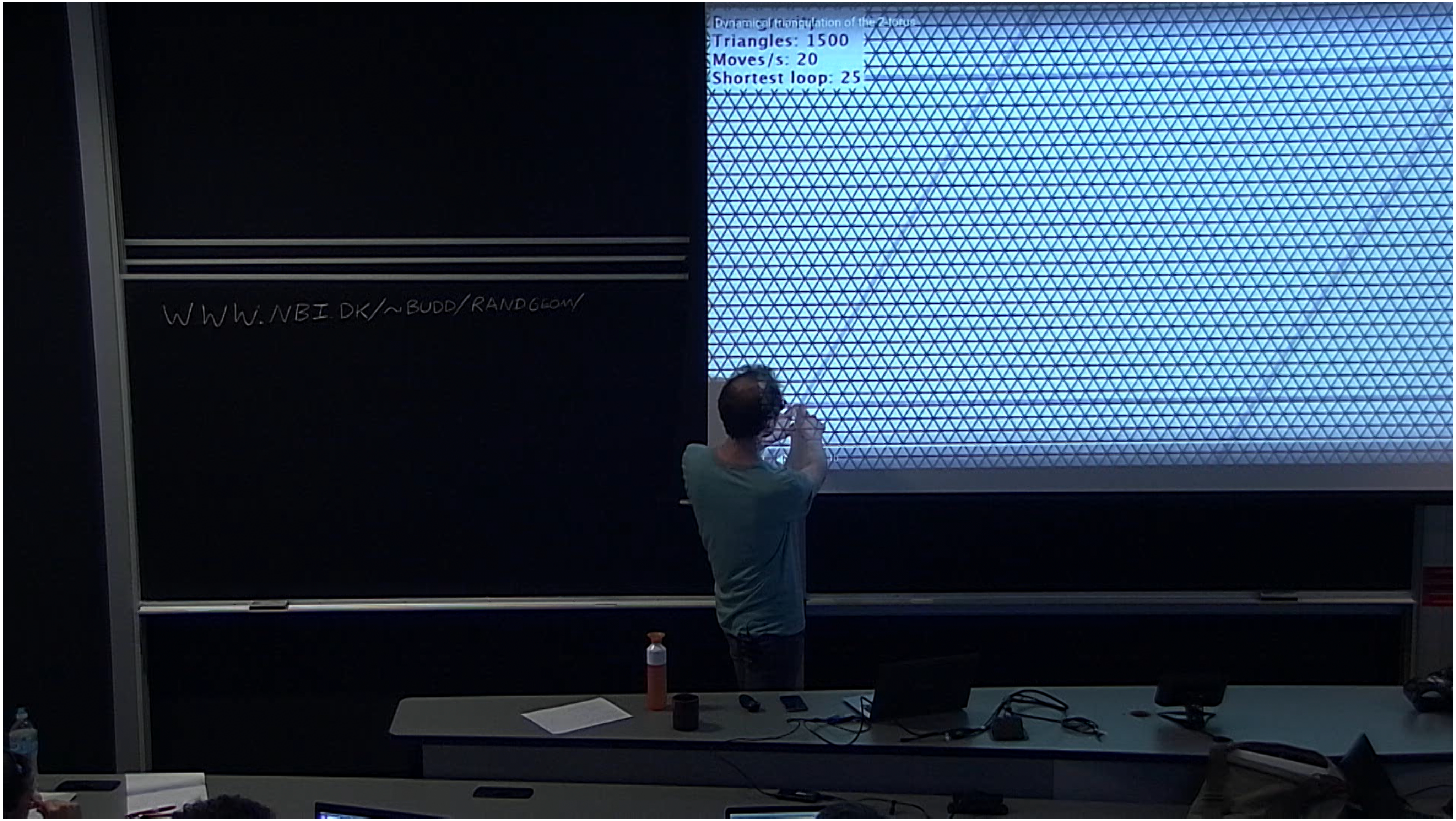
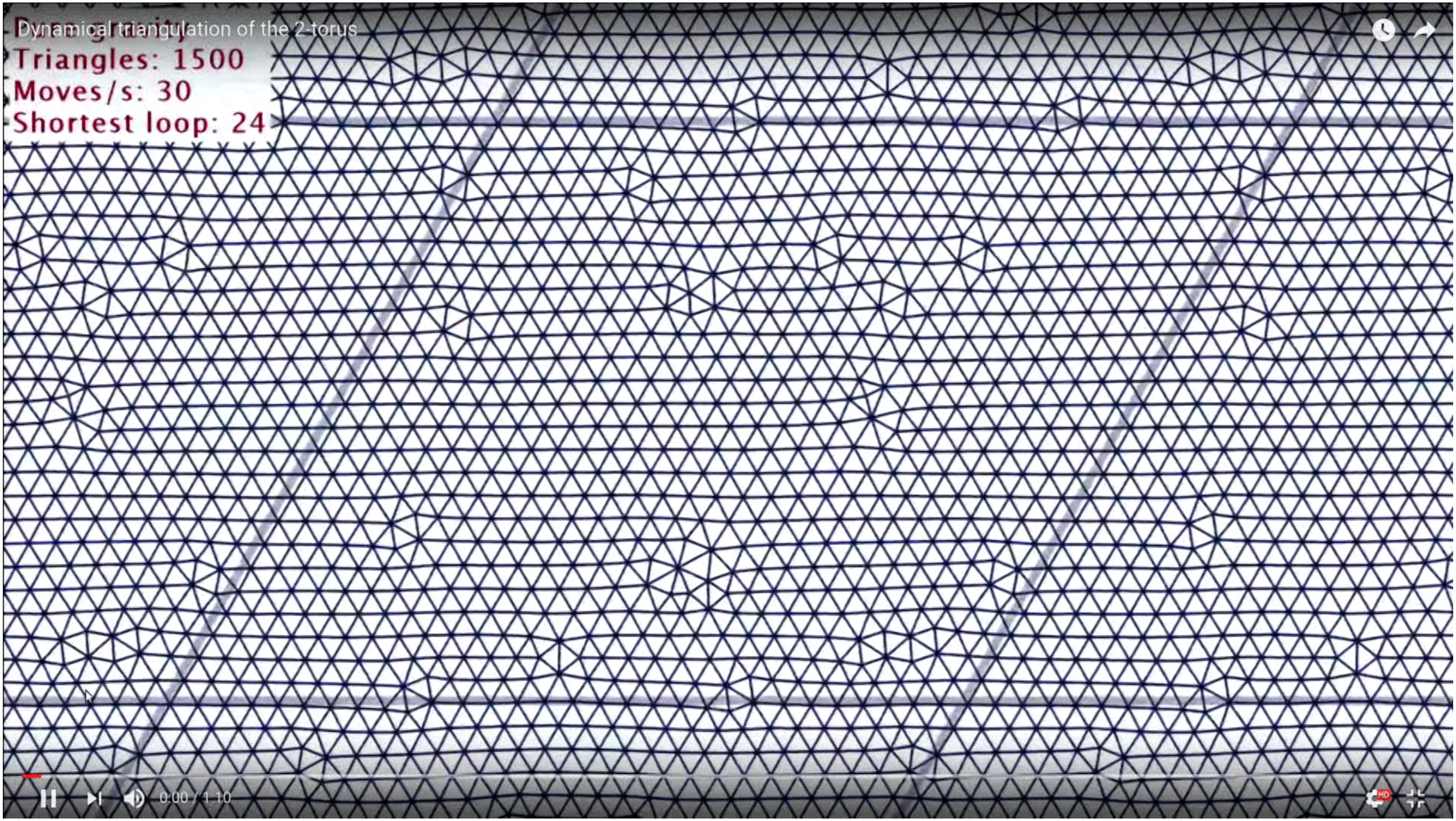
Title: Monte Carlo methods in Dynamical Triangulations - 2

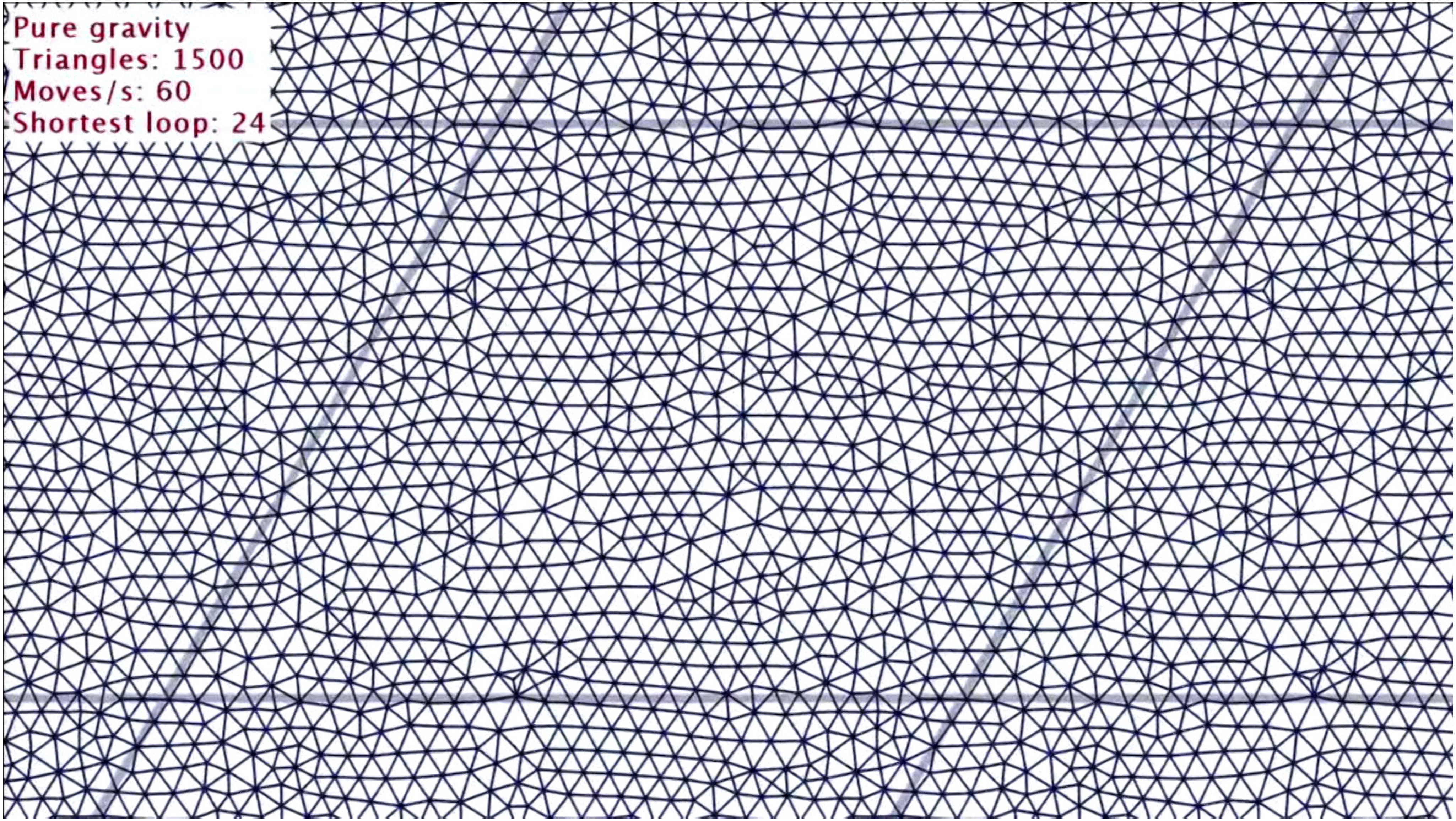
Date: Jun 21, 2017 09:45 AM

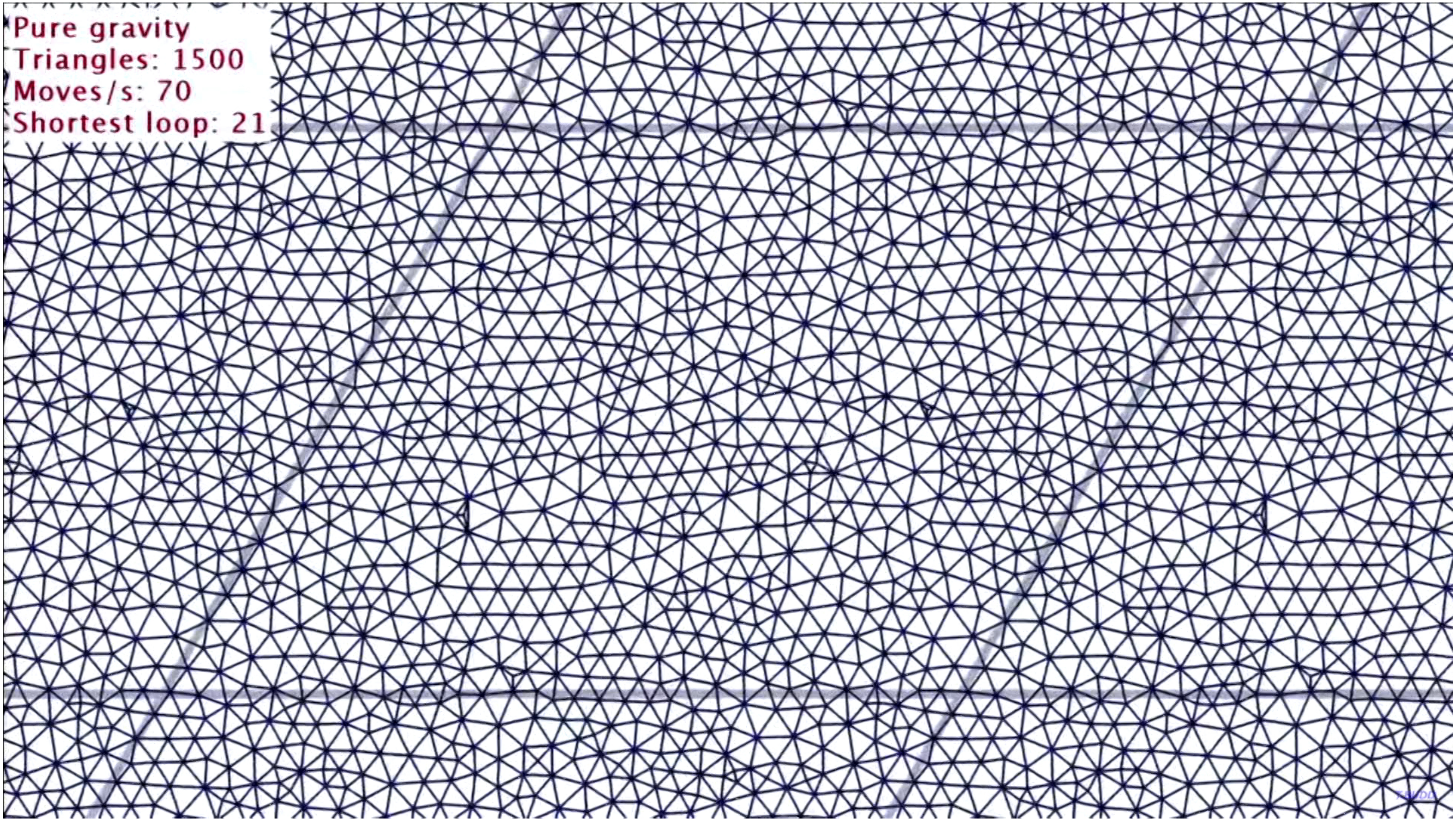
URL: http://pirsa.org/17060076

Abstract:


