Title: Monte Carlo methods in Dynamical Triangulations - 1

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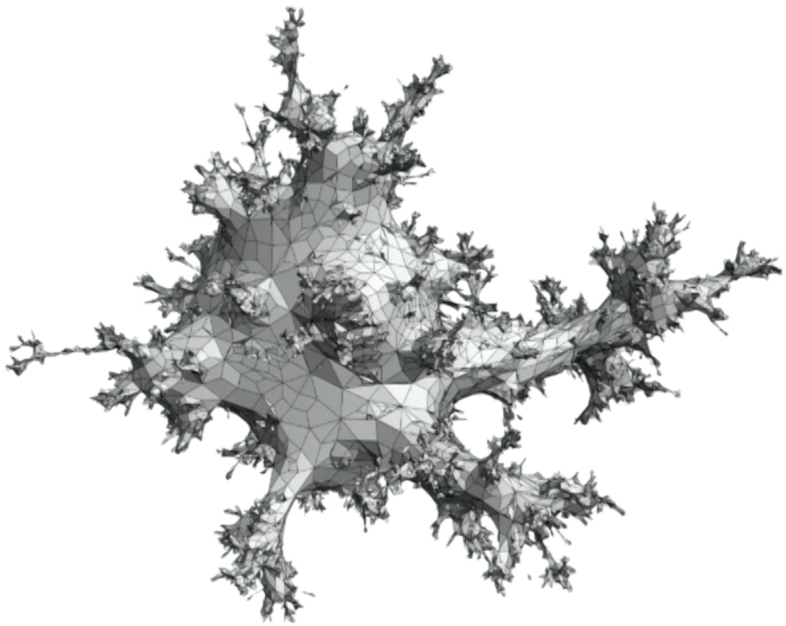
Abstract:

Making Quantum Gravity Computable, 21-06-2017

Monte Carlo methods in Dynamical Triangulations

Part I: 2D random geometry

Timothy Budd



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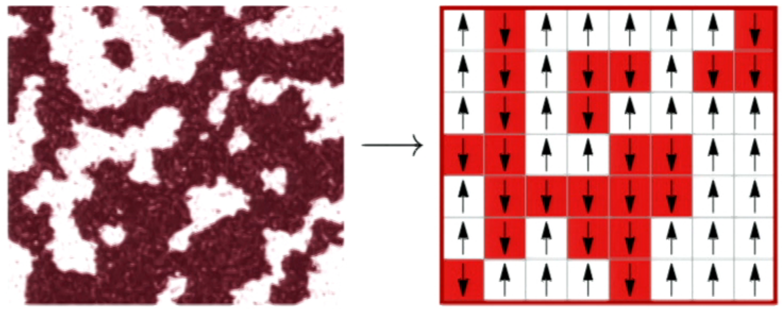
Outline

- Day 1: 2D random geometry
  - Combinatorial representation
  - Markov Chain Monte Carlo (MCMC) methods
  - Matter coupling
  - Observables
- Day 2: Dynamical Triangulations in higher dimensions
  - Quantum gravity
  - Combinatorial representation
  - MCMC methods
  - Phase diagram
  - Causal Dynamical Triangulations
- Tutorials: numerical analysis of various 2D random geometries
  - Measure observables for random geometries (produced by black box)
  - Extract critical exponents.
  - Experiment with (new?) observables.
  - Conclusions will be collected at the end and be discussed.

Discrete geometry?

Discretization in . . .

- . . . the Ising model: (Barkema's course)



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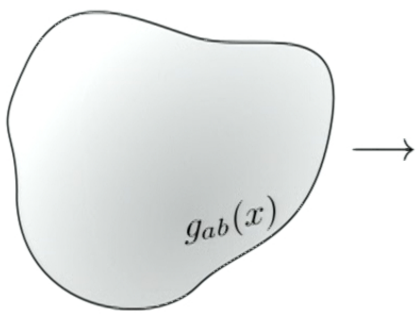

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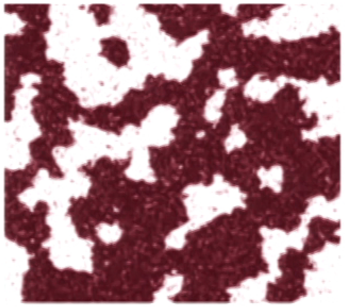

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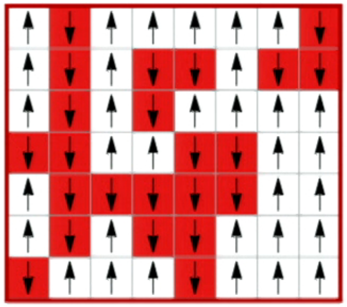

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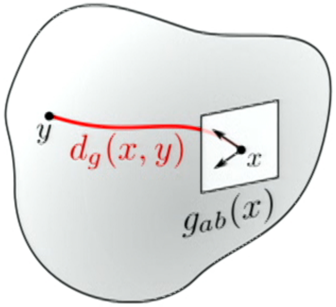

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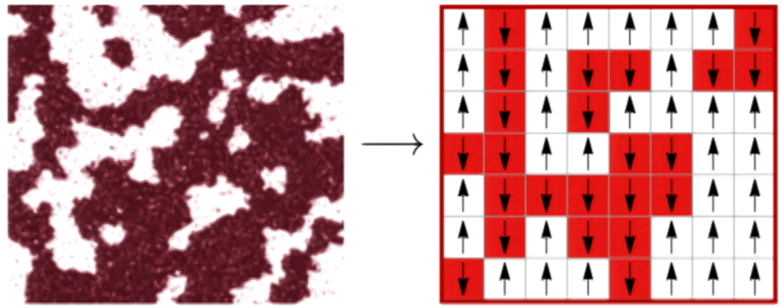

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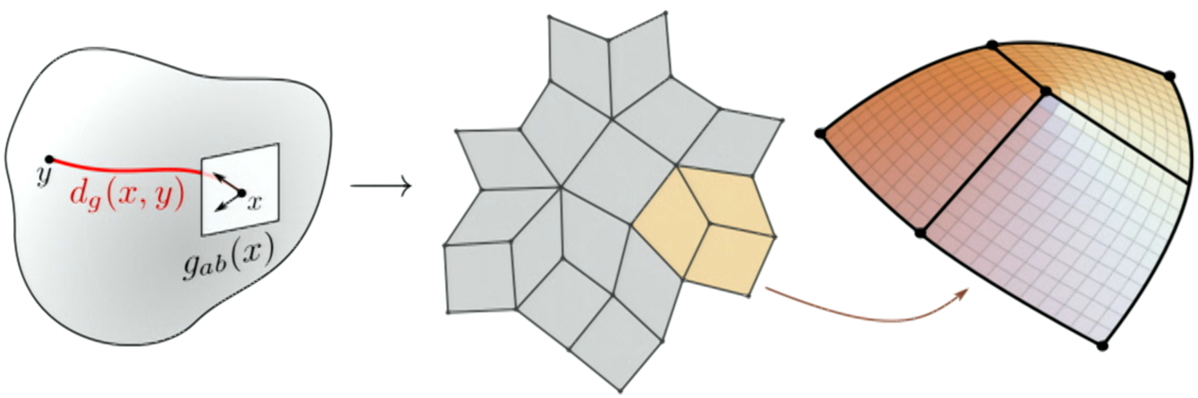

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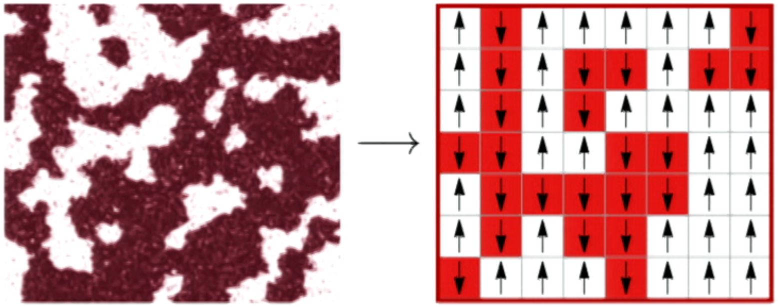

