Title: Monte Carlo methods in Dynamical Triangulations - 4

Date: Jun 22, 2017 09:45 AM

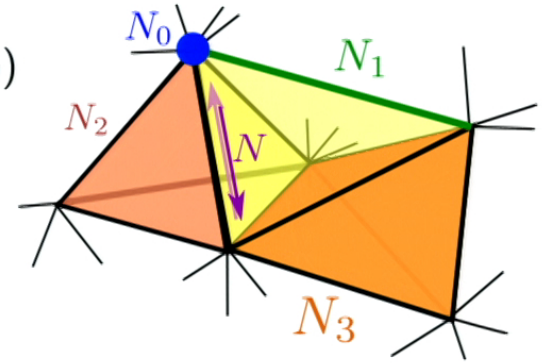
URL: http://pirsa.org/17060083

Abstract:

Triangulation size?

- What is the "size" of a D-triangulation?

$N = \#$ of half-edges (size of n, a_d)
$N_0 = \#$ of vertices
$N_1 = \#$ of edges
$\vdots$
$N_D = \#$ of D-simplices



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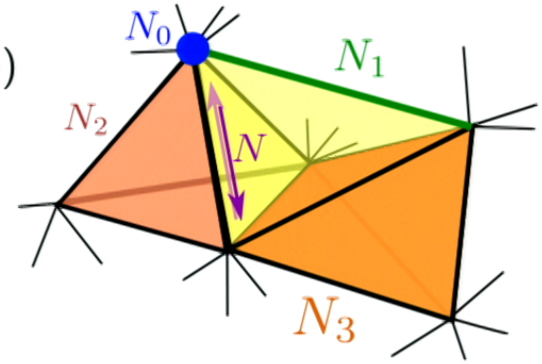

- Relations: $N = N_D(D+1)!/2$, $2N_{D-1} = N_D(D+1)$, $\sum_{k=0}^{d}(-1)^k N_k = \chi$ (Euler characteristic). In $D \geq 4$ more linear (Dehn-Sommerfield) relations.

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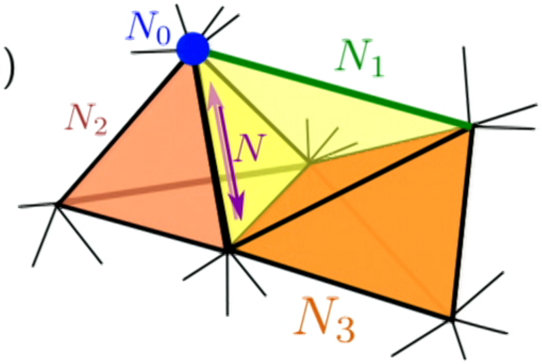

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- Only $\lfloor \frac{D+1}{2} \rfloor$ independent numbers. In 3D and 4D these are usually taken to be N_D and N_{D-2}, or N_D and N_0.
- Recall the EH action $S[N_D, N_{D-2}] = \kappa_D N_D - \kappa_{D-2} N_{D-2}$ is exactly a linear combination of these.
- As we will see: for fixed N_D, varying the ratio N_{D-2}/N_D has a large effect on the random geometries!

Labeling & symmetry

- Recall from yesterday: in 2D for fixed N_2 a uniform labeled triangulation $\mathfrak{t}$ with N_2 triangles is equivalent to an unlabeled triangulation $\tilde{\mathfrak{t}}$ with probability proportional to $1/|\mathrm{Aut}(\tilde{\mathfrak{t}})|$:

$$Z_{N_2} = \sum_{\substack{\text{labeled} \\ \text{triangulations } \mathfrak{t}}} 1 = (3N_2)! \sum_{\substack{\text{unlabeled} \\ \text{triangulations } \tilde{\mathfrak{t}}}} \frac{1}{|\mathrm{Aut}(\tilde{\mathfrak{t}})|}$$

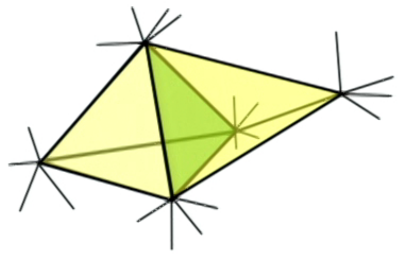
- No longer equivalent if N_2 (or N_D in dimension D) is allowed to vary.
- Settle upon convention that $S[N_D, N_0]$ is action for unlabeled triangulations:

$$Z = \sum_{\substack{\text{labeled} \\ \text{triangulations } \mathfrak{t}}} \frac{e^{-S[N_D,N_0]}}{(\#\text{labels})!} = \sum_{\substack{\text{unlabeled} \\ \text{triangulations } \tilde{\mathfrak{t}}}} \frac{e^{-S[N_D,N_0]}}{|\mathrm{Aut}(\tilde{\mathfrak{t}})|}$$

(#labels $= N_D(D+1)!/2$ for general and N_0 for simplicial triangulations)

Moves in 3D

- *23-move*: select a uniform random triangle, merge incident tetrahedra, split into 3 tetrahedra.



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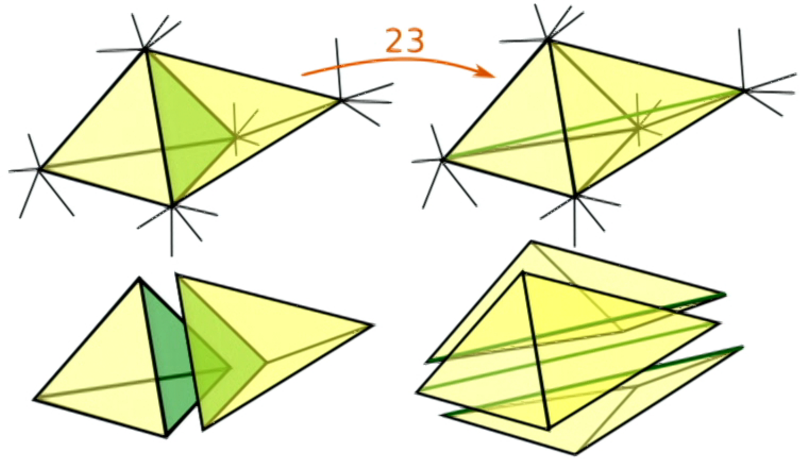
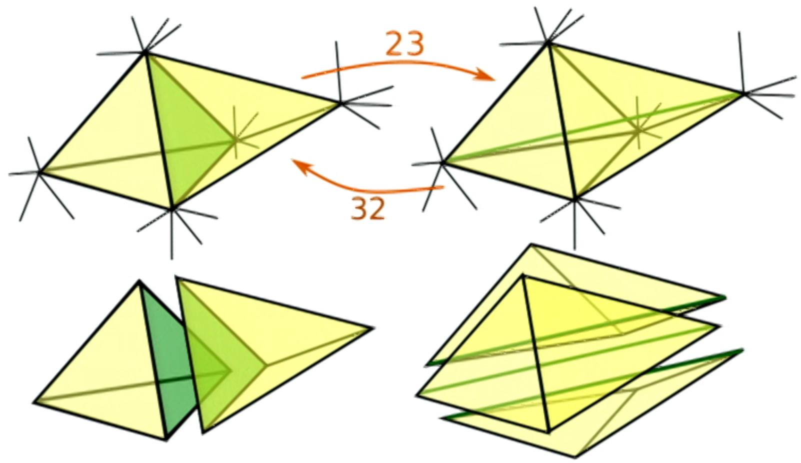