WWW.NBI.DK/~BUDD/RANDGEOM/
Dynamical triangulation of the 2-torus
Triangles: 1500
Moves/s: 20
Shortest loop: 25


Dynamical triangulation of the 2-torus
Triangles: 1500
Moves/s: 30
Shortest loop: 24
0:00 / 1:10


Pure gravity
Triangles: 1500
Moves/s: 60
Shortest loop: 24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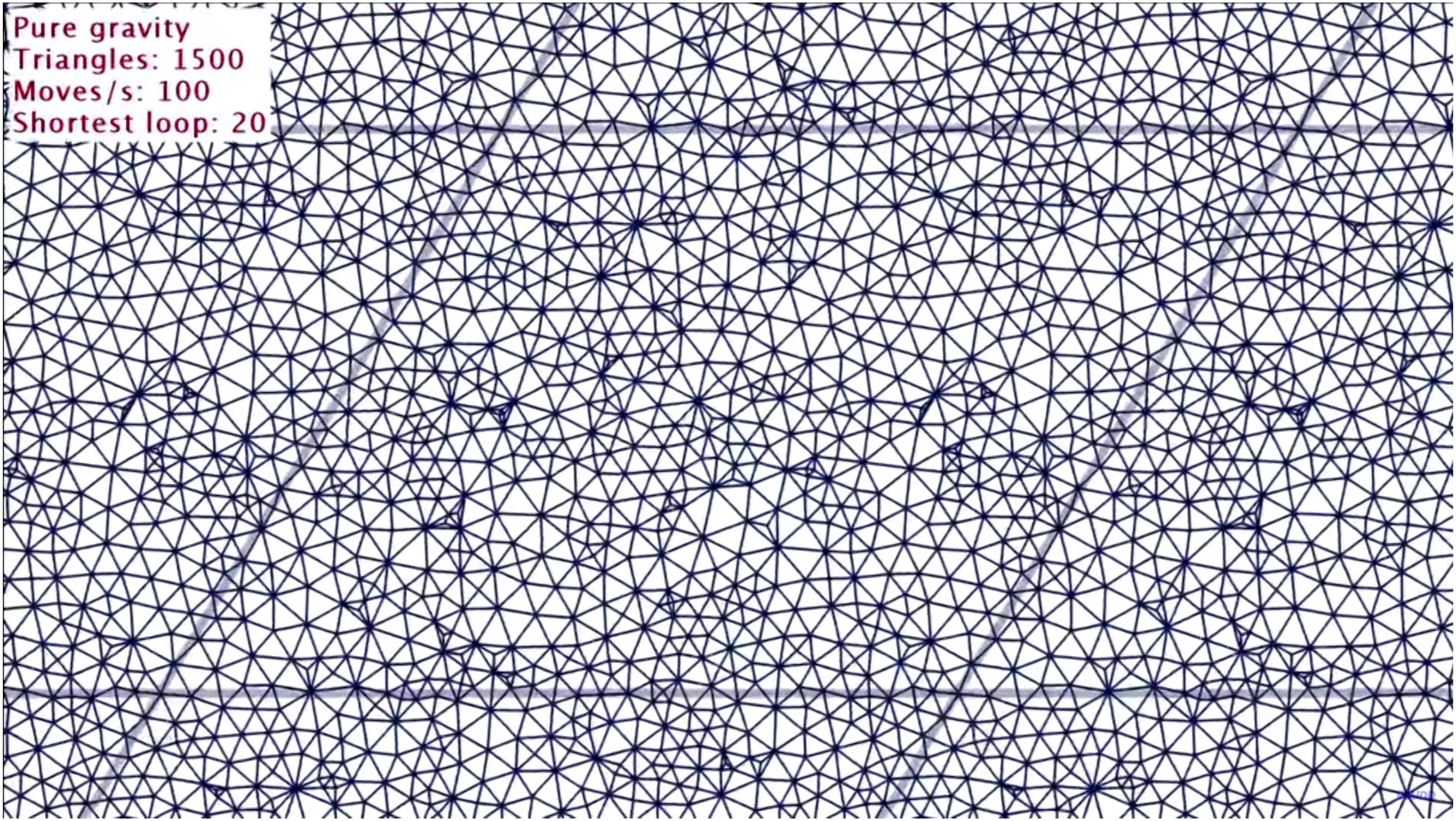
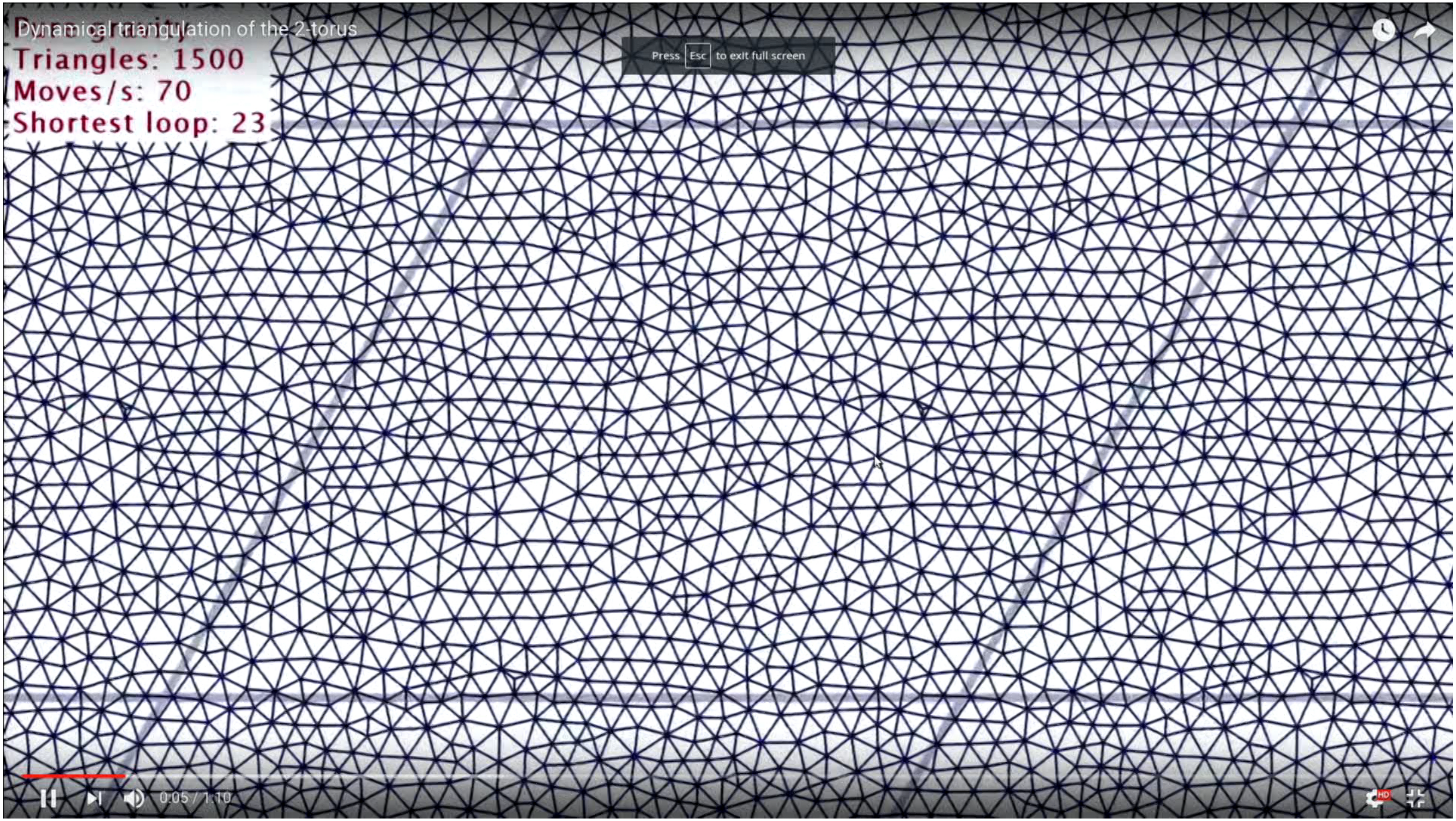
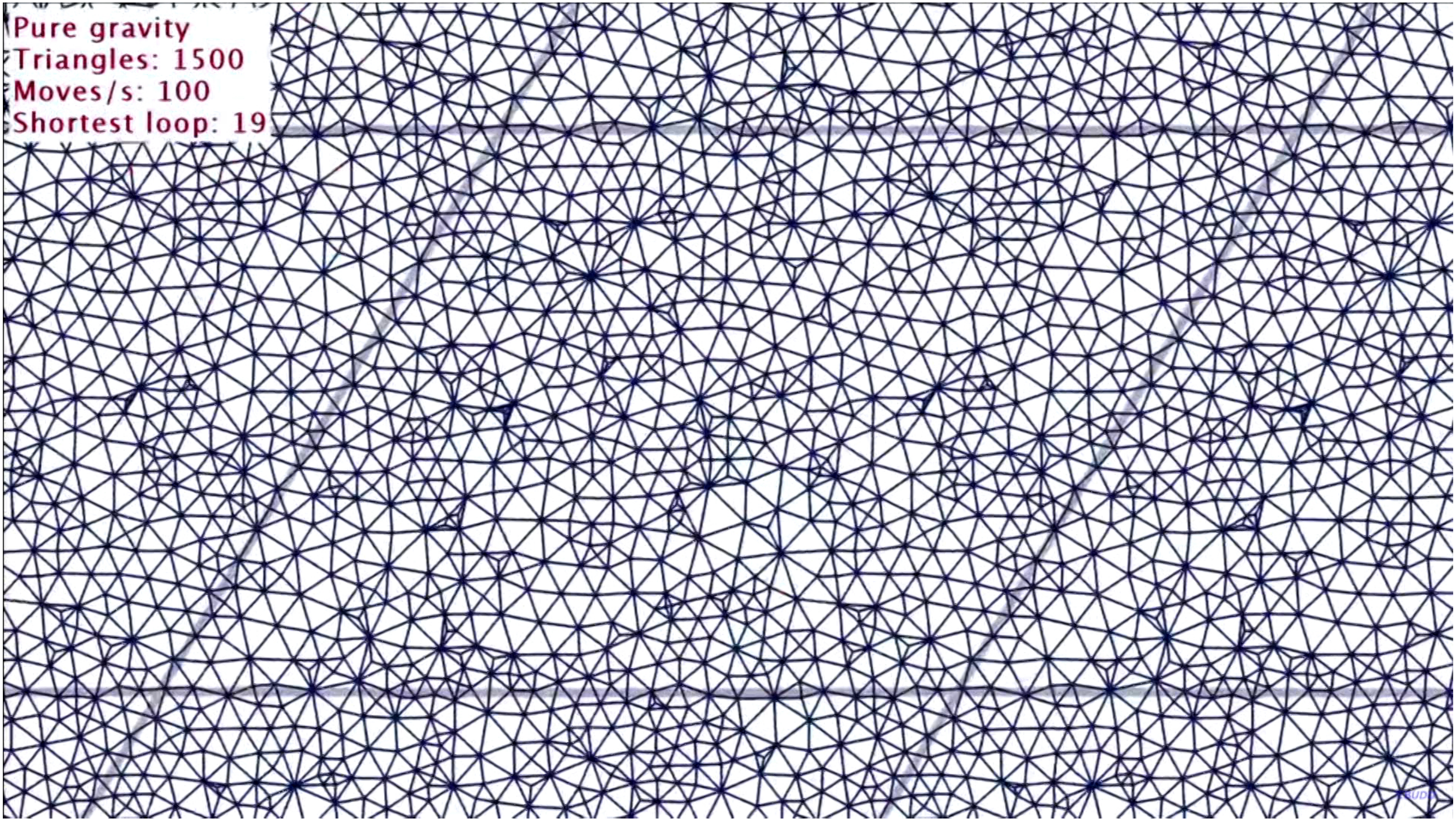
Triangles: 1500
Moves/s: 70
Shortest loop: 21


