Title: The Universal Renormalization Group Machine - 2

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Abstract: Functional renormalization group techniques based on Wetterichs equation provide a powerful tool for studying the properties of gravity in the quantum regime and its connection to the observable low-energy world. Explicit computations in this framework require the evaluation of functional traces over operator-valued functions. In these lectures I will give a pedagogical introduction to the Universal Renormalization Group Machine, a combinatorial algorithm which allows evaluating such traces in a systematic way. Moreover, I will discuss flow equations for â€œcomposite operatorsâ€• which may pave the route towards developing a microscopic description of quantum spacetime within the Asymptotic Safety approach.

The Einstein-Hilbert truncation: setup

Einstein-Hilbert truncation: two running couplings: G_k, Λ_k

$$\Gamma_k = \frac{1}{16\pi G_k} \int d^dx \sqrt{g} \{ -R + 2\Lambda_k \} + \Gamma_k^{\rm gf} + S^{\rm gh}$$

- goal: project flow onto G-$\Lambda-$plane

1. harmonic gauge fixing

$$\Gamma_k^{\rm gf} = \frac{1}{32\pi G_k} \int d^dx \sqrt{\bar{g}} F_\mu \bar{g}^{\mu\nu} F_\nu \,, \qquad F_\mu = \bar{D}^\nu h_{\mu\nu} - \tfrac{1}{2} \bar{D}_\mu h^\nu{}_\nu$$

2. field decomposition: split off the conformal mode

$$h_{\mu\nu} = \hat{h}_{\mu\nu} + \tfrac{1}{d} \, \bar{g}_{\mu\nu} \, h$$

identities for component fields

$$\bar{g}^{\mu\nu} \hat{h}_{\mu\nu} = 0 \,, \qquad h = \bar{g}^{\mu\nu} h_{\mu\nu} \,.$$

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   identities for component fields

$$\bar{g}^{\mu\nu} \hat{h}_{\mu\nu} = 0 \, , \qquad h = \bar{g}^{\mu\nu} h_{\mu\nu} \, .$$

3. choose background as d-spheres S^d

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Hessian $\Gamma_k^{(2)}$: matrix in field space

$$\left[\Gamma_k^{(2)}\right]_{\hat{h}\hat{h}} = \frac{1}{32\pi G_k} \left[\Delta - 2\Lambda_k + C_T \bar{R}\right] \mathbb{1}_{2T} \,, \qquad C_T = \frac{d^2-3d+4}{d(d-1)}$$

$$\left[\Gamma_k^{(2)}\right]_{hh} = - \frac{1}{32\pi G_k} \frac{d-2}{2d} \left[\Delta - 2\Lambda_k + C_S \bar{R}\right] \,, \qquad C_S = \frac{d-4}{d}$$

$$\left[\Gamma_k^{(2)}\right]_{\bar{C}C} = \left[\Delta + C_V \bar{R}\right] \mathbb{1}_V \,, \qquad C_V = -\frac{1}{d}$$

- Hessian is diagonal in field space
- all differential operators are of Laplacian form
- all terms have a potential $\mathbb{M} = C_s R$

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- Hessian is diagonal in field space
- all differential operators are of Laplacian form
- all terms have a potential $\mathbb{M} = C_s R$

heat-kernel techniques are applicable

The Einstein-Hilbert truncation: setup

Einstein-Hilbert truncation: two running couplings: $G(k), \Lambda(k)$

$$\Gamma_k = \frac{1}{16\pi G(k)} \int d^4x\sqrt{g}\{-R + 2\Lambda_k\} + S^{\mathrm{gf}} + S^{\mathrm{gh}}$$

explicit β-functions for dimensionless couplings $g_k := k^2 G(k)\,,\ \lambda_k := \Lambda(k)k^{-2}$

- Litim-type regulator:

$$k\partial_k g_k = (\eta_N + 2)g_k\,,$$

$$k\partial_k \lambda_k = -(2-\eta_N)\,\lambda_k - \tfrac{g_k}{2\pi}\left[5\tfrac{1}{1-2\,\lambda_k} - 4 - \tfrac{5}{6}\tfrac{1}{1-2\lambda_k}\eta_N\right]$$

- anomalous dimension of Newton's constant:

$$\eta_N = \frac{gB_1}{1-gB_2}$$

$$B_1 = \tfrac{1}{3\pi}\left[5\,\tfrac{1}{1-2\lambda} - 9\,\tfrac{1}{(1-2\lambda)^2} - 7\right]\,,\ B_2 = -\tfrac{1}{12\pi}\left[5\,\tfrac{1}{1-2\lambda} + 6\,\tfrac{1}{(1-2\lambda)^2}\right]$$

Einstein-Hilbert truncation: Fixed Point structure

β-functions for $g_k := k^2 G(k)\,,\ \lambda_k := \Lambda(k) k^{-2}$

$$k\partial_k g_k = (\eta_N + 2) g_k\,,$$

$$k\partial_k \lambda_k = -\left(2 - \eta_N\right)\lambda_k - \frac{g_k}{2\pi}\left[5\,\frac{1}{1-2\,\lambda_k} - 4 - \frac{5}{6}\,\frac{1}{1-2\lambda_k}\eta_N\right]$$