Moves in 3D

- *23-move*: select a uniform random triangle, merge incident tetrahedra, split into 3 tetrahedra.
- *32-move*: select uniform random tetrahedron and one of its edges, check edge has degree 3, merge tetrahedra, split into 2 tetrahedra.



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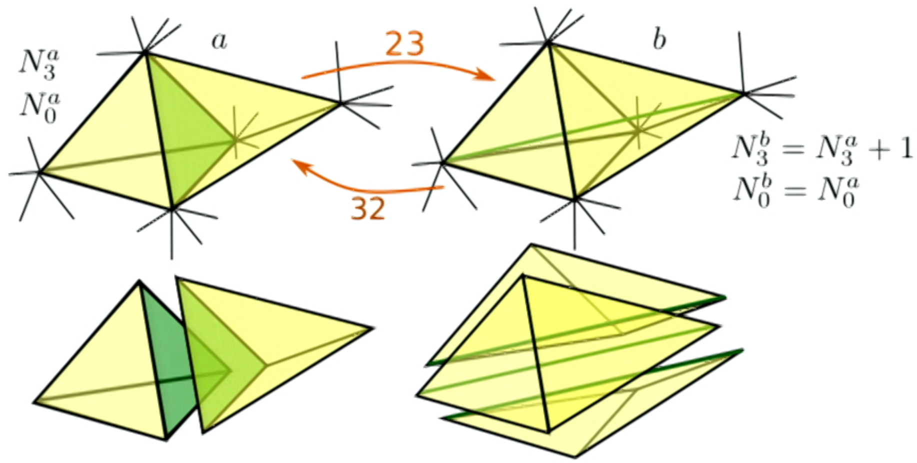
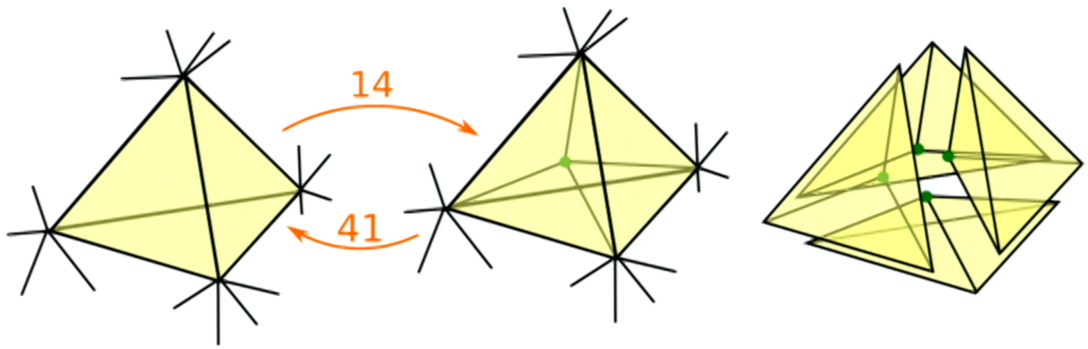
- *32-move*: select uniform random tetrahedron and one of its edges, check edge has degree 3, merge tetrahedra, split into 2 tetrahedra.
- Always valid for general triangulations, provided tetrahedra are distinct. For simplicial triangulations need to check no "double" edges or triangles created.
- Detailed balance: $\frac{P(a \to b)}{P(b \to a)} = \frac{\text{SelectProb}(a \to b)}{\text{SelectProb}(b \to a)} \frac{\text{AcceptProb}(a \to b)}{\text{AcceptProb}(b \to a)}$



Moves in 3D

- *14-move*: select a uniform tetrahedron, split into 4 tetrahedra.
- *41-move*: select a uniform tetrahedron and one of its vertices, check configuration, remove vertex.
- Always valid both for general and simplicial triangulations.



Grand canonical?

- The Markov step that attempts 23-, 32-, 14-, 41-move with probabilities $\frac{p}{2}, \frac{p}{2}, \frac{1-p}{2}, \frac{1-p}{2}$ $(0 < p < 1)$ satisfies detailed balance (w.r.t. Boltzmann weight $e^{-S[N_3,N_0]}$).

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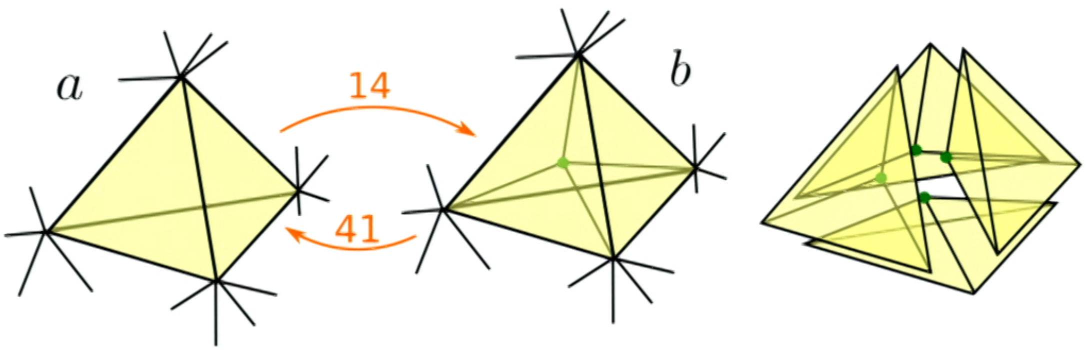

$N_3^b = N_3^a + 3$
$N_0^b = N_0^a + 1$

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- Ergodic, provided we do not restrict N_3 or N_0! [Pachner, '91]
- To ensure ergodicity for $N_3 \leq n$, must allow intermediate triangulations of size $N_3 \leq f(n)$.
  - Theoretically: $f(n) < e^{cn^2}$ [Mijatović,'03]
  - In practice: $f(n) \leq n + 2$ for all $n \leq 9$ (10^8 triangulations) [Burton,'11]

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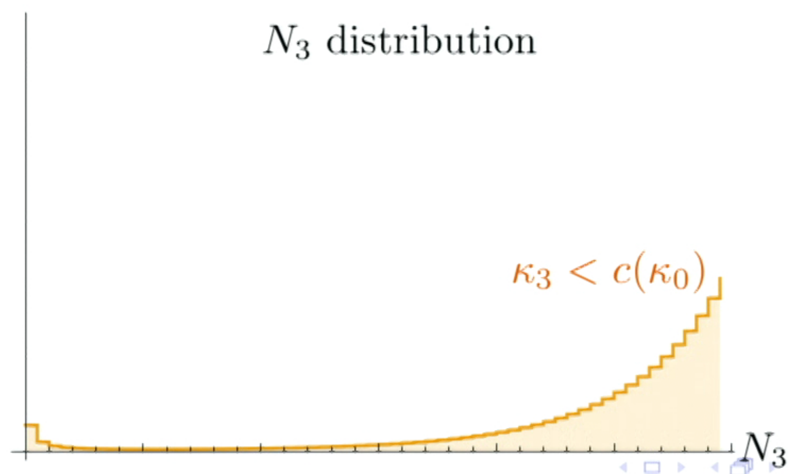
- Need to use a grand-canonical ensemble in 3D/4D (contrary to 2D)!

- Why not just

$$Z = \sum_{\text{triang. } t} \frac{1}{|\text{Aut}(t)|} e^{-S[N_3, N_0]} = \sum_{N_3} Z_{N_3} e^{-\kappa_3 N_3}, \ S[N_3, N_0] = \kappa_3 N_3 - \kappa_0 N_0?$$

- Typically $Z_{N_3} = \sum \frac{1}{|\text{Aut}(t)|} e^{\kappa_0 N_0} \sim f(N_3) e^{c(\kappa_0) \cdot N_3}$ as $N_3 \to \infty$, $f(N_3) \to 0$ subexponentially.
  - $\kappa_3 < c(\kappa_0)$: $Z[\kappa_3, \kappa_0] = \infty$



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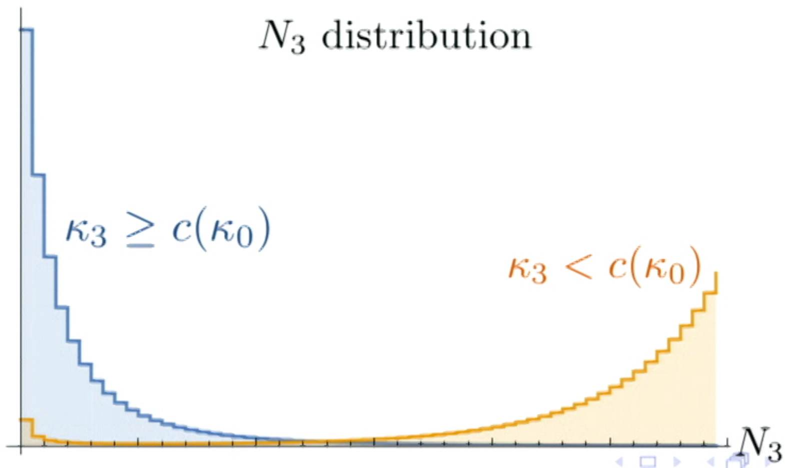

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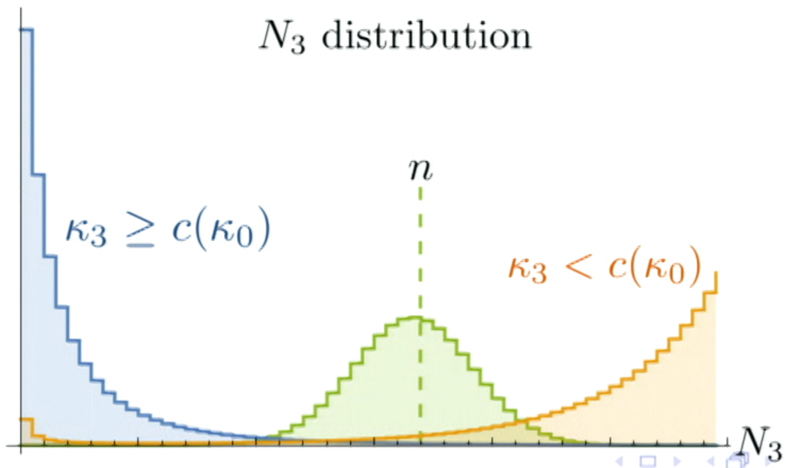

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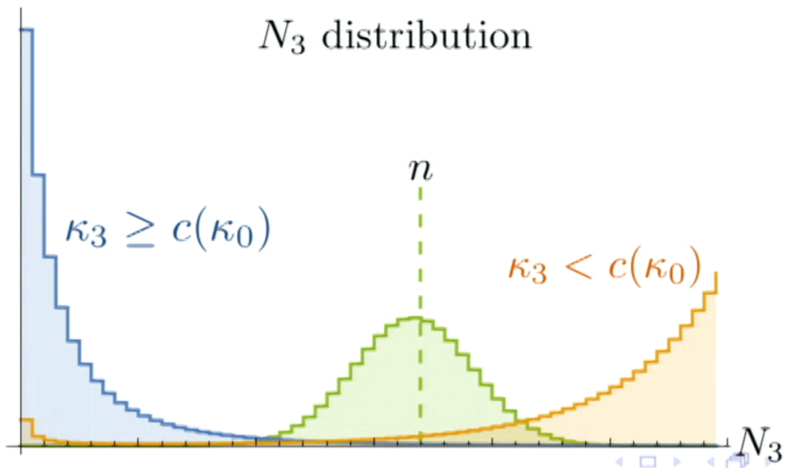
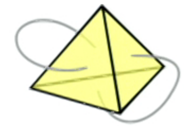
    - Rejection sampling of MCMC: effectively simulate $Z_{N_3=n}[\kappa_0] = \sum e^{\kappa_0 N_0}$. Need ϵ not too small.
    - Need ϵ not too large for ergodicity.



MCMC overview

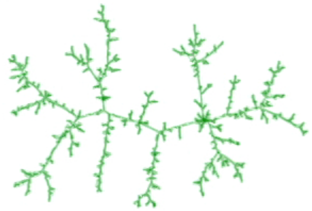
- Read parameters: desired size n, coupling κ_0.
- Initialize configuration: correct topology is sufficient.



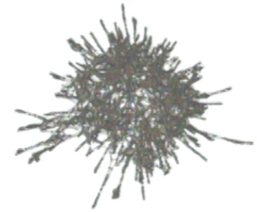
- Start performing Monte Carlo moves indefinitely
  - Thermalization phase
    - Parameter tuning (ϵ, κ_D, relative move frequency p)
    - Monitor thermalization with suitable observables.
  - Measurement phase
    - With predetermined frequency attempt measurement.
    - If desired, reject configuration if size outside window around n.
    - Add measurement data to list or histogram.

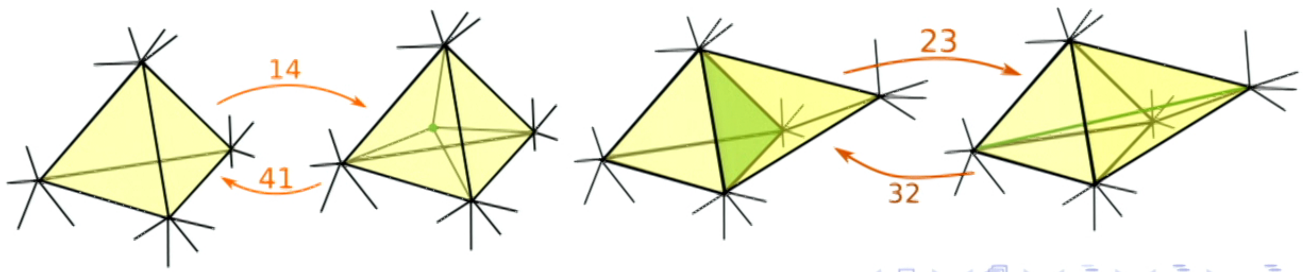
Phases

- By examining the moves we can already get an idea what the geometries will look like for κ_0 very small/large.
  - κ_0 large, maximize N_0 for fixed N_3: many 14-moves $\to$ tree-like structure.



  - κ_0 small, minimize N_0 for fixed N_3: many 23-moves $\to$ highly connected





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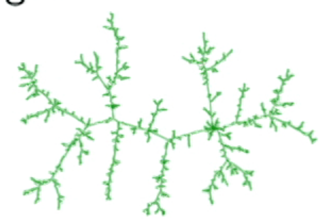
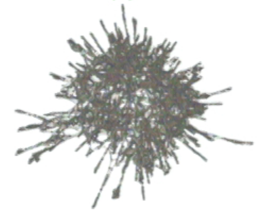
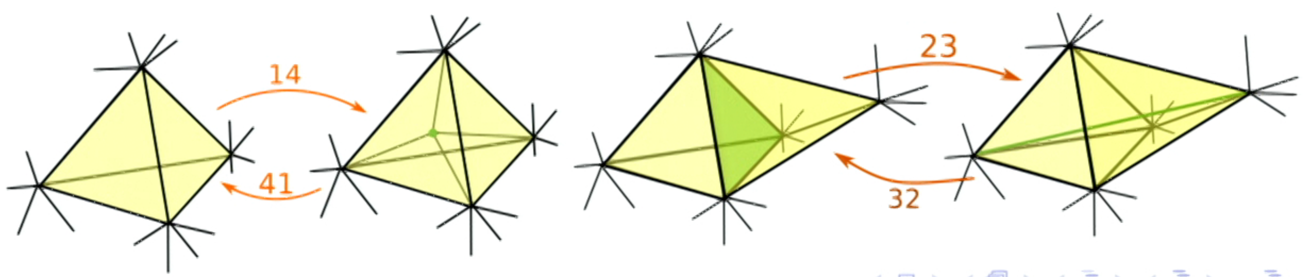
  - κ_0 large, maximize N_0 for fixed N_3: many 14-moves $\rightarrow$ tree-like structure. "Branched polymer phase" $d_\mathrm{H} = 2$, $d_s = 4/3$



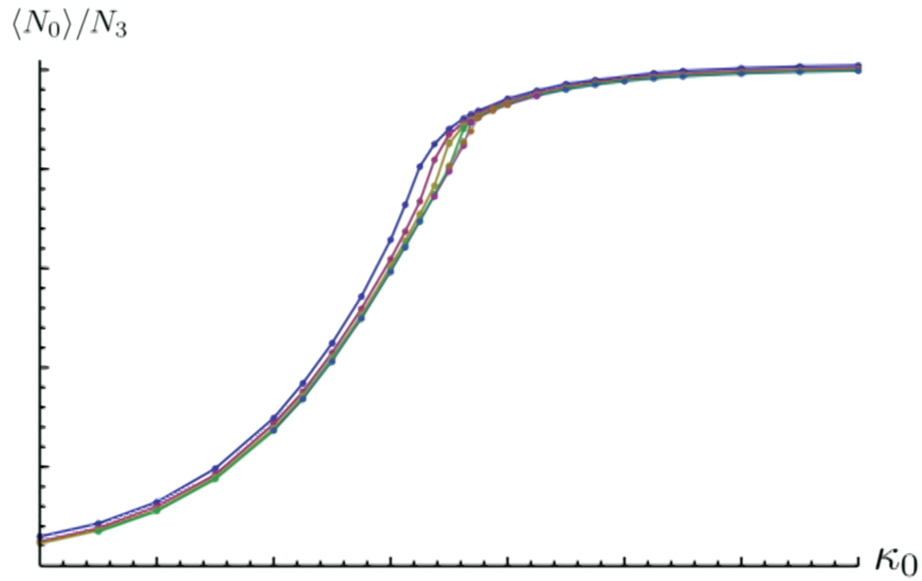
  - κ_0 small, minimize N_0 for fixed N_3: many 23-moves $\rightarrow$ highly connected "Crumpled phase" no conclusive scaling ($d_\mathrm{H} = d_s = \infty$?)



- Indeed these structures are characteristic for the two phases of DT in 3D and 4D. [Boulatov, Krzywicki, Ambjørn, Varsted, Agishtein, Migdal, Jurkiewicz, Renken, Catterall, Kogut, Thorleifsson, Bialas, Burda, Bilke, Thorleifsson, Petersson,. . . ,'90s]



Phase transition



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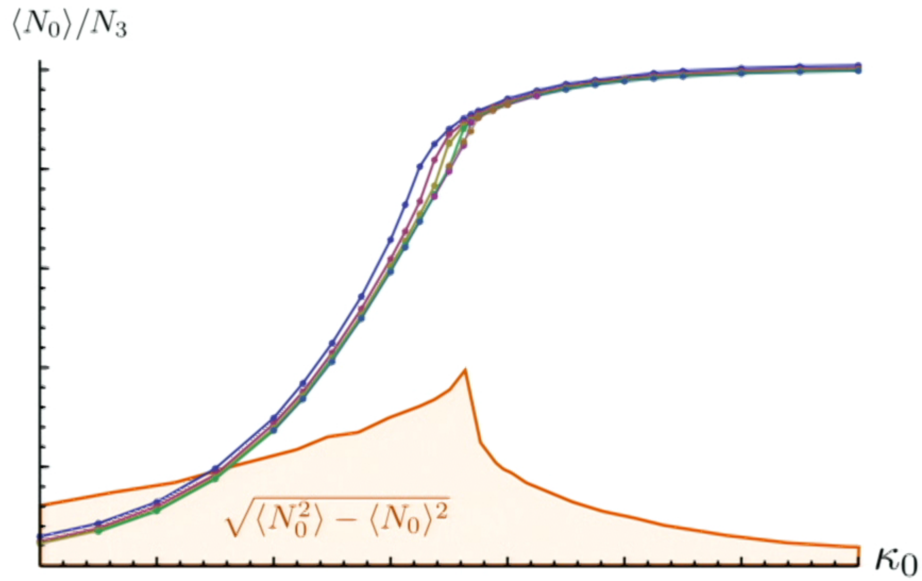

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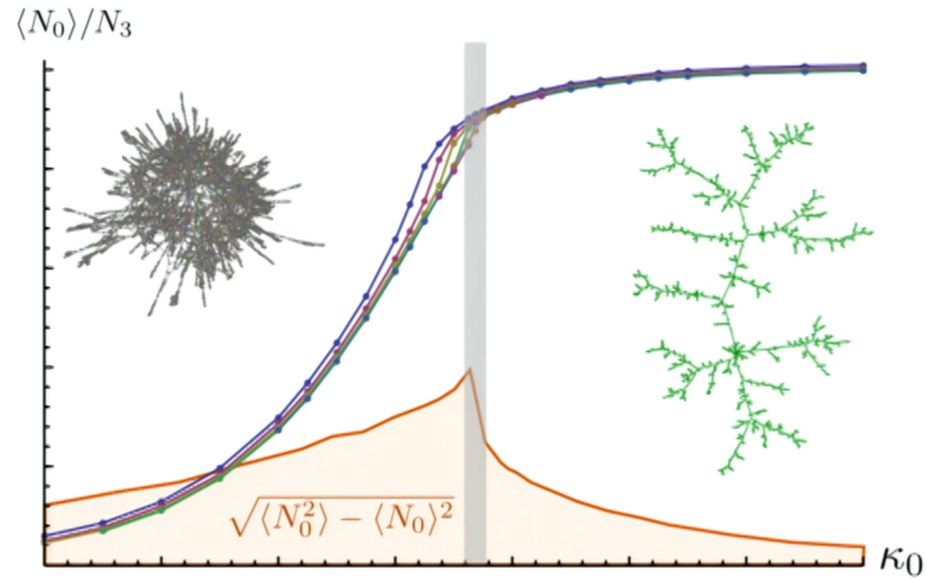
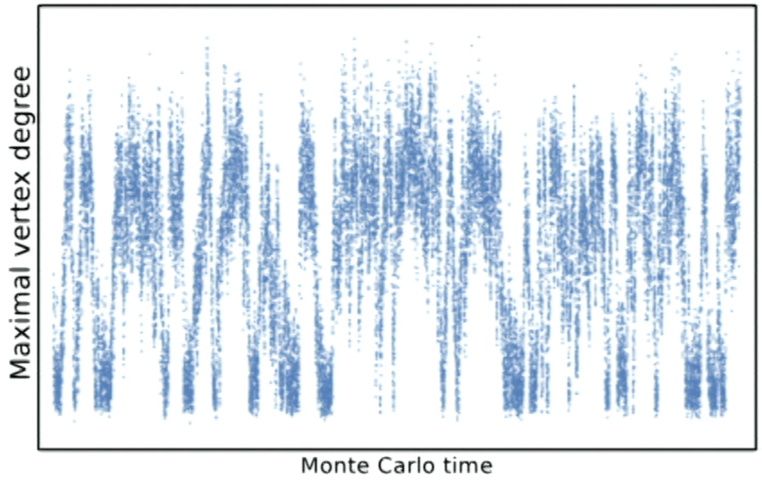


- All is not lost: perhaps enhanced scaling at the phase transition?
- Not clear from this plot whether transitions is discontinuous (1st order) or continuous (higher order).

Double peak structure



- When κ_0 is tuned to critical value: MCMC jumps between two meta-stable states.

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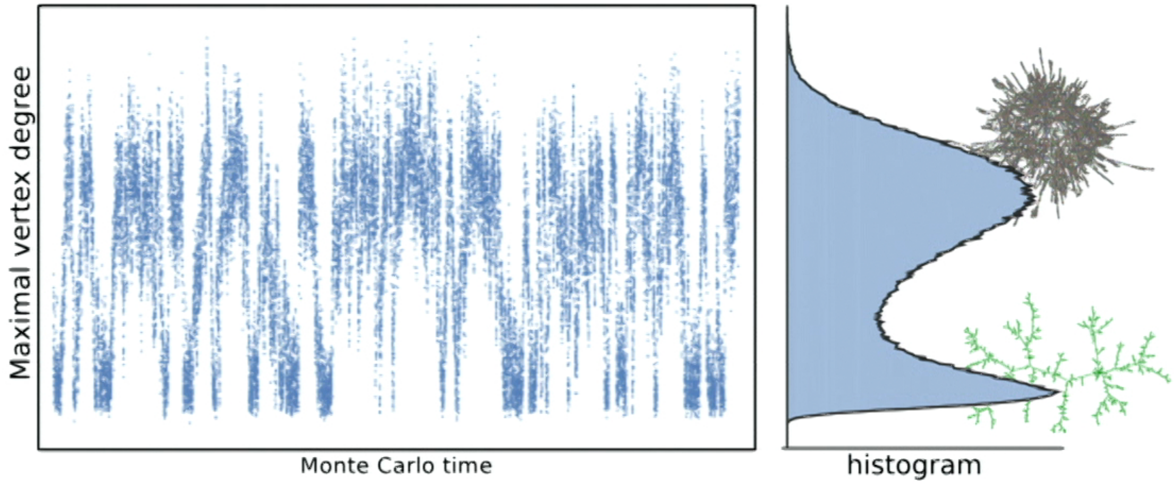

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- If double peak in histogram becomes more pronounced as $N_4 \to \infty$ then transition is discontinuous.

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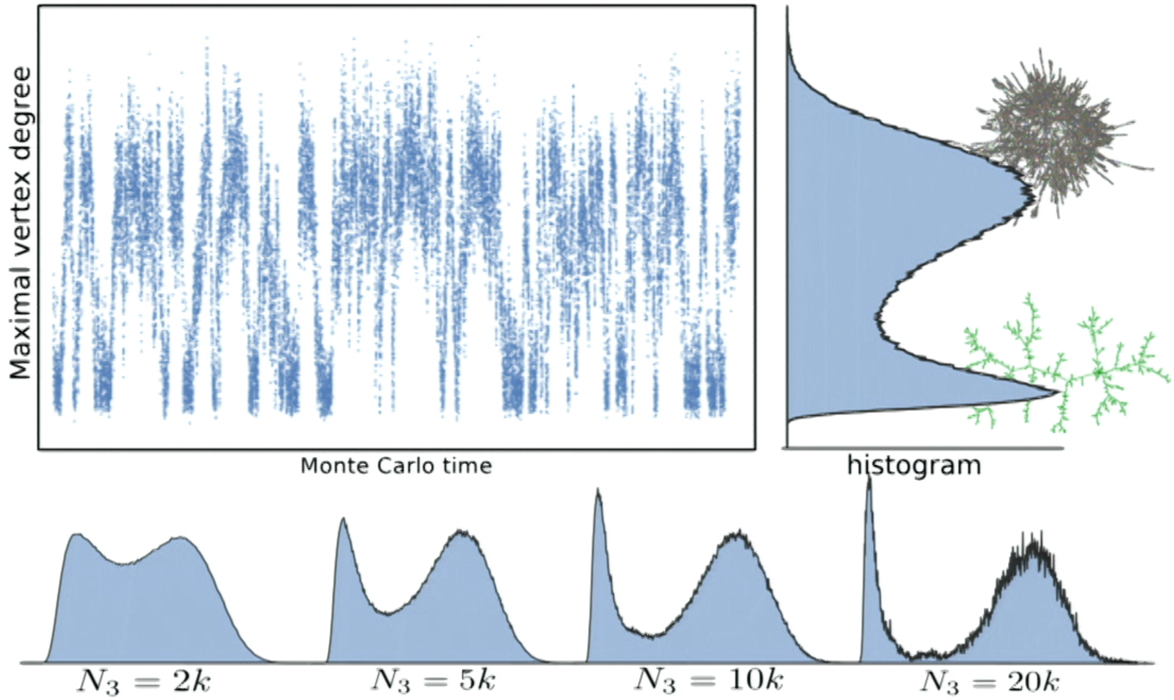

- When κ_0 is tuned to critical value: MCMC jumps between two meta-stable states.
- If double peak in histogram becomes more pronounced as $N_4 \to \infty$ then transition is discontinuous.
- It does. No hope of new scaling at transition.

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- Enlarge phase diagram with extra couplings or matter fields.
  - Higher curvature terms.
  - Non-trivial measure: $e^{-S} \to e^{-S} \prod_{\sigma_{D-2}} |\deg(\sigma_{D-2})|^{\beta}$.
  - Gauge fields, Gaussian fields, Ising models.

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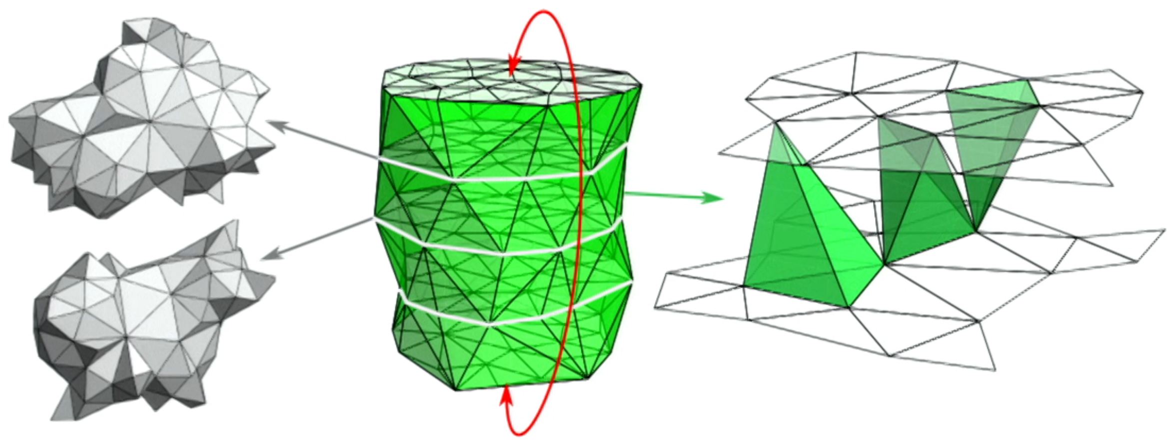
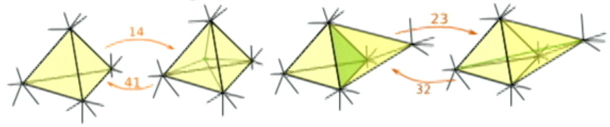
  - Gauge fields, Gaussian fields, Ising models.
- Change the ensemble of geometries.
  - Change topology.
  - Different polyhedra as building blocks.
  - Introduce foliation: Causal Dynamical Triangulations (CDT).

Causal Dynamical Triangulations in 3D

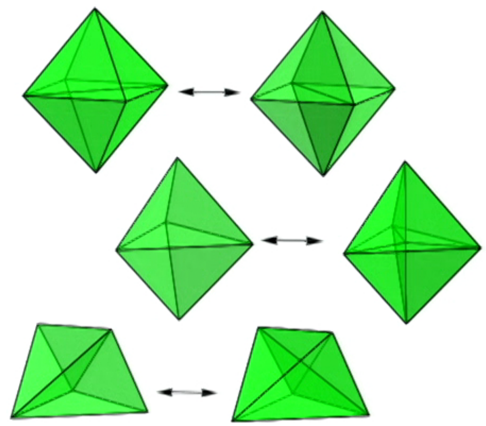
- Consider a (general or simplicial) 3-Triangulation of topology $S^1 \times S^2$.
- It is *causal* if it is "foliated" by triangulations of S^2 and all tetrahedra of two types (31-, 22-simplex).



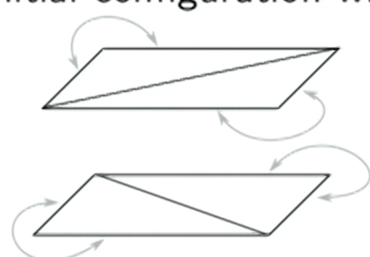
Adaption to Causal triangulations

- Replace moves 

with a set that preserves the foliation and is ergodic in causal triangulations (with fixed T).



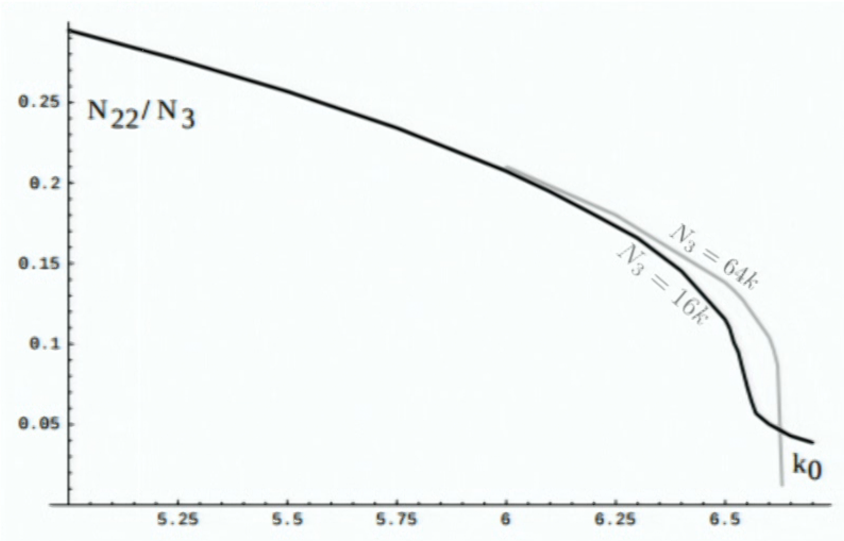
- Update detailed balance conditions.
- Construct by hand an initial configuration with correct topology.



Phase diagram of CDT in 3D

- For fixed N_3
  - κ_0 large, maximize N_0, few 22-simplices
  - κ_0 small, minimize N_0, many 22-simplices

[Ambjorn, Jurkiewicz, Loll, hep-th/0011276]



Phase diagram of CDT in 3D

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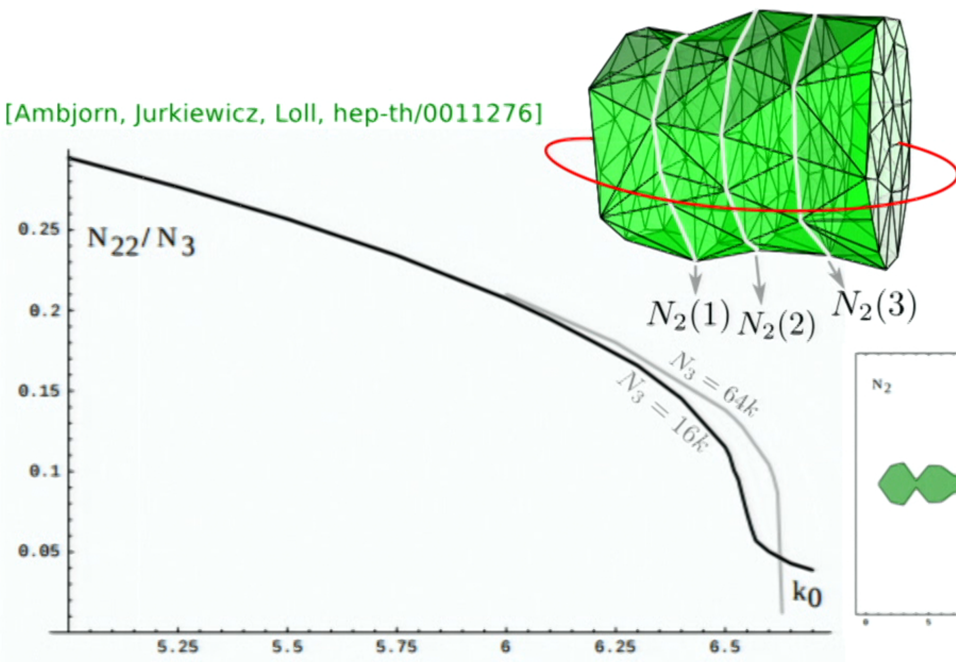
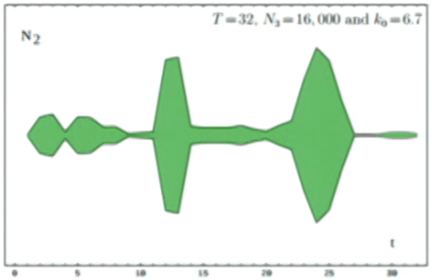
  - κ_0 large, maximize N_0, few 22-simplices
    Weak correlation between slices; collection of 2d random geometries
  - κ_0 small, minimize N_0, many 22-simplices

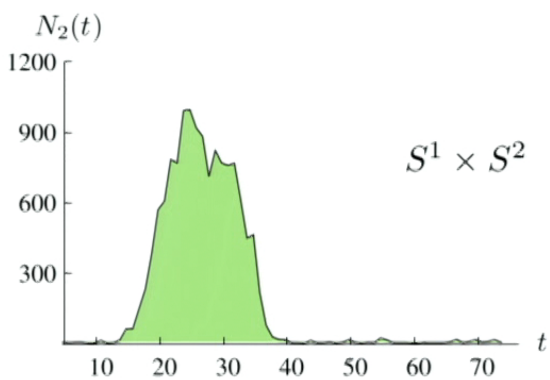
[Ambjorn, Jurkiewicz, Loll, hep-th/0011276]





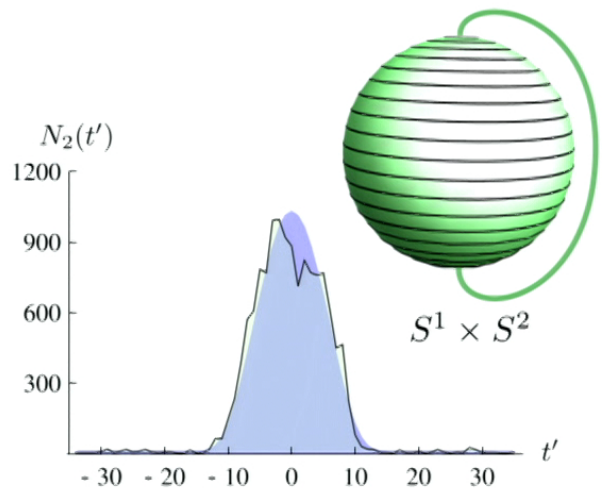
A closer look at the condensation phase

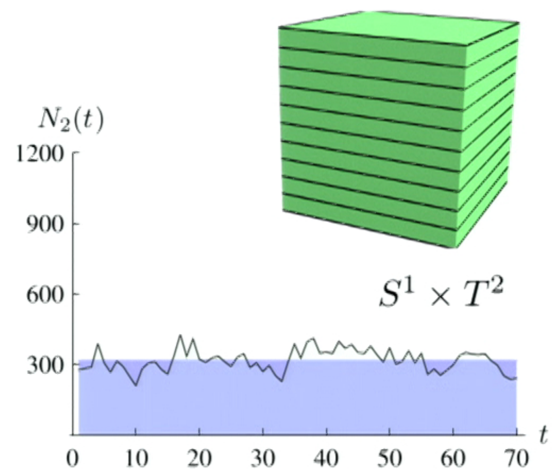
- As $N_3 \to \infty$ the relative fluctuations of $N_2(t')$ w.r.t $\langle N_2(t') \rangle$ decrease to 0.



A closer look at the condensation phase

- As $N_3 \to \infty$ the relative fluctuations of $N_2(t')$ w.r.t $\langle N_2(t')\rangle$ decrease to 0.
- $\langle N_2(t')\rangle$ accurately matches $a \cdot \cos^2(b \cdot t')$ (which happens to match the volume profile of S^3).
- Spectral dimension $d_\mathrm{s} \approx 3$.

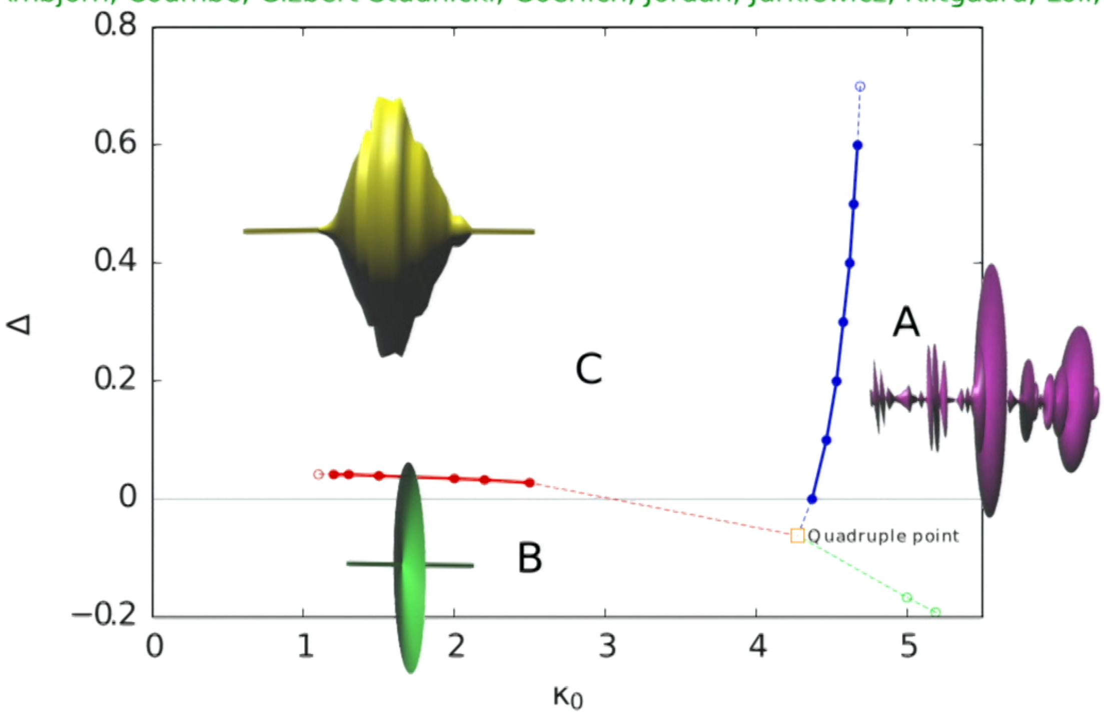




CDT in 4D: the state of the art

- A richer phase diagram in 4D: similar phase C with semi-classical volume profile and $d_s \approx 4$.

[Ambjorn, Coumbe, Gizbert-Studnicki, Goerlich, Jordan, Jurkiewicz, Klitgaard, Loll, ...]



Take-home messages

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- Continuous phase transitions are essential to model sub-Planckian geometry.
- The possession of a semi-classical thermodynamic limit is a highly non-trivial property in the case of (background-independent) random geometries.