Pure gravity
Triangles: 1500
Moves/s: 100
Shortest loop: 20


Dynamical triangulation of the 2-torus
Triangles: 1500
Moves/s: 70
Shortest loop: 23
Press Esc to exit full screen
0:05 / 1:10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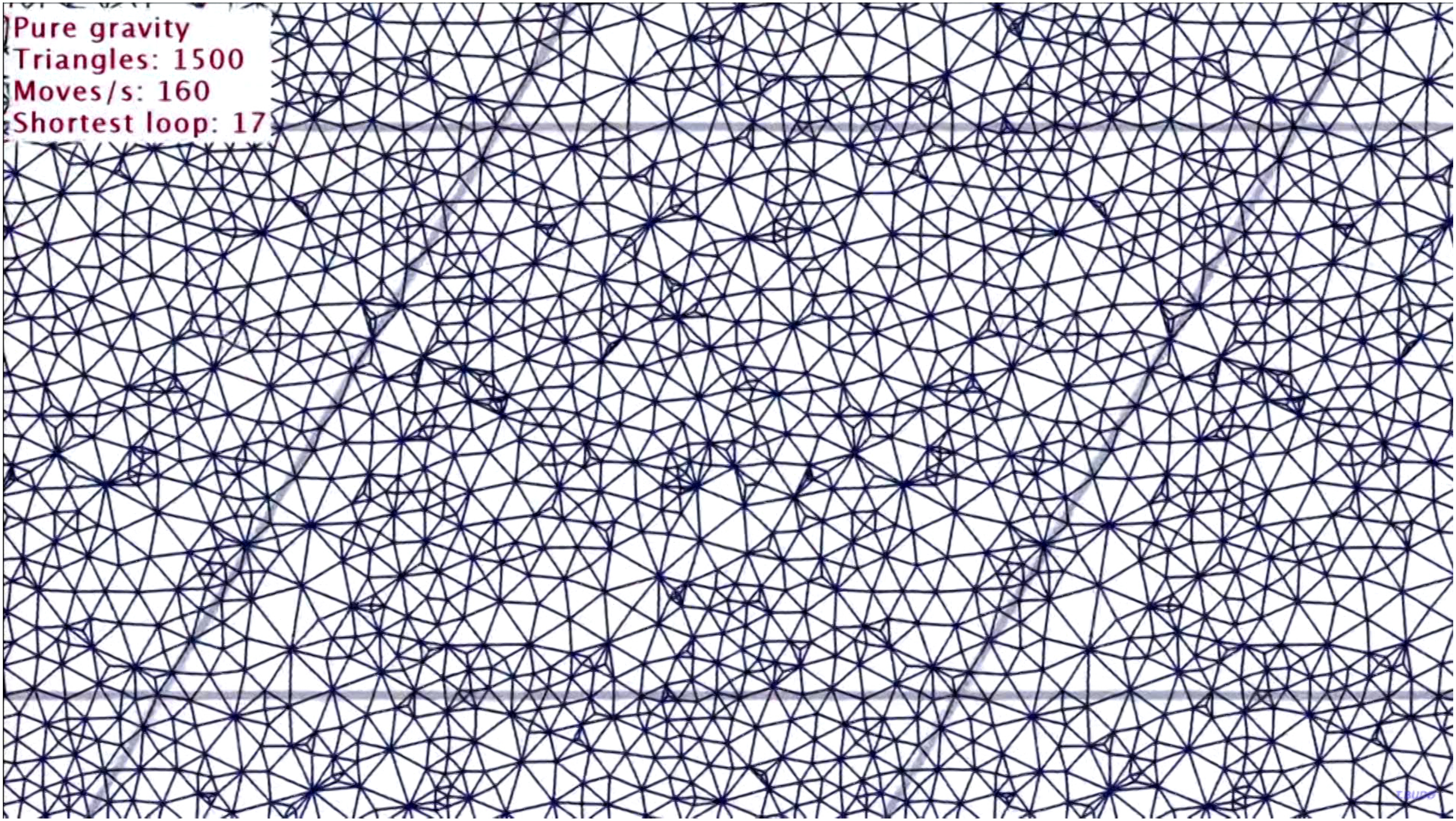
Triangles: 1500
Moves/s: 100
Shortest loop: 19