microscopic theory $\Longleftrightarrow$ fixed points of the β-functions

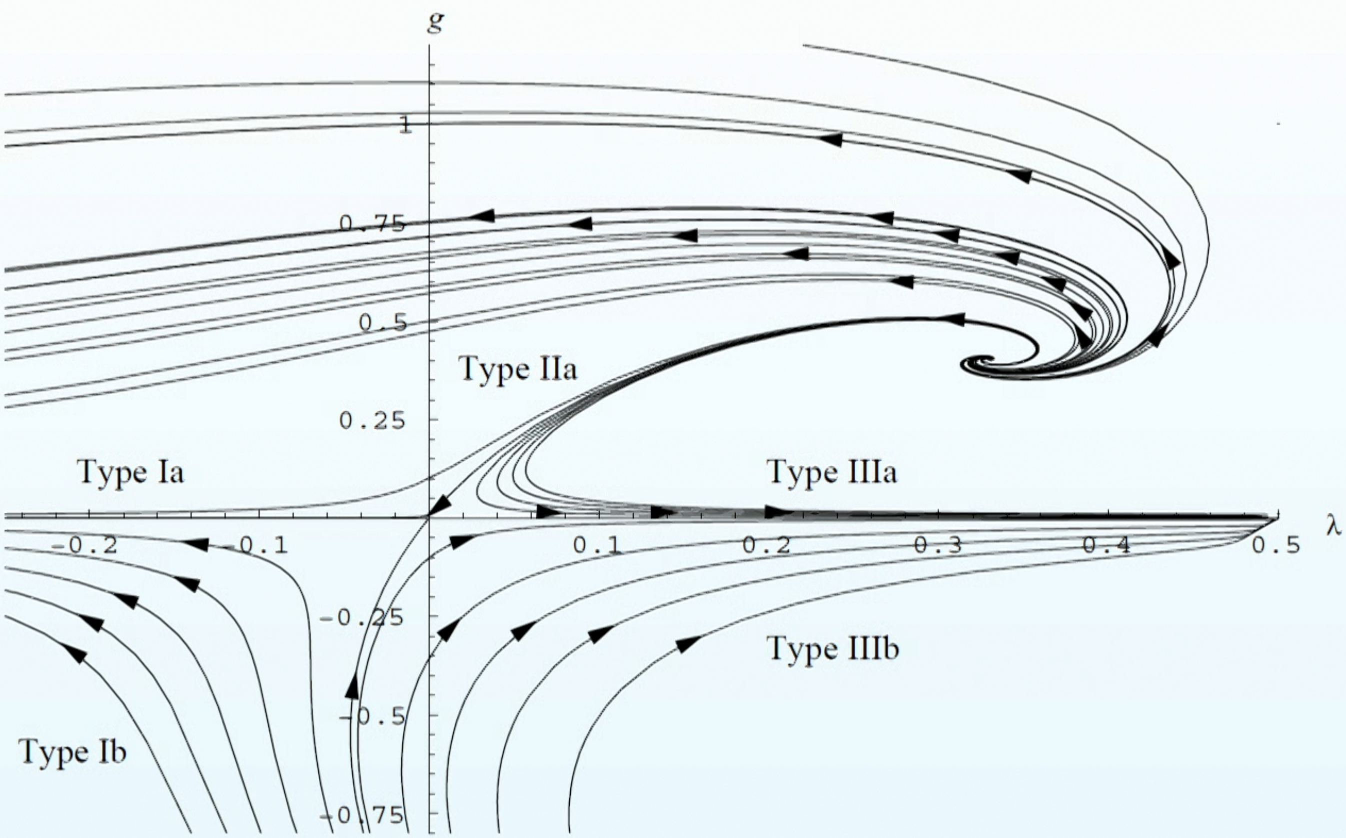
$$\beta_g(g^*, \lambda^*) = 0\,, \qquad \beta_\lambda(g^*, \lambda^*) = 0$$

- Gaussian Fixed Point:
  - at $g^* = 0, \lambda^* = 0 \Longleftrightarrow$ free theory
  - UV-repulsive for $g > 0$
- non-Gaussian Fixed Point ($\eta_N^* = -2$):
  - at $g^* > 0, \lambda^* > 0 \Longleftrightarrow$ "interacting" theory
  - UV attractive in g_k, λ_k

Asymptotic safety: non-Gaussian Fixed Point is UV completion for gravity

Einstein-Hilbert-truncation: the phase diagram

M. Reuter and F. Saueressig, Phys. Rev. D **65** (2002) 065016



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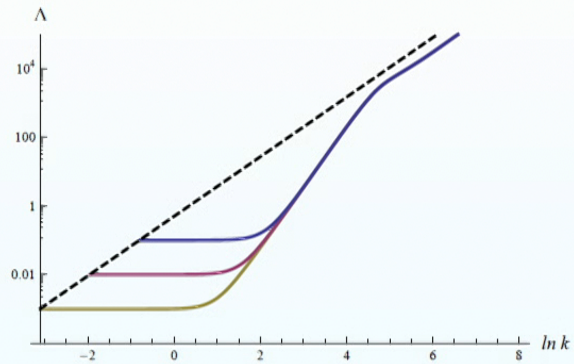
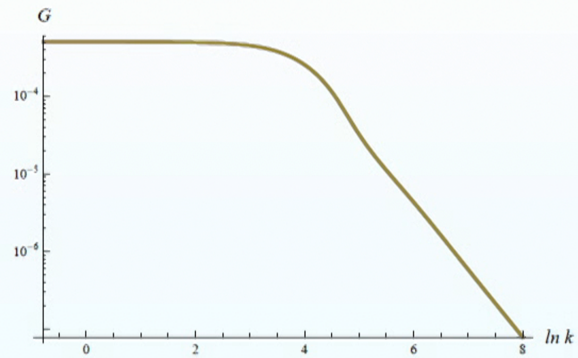
Connecting the quantum to classical gravity