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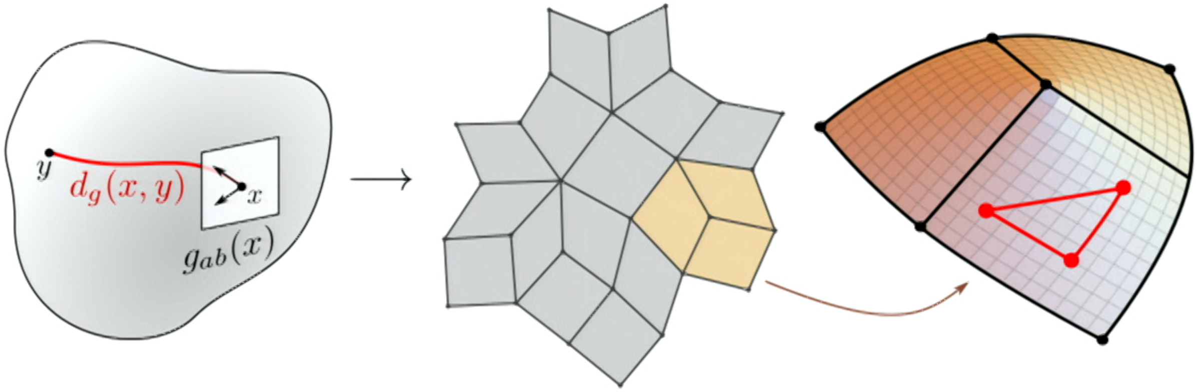

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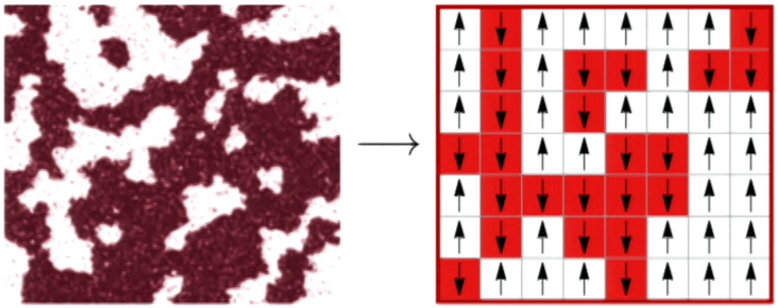

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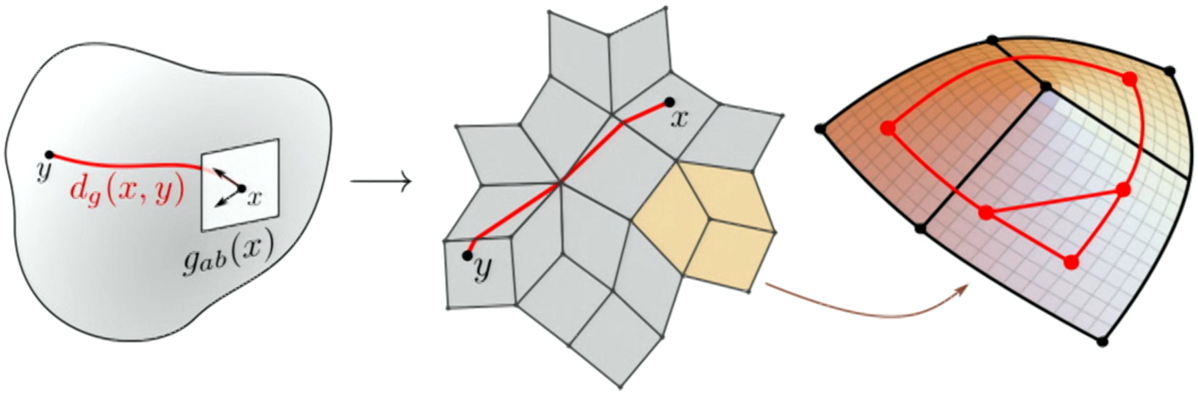
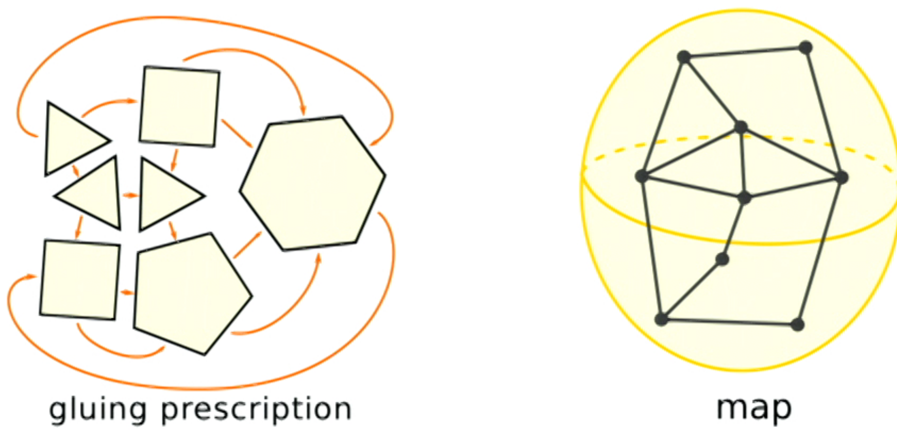


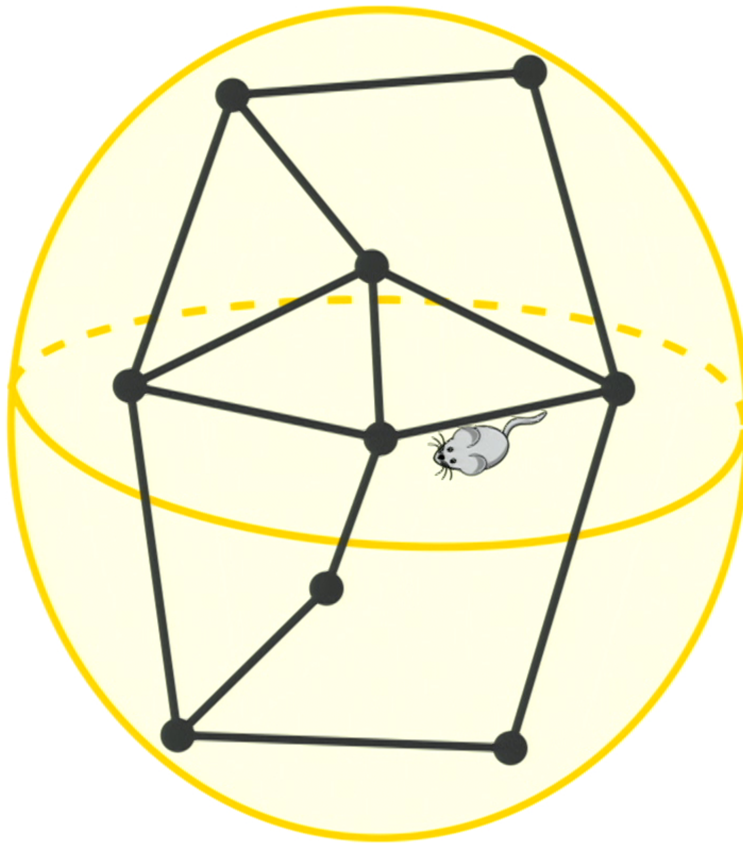
Geometry from polygons

- To change a discrete geometry, one may change ...
  - ...shape of the polygons, ("Regge calculus")
  - ...or the connectivity. ("Dynamical triangulation")
- Fix once and for all the geometry of each polygon of degree k to be that of the regular k-gon in Euclidean space with sides of length 1.
- Then can represent geometry equivalently by
  - a "gluing prescription" on a collection of polygons.
  - a "map": a proper embedding of a graph in a surface;



Combinatorial description of a map

- Navigate map using "next" and "adjacent".



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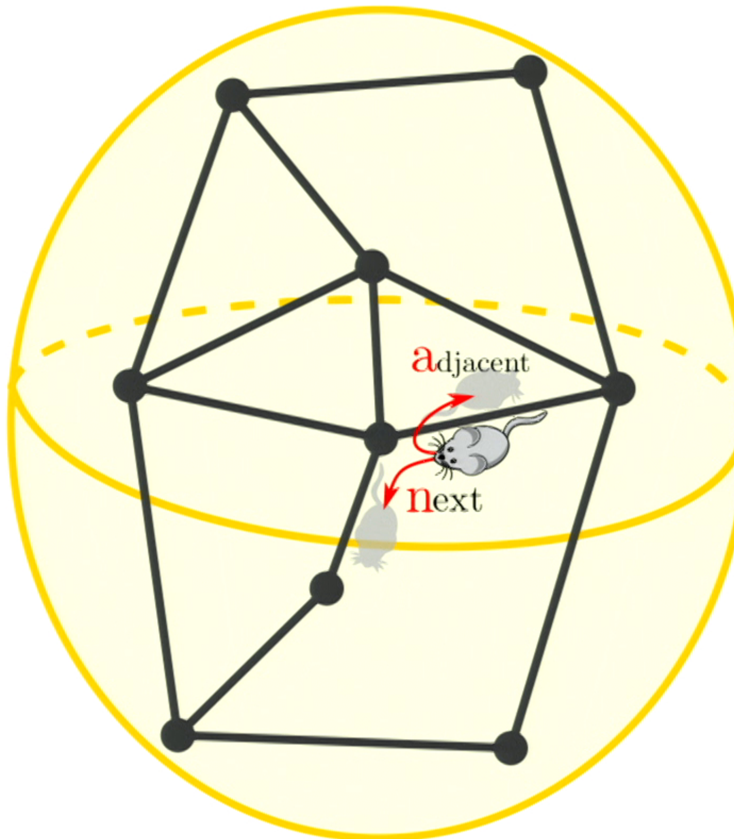

