature T_c such that the Gibbs state of the RBH model is 1-form SPT ordered for $0 \leqslant T < T_c \simeq 0.3\Delta$

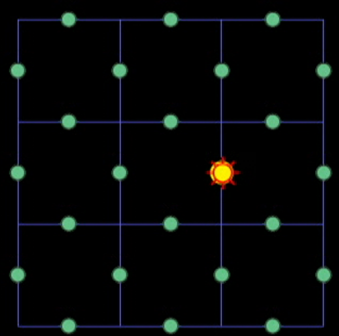
Dennis *et al.* 0110143

Putting it all together

- Measurement errors $\mapsto$ thermal errors according to

$$\frac{p}{1-p} = e^{-2\beta}$$

Thermal SPT order $\rightarrow$ fault tolerance / single shot error correction



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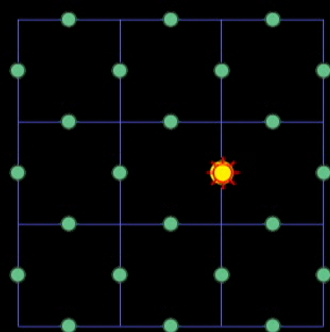


- Definition of SPT is protocol independent as one can use optimal decoder i.e. maximum likelihood decoding...
- ...but notions of (symmetry protected) topological order seem to be intimately intertwined with error correction

Conclusion: in this talk

1. Thermal fragility of SPT models protected by onsite symmetries
2. Robustness of SPT in the 3D cluster scheme and its computational aspects (distilling entanglement, error correction, fault tolerant gates*)

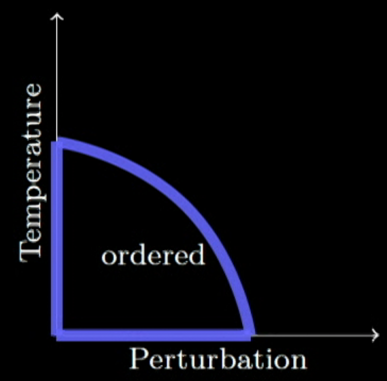
- Steps toward understanding what is possible: thermally stable computational phases of matter

Take home: The importance of higher form symmetry for stability and fault tolerance in MBQC. Correspondence:

$$\text{SPT at } T>0 \rightarrow \text{Fault-tolerant MBQC}$$

Further questions

1. The relationship between thermal SPT non triviality and computational power (in MBQC)
   $\Longrightarrow$ Analogous to the question of thermal topological order and its relationship to self-correcting quantum memories
2. Other higher-form SPTs as resources for MBQC



3. Other defects in the topological MBQC scheme (e.g. twists)
4. Foliation/clusterization of other codes with better transversal gates
5. Physics away from equilibrium, prethermalization, floquet SPTs
6. Symmetry principles to understand 3D gauge color code physics

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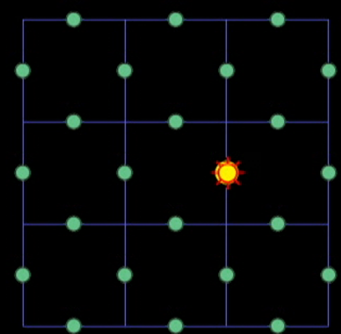
Thermal SPT order $\rightarrow$ fault tolerance / single shot error correction



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