Title: Monte Carlo methods in Dynamical Triangulations - 3

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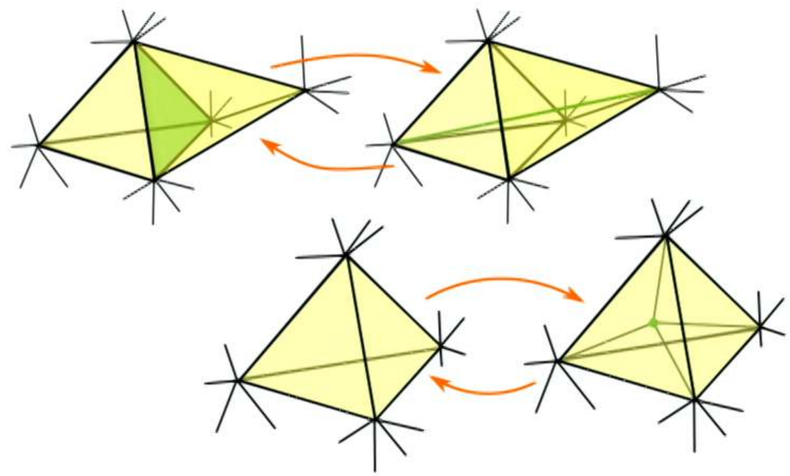
Abstract:

Making Quantum Gravity Computable, 22-06-2017

# Monte Carlo methods in Dynamical Triangulations

## Part II: Higher dimensions


Timothy Budd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# Outline

- Day 1: 2D random geometry
  - Combinatorial representation
  - Markov Chain Monte Carlo (MCMC) methods
  - Matter coupling
  - Observables
- Day 2: Dynamical Triangulations in higher dimensions
  - Quantum gravity
  - Combinatorial representation
  - MCMC methods
  - Phase diagram
  - Causal Dynamical Triangulations
- Tutorials: numerical analysis of various 2D random geometries
  - Measure observables for random geometries (produced by black box)
  - Extract critical exponents.
  - Experiment with (new?) observables.
  - Conclusions will be collected at the end and be discussed.

# A space-time path integral?

$$\int_{\substack{\text{Lorentzian}\\ \text{metrics}}} \frac{\mathrm{d}\mu(g_{\alpha\beta})}{\mathrm{Diff}} e^{iS[g_{\alpha\beta}]}$$

Difficulties:

- QFT in perturbative regime: non-renormalizable
- Infinite-dimensional integral
- What is a good diffeo-invariant measure?
- Destructive interference is delicate
- How to interpret integrand?
- Numerical evaluation is hard

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Lorentzian → Euclidean

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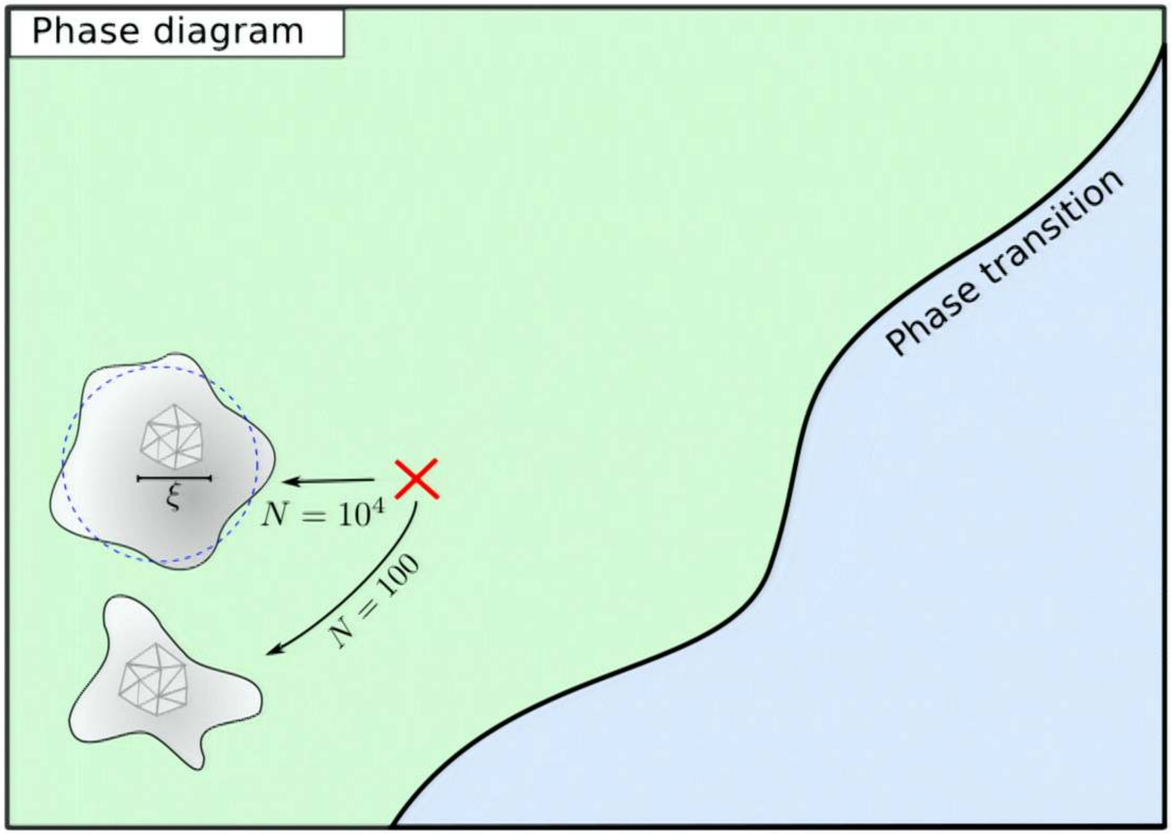