- flow interpolates between
  - fixed point regime: $k \gg 1$
  - classical regime: $k \approx 1$
- compatible with tiny cosmological constant (free parameter)
- break-down at Hubble-horizon scales

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Evaluating the FRGE

uses tabulated heat-kernel coefficients for minimal second-order operators

requirement: reduce operator structure of $\Gamma_k^{(2)}$ to suitable form

- clever choice of background $\bar{g}_{\mu\nu}$
- suitable gauge-fixing
- field decompositions of the fluctuation field

Transverse-traceless decomposition of fluctuations

- vector: $v_\mu = v_\mu^{\rm T} + D_\mu v \quad , \qquad D^\mu v_\mu^{\rm T} = 0$
- graviton: $h_{\mu\nu} = h_{\mu\nu}^{\rm T} + D_\mu \xi_\nu + D_\nu \xi_\mu - \frac{2}{d} g_{\mu\nu} D^\alpha \xi_\alpha + \frac{1}{d} g_{\mu\nu} h$

$$D^\mu h_{\mu\nu}^{\rm T} = 0\,, \qquad g^{\mu\nu} h_{\mu\nu}^{\rm T} = 0\,, \qquad g^{\mu\nu} h_{\mu\nu} = h$$

Exploring the theory space spanned by $\Gamma_k^{\mathrm{grav}}[g]$

Einstein-Hilbert truncation

$\vdots$			
R^8	$\ldots$		
R^7	$\ldots$		
R^6	$\ldots$		
R^5	$\ldots$		
R^4	$\ldots$		
R^3	$C_{\mu\nu}{}^{\rho\sigma} C_{\rho\sigma}{}^{\kappa\lambda} C_{\kappa\lambda}{}^{\mu\nu}$	$R \Box R$	+ 7 more
R^2	$C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$	$R_{\mu\nu} R^{\mu\nu}$	
R			
$\mathbb{1}$			

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Exploring the theory space spanned by $\Gamma_k^{\mathrm{grav}}[g]$

Einstein-Hilbert truncation

polynomial $f(R)$-truncation

$\vdots$			
R^8	$\dots$		
R^7	$\dots$		
R^6	$\dots$		
R^5	$\dots$		
R^4	$\dots$		
R^3	$C_{\mu\nu}{}^{\rho\sigma} C_{\rho\sigma}{}^{\kappa\lambda} C_{\kappa\lambda}{}^{\mu\nu}$	$R \Box R$	+ 5 more
R^2	$C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$	$R_{\mu\nu} R^{\mu\nu}$	
R			
$\mathbb{1}$			

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Einstein-Hilbert projection:

- background sphere S^d + harmonic gauge condition
- background sphere S^d + TT decomposition

polynomial $f(R)$-truncation:

- background sphere S^d + TT decomposition + geometrical gauge condition
  $\Longrightarrow$ generating functional for beta functions $f(R) = u_0 + u_1 R + u_2 R^2 + \ldots$

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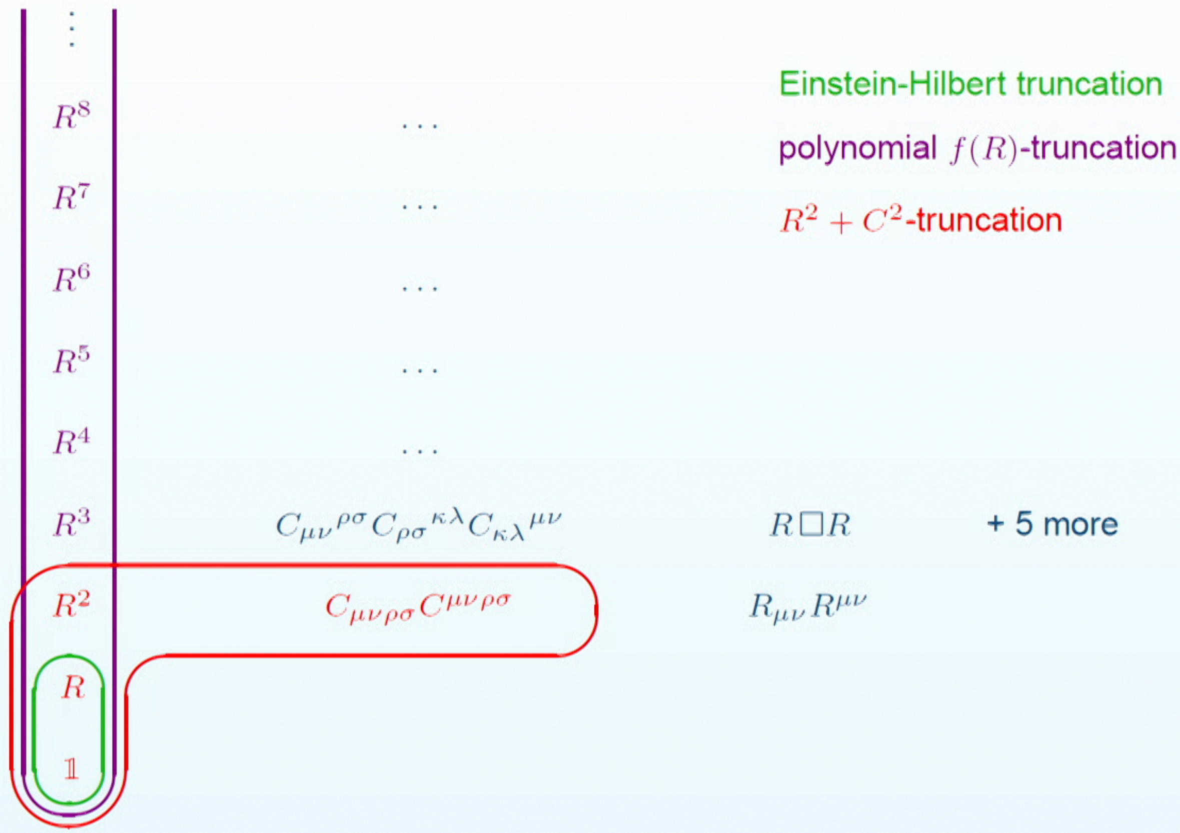
Exploring the theory space spanned by $\Gamma_k^{\text{grav}}[g]$