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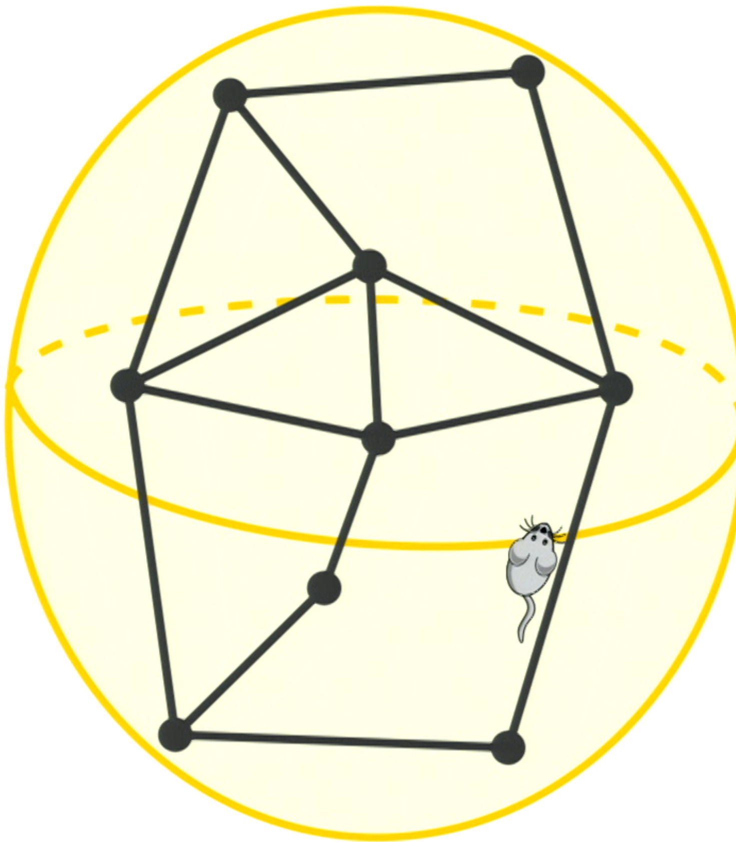

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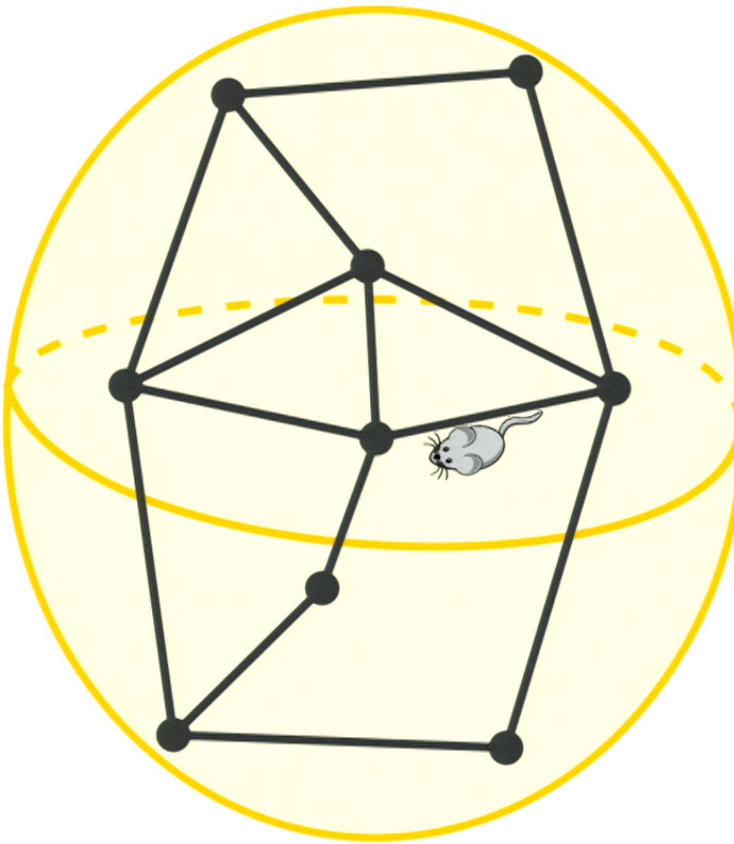

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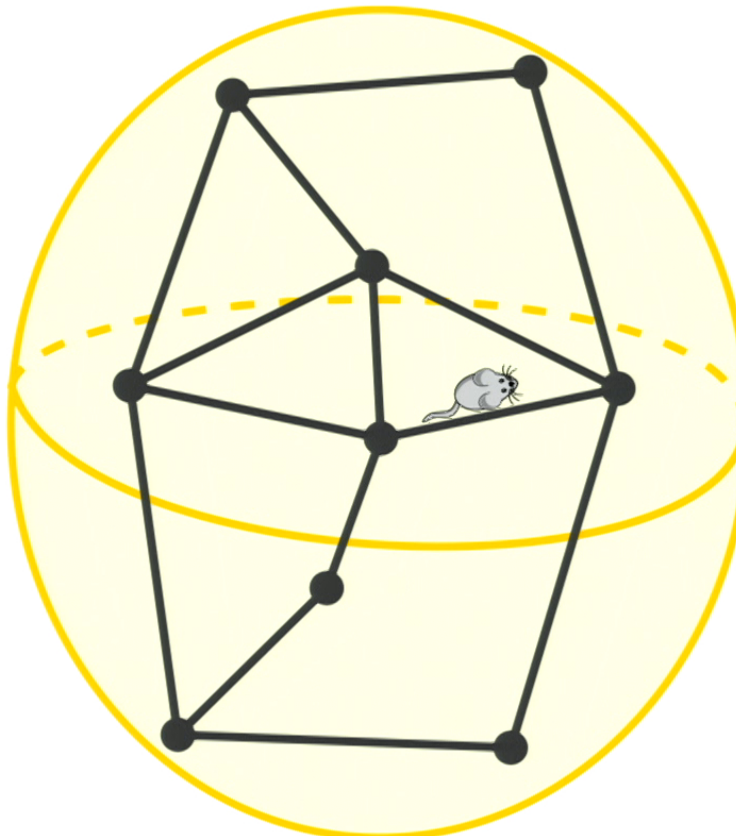

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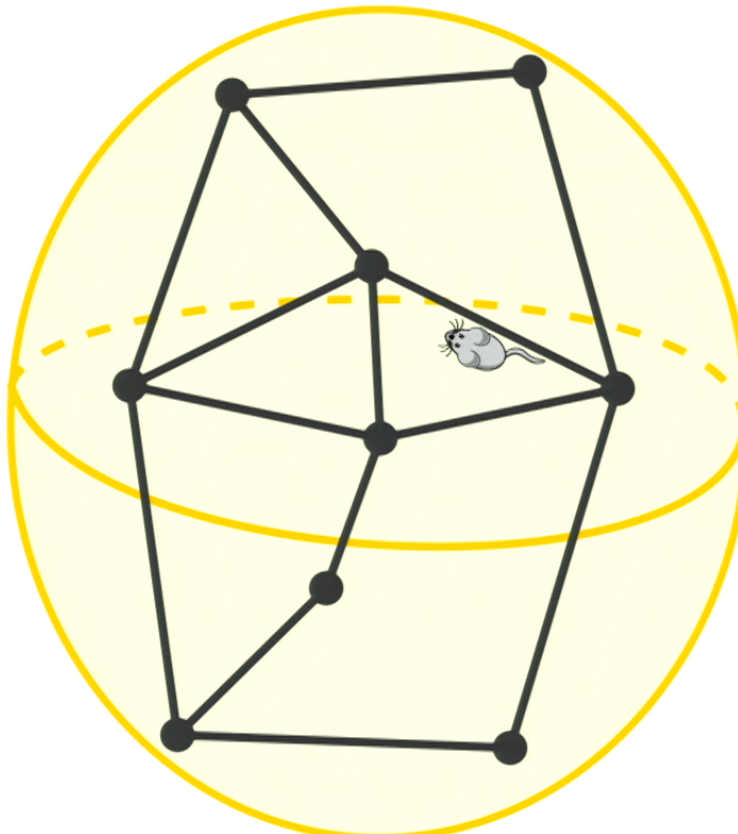