Pure gravity
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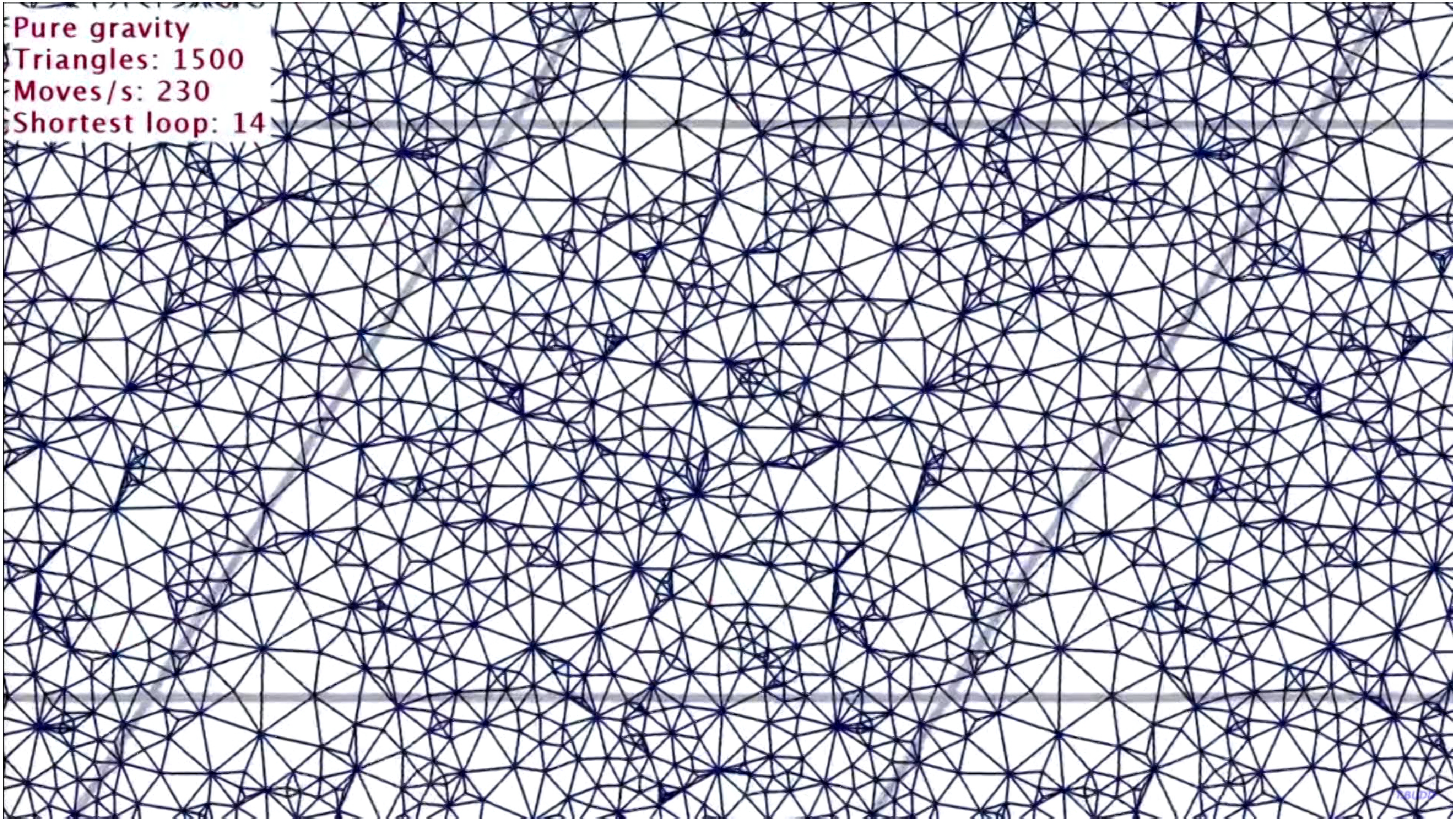
Moves/s: 160
Shortest loop: 17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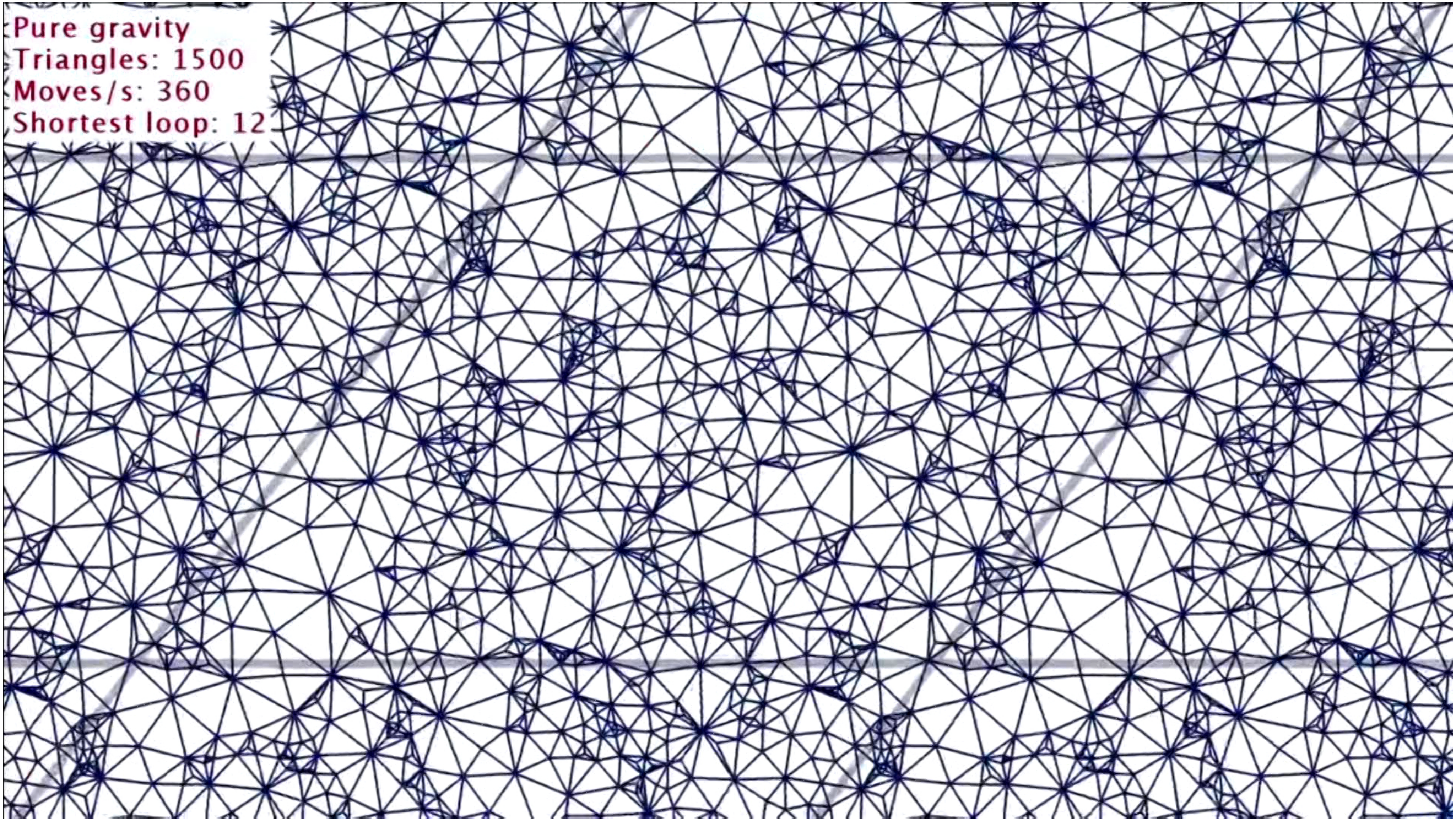
Moves/s: 230
Shortest loop: 14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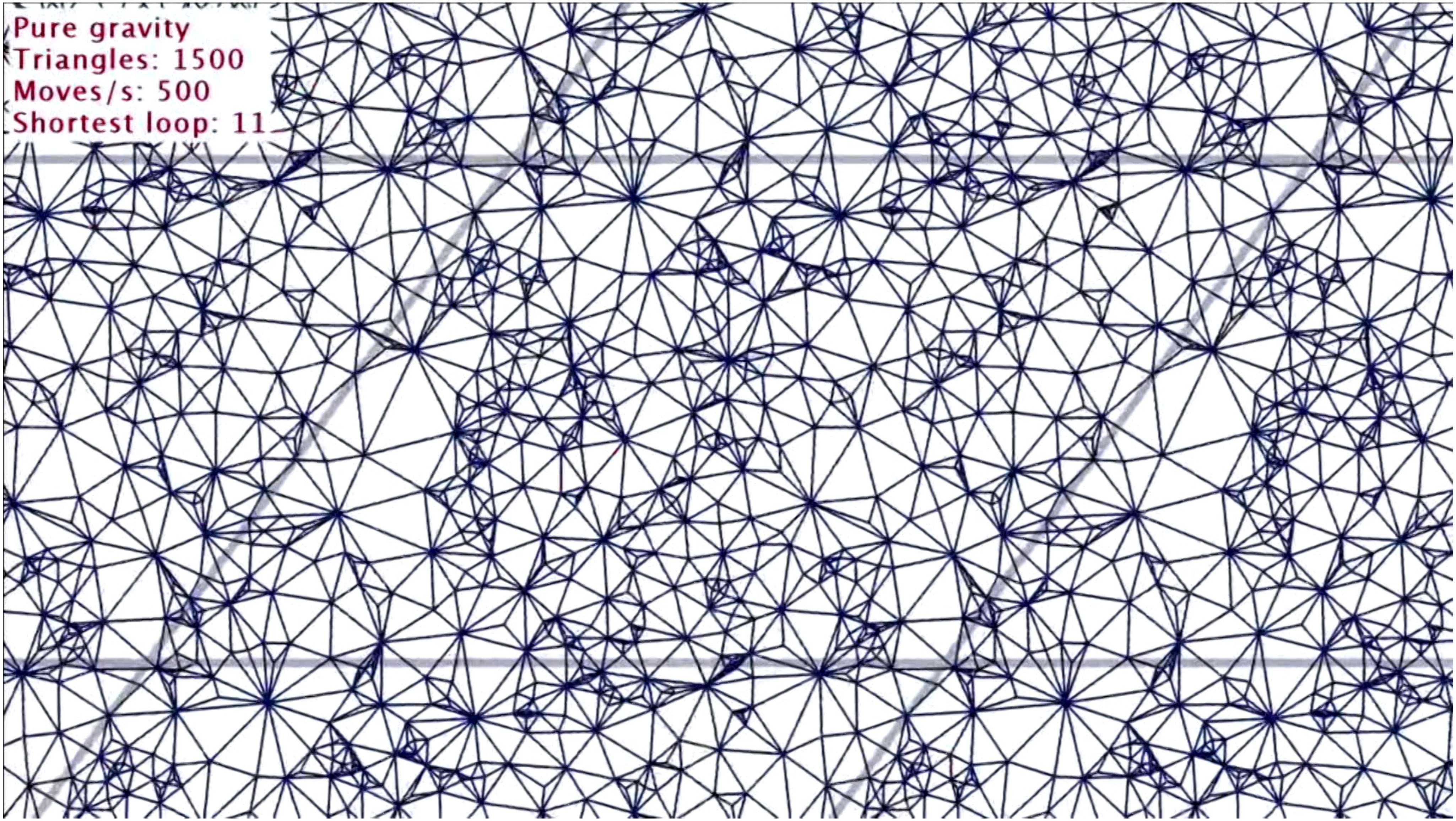
Triangles: 1500
Moves/s: 360
Shortest loop: 12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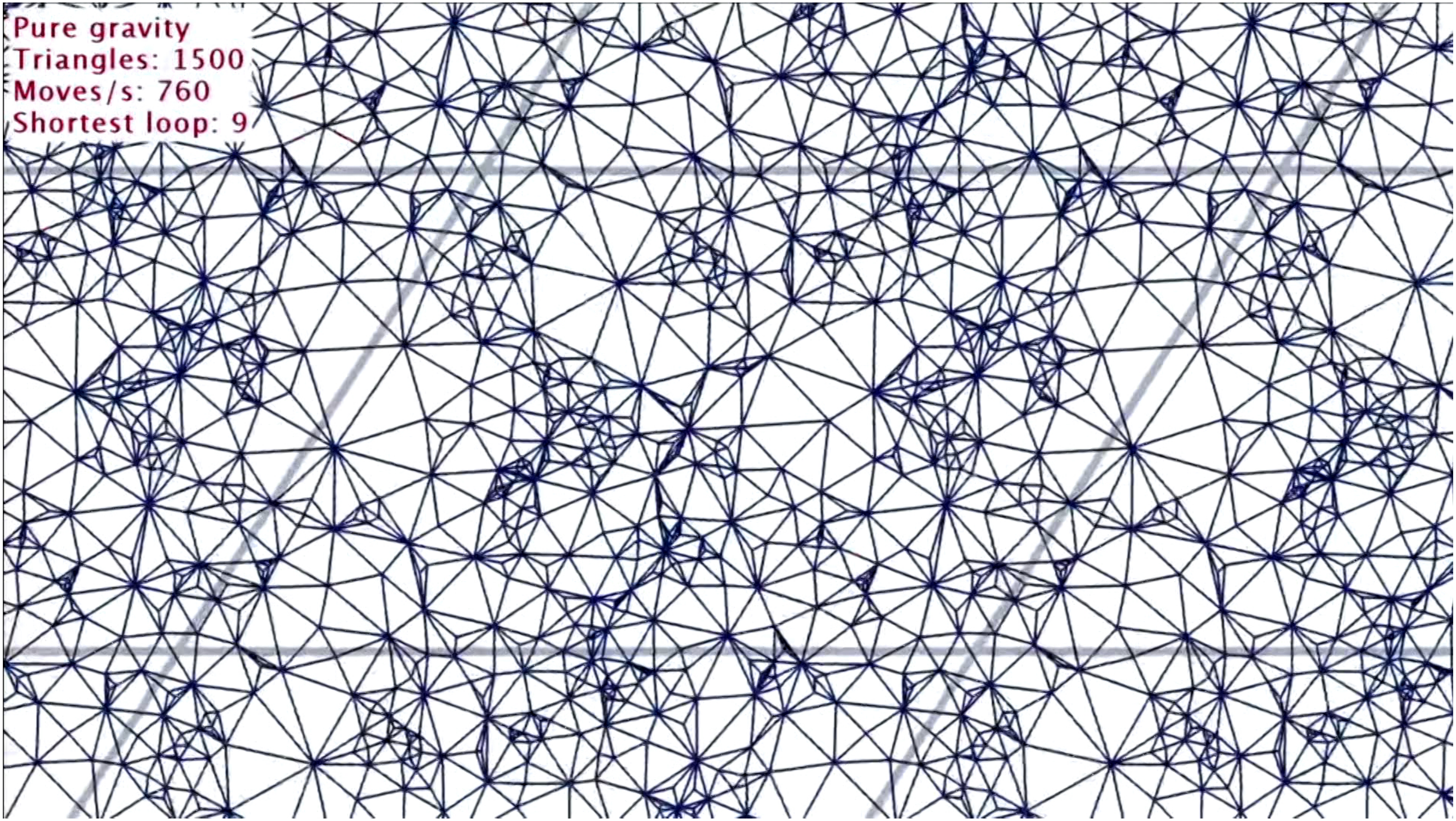
Triangles: 1500
Moves/s: 500
Shortest loop: 11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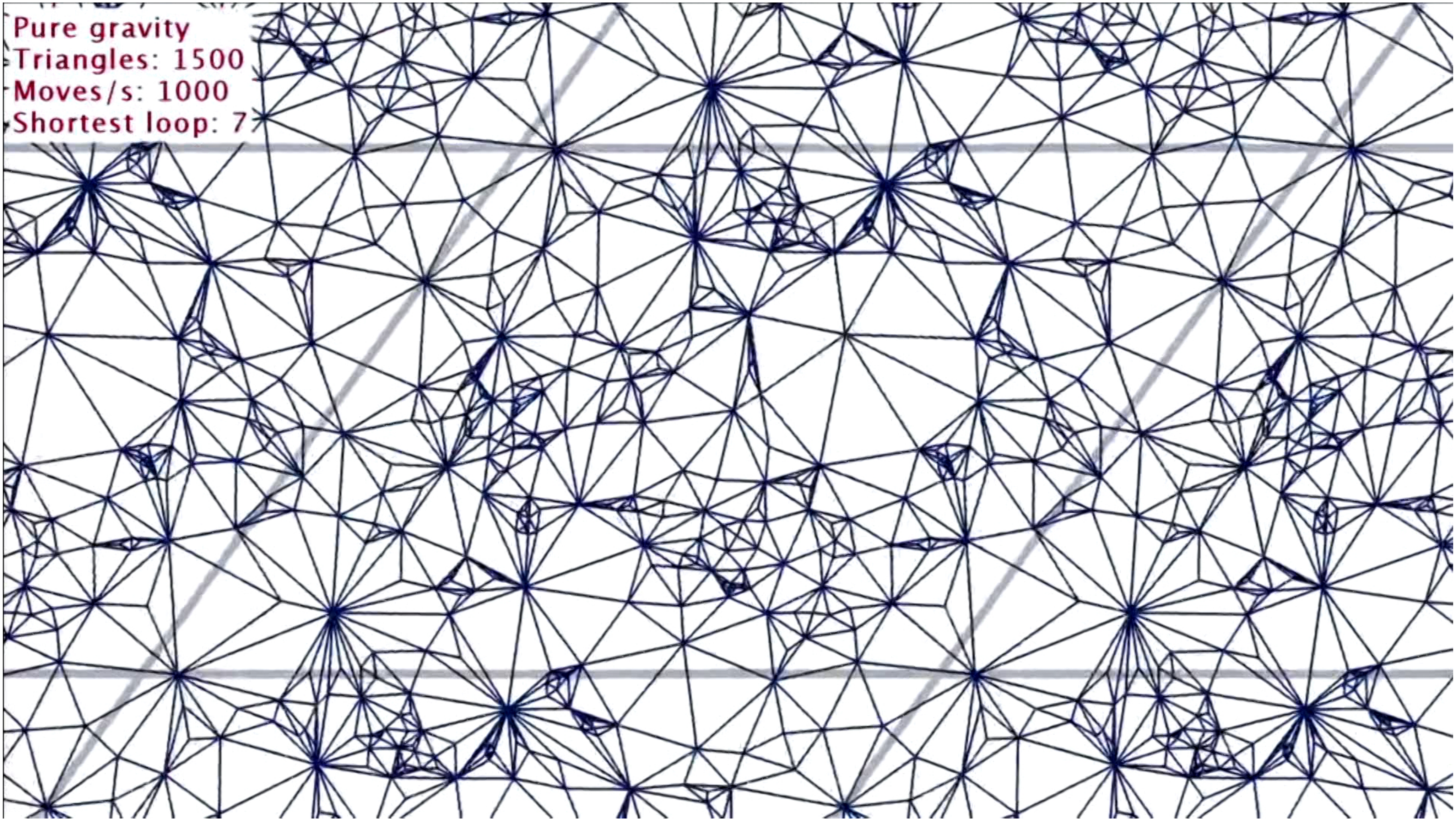
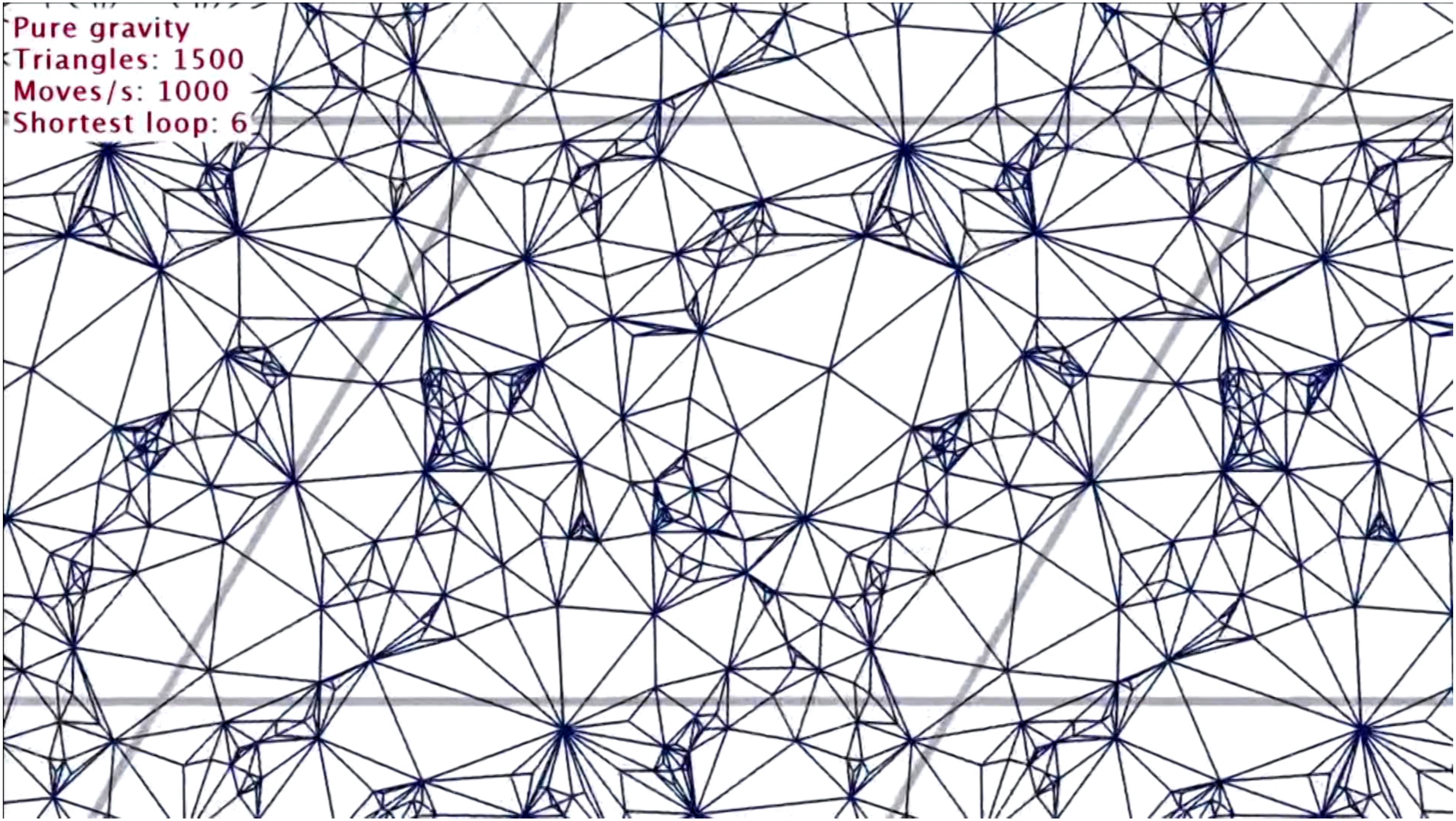
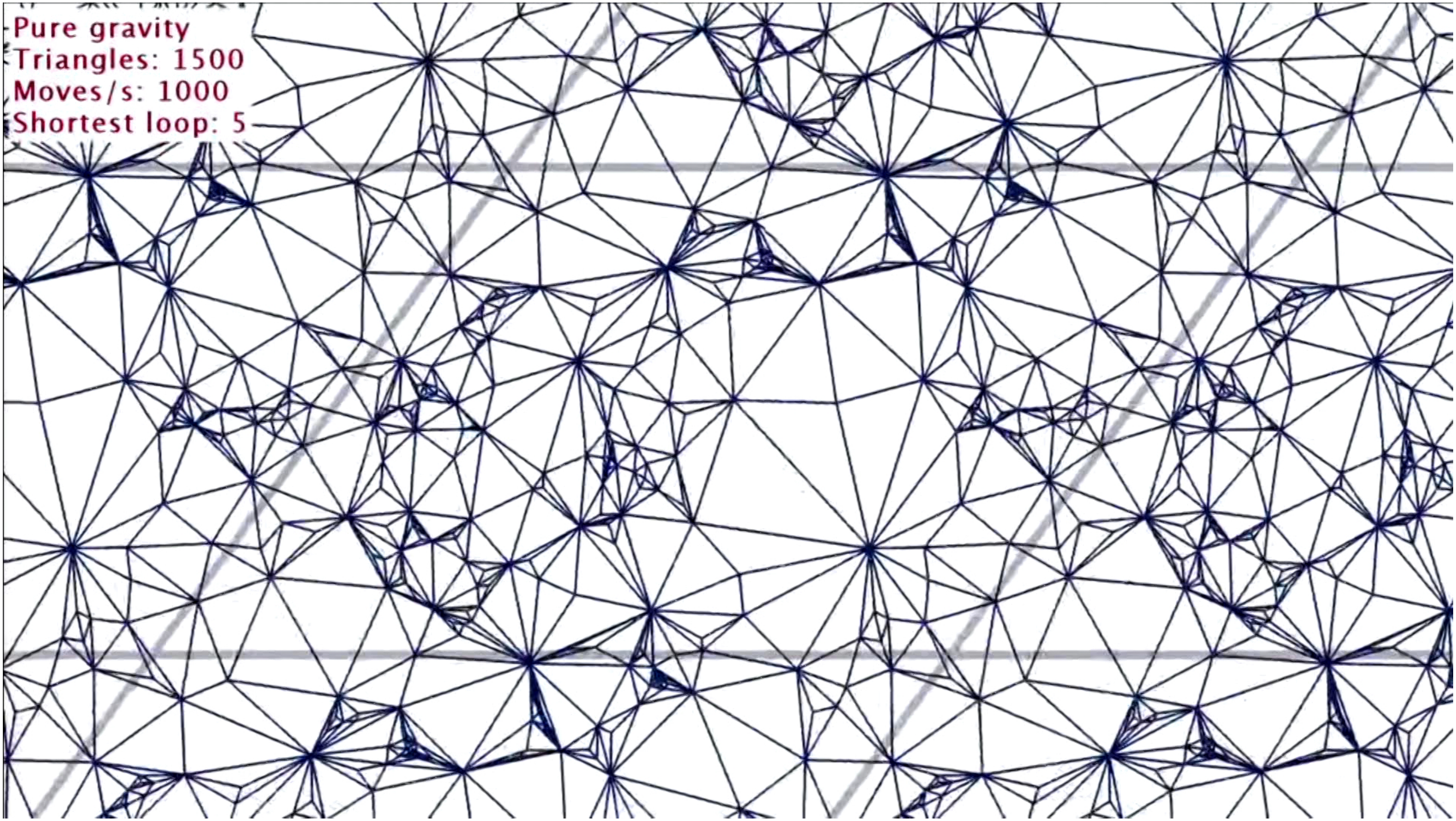
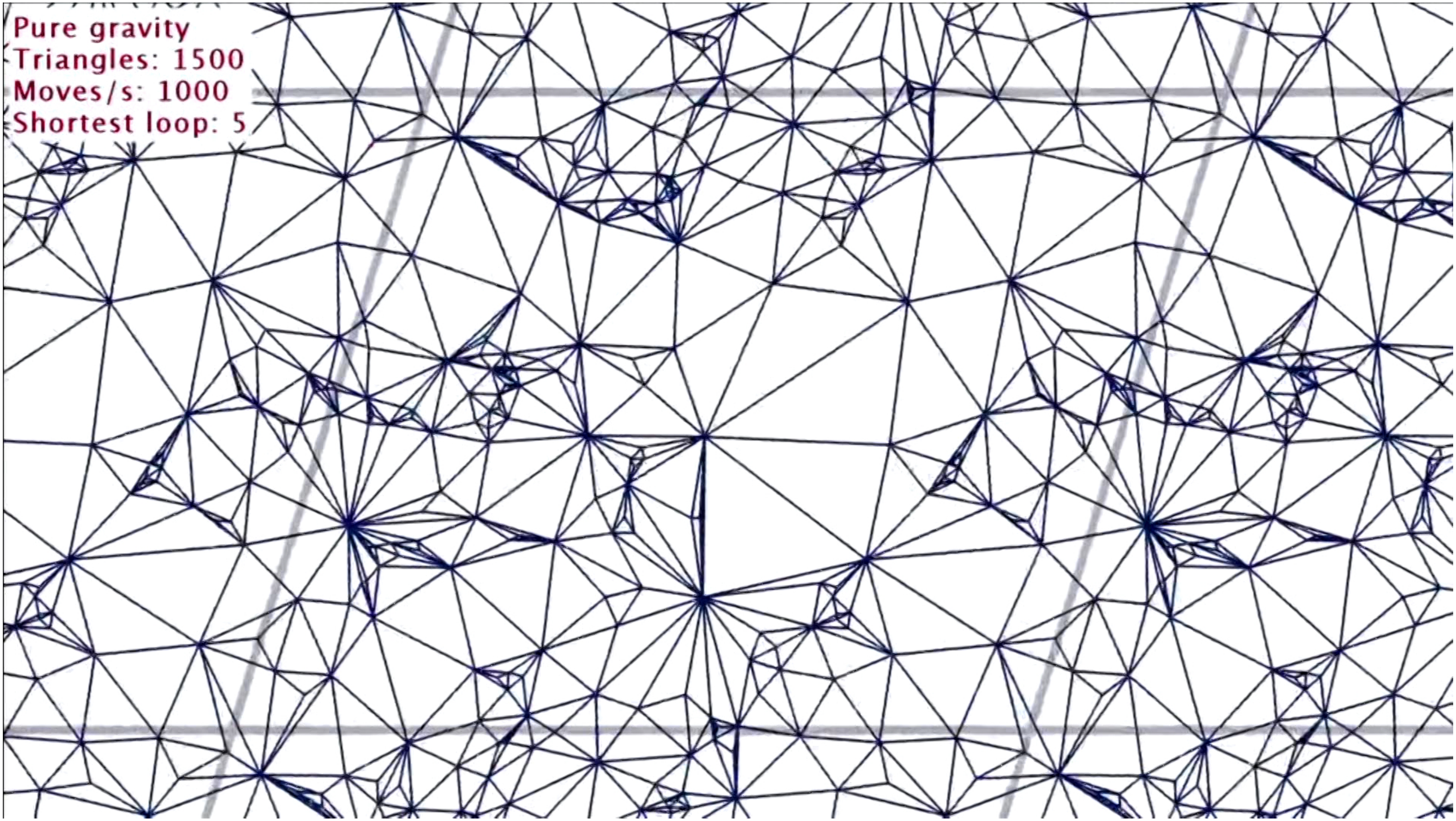
Moves/s: 760
Shortest loop: 9