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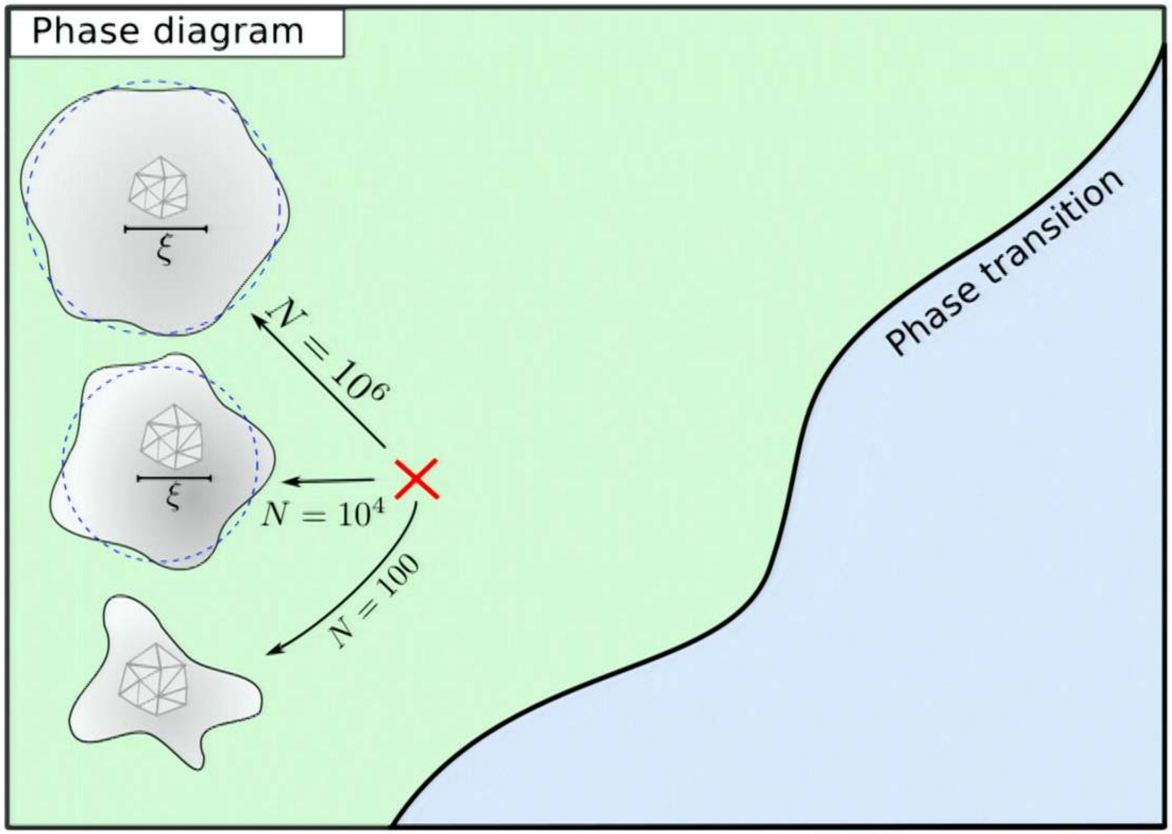

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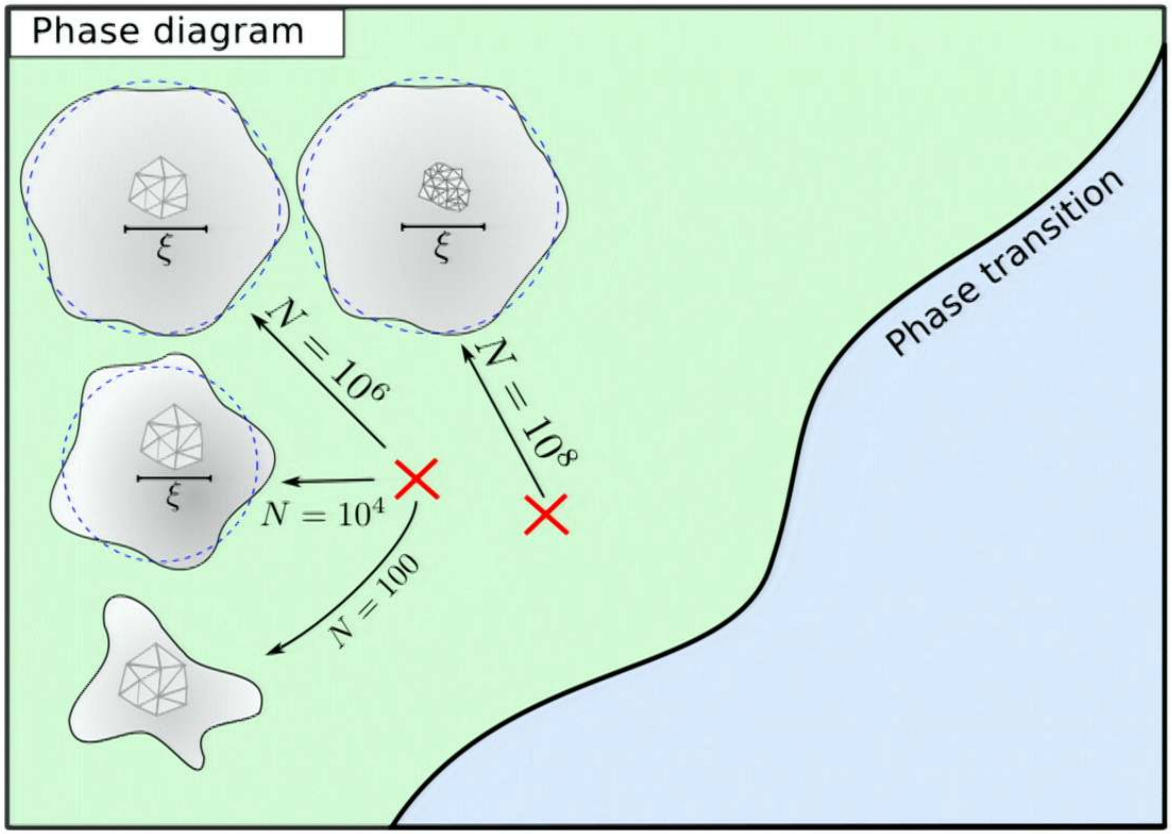