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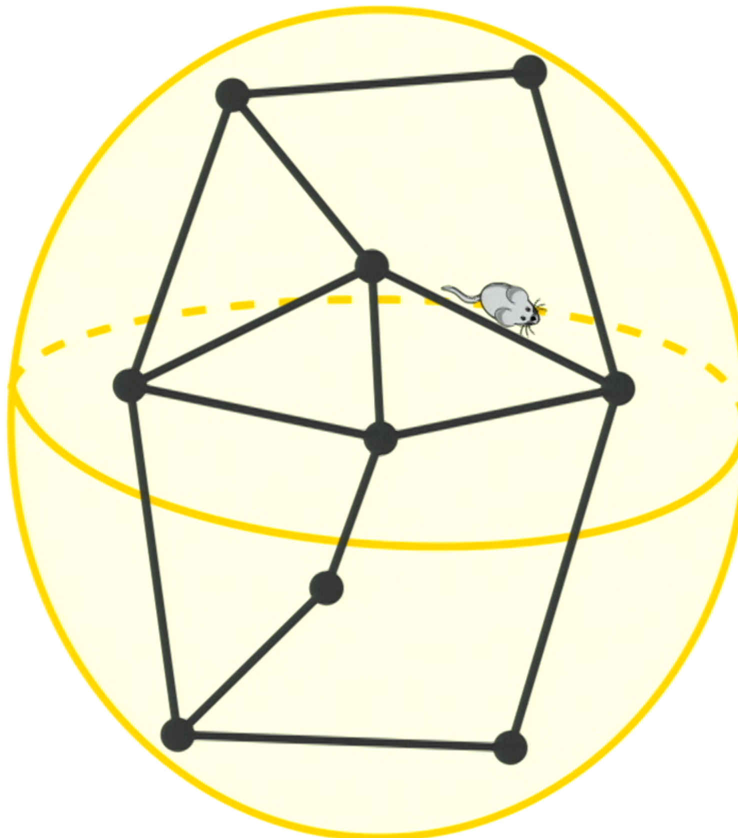

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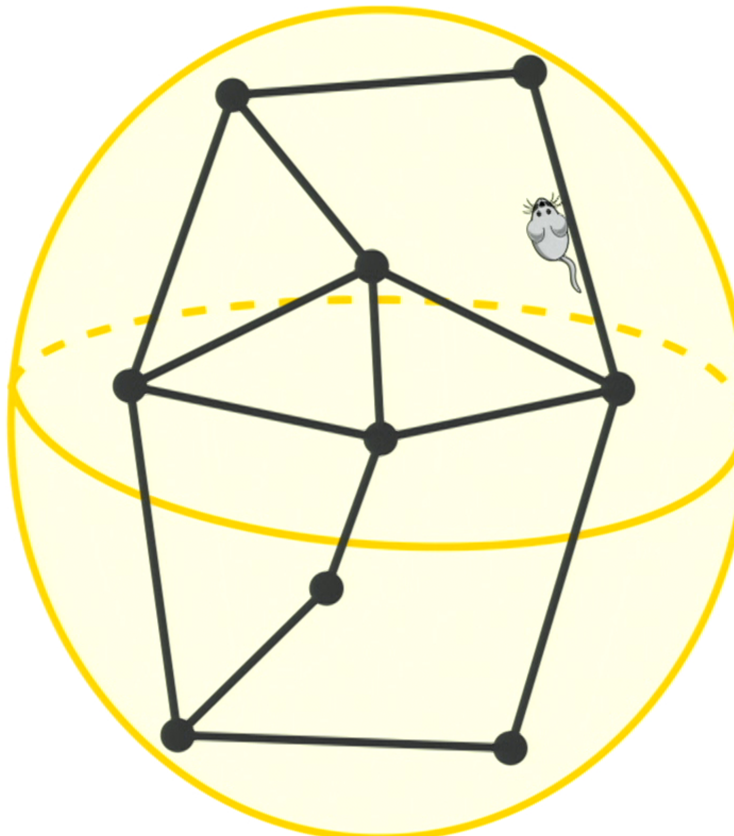

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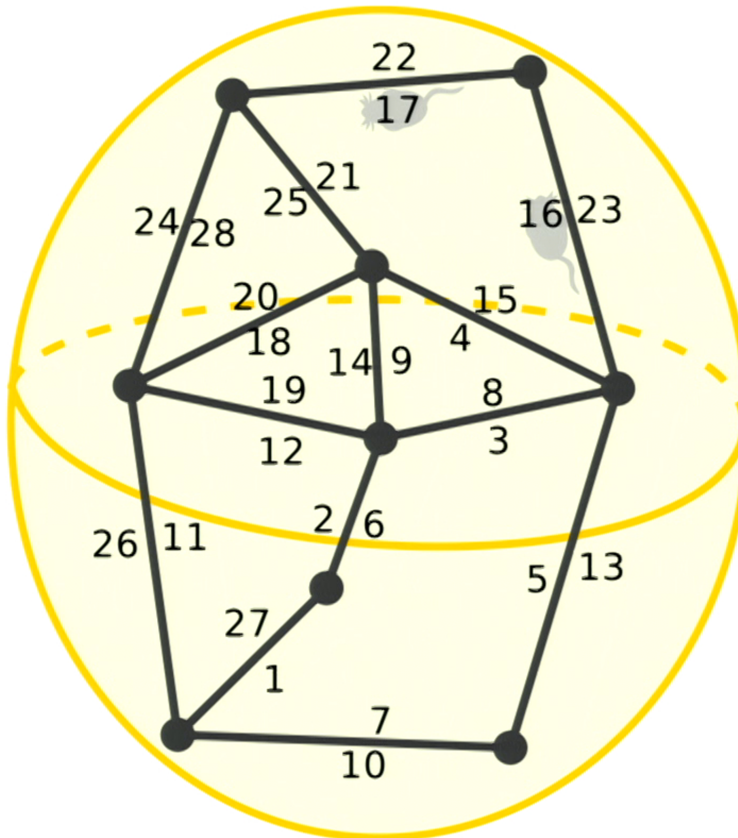

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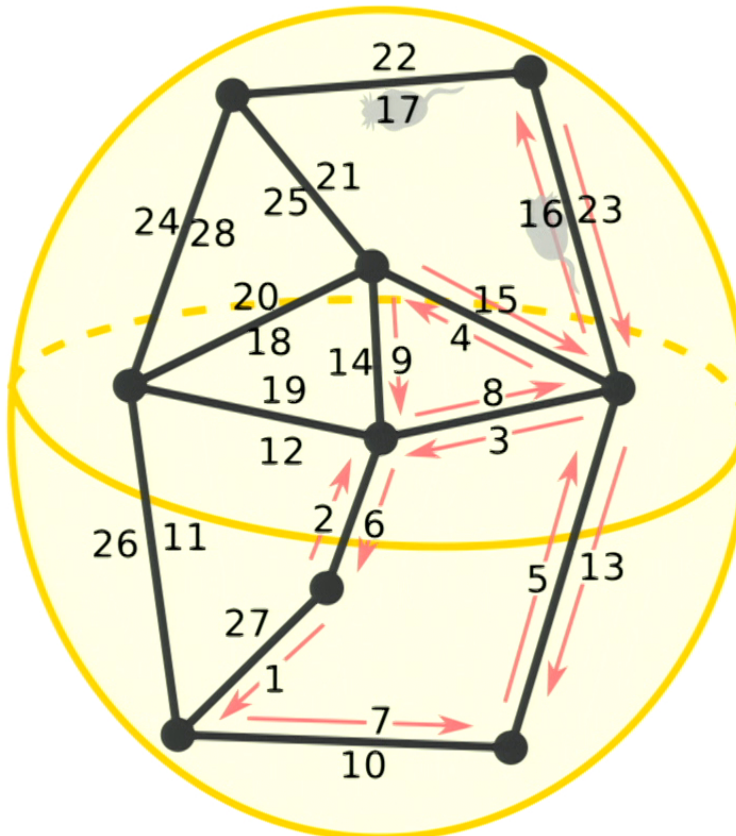
- These define permutations on the half-edge labels, $1 \cdots 28$:

$$n = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & 28 \\ 7 & 12 & 6 & 9 & 3 & \cdots & 20 \end{pmatrix}$$

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & 28 \\ 27 & 6 & 8 & 15 & 13 & \cdots & 24 \end{pmatrix}$$



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- Cycles of n represent faces:

$$n = (1\,7\,5\,3\,6)(4\,9\,8)\cdots$$

- Cycles of a represent edges:

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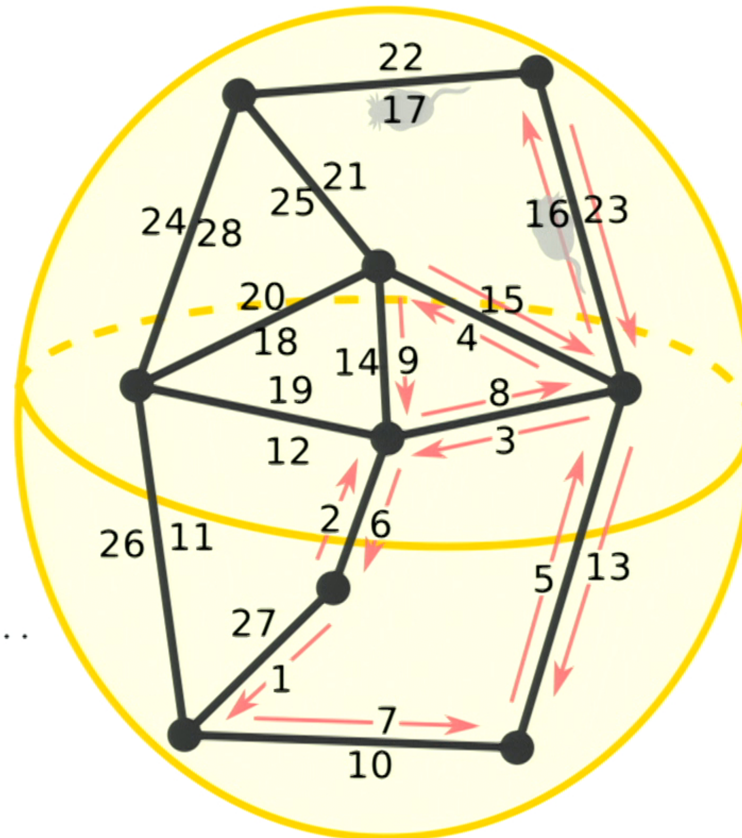

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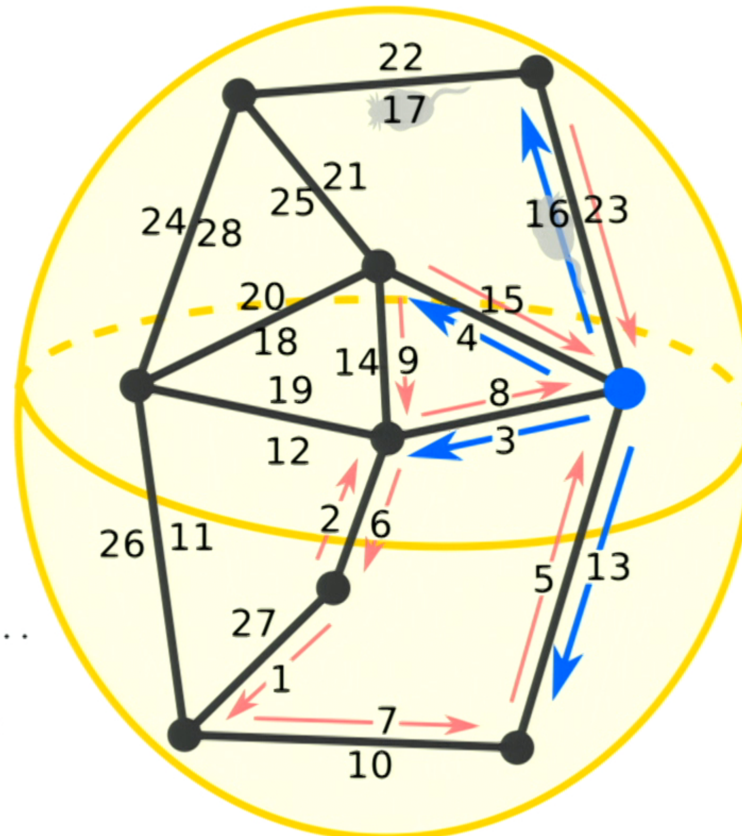
- Cycles of $n \circ a$ represent vertices:

$$n \circ a = (1\,2)(3\,4\,16\,13)(5\,10)\cdots$$



- Any pair (n, a) of permutations on $[2N] := \{1, 2, \ldots, 2N\}$ with $a \circ a = \mathbf{1}$ and $a(x) \neq x$ determines a (labeled) map with N edges, hence a piecewise flat geometry on an (oriented!) surface.
- Connected iff n, a act transitively.
- Topology? Euler's formula for the genus $g = g(n, a)$:

$$2 - 2g = V - E + F = \#\mathrm{Cyc}(n \circ a) - \#\mathrm{Cyc}(a) + \#\mathrm{Cyc}(n)$$

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- The set $\mathcal{T}_N$ of *labeled triangulations* of S^2 with N edges can be described combinatorially by

$$\mathcal{T}_N \equiv \{(n, a) : \text{transitive}, g(n, a) = 0, \text{all cycles of } n \text{ of length } 3\}.$$

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etc.
- In particular, $|\mathcal{T}_N|, |\mathcal{Q}_N| < ((2N)!)^2 < \infty$.

Random discrete geometries

- The *uniform random* labeled triangulation of S^2 with N edges is an element of $\mathcal{T}_N$ chosen with probability $1/|\mathcal{T}_N|$ each.

$N = 60$

- In statistical physics terminology: this is a canonical ensemble with partition function

$$Z_N = \sum_{\mathfrak{m} \in \mathcal{T}_N} 1 = |\mathcal{T}_N|$$

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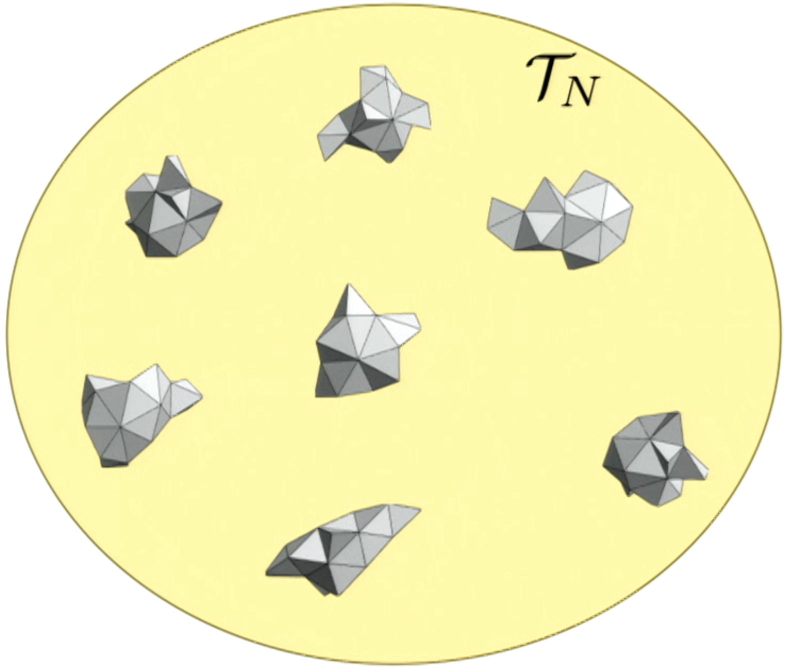
- How to sample from this ensemble? And compute $\langle \mathcal{O} \rangle_N$?
  - The analytic way: combinatorial algorithms; direct random generation.
  - Markov Chain Monte Carlo methods.

A Markov chain on triangulations



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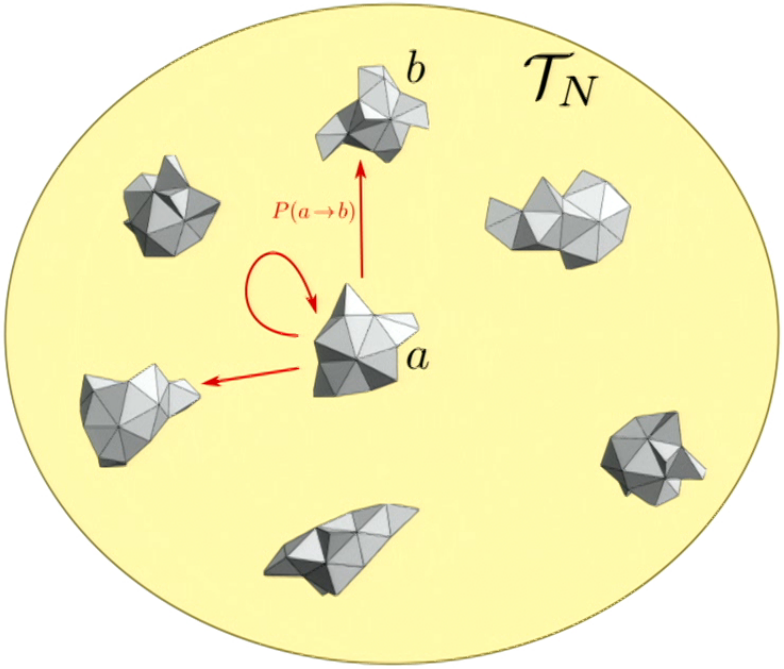