Pure gravity
Triangles: 1500
Moves/s: 1000
Shortest loop: 7


Pure gravity
Triangles: 1500
Moves/s: 1000
Shortest loop: 6


Pure gravity
Triangles: 1500
Moves/s: 1000
Shortest loop: 5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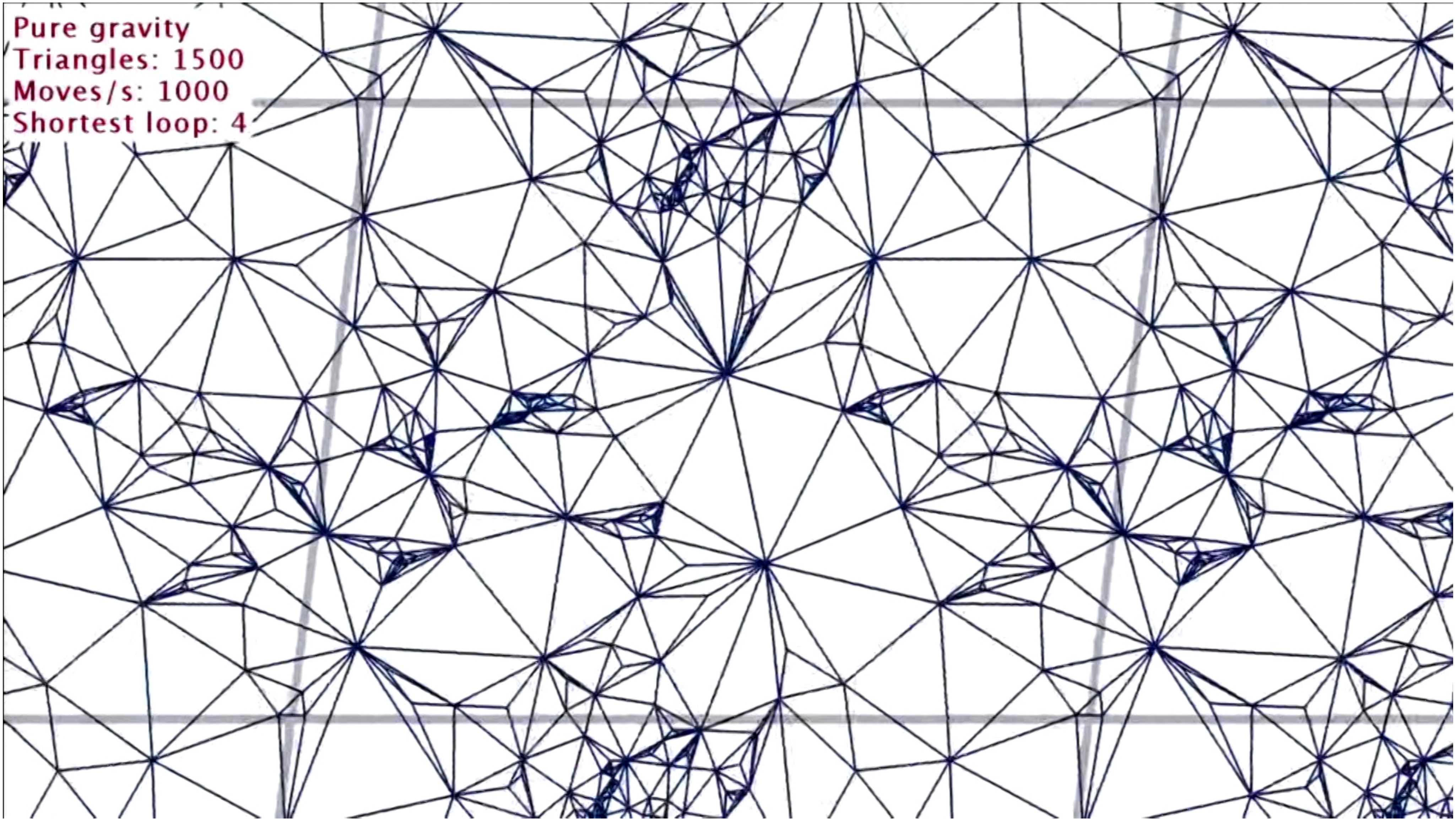
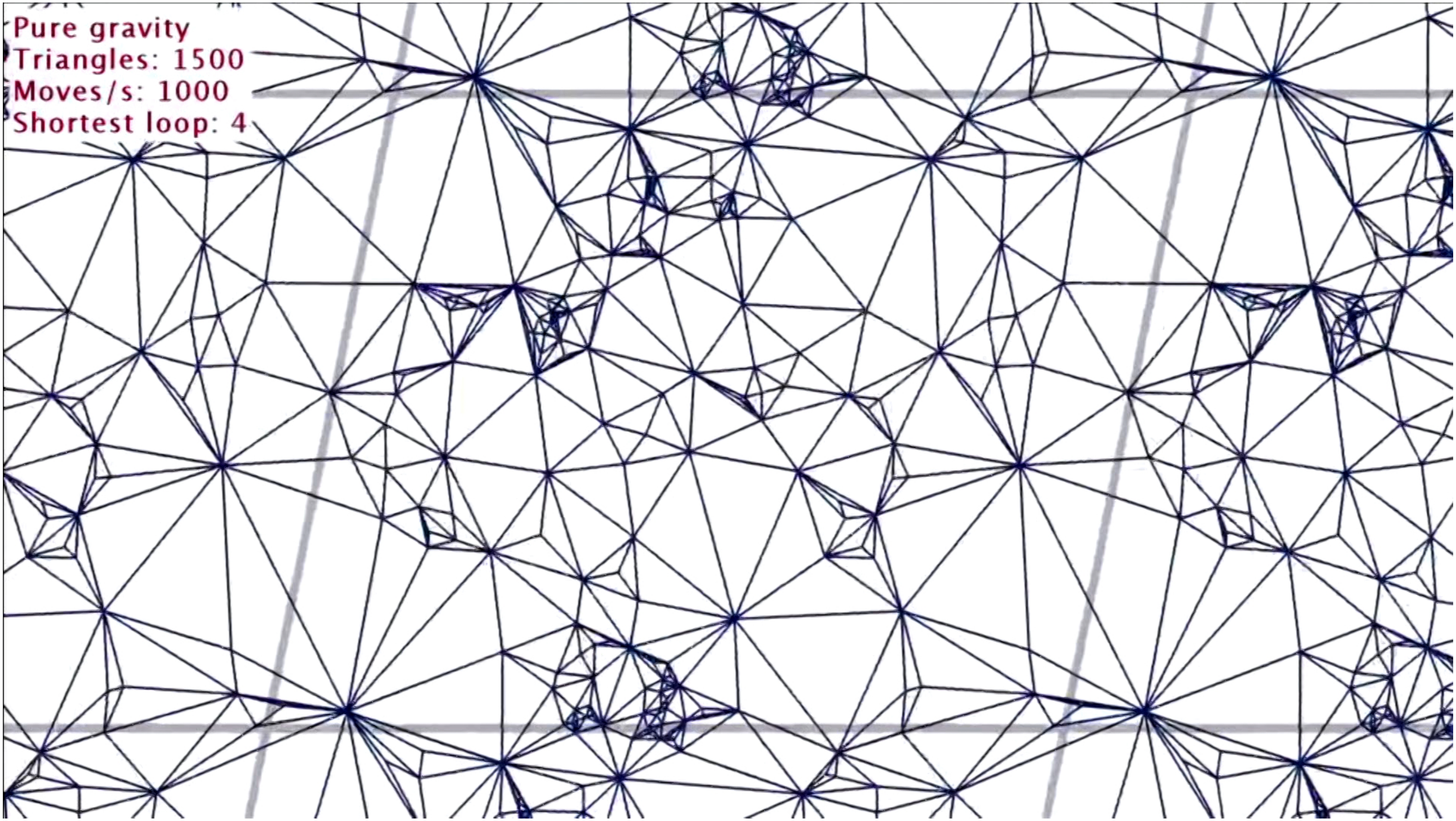