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$R^2 + C^2$-projection:

- background: Einstein space + TT decomposition + geometrical gauge

Evaluating the FRGE

uses tabulated heat-kernel coefficients for minimal second-order operators

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polynomial $f(R)$-projection:

- background sphere S^d + TT decomposition + geometrical gauge
  $\Longrightarrow$ generating functional for beta functions $f(R) = u_0 + u_1 R + u_2 R^2 + \ldots$

$R^2 + C^2$-projection:

- background: Einstein space + TT decomposition + geometrical gauge running the renormalization group

industrial computations

What if one could track the flow of 20, 100, . . . couplings???

Including the two-loop counterterm in Γ_k

$\vdots$			
R^8	$\ldots$		
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R^6	$\ldots$		
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R^4	$\ldots$		
R^3	$C_{\mu\nu}{}^{\rho\sigma} C_{\rho\sigma}{}^{\kappa\lambda} C_{\kappa\lambda}{}^{\mu\nu}$	$R \Box R$	+ 5 more
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R			
$\mathbb{1}$			

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Two-loop counterterm: setting up the stage

$$k\partial_k \Gamma_k = \frac{1}{2}\mathrm{Tr}\left[(\Gamma_k^{(2)} + \mathcal{R}_k)^{-1}\partial_t \mathcal{R}_k\right]$$

Supplement Einstein-Hilbert action by Goroff-Sagnotti term

$$\Gamma_k^{\mathrm{grav}} = \frac{1}{16\pi G_k}\int d^4x\sqrt{g}\{2\Lambda_k - R\} + \bar{\sigma}\int d^4x\sqrt{g}\, C_{\mu\nu}{}^{\rho\sigma} C_{\rho\sigma}{}^{\kappa\lambda} C_{\kappa\lambda}{}^{\mu\nu}$$

Examine the gravitational contribution to Hessian $\Gamma_k^{(2)}$

$$\Gamma_k^{\mathrm{grav},(2)} = \Gamma_k^{\mathrm{EH},(2)} + \Gamma_k^{\mathrm{GS},(2)}$$

$$\left[\Gamma_k^{\mathrm{GS},(2)}\right]_{\alpha\beta}{}^{\mu\nu} = \bar{\sigma}\left[C^{\cdots\cdot} D.D.D.D. + C^2 D.D. + C^3\right]_{\alpha\beta}{}^{\mu\nu}$$

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Examine the gravitational contribution to Hessian $\Gamma_k^{(2)}$

$$\Gamma_k^{\rm grav,(2)} = \Gamma_k^{\rm EH,(2)} + \Gamma_k^{\rm GS,(2)}$$

$$\left[\Gamma_k^{\rm GS,(2)}\right]_{\alpha\beta}{}^{\mu\nu} = \bar{\sigma}\left[C^{\cdots} D.D.D.D. + C^2 D.D. + C^3\right]_{\alpha\beta}{}^{\mu\nu}$$

- GS-part has the form of a potential $\mathbb{M}^{\rm GS}$
  $\Longrightarrow$ expand in $\mathbb{M}^{\rm GS}$ up to third order
- uncontracted derivatives can not be removed
  $\Longrightarrow$ insertions $\mathbb{M}^{\rm GS}$ are beyond standard heat-kernel techniques

Step 4b: Evaluate the operator traces including insertions $\mathbb{M}$

1. use commutators to bring trace argument into standard form:
   - contracted cov. derivatives: $\Longrightarrow$ collected into a single function $W(\Delta)$
   - remainder: $\Longrightarrow$ matrixvalued insertion $\mathcal{O}$

2. Laplace transform $W(\Delta) \to \tilde{W}(s)$

$$\mathrm{Tr}\left[\,\mathcal{O}\,W(\Delta)\,\right] = \int_0^\infty ds\,\tilde{W}(s)\,\langle x|\,\mathcal{O}\,\mathrm{e}^{-s\Delta}\,|x\rangle$$

3. Evaluate trace using off-diagonal Heat-kernel (act $\mathcal{O}$ on H)

$$\langle x|\,\mathcal{O}\,\mathrm{e}^{-s\Delta}|x\rangle = \langle x|\,\mathcal{O}\,|x'\rangle\langle x'|\mathrm{e}^{-s\Delta}|x\rangle = \int d^4x\sqrt{g}\,\mathrm{tr}_s\left[\mathcal{O}H(s,x,x')\right]_{x=x'}$$

$$H(s,x,x') := \langle x'|\mathrm{e}^{-s\Delta}|x\rangle = \frac{1}{(4\pi s)^2}\,\mathrm{e}^{-\frac{\sigma(x,x')}{2s}}\sum_{n=0}^{\infty} s^n A_{2n}(x,x')$$

- $A_{2n}(x,x')$: heat-coefficients at non-coincident point
- $2\sigma(x,x')$: geodesic distance between x, x'

Step 4b: Evaluate the operator traces including insertions $\mathbb{M}$

Evaluate trace using off-diagonal Heat-kernel (act $\mathcal{O}$ on H)

$$\mathrm{Tr}\left[\mathcal{O}^{\mu_1\cdots\mu_n} D_{\mu_1}\cdots D_{\mu_n}\, e^{-s\Delta}\right] = \mathrm{tr}\int d^dx\sqrt{g}\,\mathcal{O}^{\mu_1\cdots\mu_n}\, H_{\mu_1\cdots\mu_n}$$

The tensors $H_{\mu_1\cdots\mu_n}$ are derivatives of $H(s,x,x')$ at coincidence point:

$$H_{(\mu_1\cdots\mu_n)} \equiv \left[D_{(\mu_1}\cdots D_{\mu_n)} H(s,x,x')\right]_{x=x'}$$

H is the standard heat-kernel

$$H = \frac{1}{(4\pi s)^{d/2}} \sum_{n=0}^{\infty} s^n\, a_n$$

$H_{(\mu_1\cdots\mu_n)}$ obey a similar expansion as the heat-kernel coefficients

$$H_{(\mu\nu)} = \frac{1}{(4\pi s)^{d/2}} \left(-\tfrac{1}{2s} g_{\mu\nu} + \tfrac{1}{6}\left(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R\right) + \ldots\right)$$

$$H_{(\alpha\beta\mu\nu)} = \frac{1}{(4\pi s)^{d/2}} \left(\tfrac{1}{4s^2}\left(g_{\mu\nu}g_{\alpha\beta} + g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha}\right) + \ldots\right)$$

$H_{(\mu_1\cdots\mu_n)}$ can be obtained recursively to any order in n, R

Example I: operator insertion $\mathcal{O} = R^{\mu\nu} D_\mu D_\nu$

goal: compute scalar curvature contribution from

$$\mathcal{O} = R^{\mu\nu} D_\mu D_\nu$$

input:

$$H_{(\mu\nu)} = \frac{1}{(4\pi s)^{d/2}} \left(-\frac{1}{2s}\right) g_{\mu\nu} + \dots$$

apply master formula:

$$\mathrm{Tr}\left[\mathcal{O}^{\mu_1\cdots\mu_n} D_{\mu_1}\cdots D_{\mu_n}\, e^{-s\Delta}\right] = \mathrm{tr}\int d^dx\sqrt{g}\,\mathcal{O}^{\mu_1\cdots\mu_n} H_{\mu_1\cdots\mu_n}$$

obtain result:

$$\begin{aligned}\mathrm{Tr}_0\left[R^{\mu\nu} D_\mu D_\nu\, e^{-s\Delta}\right] &= \int d^dx\sqrt{g}\, R^{\mu\nu} H_{(\mu\nu)} \\ &= \frac{1}{(4\pi s)^{d/2}}\left(-\frac{1}{2s}\right)\int d^dx\sqrt{g}\, R\end{aligned}$$

Step 4b: Evaluate the operator traces including insertions $\mathbb{M}$

Evaluate trace using off-diagonal Heat-kernel (act $\mathcal{O}$ on H)

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$H_{(\mu_1\cdots\mu_n)}$ obey a similar expansion as the heat-kernel coefficients

$$\begin{aligned} H_{(\mu\nu)} &= \frac{1}{(4\pi s)^{d/2}}\left(-\frac{1}{2s}g_{\mu\nu} + \frac{1}{6}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + \ldots\right) \\ H_{(\alpha\beta\mu\nu)} &= \frac{1}{(4\pi s)^{d/2}}\left(\frac{1}{4s^2}\left(g_{\mu\nu}g_{\alpha\beta} + g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha}\right) + \ldots\right) \end{aligned}$$

> $H_{(\mu_1\cdots\mu_n)}$ can be obtained recursively to any order in n, R

Example II: $\bar{\sigma}^3$-term from Goroff-Sagnotti

goal: C^3 contribution from

$$\mathcal{O} = C^{\cdots\cdot} C^{\cdots\cdot} C^{\cdots\cdot} D_{(\mu_1} \ldots D_{\mu_{10})}$$

input:

$$H_{(\mu_1 \cdots \mu_{10})} = \frac{1}{(4\pi s)^{d/2}} \left(-\frac{1}{2s}\right)^5 945 \underbrace{g_{(\mu_1\mu_2} \cdots g_{\mu_9\mu_{10})}}_{945 \text{ index structures}} + \ldots$$

apply master formula:

$$\mathrm{Tr}\left[\mathcal{O}^{\mu_1\cdots\mu_n} D_{\mu_1} \cdots D_{\mu_n} e^{-s\Delta}\right] = \mathrm{tr} \int d^4x \sqrt{g}\, \mathcal{O}^{\mu_1\cdots\mu_n} H_{\mu_1\cdots\mu_n}$$

obtain result:

$$\mathrm{Tr}_2\left[\mathcal{O} e^{-s\Delta}\right] = \frac{1}{(4\pi s)^2} \left(-\frac{1}{2s}\right)^5 30240 \int d^4x \sqrt{g}\, C_{\mu\nu}{}^{\rho\sigma} C_{\rho\sigma}{}^{\kappa\lambda} C_{\kappa\lambda}{}^{\mu\nu}$$

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requires geometrical identity (special to $d = 4$)

$$C^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} C^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma} C^{\rho}{}_{\alpha}{}^{\sigma}{}_{\beta} = \frac{1}{2} C_{\mu\nu}{}^{\rho\sigma} C_{\rho\sigma}{}^{\kappa\lambda} C_{\kappa\lambda}{}^{\mu\nu}$$

Including the two-loop counterterm in Γ_k

Supplement Einstein-Hilbert action by Goroff-Sagnotti term: $\Gamma_k^{\text{grav}} = \Gamma_k^{\text{EH}} + \Gamma_k^{\text{GS}}$

$$\Gamma_k^{\text{EH}} = \frac{1}{16\pi G_k} \int d^4x \sqrt{g} \{2\Lambda_k - R\}$$

$$\Gamma_k^{\text{GS}} = \bar{\sigma}_k \int d^4x \sqrt{g}\, C_{\mu\nu}{}^{\rho\sigma} C_{\rho\sigma}{}^{\kappa\lambda} C_{\kappa\lambda}{}^{\mu\nu}$$

dimensionless coupling constants

$$\lambda = \Lambda_k\, k^{-2}\,, \qquad g = G_k\, k^2\,, \qquad \sigma = \bar{\sigma}_k\, k^2$$

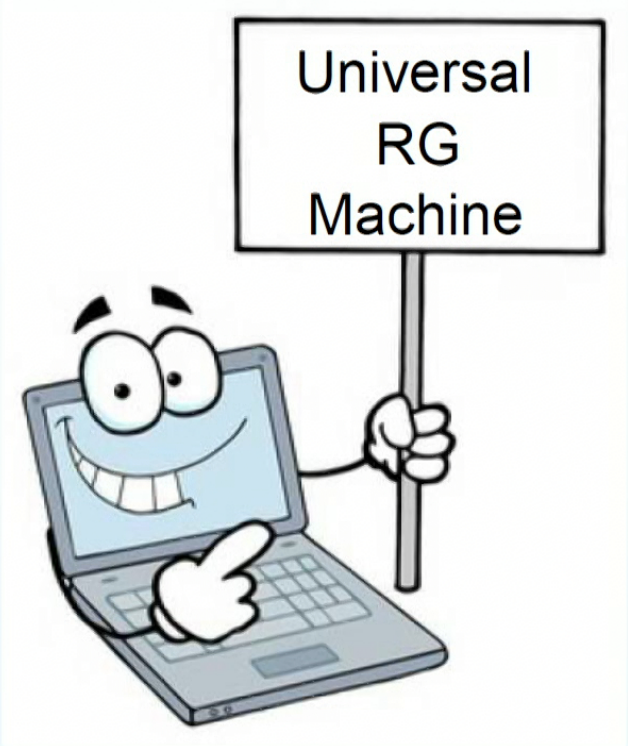
1. the counterterm does not feed back into the Einstein-Hilbert sector

flow of g, λ is not changed $\Longleftrightarrow$ EH-sector has NGFP

2. the beta-function for σ_k is cubic

$$\beta_\sigma = c_0 + (2 + c_1)\,\sigma + c_2\,\sigma^2 + c_3\,\sigma^3$$

Compute the coefficients c_i



After one month of CPU time crunching 900 vertex-insertions

Beta functions of the Goroff-Sagnotti projection

flow in the Einstein-Hilbert sector is unchanged:

$$\beta_g = (2 + \eta_N)\; g\,,$$

$$\beta_\lambda = (\eta_N - 2)\;\lambda + \tfrac{g}{2\pi}\left(\tfrac{5}{1-2\lambda} - 4 - \tfrac{5}{6}\eta_N\,\tfrac{1}{1-2\lambda}\right).$$

Beta function for the Goroff-Sagnotti coupling

$$\beta_\sigma = c_0 + (2 + c_1)\,\sigma + c_2\,\sigma^2 + c_3\,\sigma^3$$

Coefficients c_i are functions of g, λ:

$$c_0 = \tfrac{1}{64\pi^2(1-2\lambda)}\left(\tfrac{2-\eta_N}{2(1-2\lambda)} + \tfrac{6-\eta_N}{(1-2\lambda)^3} - \tfrac{5\eta_N}{378}\right),$$

$$c_1 = \tfrac{3g}{16\pi(1-2\lambda)^2}\left(5(6-\eta_N) + \tfrac{23(8-\eta_N)}{8(1-2\lambda)} - \tfrac{7(10-\eta_N)}{10(1-2\lambda)^2}\right),$$

$$c_2 = \tfrac{g^2}{2(1-2\lambda)^3}\left(\tfrac{233(12-\eta_N)}{10} - \tfrac{9(14-\eta_N)}{7(1-2\lambda)}\right),$$

$$c_3 = \tfrac{6\pi g^3(18-\eta_N)}{(1-2\lambda)^4} \neq 0\,.$$

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Beta functions of the Goroff-Sagnotti projection

flow in the Einstein-Hilbert sector is unchanged:

$$\beta_g = (2 + \eta_N)\, g\,,$$

$$\beta_\lambda = (\eta_N - 2)\, \lambda + \tfrac{g}{2\pi}\left(\tfrac{5}{1-2\lambda} - 4 - \tfrac{5}{6}\eta_N \tfrac{1}{1-2\lambda}\right).$$

Beta function for the Goroff-Sagnotti coupling

$$\beta_\sigma = c_0 + (2 + c_1)\,\sigma + c_2\,\sigma^2 + c_3\,\sigma^3$$

Coefficients c_i are functions of g, λ:

$$c_0 = \tfrac{1}{64\pi^2(1-2\lambda)}\left(\tfrac{2-\eta_N}{2(1-2\lambda)} + \tfrac{6-\eta_N}{(1-2\lambda)^3} - \tfrac{5\eta_N}{378}\right),$$

$$c_1 = \tfrac{3g}{16\pi(1-2\lambda)^2}\left(5(6-\eta_N) + \tfrac{23(8-\eta_N)}{8(1-2\lambda)} - \tfrac{7(10-\eta_N)}{10(1-2\lambda)^2}\right),$$

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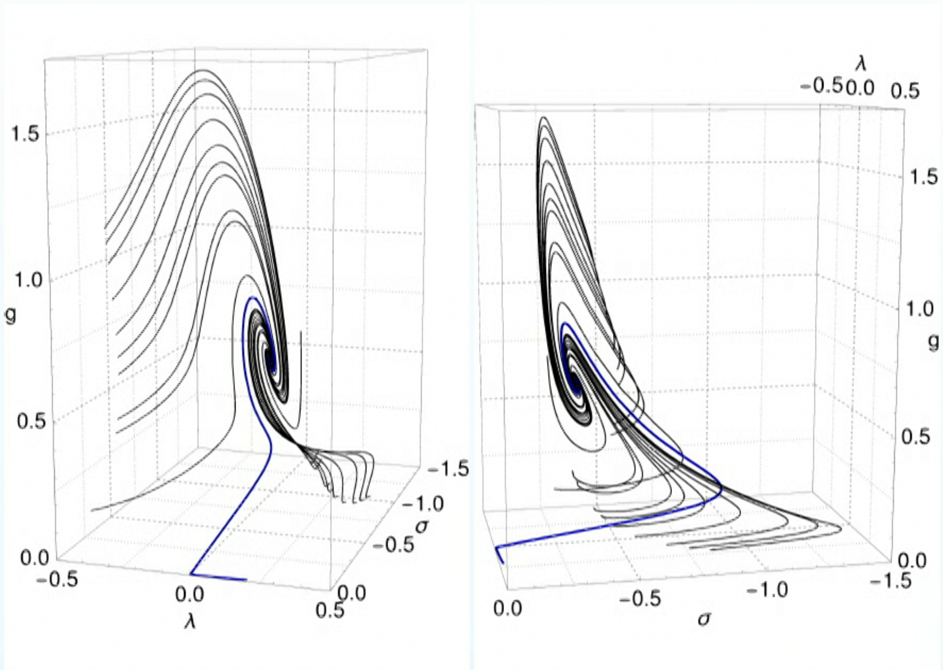
$g_* > 0$ implies $c_3 > 0 \Longleftrightarrow$ NGFP extents to GS-projection

Phase portrait of the Goroff-Sagnotti projection

H. Gies, B. Knorr, S. Lippoldt and F. Saueressig, arXiv:1601.01800



blue trajectory: crossover to classical regime intact evaluating flow equations

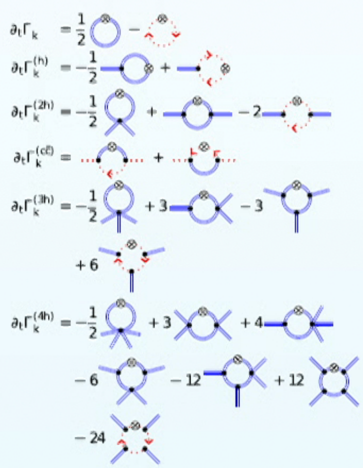
vertex expansion

vertex expansion in a flat space background

N. Christiansen, B. Knorr, J. Meibohm, J. Pawlowski, M. Reichert, Phys. Rev. D92 (2015) 121501
P. Dona, A. Eichhorn, P. Labus, R. Percacci, Phys. Rev. D **93** (2016) 044049
J. Meibohm, J. Pawlowski, M. Reichert, Phys. Rev. D93 (2016) 084035
T. Denz, J. Pawlowski, M. Reichert, arXiv:1612.07315

$$\Gamma_k[h;\bar{g}] = \sum_n \Gamma_k^{n,0}[\bar{g}]_{\mu_1\nu_1\ldots\mu_n\nu_n}\, h^{\mu_1\nu_1}\ldots h^{\mu_n\nu_n}$$

- encodes information on curvature tensors in tensor structures



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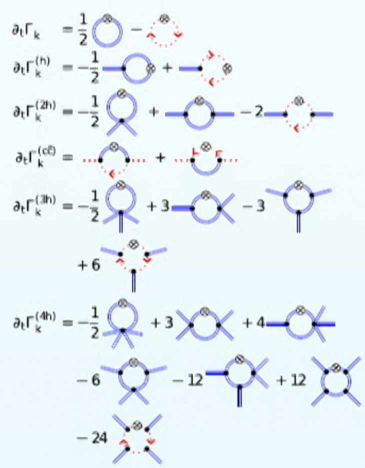

Massive use of computer algebra resources

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characterizing quantum spacetime

Expectation values for composite operators

C. Pagani, M. Reuter, Phys. Rev. D95 (2017) 066002

idea: characterize quantum geometry by expectation values of operators:

- interesting operators are composite functions of fundamental fields

expectation values: introduce new source for composite operator

$$\langle \mathcal{O}(x) \rangle \propto \int \mathcal{D}\chi \, \mathcal{O} \, e^{-S} \propto \frac{\delta}{\delta\epsilon} \int \mathcal{D}\chi \, \mathcal{O} \, e^{-S+\epsilon\mathcal{O}}$$

anomalous scaling dimension of $\mathcal{O}$:

$$\gamma_{\mathcal{O}} \equiv \partial_t \ln Z_k$$

obeys flow equation

$$\gamma_{\mathcal{O}} \, \mathcal{O} = -\frac{1}{2} \mathrm{Tr} \left[\left(\Gamma_k^{(2)} + \mathcal{R}_k \right)^{-1} \mathcal{O}^{(2)} \left(\Gamma_k^{(2)} + \mathcal{R}_k \right)^{-1} \partial_t \mathcal{R}_k \right]$$

- evaluated on an approximate solution of the FRGE

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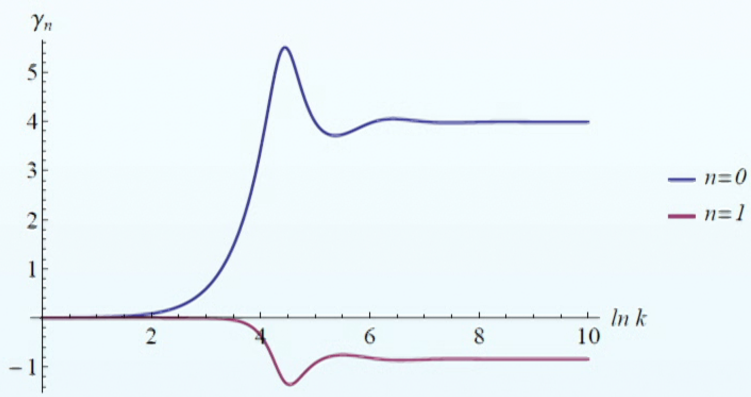
practical implementation

infinite class of diffeomorphism invariant composite operators

$$\mathcal{O}_n = \int d^4x \sqrt{g}\, R^n$$

- flow of Γ_k from Einstein-Hilbert truncation



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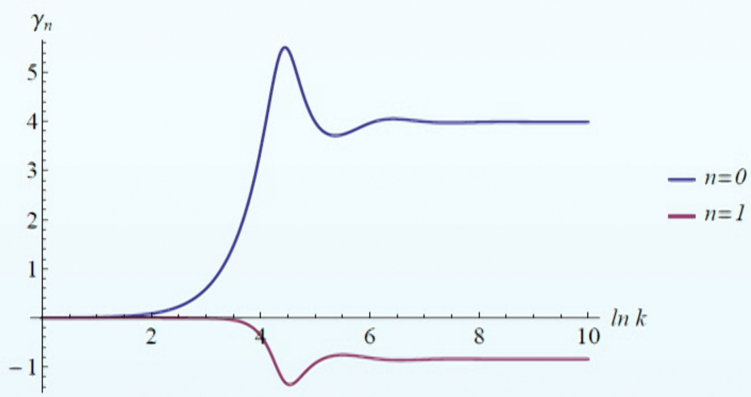

geometry undergoes transition from quantum to classical

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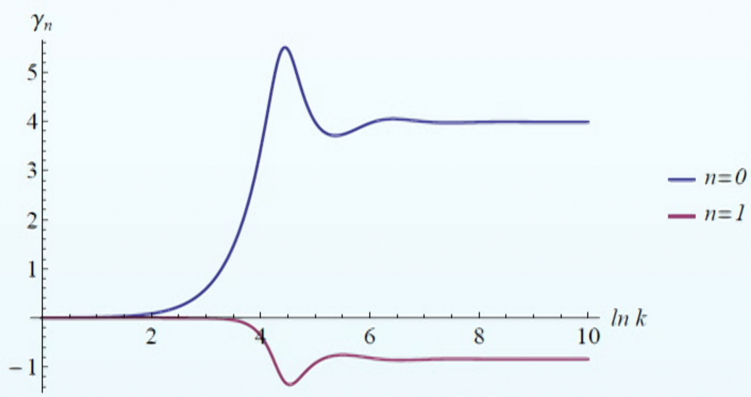

geometry undergoes transition from quantum to classical

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summary and outlook

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summary

pure gravity:

- non-Gaussian fixed point established in a wide range of approximations
  - the two-loop counterterm is asymptotically safe
  - low number of relevant parameters ($\simeq$ 3-4)
  - canonical power-counting still determines relevance of an operator

construction of beta functions reduced to a combinatorial problem

summary

pure gravity:

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  - the two-loop counterterm is asymptotically safe
  - low number of relevant parameters ($\simeq$ 3-4)
  - canonical power-counting still determines relevance of an operator

construction of beta functions reduced to a combinatorial problem

gravity coupled to matter:

- Asymptotic Safety is operative upon adding suitable matter systems

[P. Dona, A. Eichhorn, R. Percacci, arXiv:1311.2898]
[J. Meibohm, J. M. Pawlowski, M. Reichert, arXiv:1510.07018]
[T. Denz, J. Pawlowski, M. Reichert, arXiv:1612.07315]
[A. Platania, J. Biemans, F.S., arXiv:1702.06539]
[Y. Hamada, M. Yamada, arXiv:1703.09033]

  - compatibility with matter content of the standard model
  - prediction of the Higgs mass $m_H \simeq 126$ GeV

[M. Shaposhnikov, C. Wetterich, arXiv:0912.0208]

outlook

exciting explorations ahead:

- develop the spacetime picture of the fixed point theory
  - determine: correlation functions, generalized dimensions, ...
- systematic connection to experimental data
  - including the matter content of the standard model including interactions
  - cosmological model building from first principles

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computer power needed

but this does not lift the burden of formulating good questions