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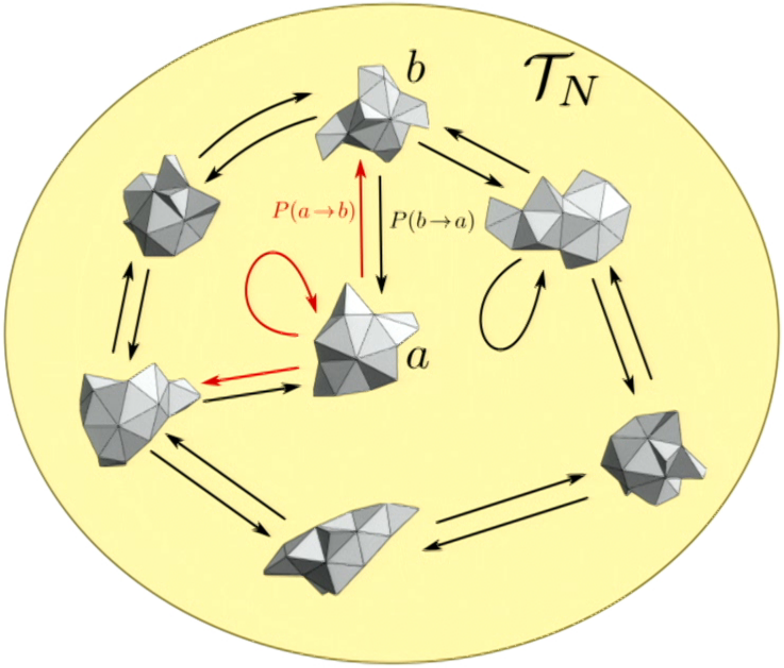

- To obtain a Markov process converging to the uniform distribution on $\mathcal{T}_N$ (from any starting point) it suffices to select an update algorithm that...
  - preserves topology and size of the map;
  - satisfies *Detailed balance*: $P(a \to b) = P(b \to a)$;

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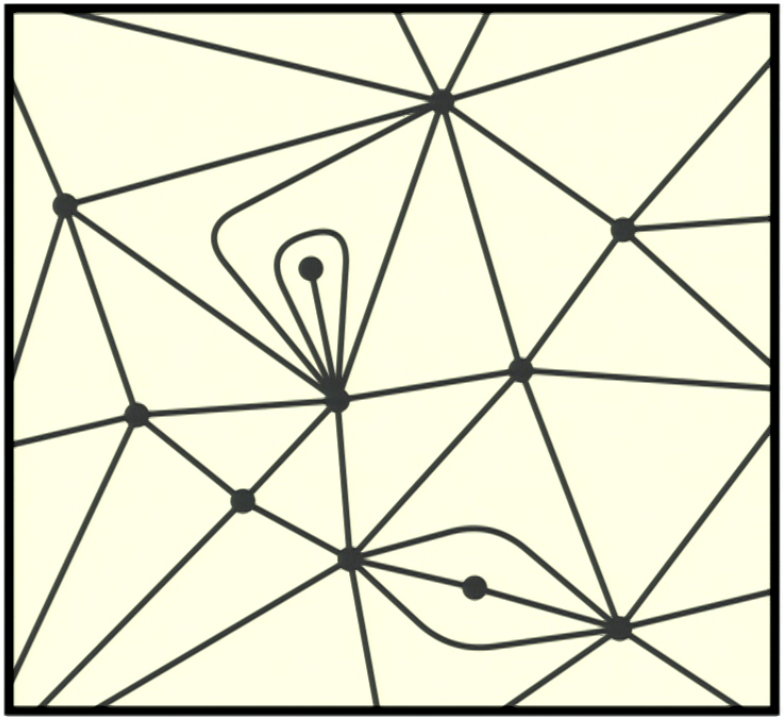

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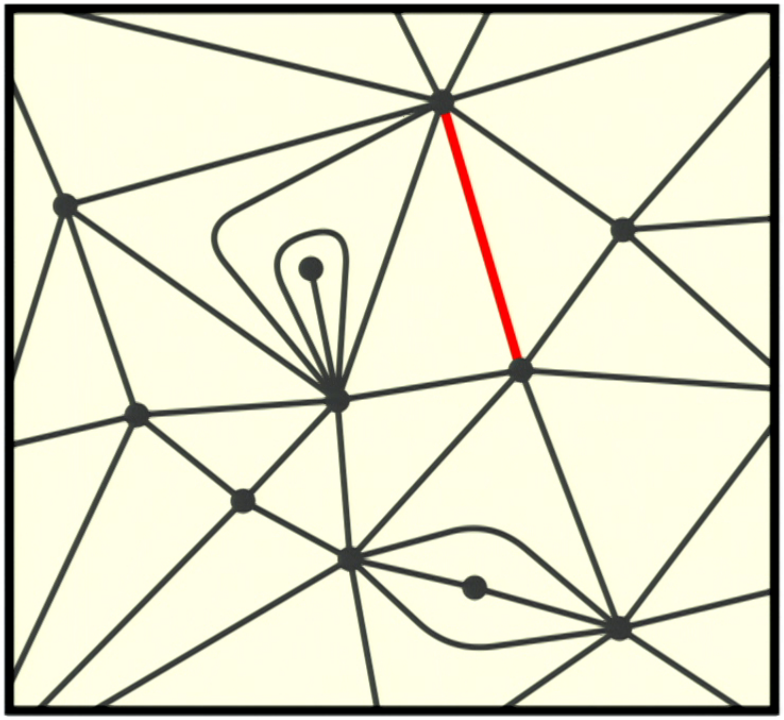

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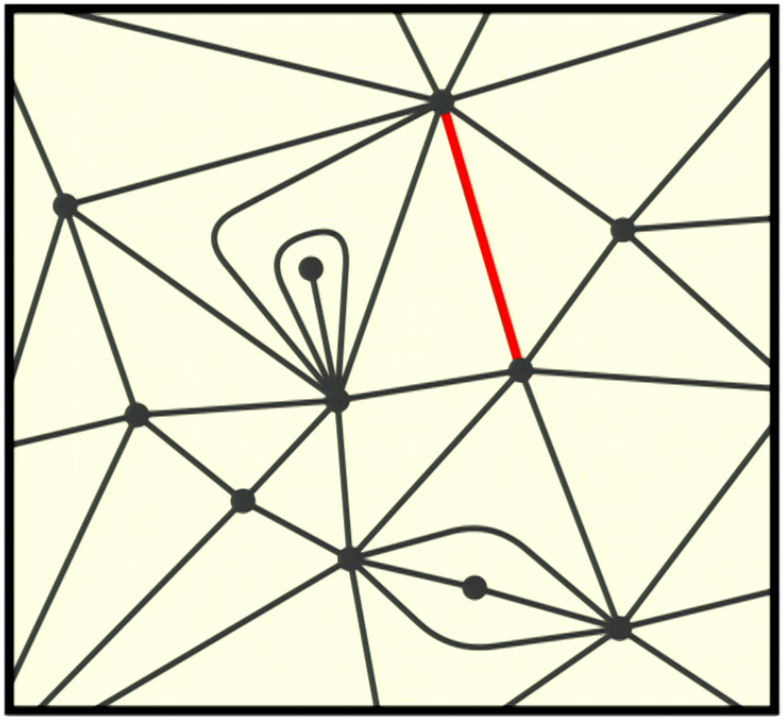

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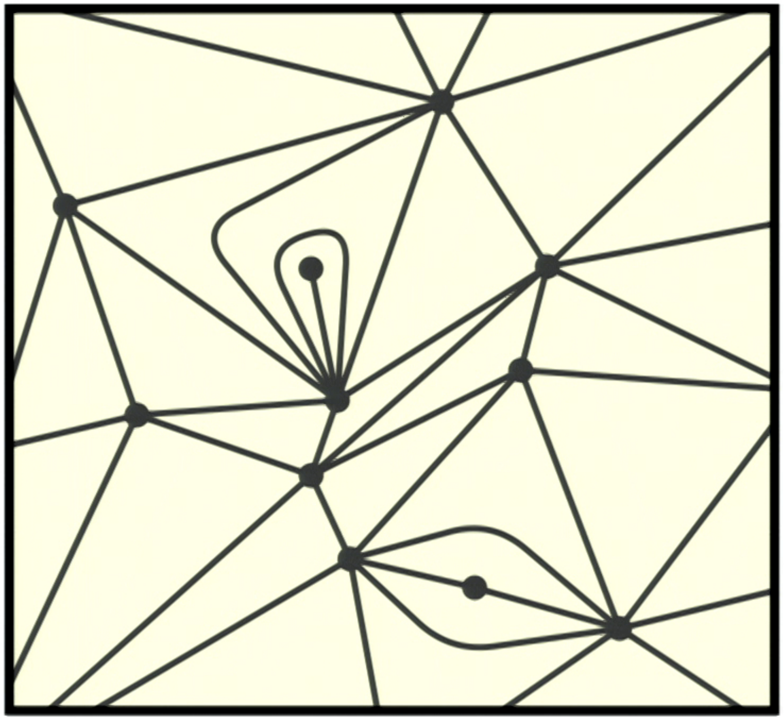

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- *Flip* it: Delete edge and draw the other diagonal of the resulting quadrangle.