Pure gravity
Triangles: 1500
Moves/s: 1000
Shortest loop: 4


Pure gravity
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Moves/s: 1000
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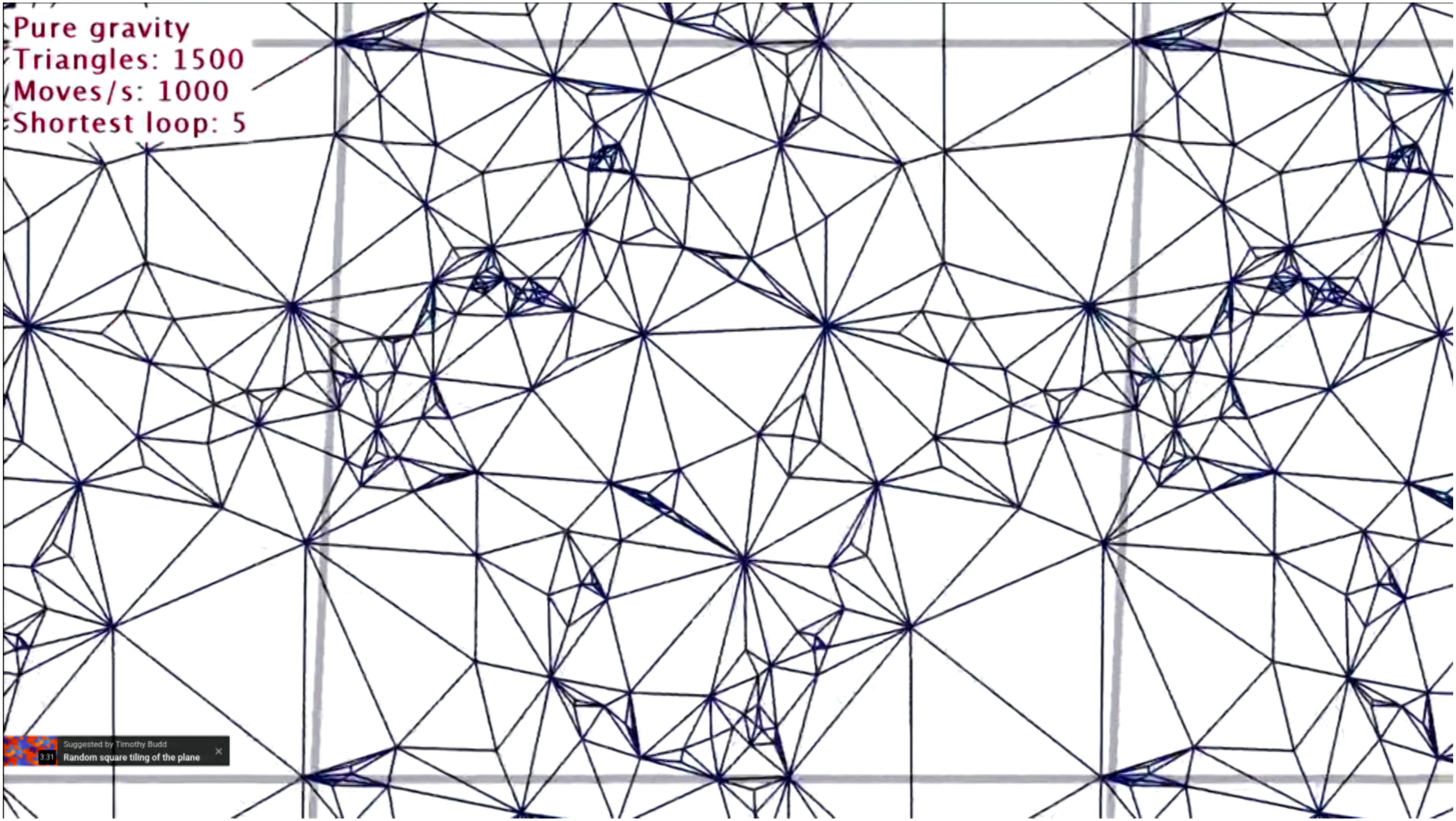
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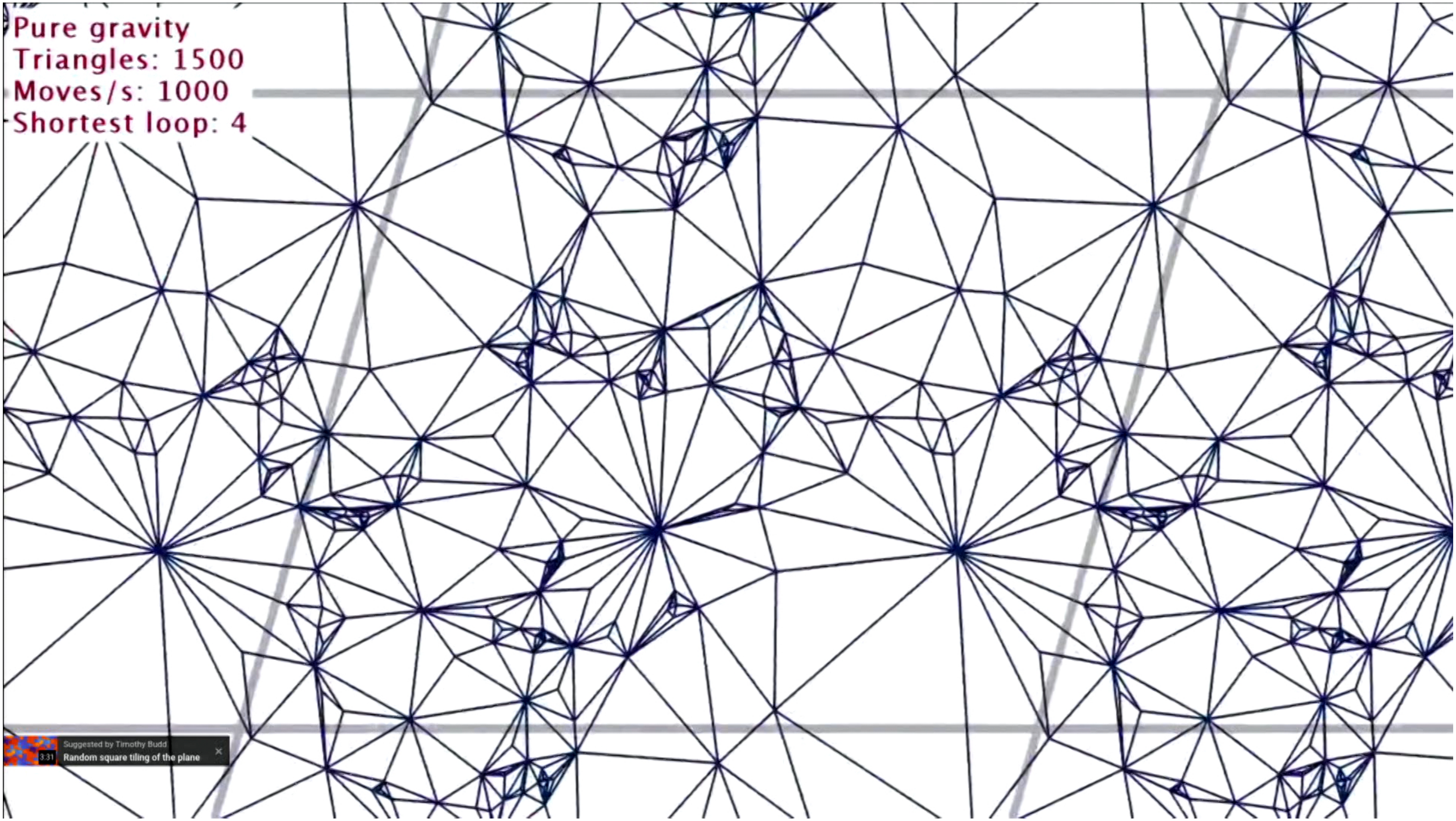
Moves/s: 1000
Shortest loop: 5
Suggested by Timothy Budd
3:31
Random square tiling of the plane


Pure gravity
Triangles: 1500
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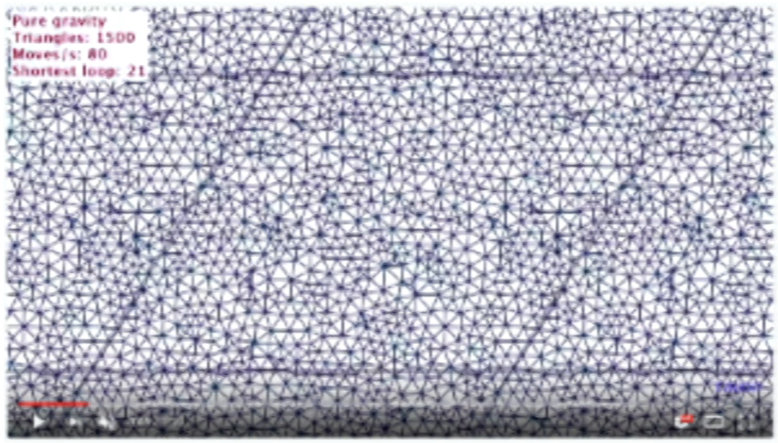
Video of the Markov process on a torus



https://www.youtube.com/watch?v=c3NdgSIe030

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- Interested in critical phenomena of the system: size $N \to \infty$, and large-range correlation.
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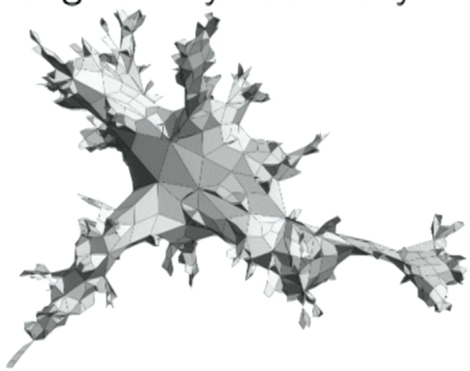
$N = 2000$: 

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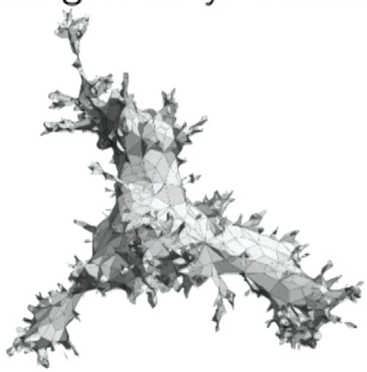
$N = 5000$: 

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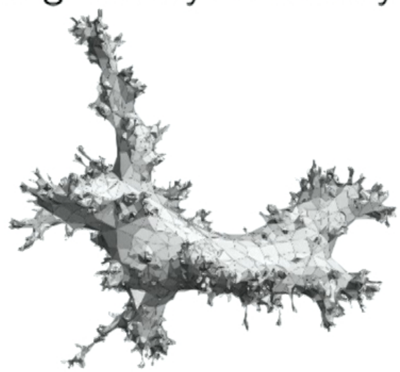
$N = 10000$:



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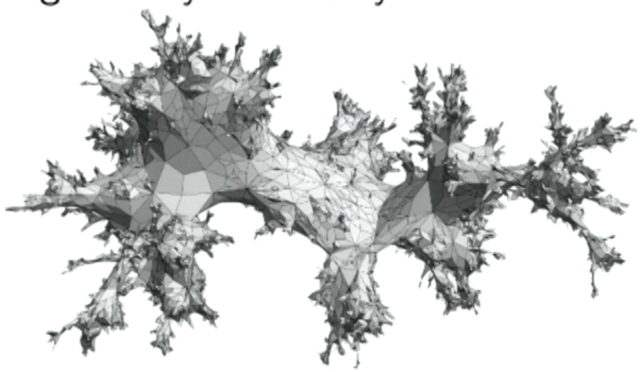
$N = 20000$: 

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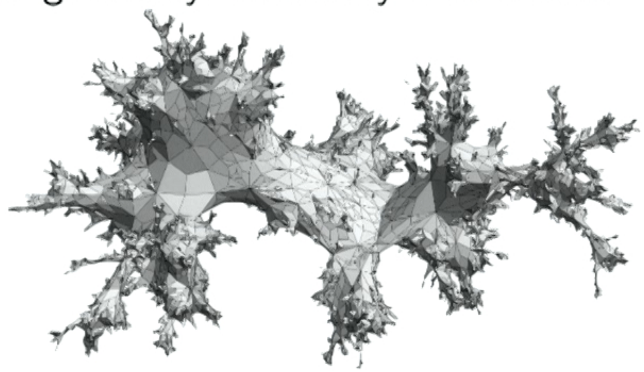
$N = 20000$: 

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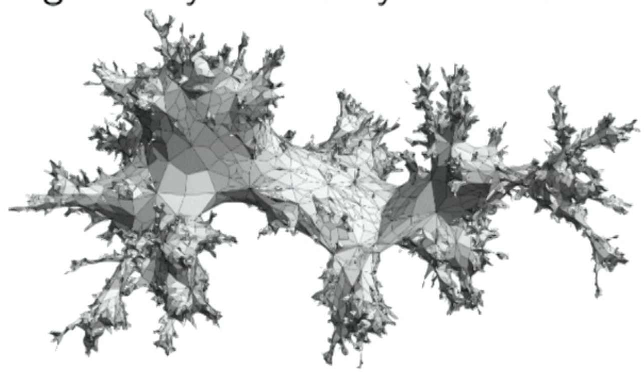
$N = 20000$: 

- The large-scale properties are universal as $N \to \infty$: independent of precise ensemble of maps used.
- To determine critical exponents of this universality class: need a family $\mathcal{O}_n$ of observables, with n related to the scale at which the system is probed. Measure $\langle \mathcal{O}_n \rangle_N$ and analyze $N, n \to \infty$ limit.

Matter coupling

- Could introduce a non-trivial action $S[\mathfrak{m}]$ (or energy $\beta E[\mathfrak{m}]$)

$$Z_N = \sum_{\mathfrak{m} \in \mathcal{T}_N} 1 \quad \mapsto \quad Z_N = \sum_{\mathfrak{m} \in \mathcal{T}_N} e^{-S[\mathfrak{m}]}.$$

  Introduce acceptance probability in MCMC to ensure detailed balance.

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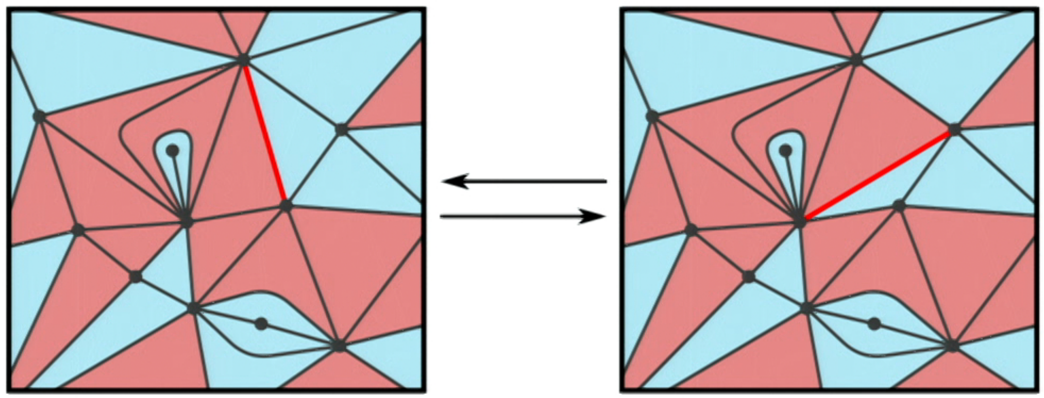
- If the matter is tuned to criticality, the critical exponents of the geometry are affected!
- Widely believed there is a 1-parameter family of universality classes of 2d random geometry, parametrized by total central charge c of coupled matter fields. (no matter: $c = 0$; Ising: $c = 1/2$)

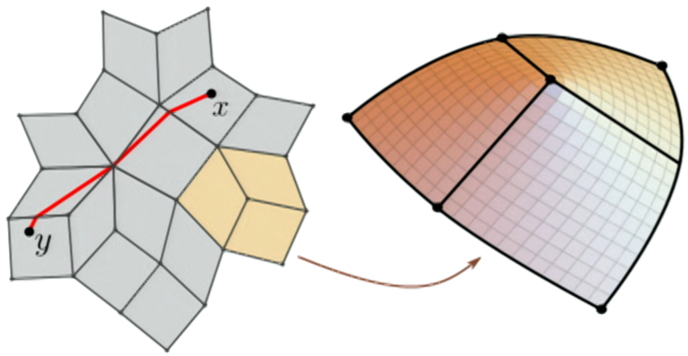
Example: random triangulations + Ising model



$$Z_{N,\mathrm{Ising}} = \sum_{\mathfrak{m}\in\mathcal{T}_N} \sum_{\{s_i\}} e^{\beta J \sum_{\langle ij \rangle} s_i s_j}$$

- For MCMC simulation: supplement triangle flip moves with standard Ising update moves (single-spin-flips, Wolff, . . . , cf. Barkema's lectures)

Observables: geodesic distances



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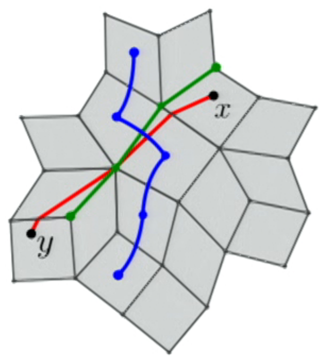

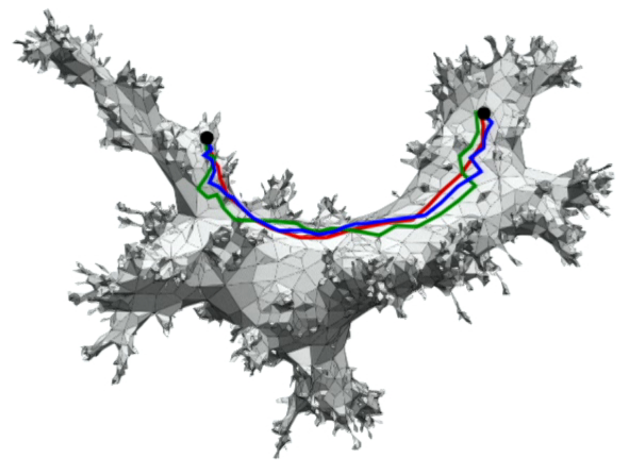

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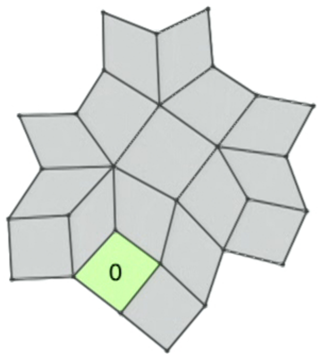

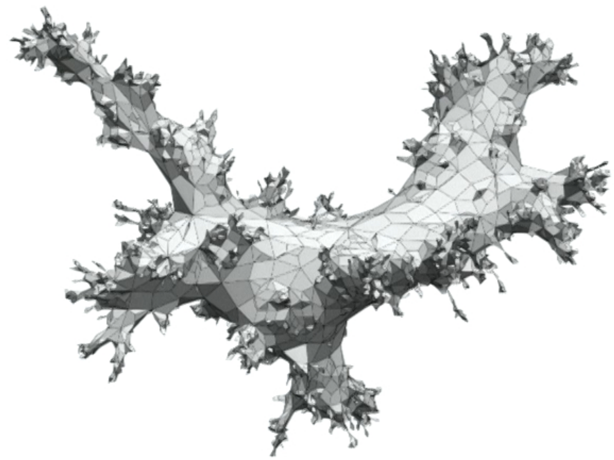

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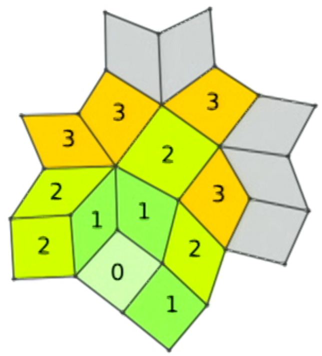

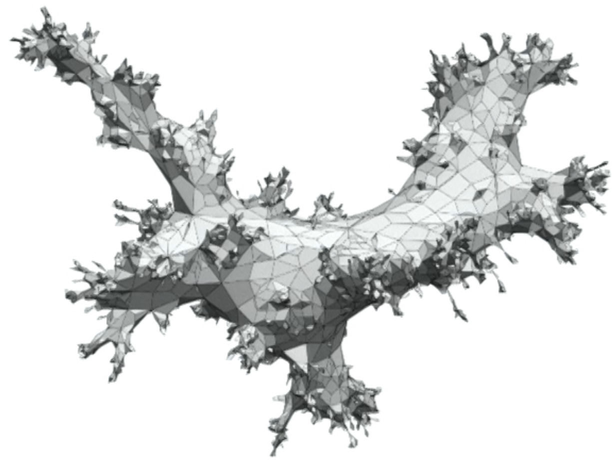

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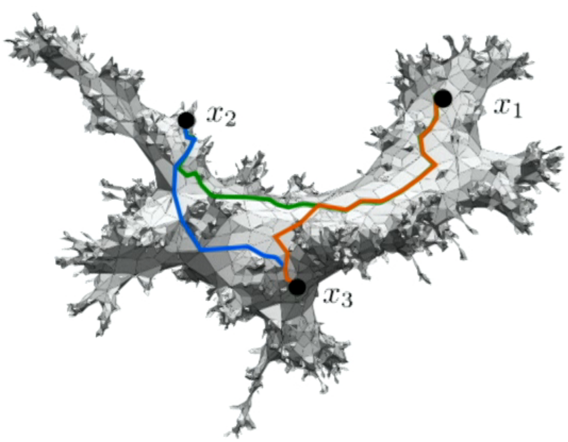
Observables: geodesic distances (continued)

- How to turn distance into (label-invariant) observable $\mathcal{O} : \mathcal{T}_N \to \mathbb{R}$?
- Geodesic n-point functions: given $f : \mathbb{R}^{n(n-1)/2} \to \mathbb{R}$,

$$\mathcal{O}_f(\mathfrak{m}) := \frac{1}{|\mathfrak{m}|^n} \sum_{x_1,\ldots,x_n} f(d(x_1,x_2), d(x_1,x_3), d(x_2,x_3), \ldots)$$

is a well-defined observable.



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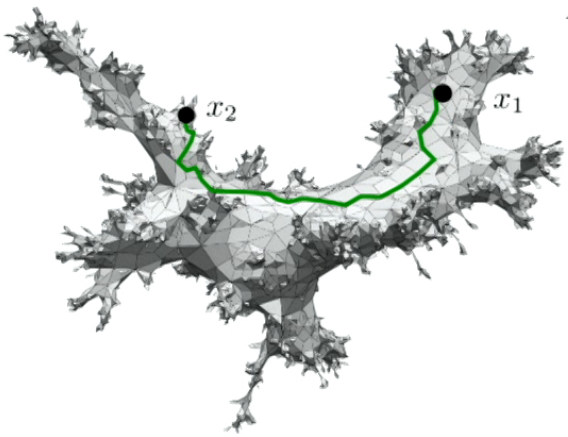

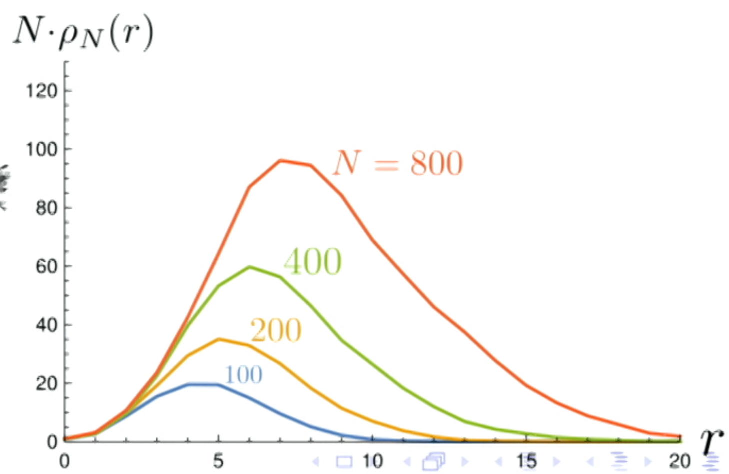

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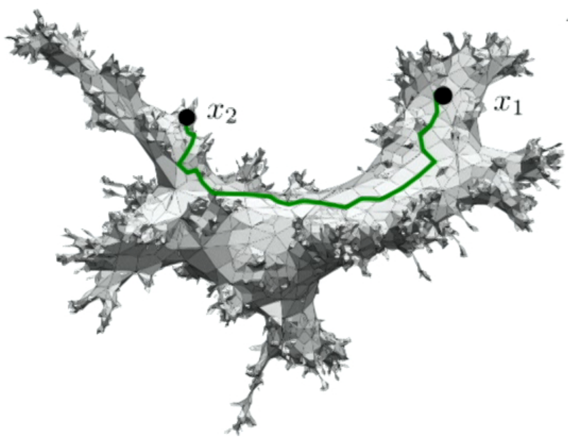
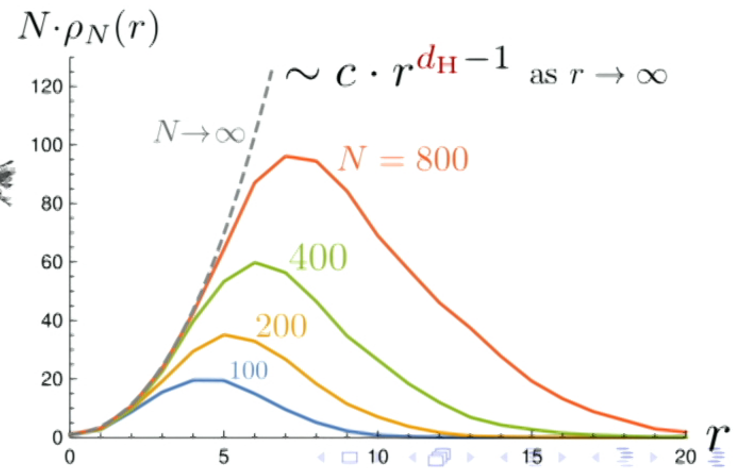
- d_{H} is Hausdorff dimension (which equals d in flat $\mathbb{R}^d$).



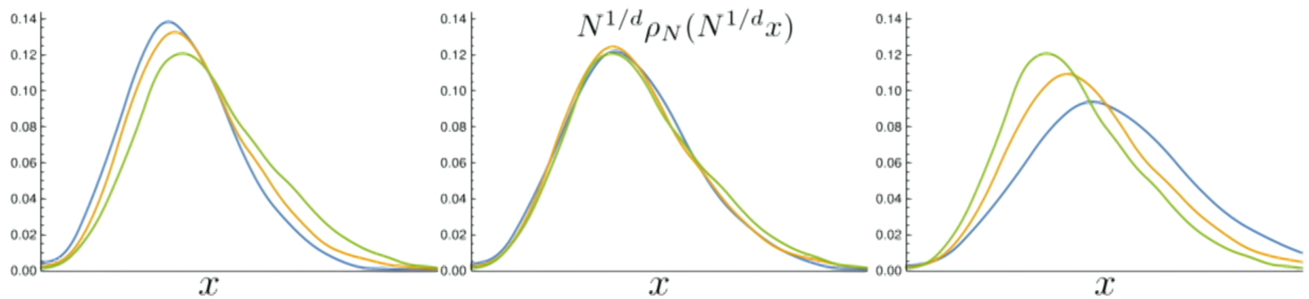


Observables: geodesic distances (continued)

- Instead of taking $N \to \infty$ and then $r \to \infty$, it is usually better to use *finite-size scaling methods*: one expects

$$N^{1/d_{\mathrm{H}}} \rho_N(N^{1/d_{\mathrm{H}}} x) \text{ to converge as } N \to \infty \text{ for any fixed } x \in \mathbb{R}.$$

- Equivalently, we expect the distribution of the distance between two random points to converge as $N \to \infty$ provided we take edge lengths $\sim N^{-1/d_{\mathrm{H}}}$.
- Estimate d_H by "collapsing curves":



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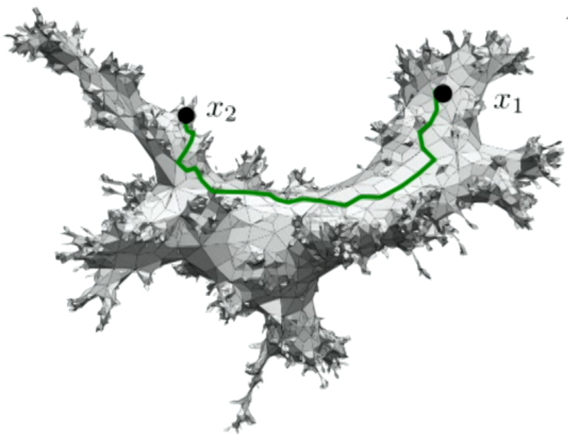

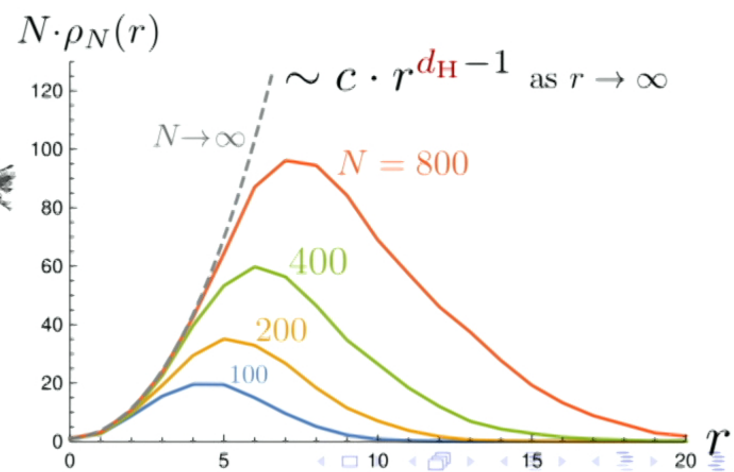

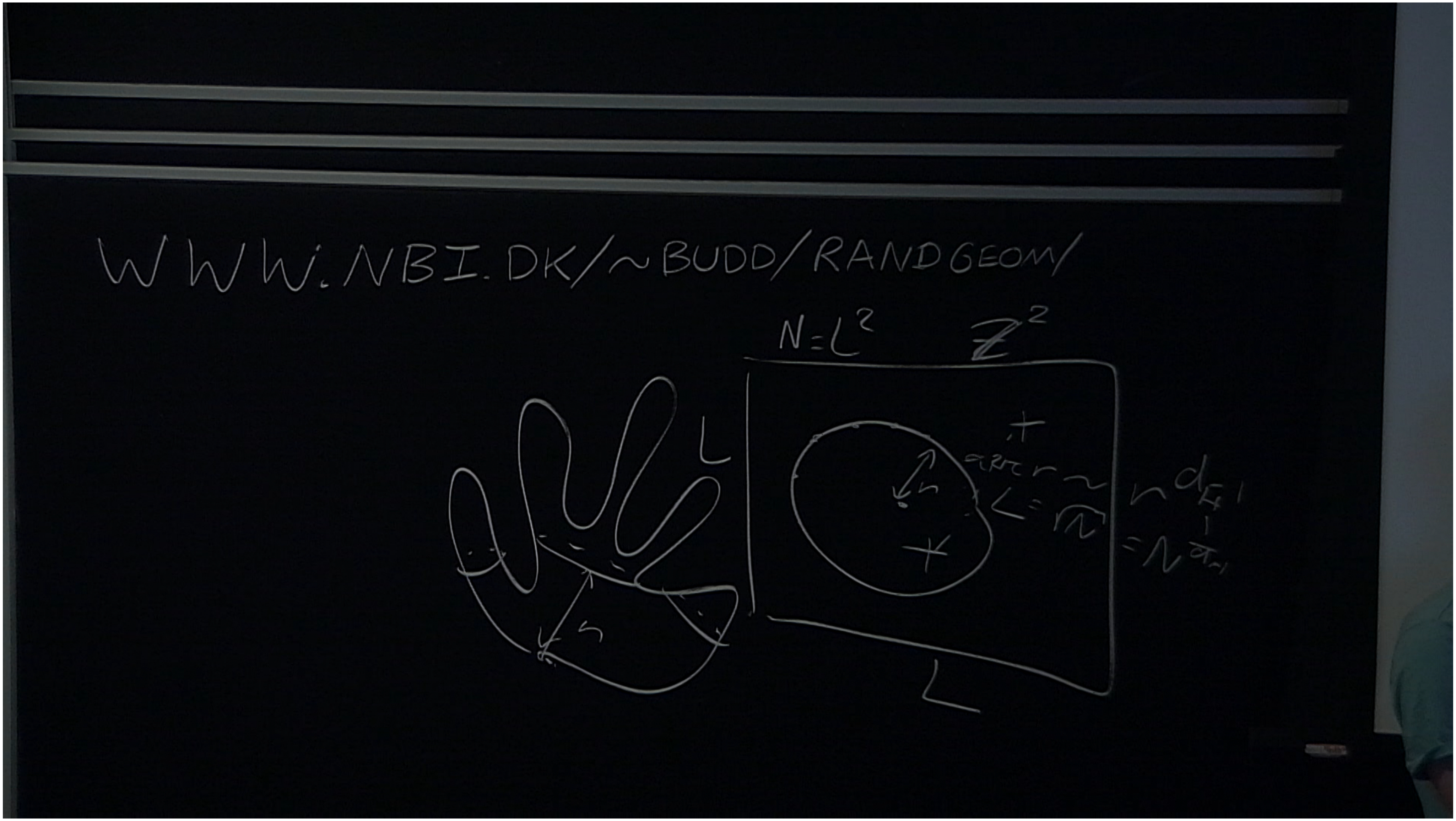
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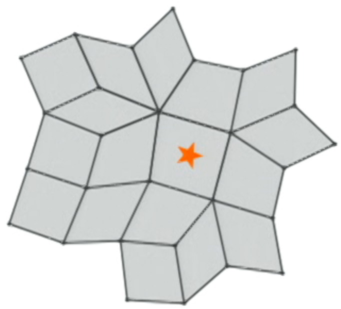
Observables: spectral dimension

- Consider a simple random walk on the faces of $\mathfrak{m}$, started at face x.



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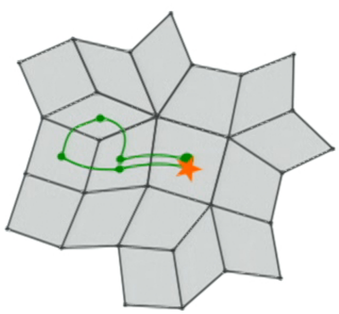
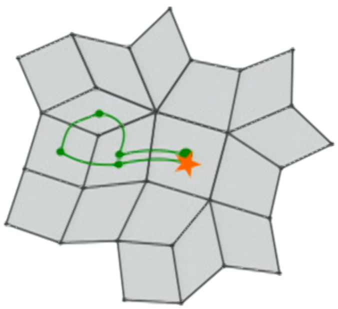
- *Return probability* $p_x(t;\mathfrak{m})$ is probability that it is back at x after t steps.

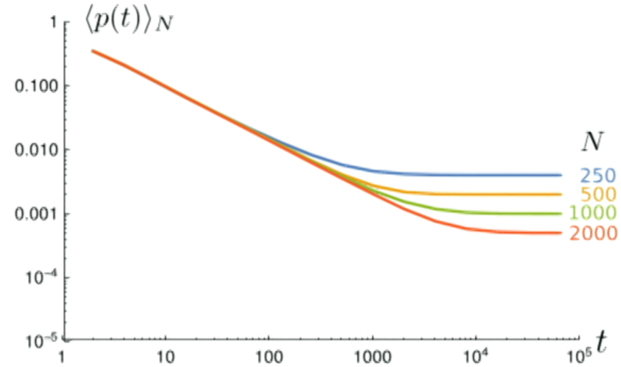


Observables: spectral dimension

- Consider a simple random walk on the faces of $\mathfrak{m}$, started at face x.
- *Return probability* $p_x(t;\mathfrak{m})$ is probability that it is back at x after t steps.
- Averaging over all starting points we get a family of observables

$$p(t;\mathfrak{m}) := \frac{1}{|\mathfrak{m}|} \sum_x p_x(t;\mathfrak{m}).$$





Observables: spectral dimension (continued)

- How to measure $p_x(t;\mathfrak{m})$, $p(t;\mathfrak{m})$, $\langle p(t;\mathfrak{m})\rangle$?
  - Perform random walk with random starting point x (no need to average over all starting points!).
  - Linear algebra.
- Define $N \times N$ (normalized) *adjacency matrix* A by

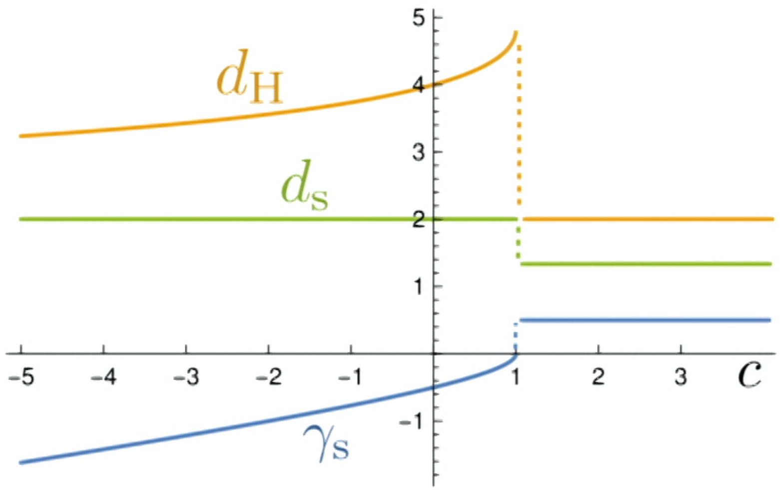
$$A_{ij} := \mathbb{P}(j \to i) = \frac{\#\text{edges shared by face } i \text{ and } j}{\text{degree of face } j}$$

Then $\quad p_x(t;\mathfrak{m}) = (A^t)_{xx} \quad$ and $\quad p(t;\mathfrak{m}) = \frac{1}{N}\mathrm{Tr}(A^t) = \frac{1}{N}\sum_{i=1}^{N}\lambda_i^t,$

where $(\lambda_1, \ldots, \lambda_N)$ is the spectrum of A (hence *spectral* dimension).

Conjectural relation d_{H}, d_{s}, γ_{s} to central charge c

$c \in (-\infty, 1]$	$c > 1$
$\gamma_{\mathrm{s}} = \dfrac{c - 1 - \sqrt{(c-1)(c-25)}}{12}$	$\gamma_{\mathrm{s}} = \dfrac{1}{2}$
$d_{\mathrm{H}} = 2\dfrac{\sqrt{25-c}+\sqrt{49-c}}{\sqrt{25-c}+\sqrt{1-c}}$	$d_{\mathrm{H}} = 2$
$d_{\mathrm{s}} = 2$	$d_{\mathrm{s}} = \dfrac{4}{3}$



Conjectural relation d_{H}, d_{s}, γ_{s} to central charge c

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$$d_{\mathrm{H}} = 2$$

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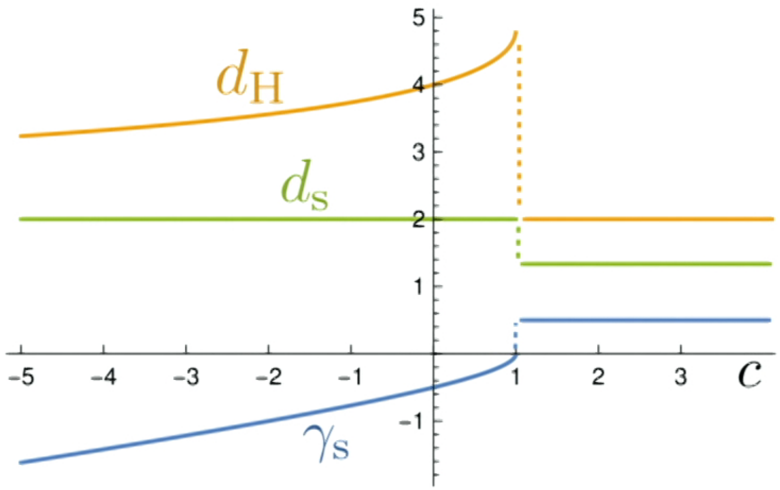

Plan for the tutorial sessions

- Team up (in groups 1, 2, 3).
- Get the program `randgeom` to work on your computer:

  `http://www.nbi.dk/~budd/randgeom/`
- Use the provided Mathematica notebook to get a feel for the random geometries produced by the various models (A, B, C, D).
- Gather data for (at least) one of the observables, and determine estimates for the corresponding critical exponent (d_{H}, d_{s} or γ_{s}) for each of the models.
- Be creative: try to think of a different observable, perform data analysis.
- Towards the end of the 2nd session: send me a brief summary of your results (nice plots?). At the end I will discuss the models and the data, and compare the latter to what we know.