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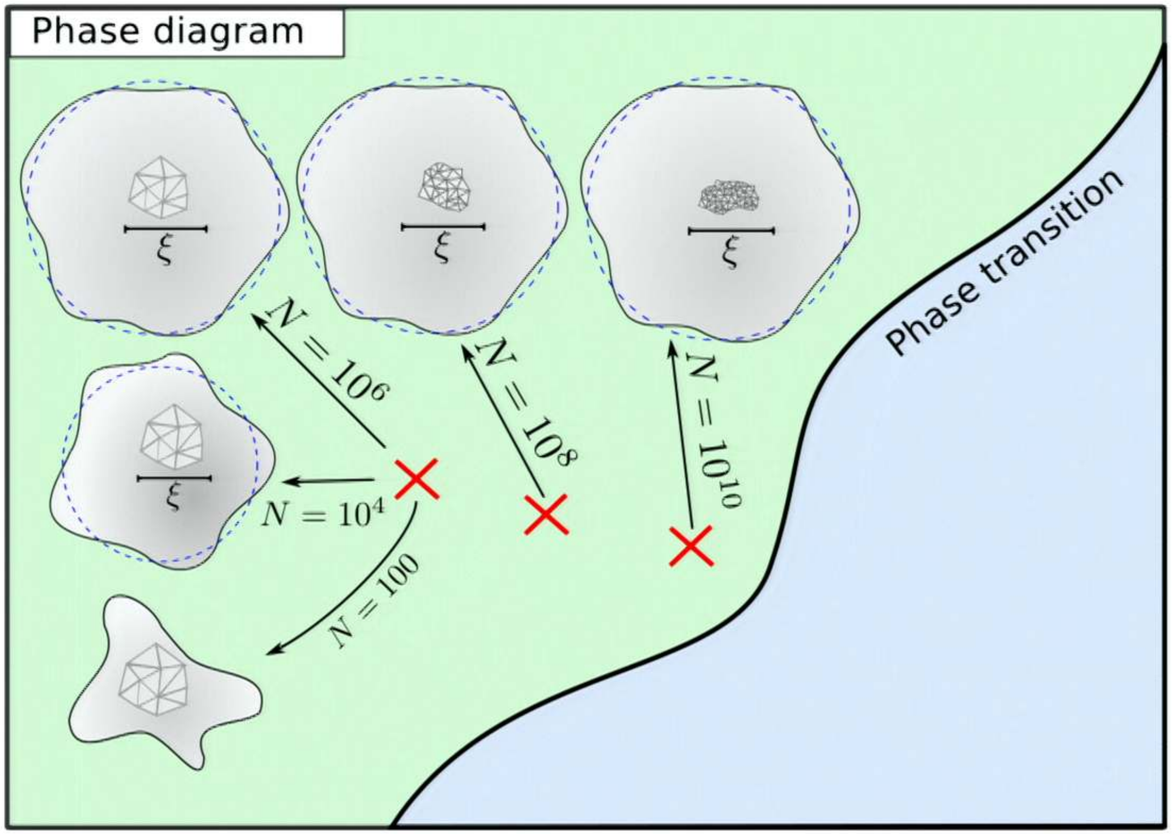

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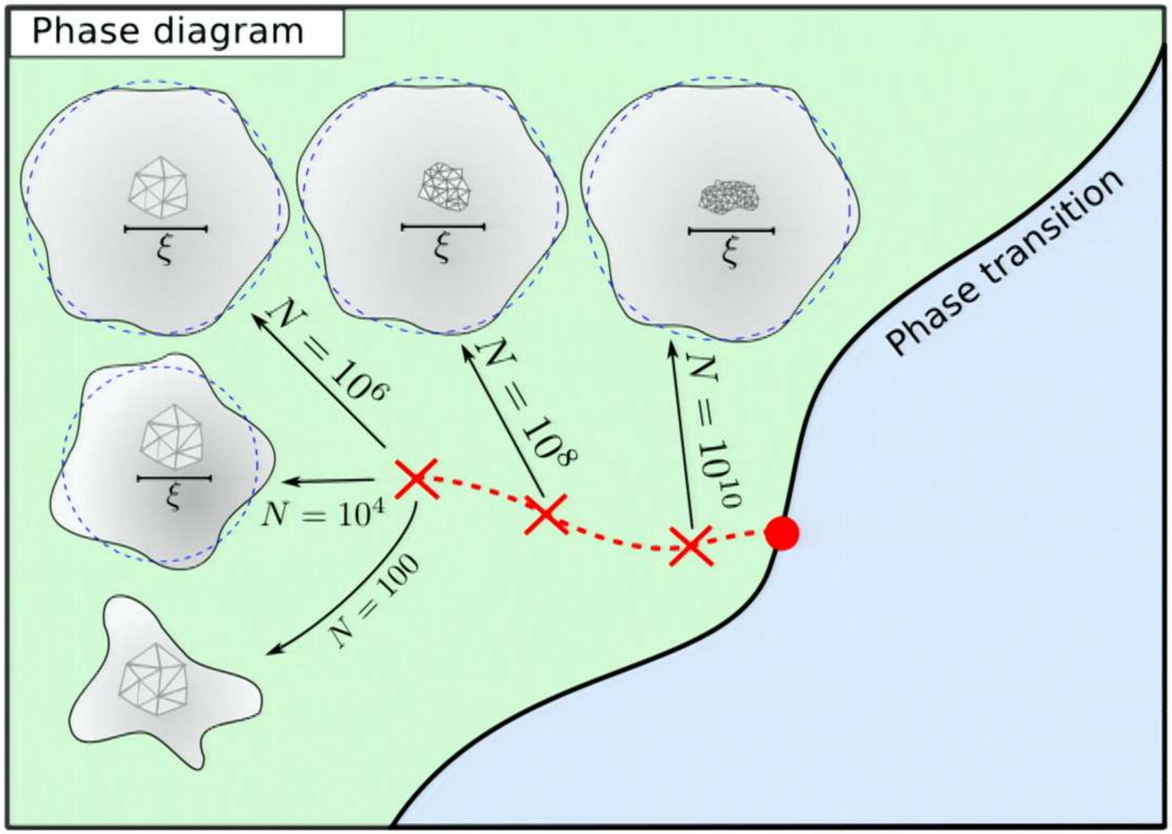

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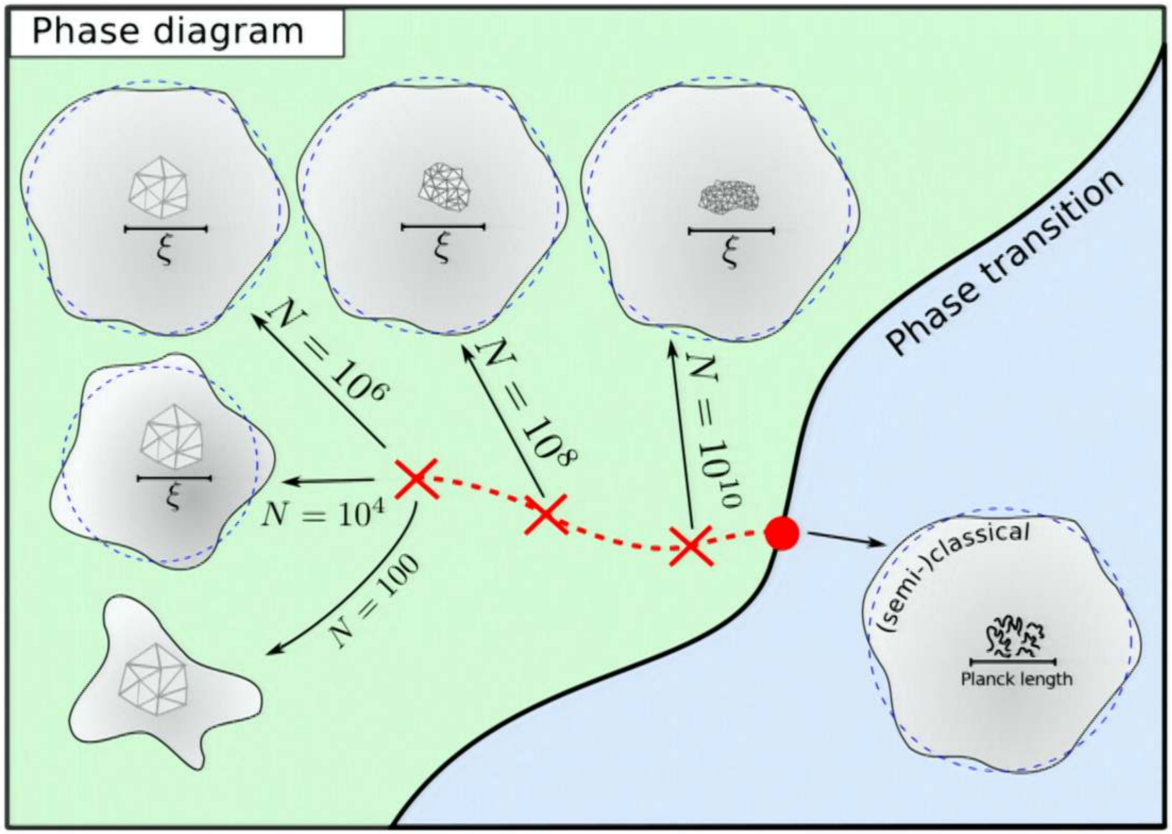
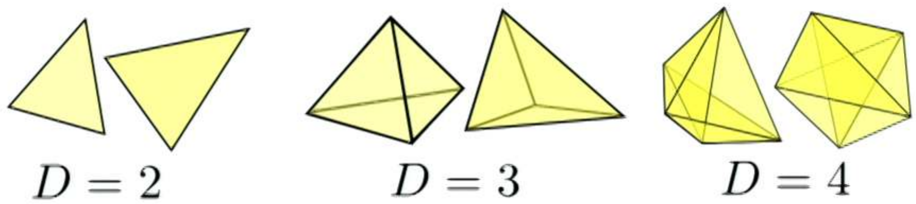


# Piecewise linear geometry

- $D$-simplex: $\{\sum_{i=0}^{D} \lambda_i \mathbf{x}_i : \lambda_i \in [0,1], \sum \lambda_i = 1\} \subset \mathbb{R}^D$ with Euclidean geometry.



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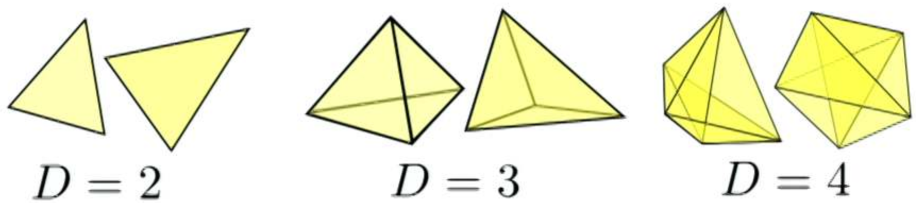
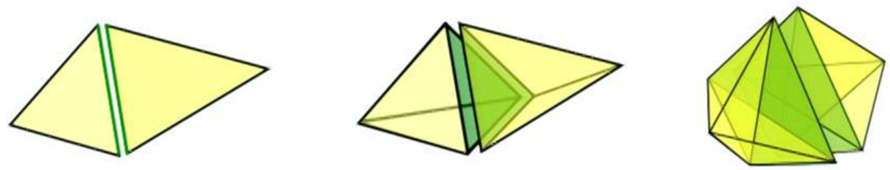



- $D$-simplices can be glued into larger metric spaces along matching $(D-1)$-simplices.



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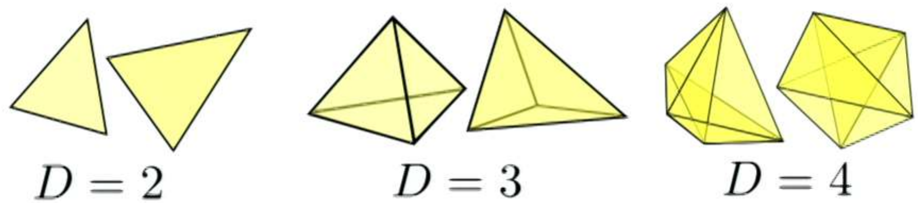


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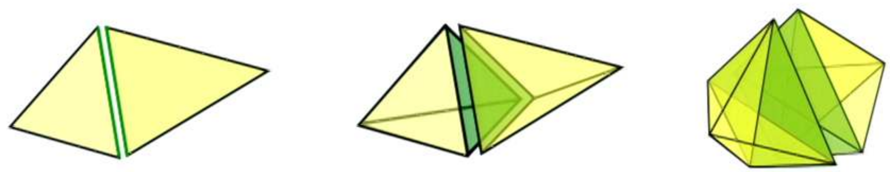
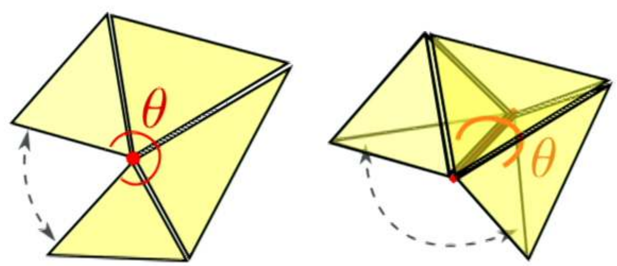


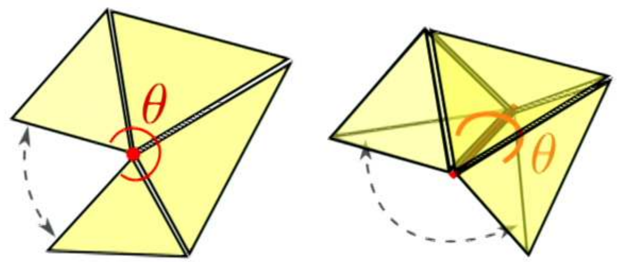
- Resulting geometry has curvature supported on $(D-2)$-simplices.



# Einstein-Hilbert action

- Integrated curvature is naturally expressed in terms of deficit angles [Regge, '61]

$$\int \mathrm{d}^D x \sqrt{g} R \longrightarrow \sum_{(D-2)-\text{simplices } \sigma} |\sigma|(2\pi - \theta_\sigma)$$



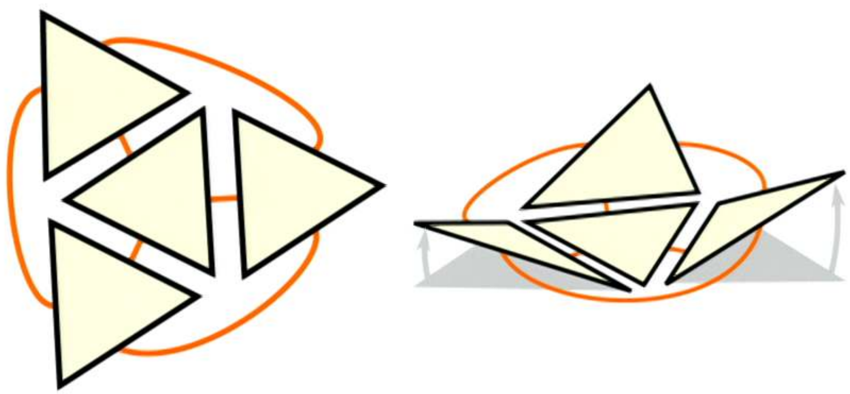

- If all simplices are taken of equal shape (say, equilateral) then linearity of Regge action implies that EH ($+ \int \mathrm{d}^D x \sqrt{g} \Lambda$) is a simple linear combination

$$\kappa_D N_D - \kappa_{D-2} N_{D-2}.$$

- Makes sense to include in MCMC at least such two terms in Boltzmann weight.

# Combinatorial representation

- Recall: need next and adjacent to navigate a map, or a polyhedron.



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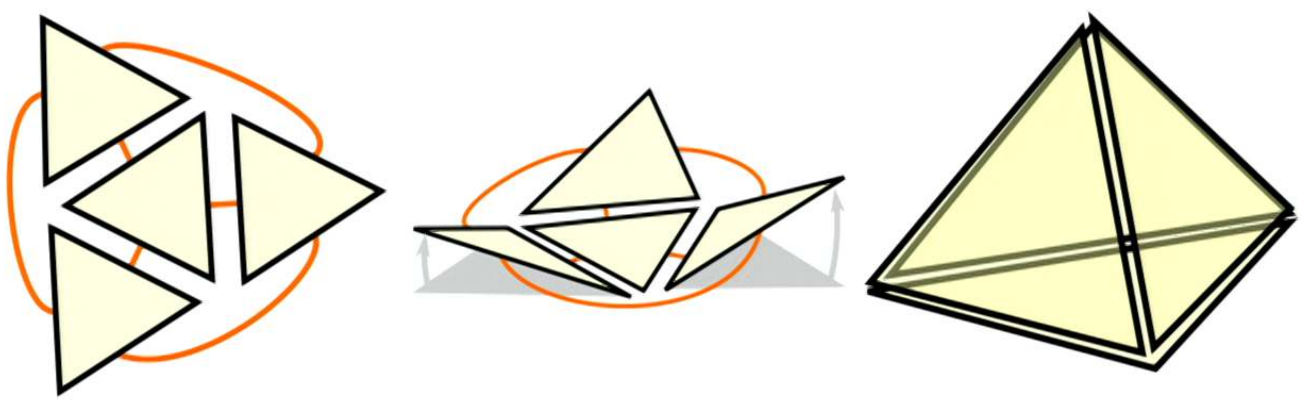

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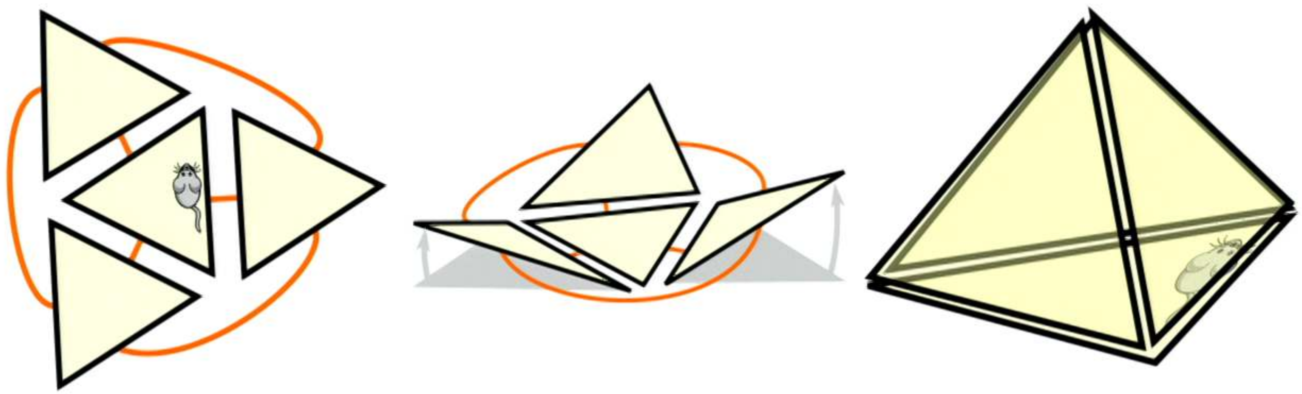

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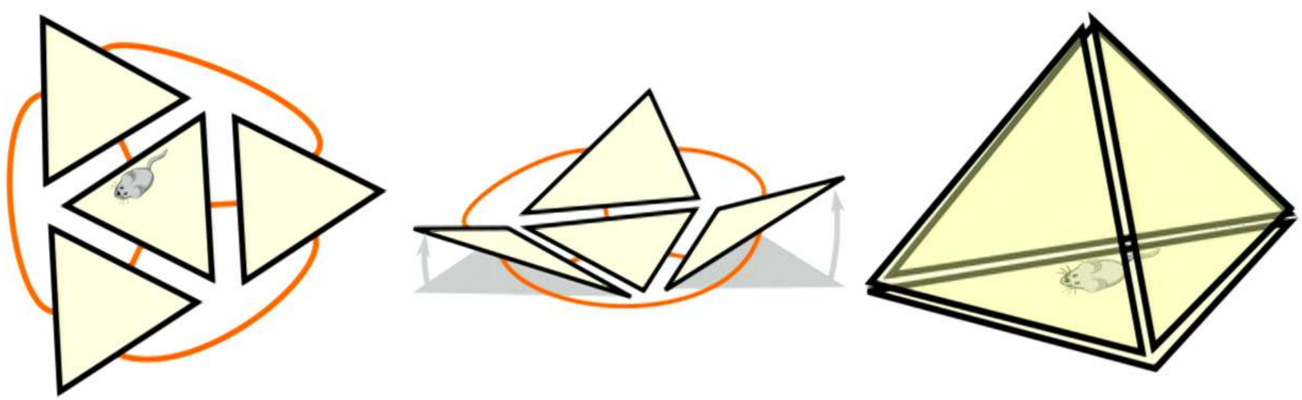

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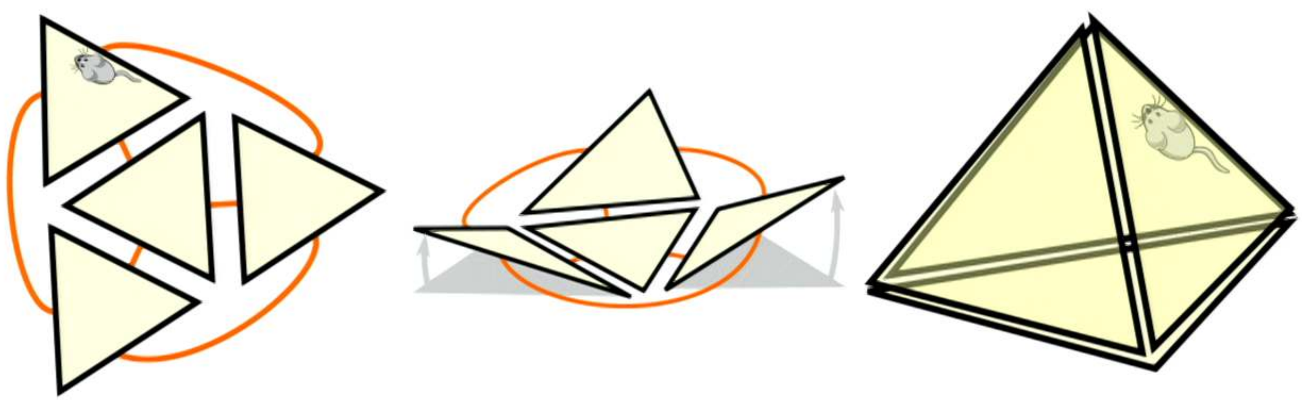

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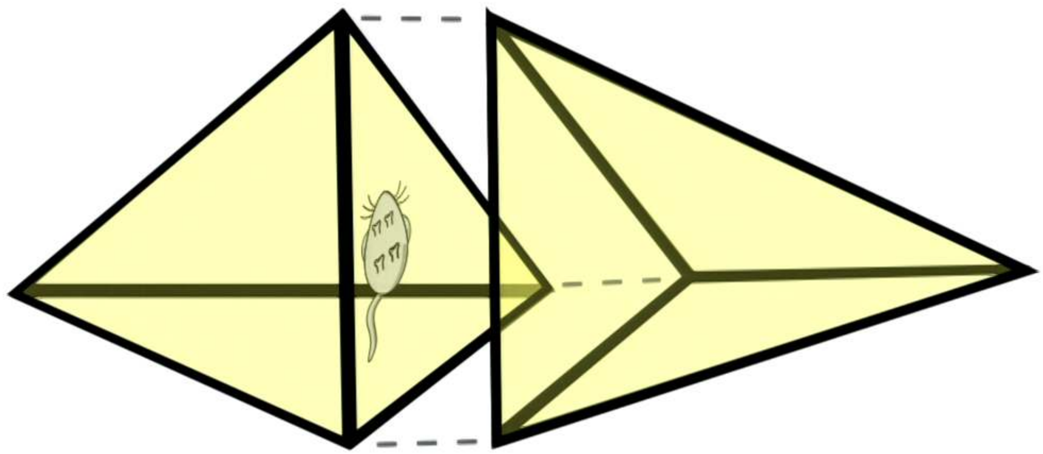
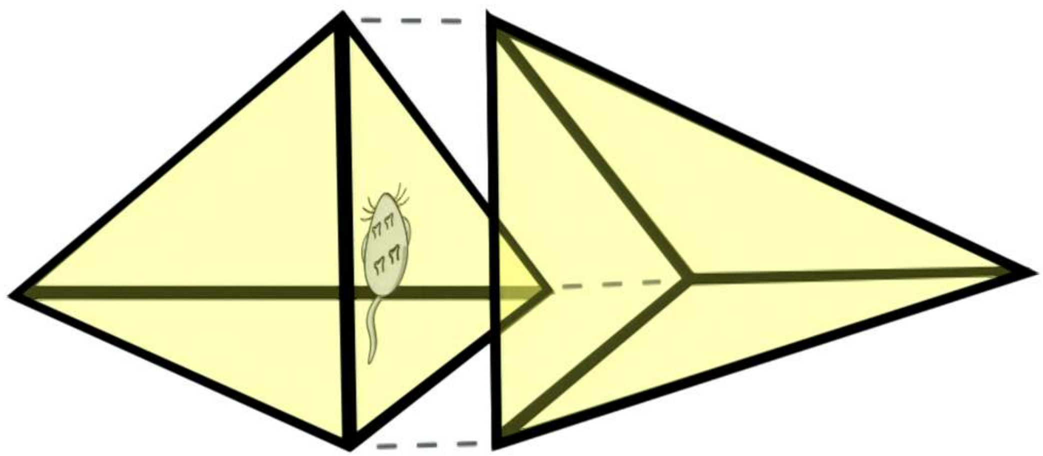


# Combinatorial representation

- Recall: need next and adjacent to navigate a map, or a polyhedron.
- Generalize: For $D$-dimensional geometry, $(n, a) \rightarrow (n, a_2, \cdots, a_D)$.
  $a_d$ maps half-edge $i$ to its $d$-dimensional neighbor $a_d(i)$.



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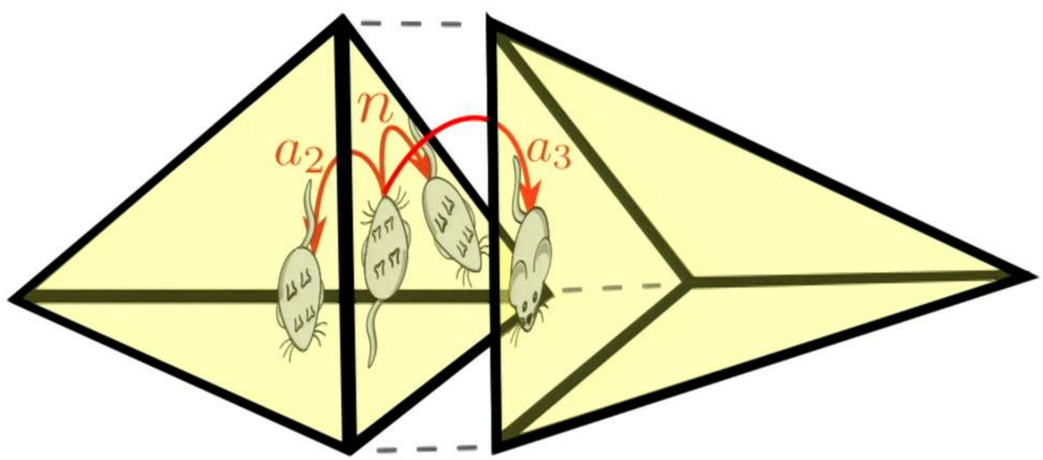

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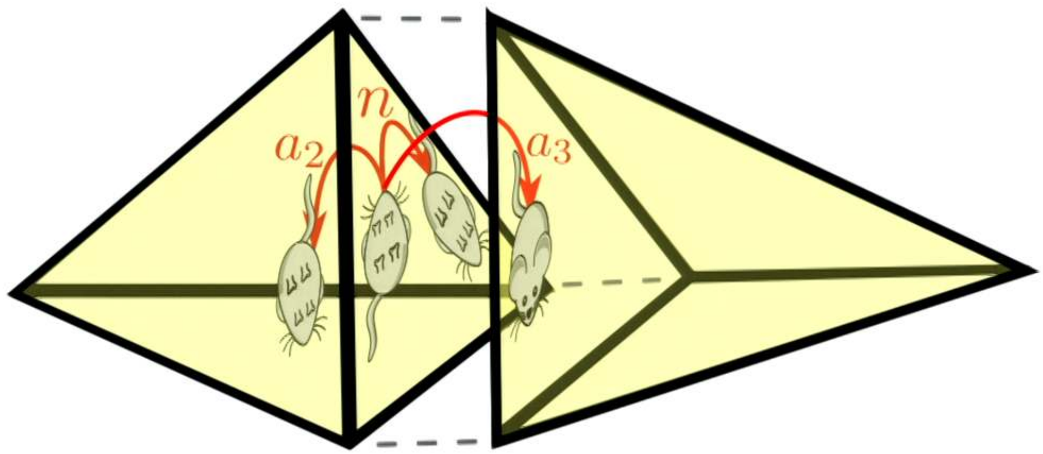
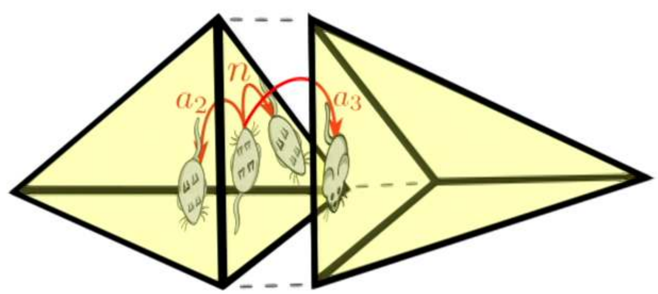
- Cells of various dimensions are identified as orbits. In 3D: $(n, a_2) \to$ polyhedra, $(n, a_3) \to$ faces, $(a_2, a_3) \to$ edges, $(n \circ a_2, n \circ a_3) \to$ vertices.



# Remarks on combinatorics in 3D

- When does a triple of permutations $(n, a_2, a_3)$ determine a topological 3-manifold?



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  - Proper gluing: $n \circ a_3 \circ n = a_3$.
  - Polyhedra (orbits under $n$, $a_2$) should have 3-ball topology (i.e. boundary $S^2$): Euler formula!
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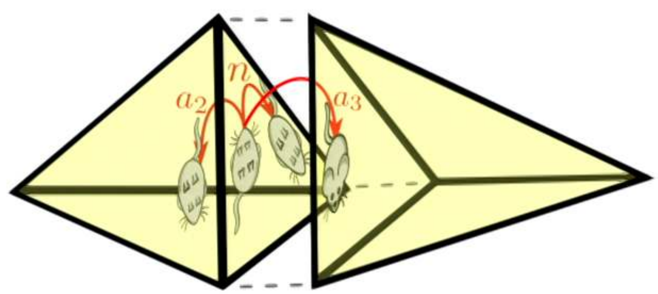

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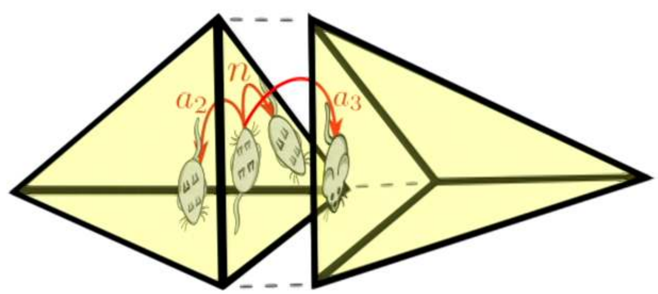

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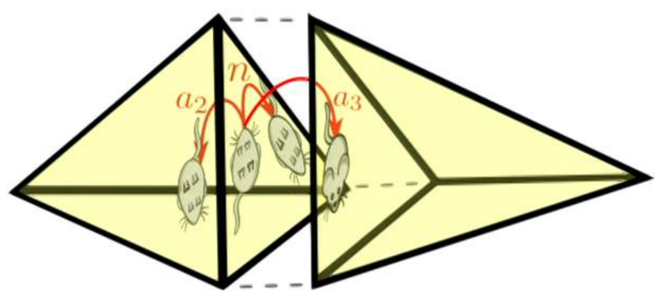
- What is the topology of the resulting 3-manifold?
  - Unfortunately, no simple combinatorial/algorithmic way to decide!
  - Luckily, any two geometries with equal topology are connected by a finite sequence of local moves!
- Situation very similar in 4D (and higher).



# Simplicial manifolds

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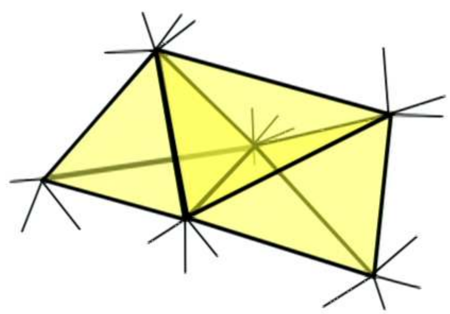

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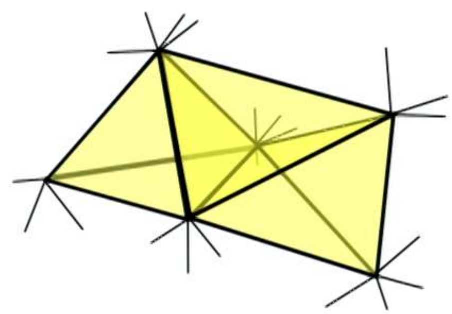

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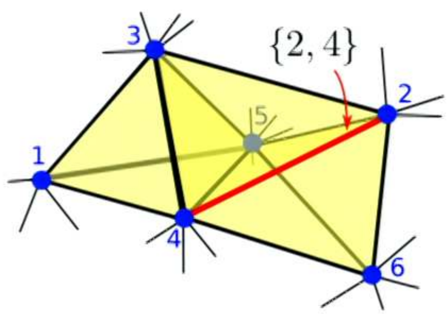

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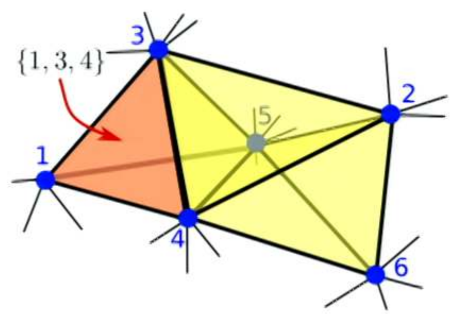

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- Knowing the set $\{\{1,3,4,5\},\{2,3,4,5\},\{2,4,5,6\},\ldots\}$ of $D$-simplices, can reproduce the triple $(n, a_2, a_3)$ up to relabeling (and orientation).