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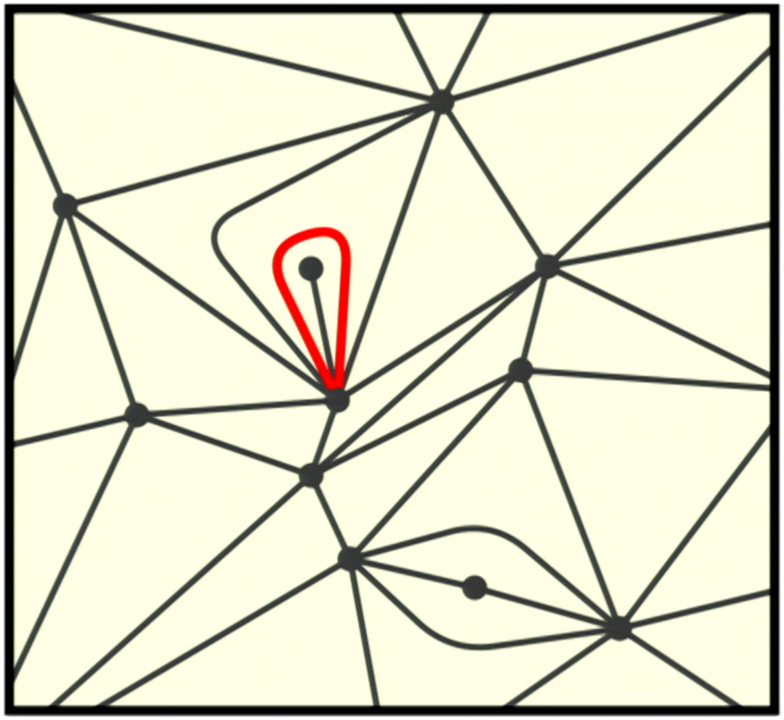

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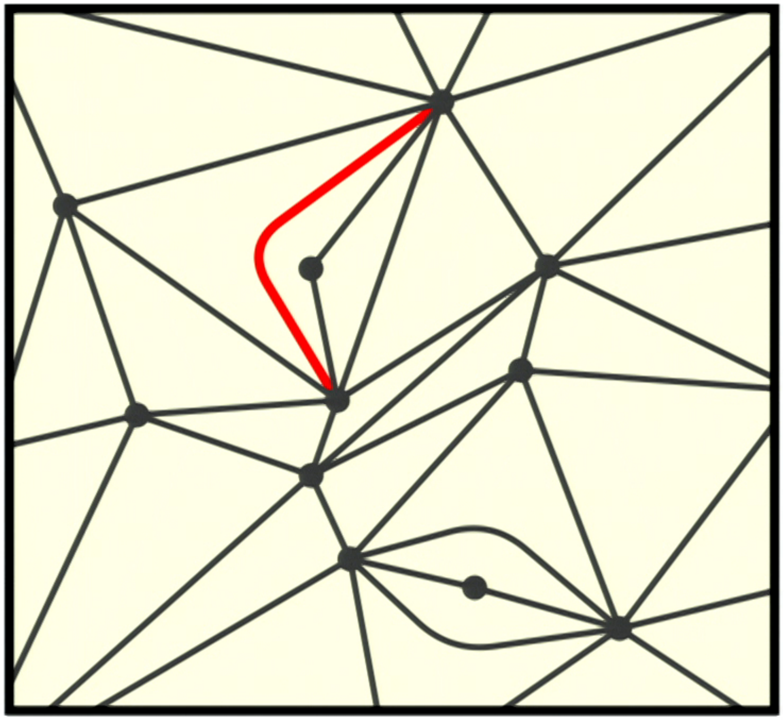

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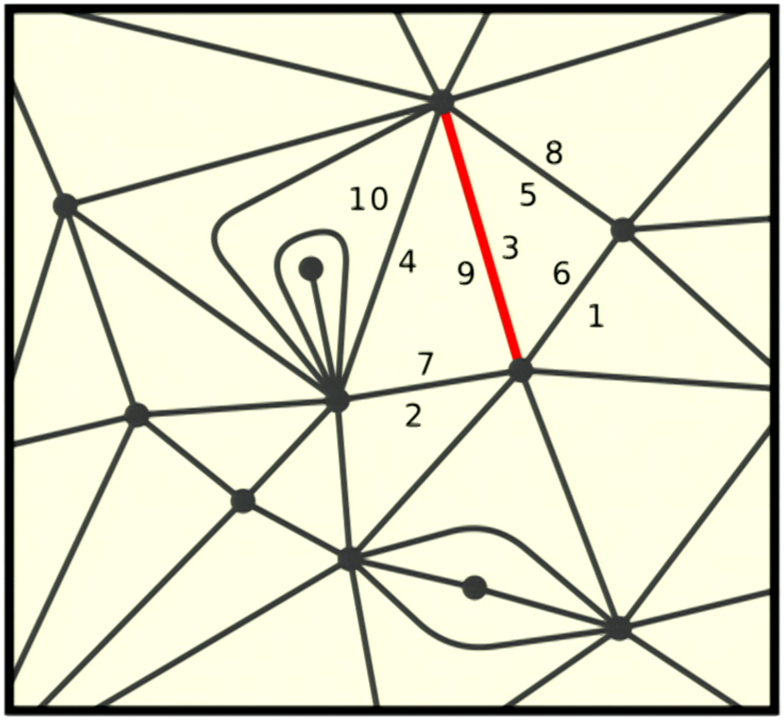

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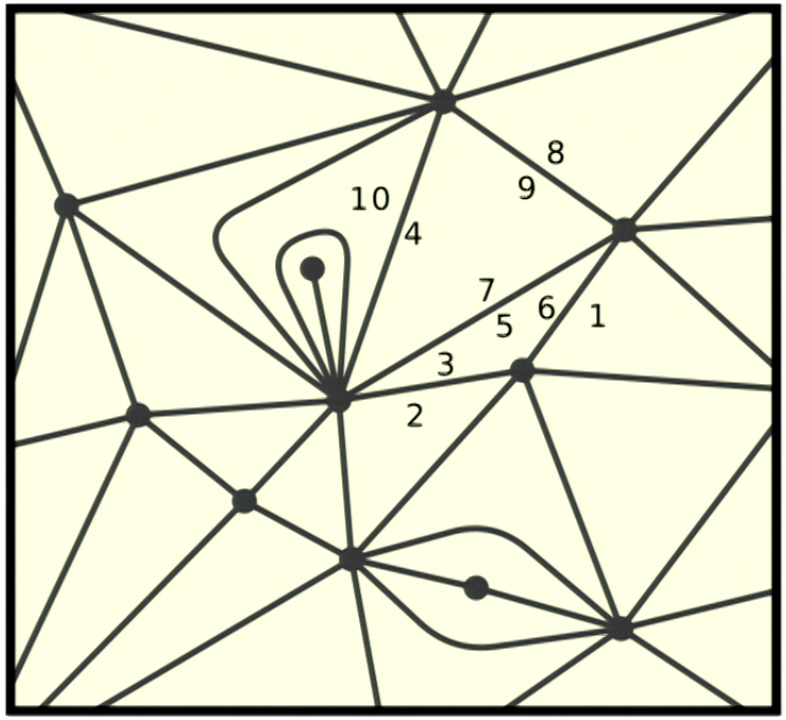

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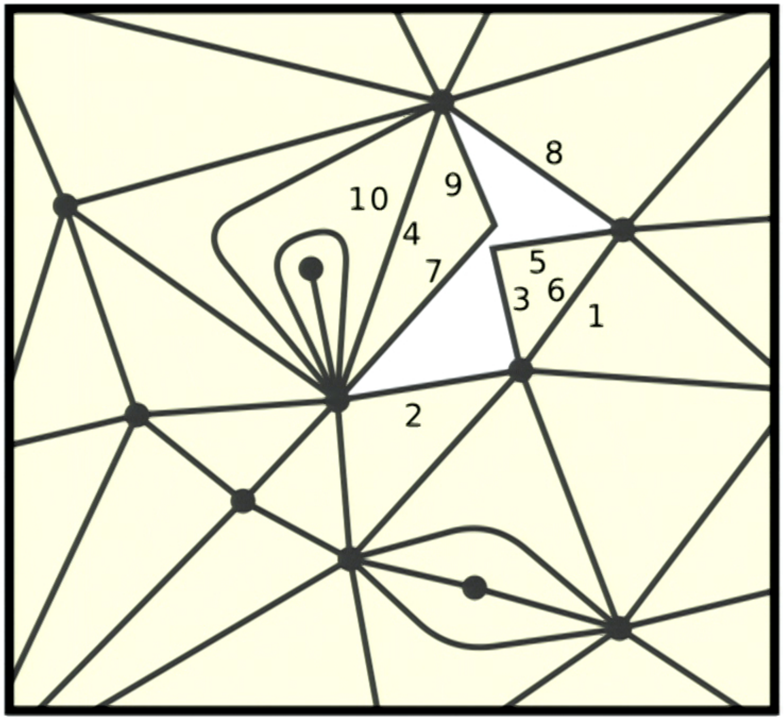

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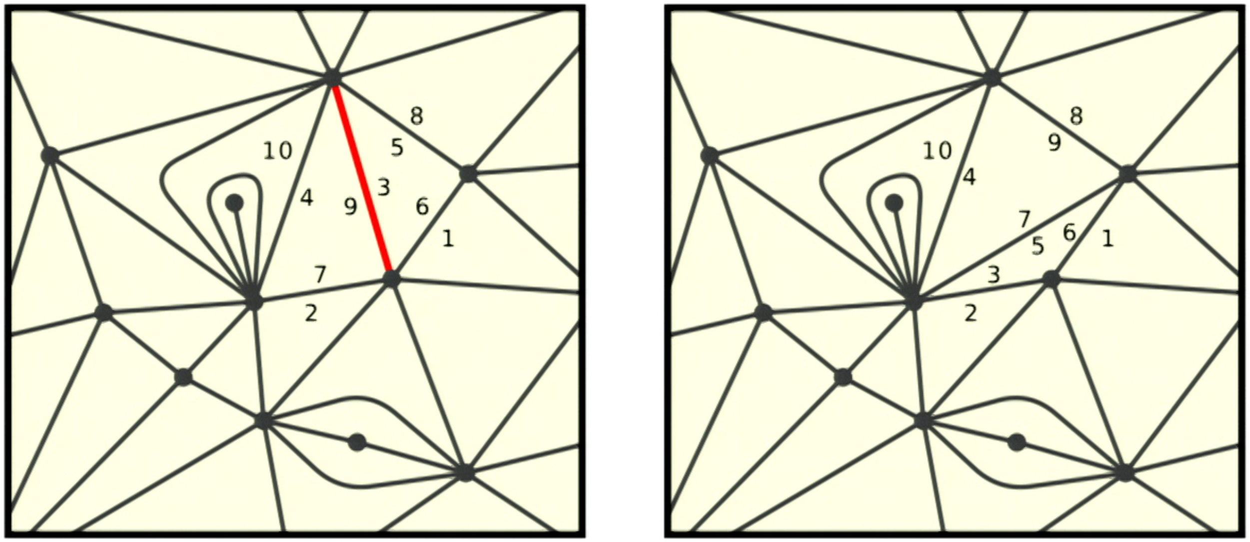

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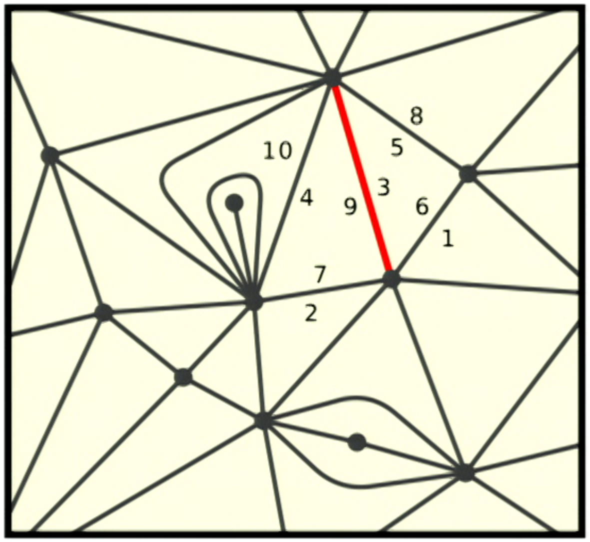

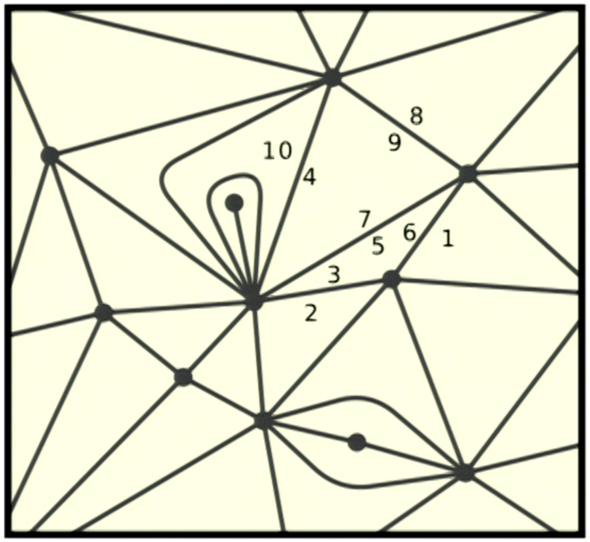

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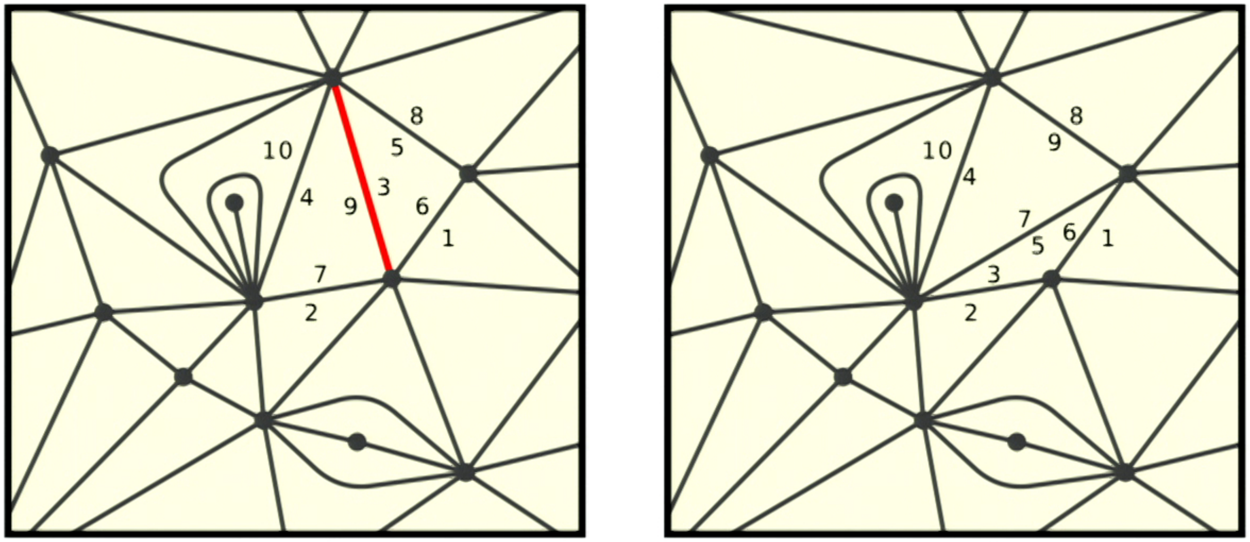

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  [Wagner, '36]
- In practice we don't permute. Why is that OK? Because flipping and permuting commute, and we may require observables $\mathcal{O} : \mathcal{T}_N \to \mathbb{R}$ to be invariant under label permutation.

Comment on labeling and symmetry

- Clearly labeling is useful when representing geometry in the computer.
- Another reason: it kills all possible symmetries, which is a good thing!

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- Let's look at unlabeled triangulations $\tilde{\mathcal{T}}_N = \mathcal{T}_n / \sim$, i.e. the set of equivalence classes of $\mathcal{T}_n$ under relabeling $\sim$.
- Sampling uniformly from $\tilde{\mathcal{T}}_N$ is not the same as sampling $\mathcal{T}_n$ and forgetting labels:

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- From the flip move point of view: