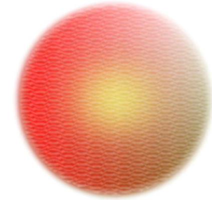
Title: Random and Fuzzy

Date: Jun 20, 2017 05:15 PM

URL: http://pirsa.org/17060074

Abstract: In this lecture I will introduce non-commutative geometry in the form of fuzzy spaces and explain how we have begun examining random fuzzy spaces using Markov Chain Monte Carlo simulations. Doing so leads us to consider a complicated matrix model, with even more complicated observables on it. A particular problem in this approach is how to define observables that can be related to physics and calculated with the computer.

Random and fuzzy
How to put NCG on the computer



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20th June

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Roadmap:

- Noncommutative geometry
- Random fuzzy spaces
- How do fuzzy spaces look?

NCG à la Connes

$$(s, \mathcal{H}, \mathcal{A}, \Gamma, J, \mathcal{D})$$

- Hilbert space
- Algebra
- Dirac operator
- signature
- Chirality
- Real structure

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Classical $(1,3)$d geometry

- $L^2(\mathcal{M}, S)$ the L^2 spinors
- Functions $C^\infty(\mathcal{M}) : f_1(x)$
- $\mathcal{D} = \partial\!\!\!/$
- $s = (q - p)$ mod $8 = 2$
- "γ^5"
- charge conjugation

The fuzzy sphere

$$\mathcal{H} = V \otimes M(n, \mathbb{C}) \qquad \mathcal{A} = SU(2)$$

V is a $(0,3)$ Clifford module, spanned by the Pauli matrices σ_i

$$\mathcal{D} = 1 + \sum_{i<j} \sigma_i \sigma_j \otimes [L_{ij}, \cdot]$$

L_{ij} a $N \times N$ rep of $SU(2)$

(Grosse Presnajder Lett. Math. Phys. 33,no.2:171–81)

Axioms for fuzzy spaces

$$(s, \mathcal{H}, \mathcal{A}, \Gamma, J, \mathcal{D})$$

- $\mathcal{H} = V \otimes M(n, \mathbb{C})$
  where V is a (p, q)-Clifford module
  p-times $(\gamma^i)^2 = 1$ and q-times $(\gamma^i)^2 = -1$
- $\mathcal{A}$ is a $*-$ algebra $M(n, \mathbb{C})$
- $s = (q - p) \bmod 8$
- $\Gamma(v \otimes m) = \gamma v \otimes m$ with γ the chirality operator on V
- $J(v \otimes m) = Cv \otimes m^*$ where C is charge conjugation on V

[illegible] spaces

$$(s, \mathcal{H}, \mathcal{A}, \Gamma, J, \mathcal{D})$$

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Dirac operator : Form

Conditions on $\mathcal{D}$

$$\mathcal{D} = \mathcal{D}^\dagger \qquad \mathcal{D}\Gamma = \pm\Gamma\mathcal{D}$$

$$\mathcal{D}J = \pm J\mathcal{D} \qquad [[\mathcal{D}, \rho(a)\triangleright], \triangleleft\rho(b)] = 0$$

Lead to:

$$\mathcal{D}(v \otimes m) = \sum_i \omega^i v \otimes \Big(\overbrace{K_i m}^{\text{left action}} + \epsilon' \overbrace{m K_i^*}^{\text{right action}} \Big)$$

(J. Barrett arXiv:1502.05383)

Examples for general fuzzy spaces

Type $(1,0)$

$$\gamma^1 = 1$$
$$D = \{H, \cdot\}$$

Type $(2,0)$

$$\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$D = \gamma^1 \otimes \{H_1, \cdot\} + \gamma^2 \otimes \{H_2, \cdot\}$$

Type $(1,1)$

$$\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$D = \gamma^1 \otimes \{H, \cdot\} + \gamma^2 \otimes [L, \cdot]$$

Examples for general fuzzy spaces

Type $(1,3)$

$$\gamma^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & I \\ 0 & 0 & -I & 0 \\ 0 & -I & 0 & 0 \\ I & 0 & 0 & 0 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\mathcal{D} = \sum_{j<k=1}^{3} \gamma^0\gamma^j\gamma^k \otimes [L_{jk}, \cdot] + \gamma^1\gamma^2\gamma^3 \otimes \{H_{123}, \cdot\}$$

$$+ \gamma^0 \otimes \{H_0, \cdot\} + \sum_{i=1}^{3} \gamma^i \otimes [L_i, \cdot]$$

The path integral over Fuzzy space

$$\langle f \rangle = \frac{\int f(\mathcal{D}) e^{-\mathcal{S}(\mathcal{D})} \mathrm{d}\mathcal{D}}{\int e^{-\mathcal{S}(\mathcal{D})} \mathrm{d}\mathcal{D}}$$

- Integral over all geometries, parametrized by D
  - Lesbegue measure on $\mathcal{D}$
  - $\mathcal{D}$ is highly constrained by conditions
- Action $\mathcal{S}$
- Need to measure functions of geometry f

The path integral over Fuzzy space

$$\langle f \rangle = \frac{\int f(\mathcal{D}) e^{-\mathcal{S}(\mathcal{D})} \mathrm{d}\mathcal{D}}{\int e^{-\mathcal{S}(\mathcal{D})} \mathrm{d}\mathcal{D}}$$
$$= \frac{\int f(\mathcal{D}(H_i, L_i)) e^{-\mathcal{S}(\mathcal{D}(H_i, L_i))} \mathrm{d}(H_i, L_i)}{\int e^{-\mathcal{S}(\mathcal{D}(H_i, L_i))} \mathrm{d}(H_i, L_i)}$$

- Integral over all geometries, parametrized by D
  - Lesbegue measure on $\mathcal{D}$
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The action

$$\mathcal{S} = g_2 \mathrm{Tr}\left(\mathcal{D}^2\right) + \mathrm{Tr}\left(\mathcal{D}^4\right)$$

(J. Barrett, LG arXiv:1510.01377)

What do we want from an action?

- physical motivation
  $\Rightarrow$ lowest order when expanding a heat kernel
- bounded from below
  $\Rightarrow$ for some g_2

The action

$$\mathcal{S} = g_2 \text{Tr}\left(\mathcal{D}^2\right) + \text{Tr}\left(\mathcal{D}^4\right)$$

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$(2,0)$ geometry

$$\mathcal{D} = \gamma^1 \otimes \{H_1, \cdot\} + \gamma^2 \otimes \{H_2, \cdot\}$$

Can write $\mathcal{D}$ as a $2N^2 \times 2N^2$ matrix using

$$[M, \cdot] = M \otimes I_N - I_N \otimes M^T \qquad \{M, \cdot\} = M \otimes I_N + I_N \otimes M^T$$

and calculate powers using

$$\{H, \{H, \cdot\}\} = \mathrm{I}_n \otimes H^T H^T + 2H \otimes H^T + HH \otimes \mathrm{I}_n$$

$$\text{Tr}\left(\{H, \{H, \cdot\}\}\right) = 2n \,\mathrm{Tr}\, H^2 + 2(\mathrm{Tr}\, H)^2$$

$$[L, [L, \cdot]] = \mathrm{I}_n \otimes L^T L^T - 2L \otimes L^T + LL \otimes \mathrm{I}_n$$

$$\text{Tr}\left([L, [L, \cdot]]\right) = 2n \,\mathrm{Tr}\, L^2 - 2(\mathrm{Tr}\, L)^2$$

The action

$$\mathcal{S} = g_2 \mathsf{Tr}\left(\mathcal{D}^2\right) + \mathsf{Tr}\left(\mathcal{D}^4\right)$$

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$(2,0)$ geometry

$$\begin{aligned}
\mathcal{D} &= \gamma^1 \otimes \{H_1, \cdot\} + \gamma^2 \otimes \{H_2, \cdot\} \\
\mathrm{Tr}\,\mathcal{D}^2 &= 4n(\mathrm{Tr}\,H_1^2 + \mathrm{Tr}\,H_2^2) + 4\left((\mathrm{Tr}\,H_1)^2 + (\mathrm{Tr}\,H_2)^2\right) \\
\mathrm{Tr}\,\mathcal{D}^4 &= 4n\Bigg(\mathrm{Tr}\,H_1^4 + \mathrm{Tr}\,H_2^4 + 4\,\mathrm{Tr}\,H_1^2 H_2^2 - 2\,\mathrm{Tr}\,H_1 H_2 H_1 H_2 \Bigg) \\
&\quad + 16\Bigg(\mathrm{Tr}\,H_1\,(\mathrm{Tr}\,H_1^3 + \mathrm{Tr}\,H_2^2 H_1) \\
&\quad + \mathrm{Tr}\,H_2\,(\mathrm{Tr}\,H_1^2 H_2 + \mathrm{Tr}\,H_2^3) + (\mathrm{Tr}\,H_1 H_2)^2 \Bigg) \\
&\quad + 12\Bigg((\mathrm{Tr}\,H_1^2)^2 + (\mathrm{Tr}\,H_2^2)^2 \Bigg) + 8\,\mathrm{Tr}\,H_1^2\,\mathrm{Tr}\,H_2^2
\end{aligned}$$

The action

$$\mathcal{S} = g_2 \mathrm{Tr}\left(\mathcal{D}^2\right) + \mathrm{Tr}\left(\mathcal{D}^4\right)$$

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$(2,0)$ geometry

$$\mathcal{D} = \gamma^1 \otimes \{H_1, \cdot\} + \gamma^2 \otimes \{H_2, \cdot\}$$

$$\mathrm{Tr}\,\mathcal{D}^2 = 4n(\mathrm{Tr}\, H_1^2 + \mathrm{Tr}\, H_2^2) + 4\left((\mathrm{Tr}\, H_1)^2 + (\mathrm{Tr}\, H_2)^2\right)$$

$$\mathrm{Tr}\,\mathcal{D}^4 = 4n\Big(\mathrm{Tr}\, H_1^4 + \mathrm{Tr}\, H_2^4 + 4\,\mathrm{Tr}\, H_1^2 H_2^2 - 2\,\mathrm{Tr}\, H_1 H_2 H_1 H_2 \Big)$$
$$+ 16\Big(\mathrm{Tr}\, H_1 \left(\mathrm{Tr}\, H_1^3 + \mathrm{Tr}\, H_2^2 H_1\right)$$
$$+ \mathrm{Tr}\, H_2 \left(\mathrm{Tr}\, H_1^2 H_2 + \mathrm{Tr}\, H_2^3\right) + (\mathrm{Tr}\, H_1 H_2)^2 \Big)$$
$$+ 12\Big((\mathrm{Tr}\, H_1^2)^2 + (\mathrm{Tr}\, H_2^2)^2 \Big) + 8\,\mathrm{Tr}\, H_1^2\,\mathrm{Tr}\, H_2^2$$

Matrix model!

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MCMC on this

Moves are easy

$$H_i \to H_i + c\delta H \qquad \text{or} \qquad L_i \to L_i + c\delta L$$

$\delta H, \delta L$ are (anti-)hermitian
$c \in \mathbb{R}$ adjusts for acceptance rate

Possible choices

- random matrices w. all elements between $-1, 1$ (uniform dist, normal dist, pick what you like)
- matrices w. single entry

MCMC on this

Moves are easy

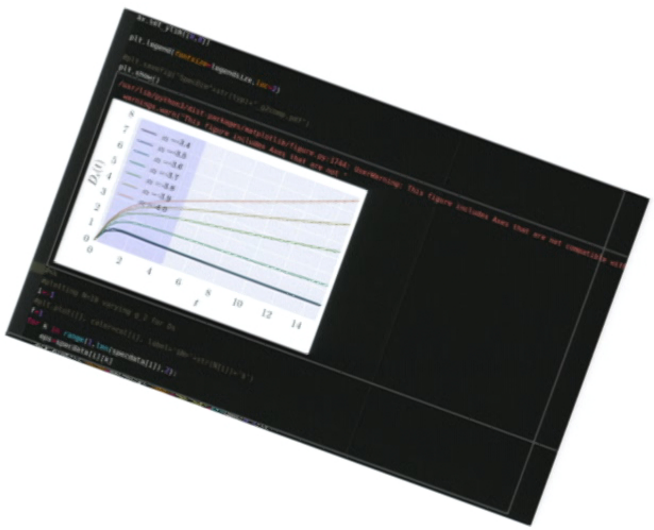
$$H_i \to H_i + c\delta H \qquad \text{or} \qquad L_i \to L_i + c\delta L$$

Update the action:

Lazy method:
Recalculate entire action for $H_i + \delta H$

My code

- C++
- Eigen library
- analysis w. Python code



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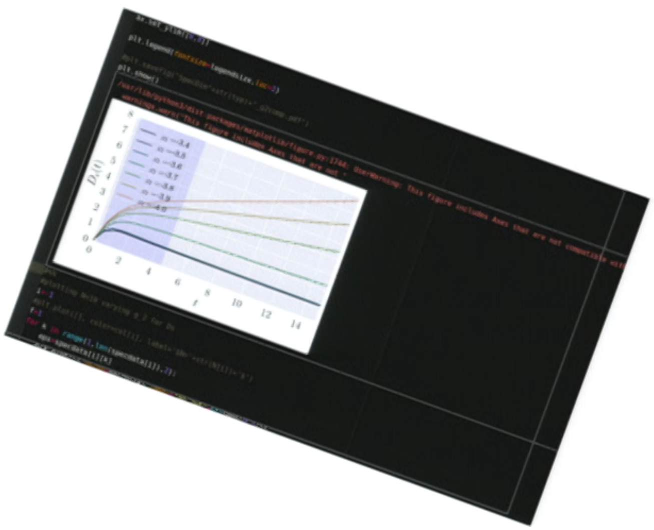

But make up your own :)

The covariant curse

MC simulations can measure $\langle f(g) \rangle$, but what are good $f(g)$?

Should be

- completly covariant
- space independent
- efficient to measure
- connect to physics?

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A few examples

- Phase transitions, Critical exponents (Thermodynamics)
- Transition amplitudes between boundary states
- Spectrum & spectral observables
  - Spectral dim
  - zeta function
  - spectral distance

Thermodynamic observables

Which geometries do I explore?

Today only Type $(2,0)$ (but results for $(1,1)$ and $(1,3)$ also available)

And observables?

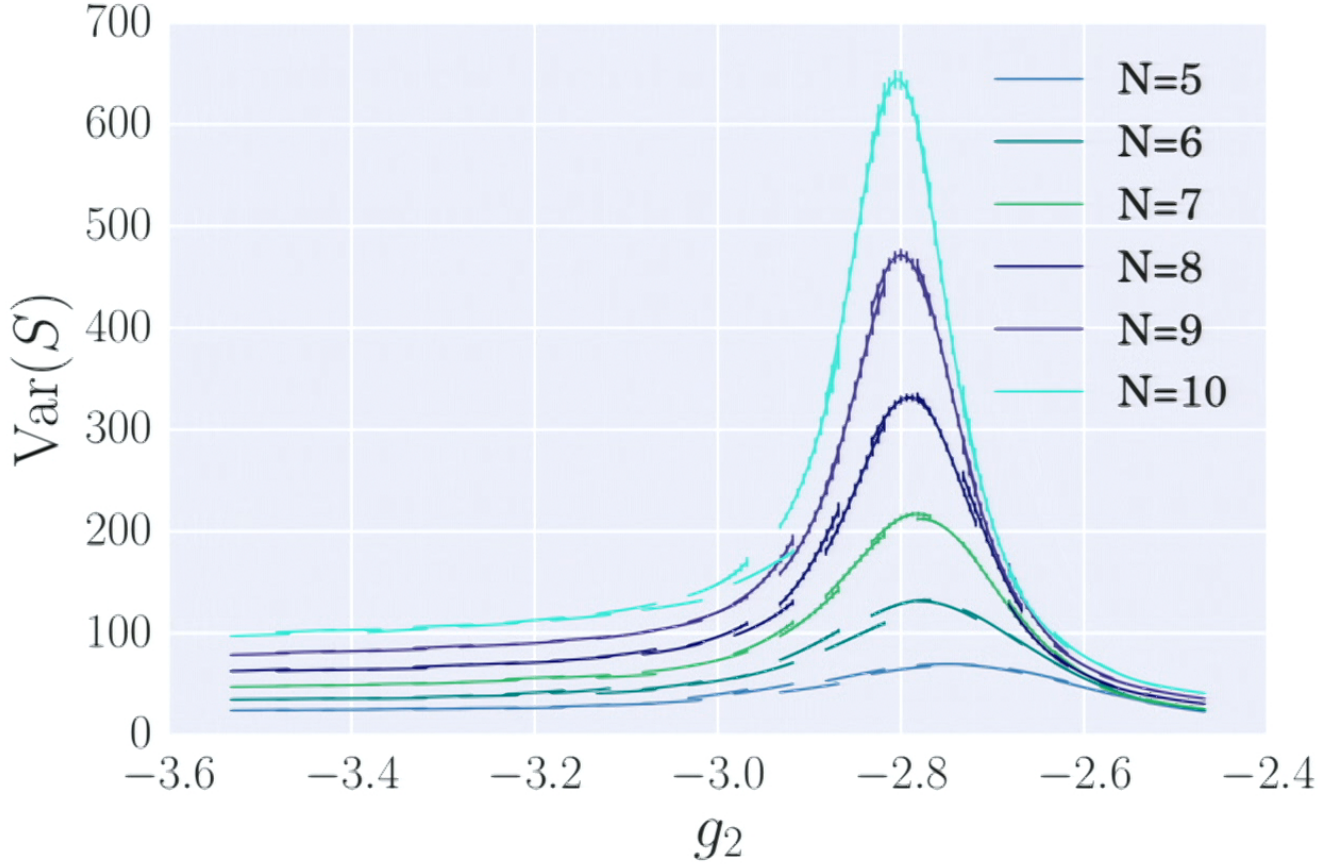
$$\langle \mathcal{S} \rangle = \sum_{MC} S \qquad \mathrm{Var}(\mathcal{S}) = \langle \mathcal{S}^2 \rangle - \langle \mathcal{S} \rangle^2$$

$$\langle \mathrm{Tr}\left(\mathcal{D}^2\right) \rangle = \sum_{MC} \mathrm{Tr}\left(\mathcal{D}^2\right) \quad \mathrm{Var}(\mathrm{Tr}\left(\mathcal{D}^2\right)) = \langle \mathrm{Tr}\left(\mathcal{D}^2\right)^2 \rangle - \langle \mathrm{Tr}\left(\mathcal{D}^2\right) \rangle^2$$

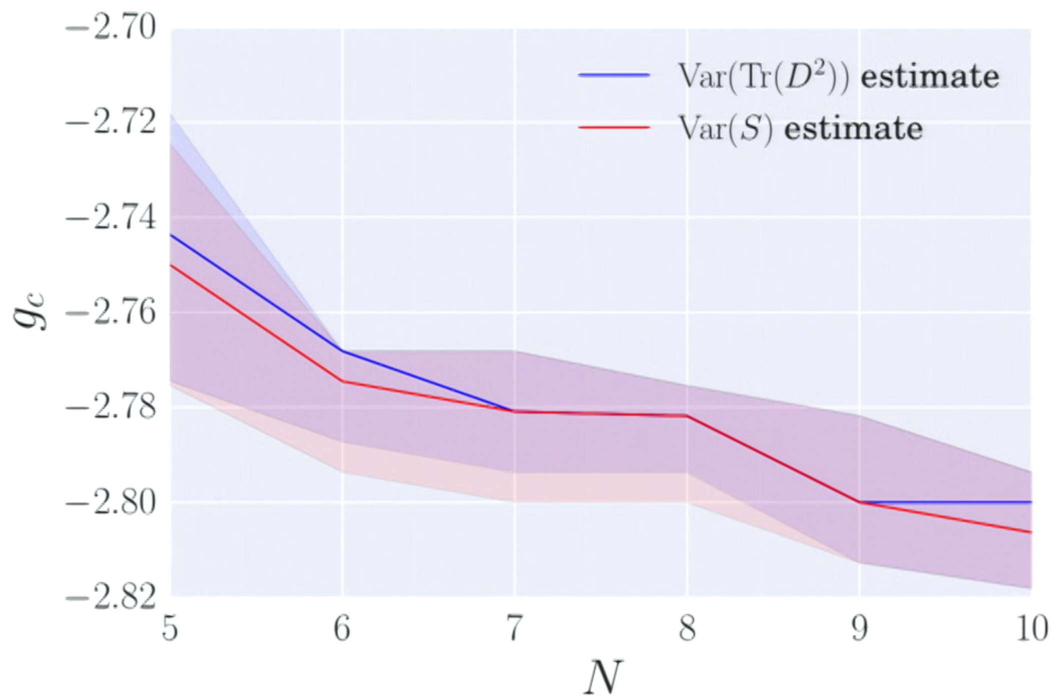
MC is the markov chain we generate with our algorithm

(Glaser arXiv:1612.00713)

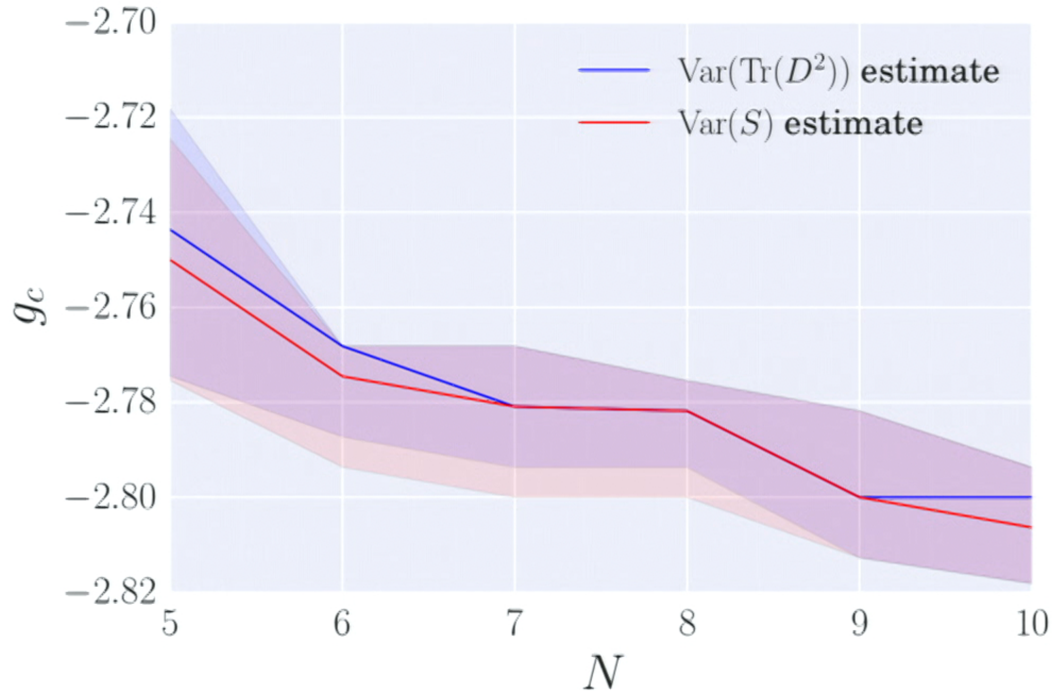
Locate the phase transition $(2,0)$



Locate the phase transition $(2, 0)$



Locate the phase transition $(2,0)$



$$g_c^{(2,0)} = -2.781 \pm 0.289$$

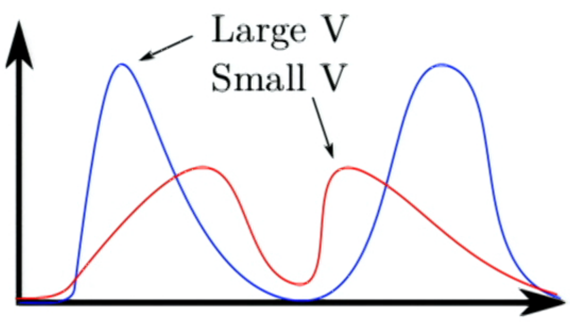
Order of a phase transition

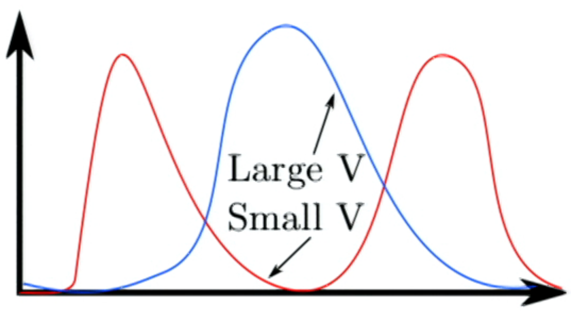
1st order transition

- no correlation growth
- discontinous 'jumps' between states, getting rarer as $V \to \infty$

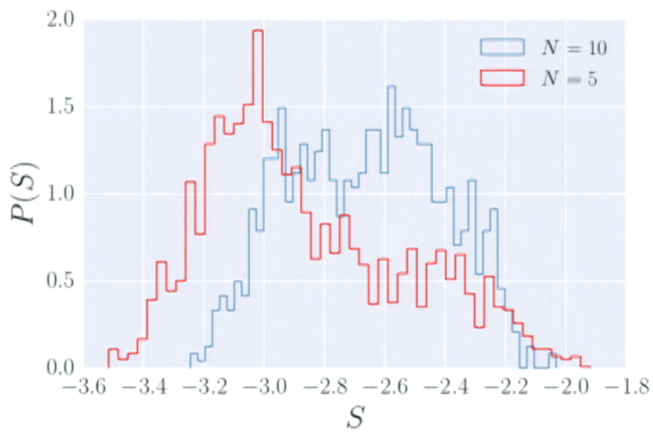
2nd order transition

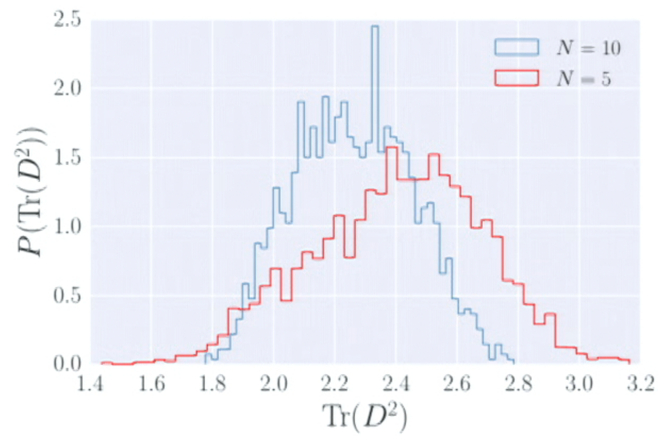
- correlation length diverges in $V \to \infty$ limit
- qualitatively new behavior exactly at transition





Order of phase transitions $(2,0)$





Order?

Two peaks, but they merge, that signals 2nd order.

Scaling with size, Variance

$$\begin{aligned}\mathrm{Var}(\mathrm{Tr}(D^2)) &= \langle \mathrm{Tr}(D^2)^2 \rangle - \langle \mathrm{Tr}(D^2) \rangle^2 \\ &= \sum_i^{n_d} \sum_j^{n_d} \langle \lambda_i^2 \lambda_j^2 \rangle - \langle \lambda_i^2 \rangle \langle \lambda_j^2 \rangle \\ &= \sum_i^{n_d} \sum_j^{n_d} \mathrm{Cov}(\lambda_i^2, \lambda_j^2)\end{aligned}$$

$\sim N^4$?

Scaling with size, Variance

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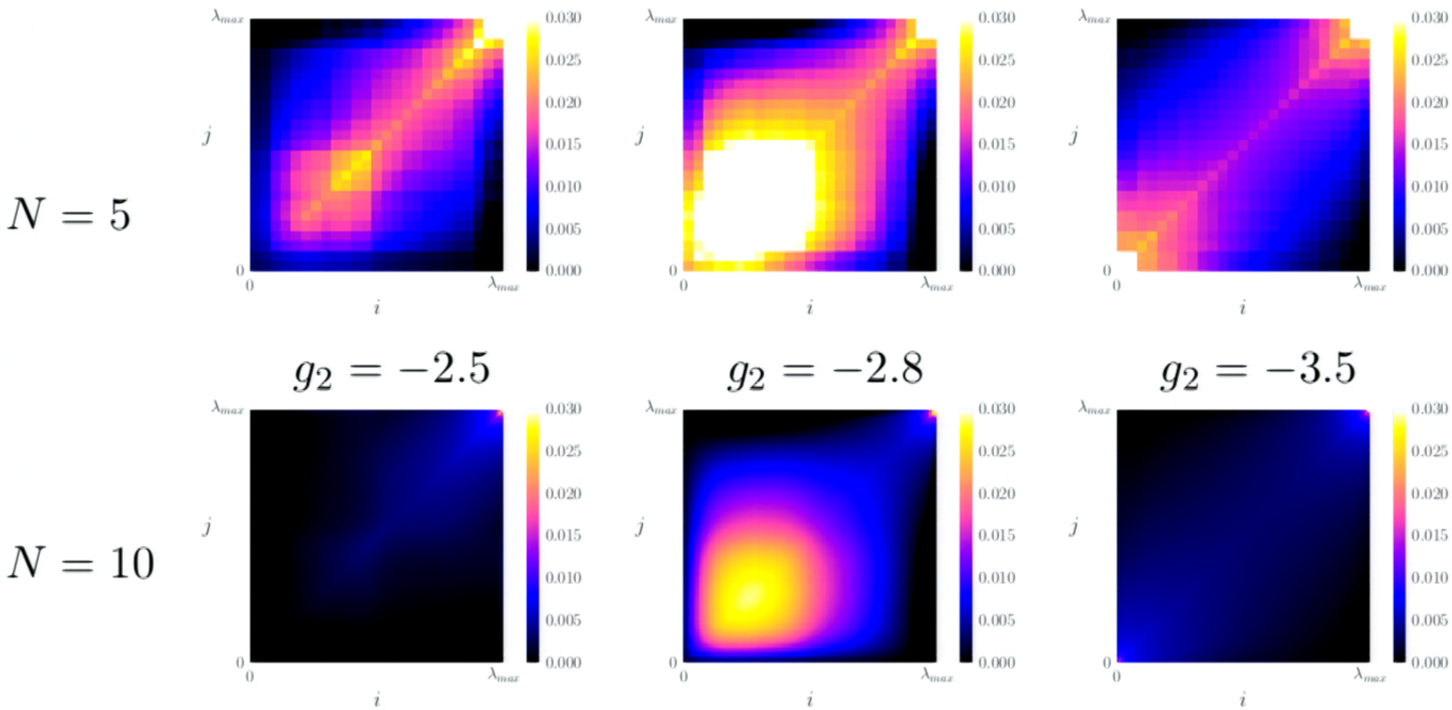
$\sim N^4$?

Not that simple, other effects arise!

$\mathrm{Cov}(\lambda_i^2, \lambda_j^2)$ **Type** $(2,0)$

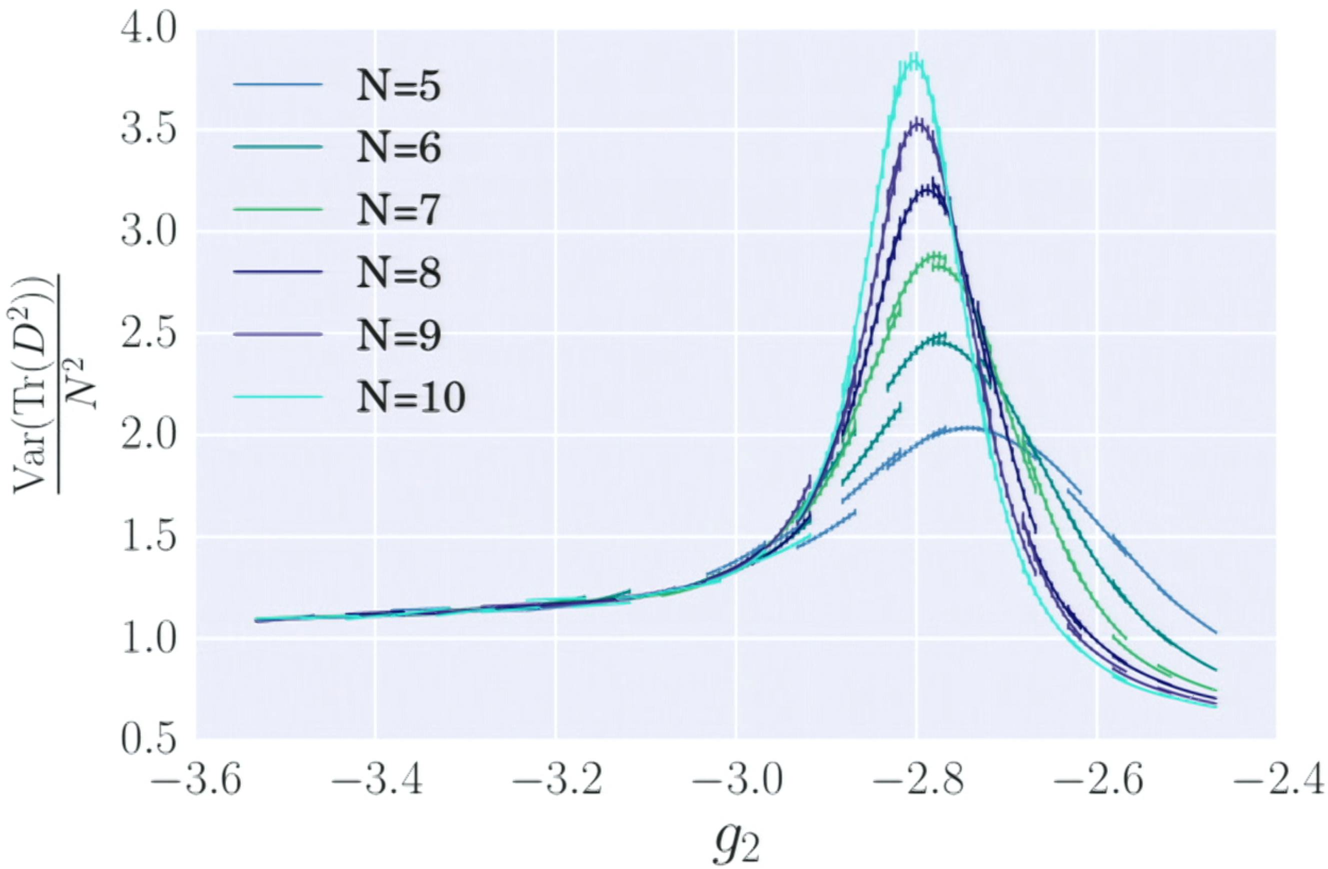


Features

- Diagonals away from PT
- Blob at PT

Away from the PT it scales as N^2!

Type $(2, 0)$



Finite size scaling

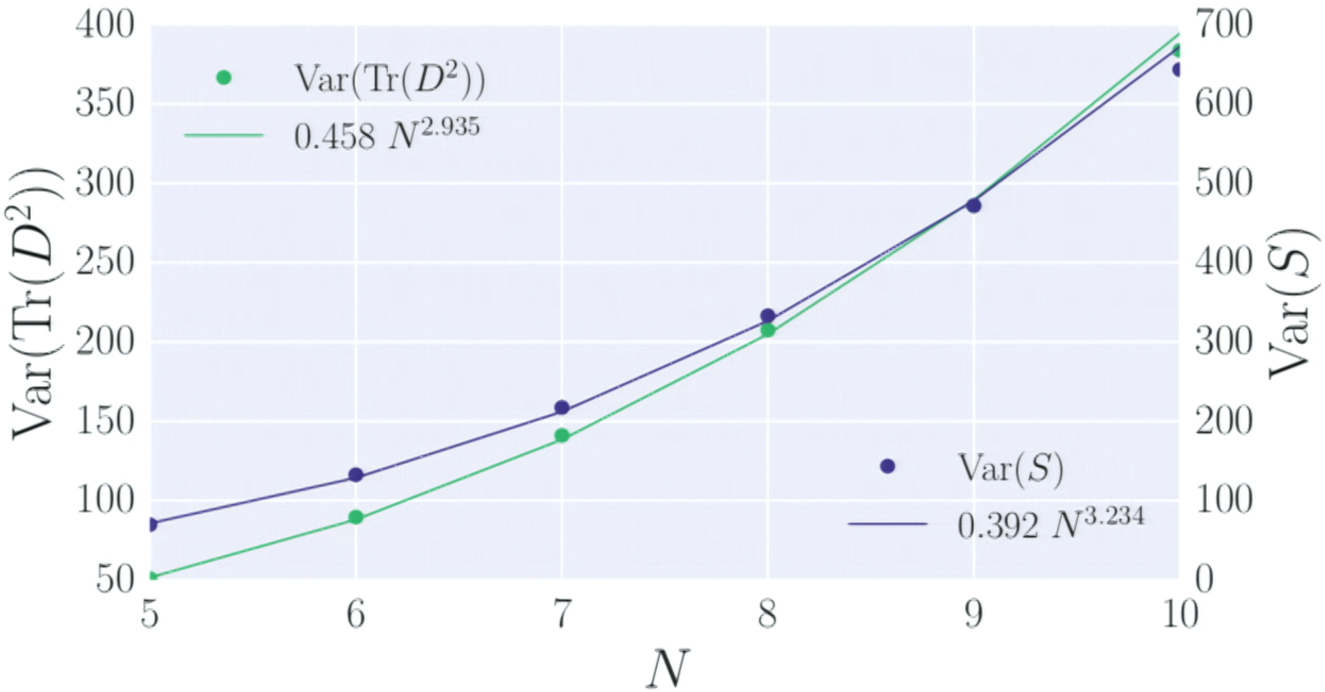
Scaling Ansatz for variance

$$\mathrm{Var}(\mathsf{Tr}\left(\mathcal{D}^2\right))(\xi, N) = N^{2-2x}\Phi(\frac{N^{\frac{2}{d}}}{\xi}) \ .$$

At PT $\xi^{-1} \to 0$
$\Rightarrow$ Variance scales as N^{2-2x}

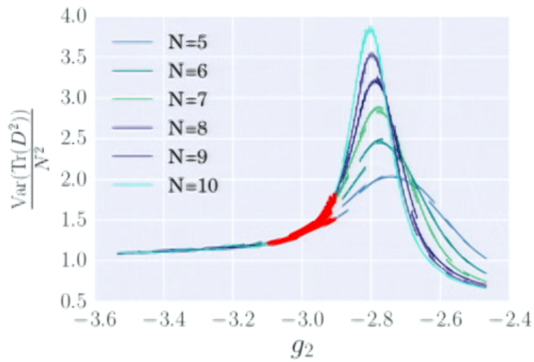
Scaling of Var at the PT

Type $(2,0) \sim N^3 \Rightarrow x = -0.5$

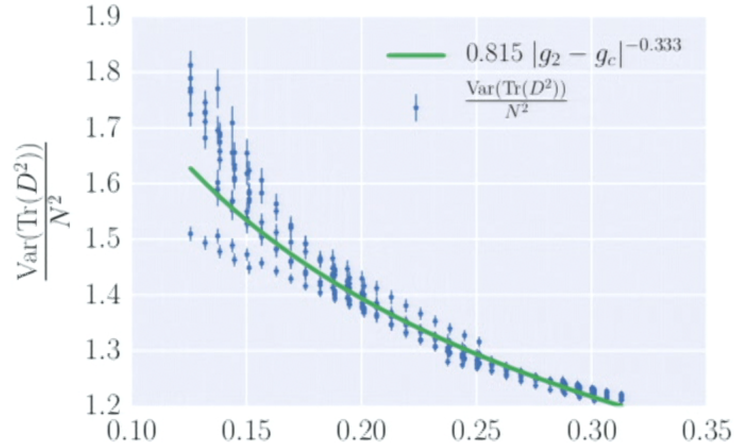


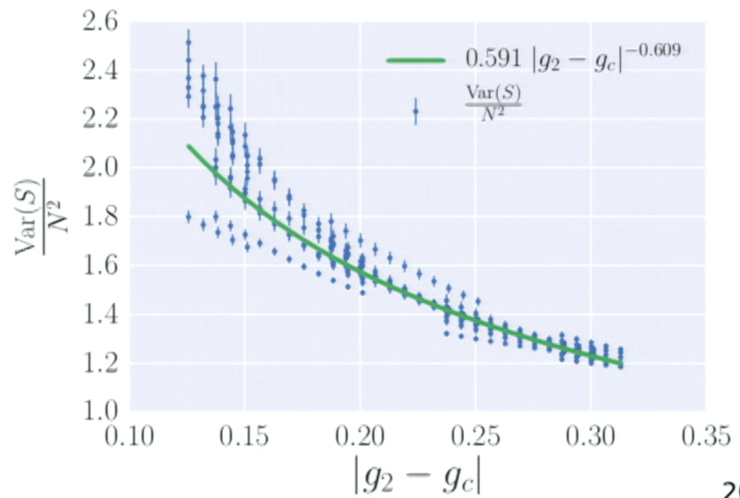
Two critical exponents

Type $(2,0)$



$$\mathrm{Var}(S(g_2)) \sim |g - g_c|^{\mu}$$
$$\mathrm{Var}(\mathrm{Tr}\,(D^2)) \sim |g - g_c|^{\nu}$$





A plot of the spectrum

Eigenvalues of $\mathcal{D}$

Advantages:

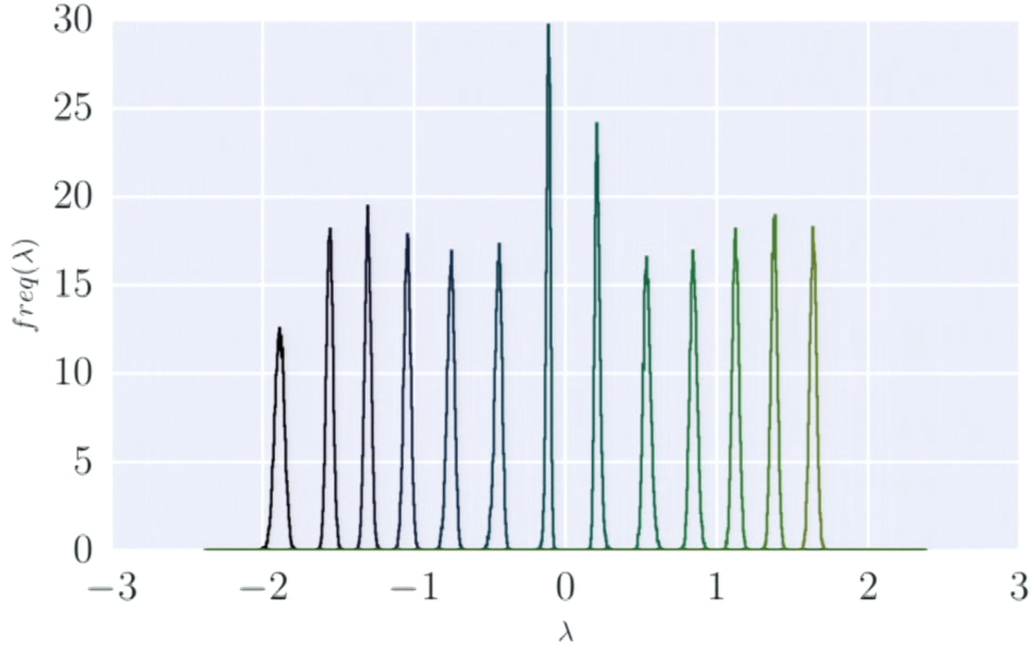
- Easy to interpret
- Ready made efficient algorithms
- Contains a lot of information

Disadvantages:

- We can't hear the shape of a drum
- contains more than just e.v. of H, L ➔ diagonalize $dim(\gamma)N^2$ dimensional matrices

A plot of the spectrum

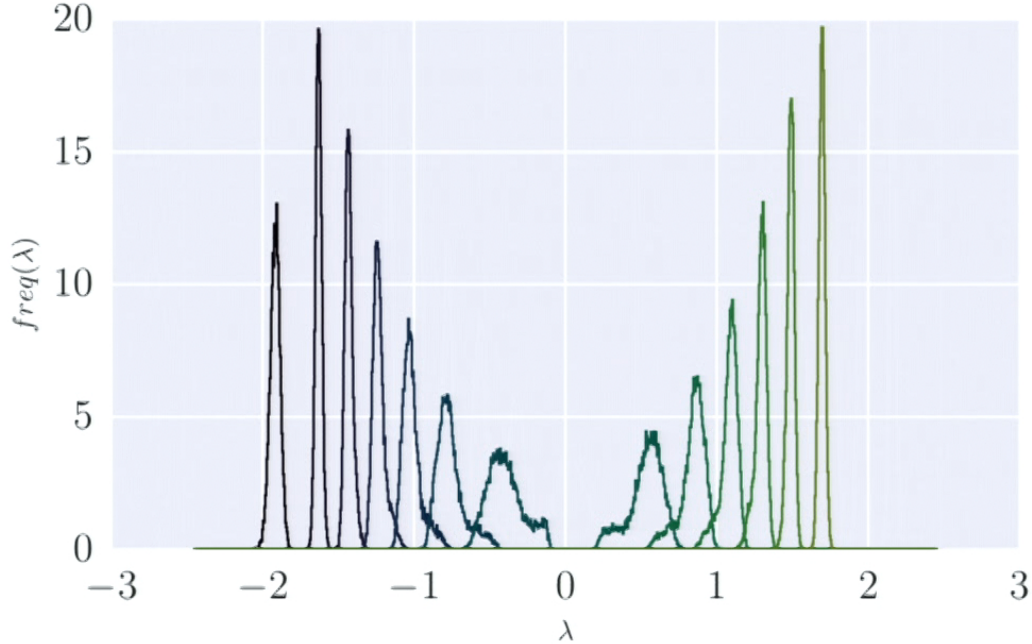
Eigenvalues of $\mathcal{D}$



$$g_2 = -3.35 \ N = 8$$

A plot of the spectrum

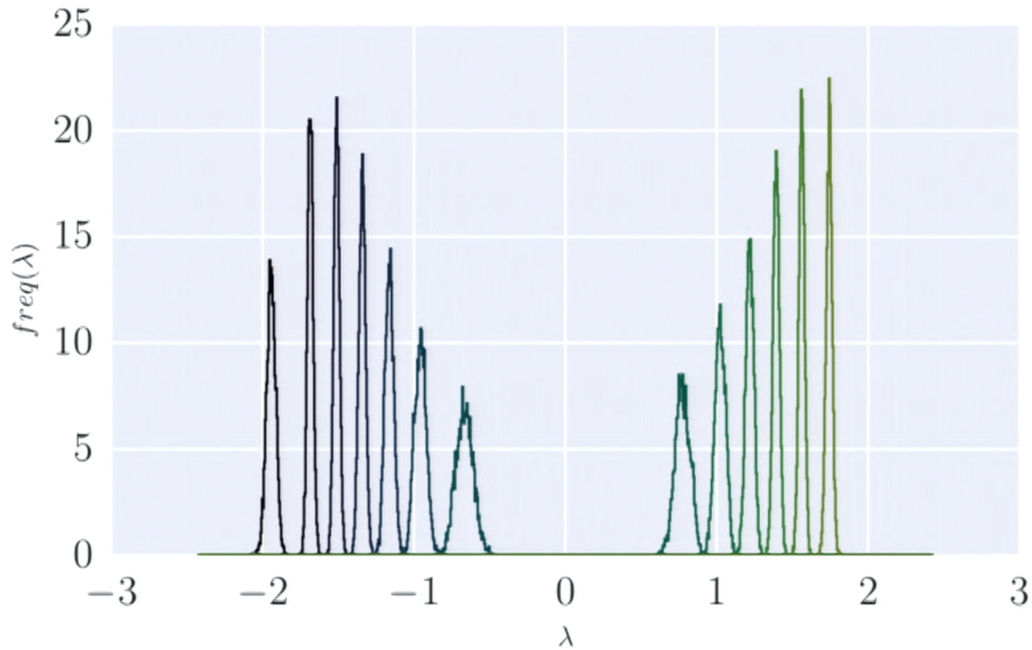
Eigenvalues of $\mathcal{D}$



$g_2 = -3.7 \; N = 8$

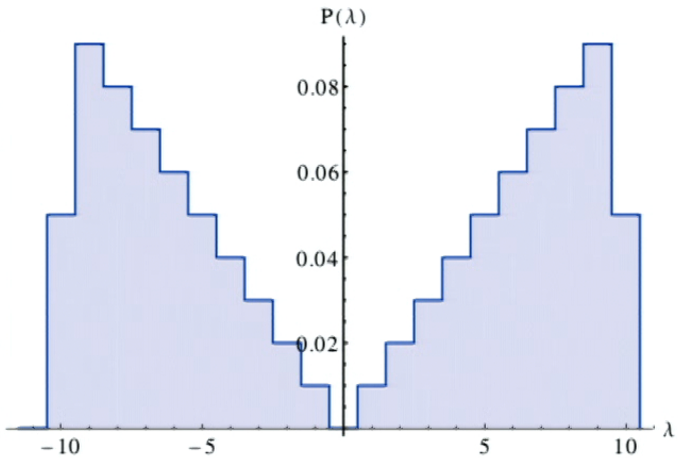
A plot of the spectrum

Eigenvalues of $\mathcal{D}$

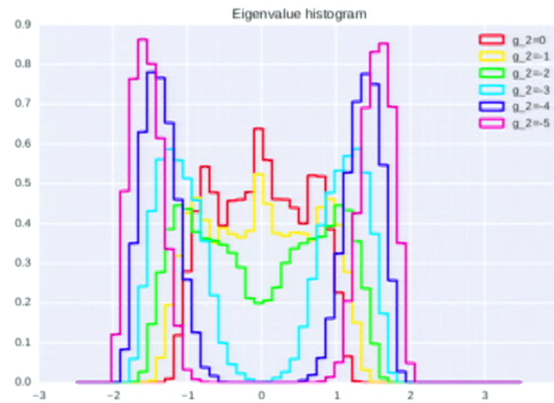


$g_2 = -4.0 \; N = 8$

Eigenvalue histograms



Fuzzy S^2



Random $(1,1$

Density of states

$$\rho(\lambda) \sim |\lambda|^{d-1}$$

The spectral dimension

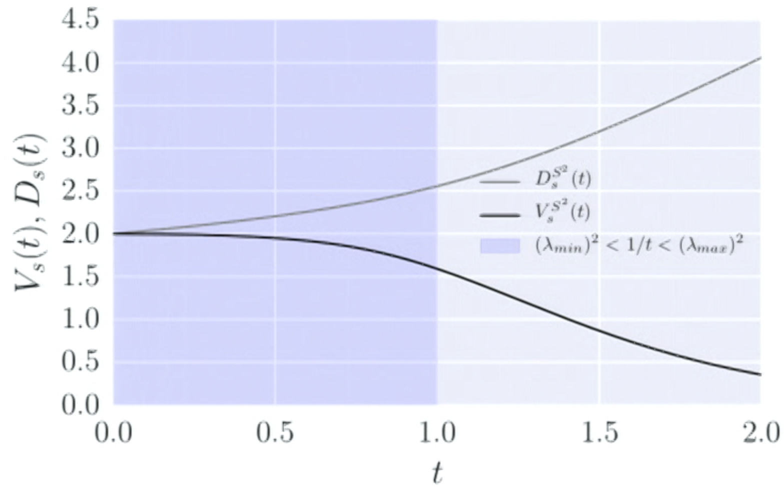
Use $\mathcal{D}^2$ to get a Δ-type operator

$$D_s(t) = 2t \, \frac{\sum_\lambda \lambda^2 e^{-t\lambda^2}}{\sum_\lambda e^{-t\lambda^2}}$$

$\sim t\lambda_0^2$ for large t

$\lambda \in Ev(\mathcal{D})$

$\lambda_0^\Delta = 0 \Rightarrow$ no problem
$\lambda_0 \neq 0 \Rightarrow$ linear mode



exact S^2

(Barrett, Druce, Glaser coming soon!)

The spectral dimension

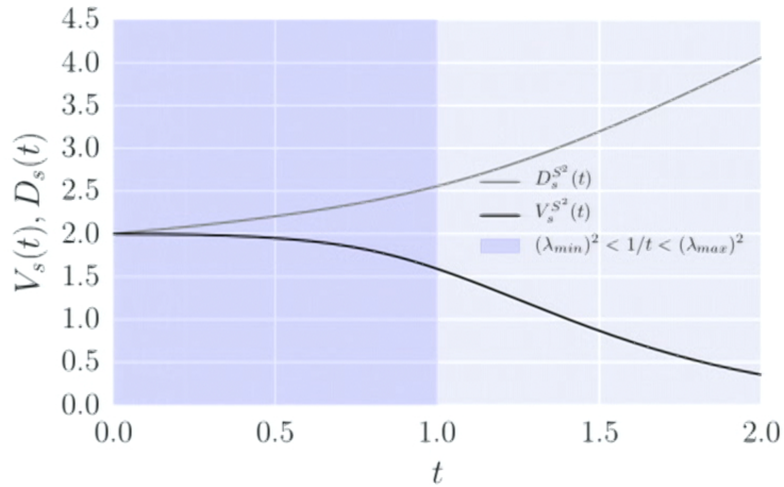
Use $\mathcal{D}^2$ to get a Δ-type operator

$$D_s(t) = 2t \frac{\sum_\lambda \lambda^2 e^{-t\lambda^2}}{\sum_\lambda e^{-t\lambda^2}}$$

$\sim t\lambda_0^2$ for large t

$\lambda \in Ev(\mathcal{D})$

$$V_s(t) = D_s(t) - t\frac{\partial D_s(t)}{\partial t}$$



exact S^2

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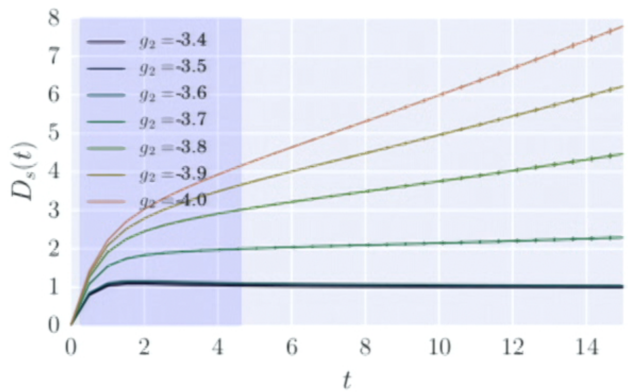
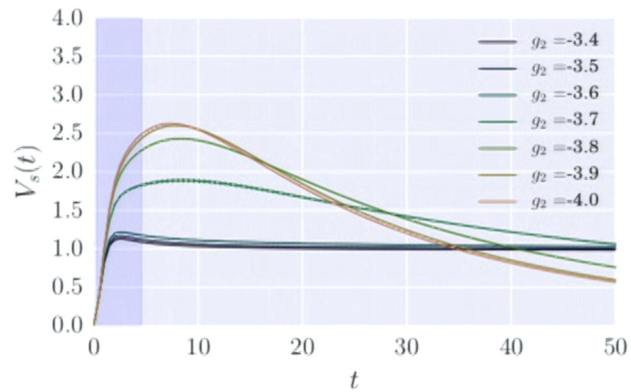
Type $(1,3)$



$D_s(t)$



$V_s(t)$

Why ζ?

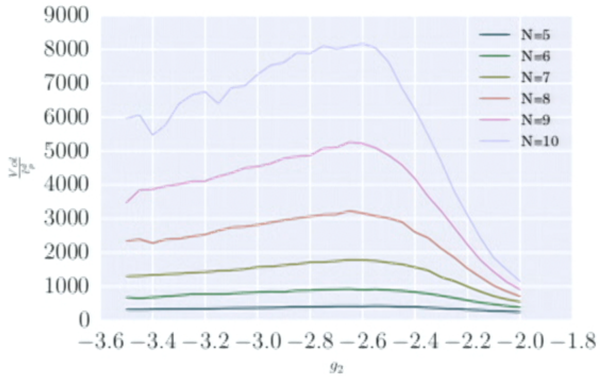
$$\zeta(s) = \sum_i \left(\lambda_i^2\right)^{-s}$$

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} K(t) \mathrm{d}t$$

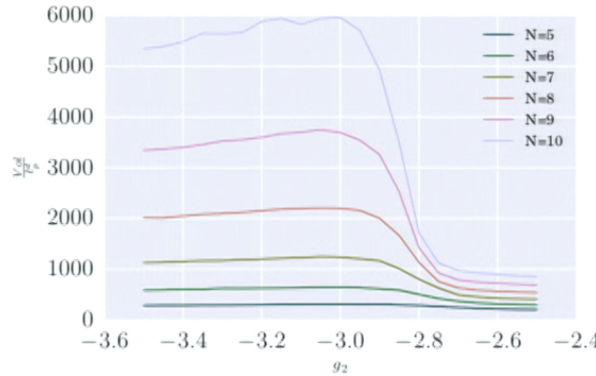
Pole at $t = \frac{d}{2}$

$$Vol(M) = \frac{(4\pi)^{d/2}}{\mathrm{Tr}\,(Id)} \frac{\zeta(d/2)\Gamma(d/2)}{log(N)}$$

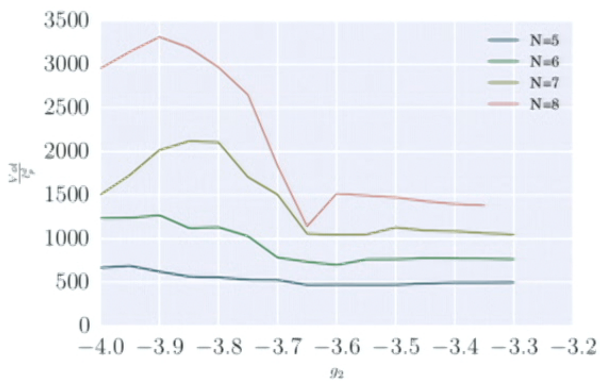
Volume from ζ



Type $(1,1)$



Type $(2,0)$



Type $(1,3)$

Read off at

$$s = d/2 = \max(V_s(t))/2$$

Normed by $l_p = \frac{1}{\lambda_{max}}$

Distance between geometries

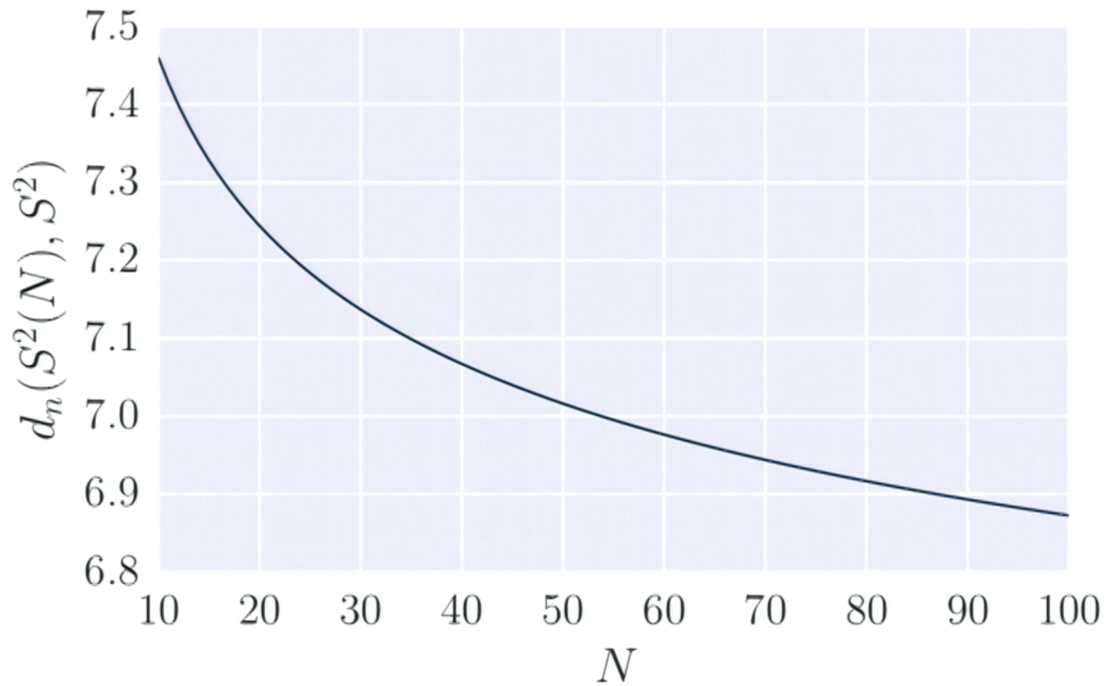
For two geometries X_1, X_2

$$d_{CK}(X_1, X_2) = \sup_{\gamma < s < \gamma+1} \left| \log \left| \frac{\zeta^{X_1}(s)}{\zeta^{X_2}(s)} \right| \right|$$

- $\gamma > d/2$ for X_1, X_2 in continuum, otherwise free

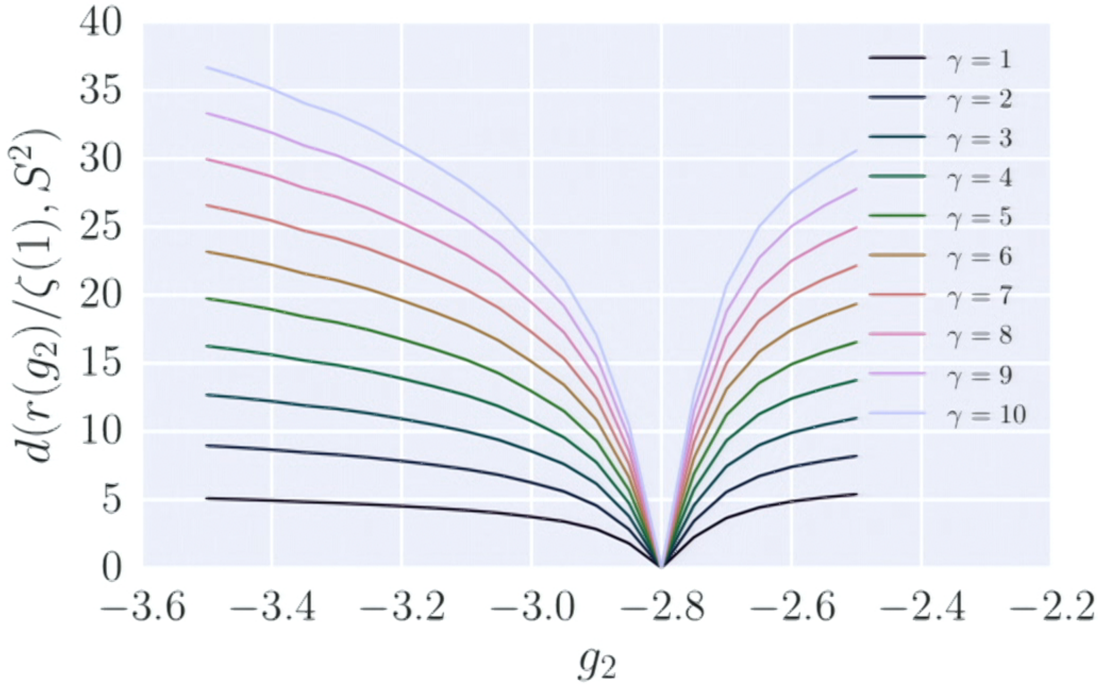
(Cornelissen, G. & Kontogeorgis, A. Lett Math Phys (2017) 107: 129)

Testing the distance measure



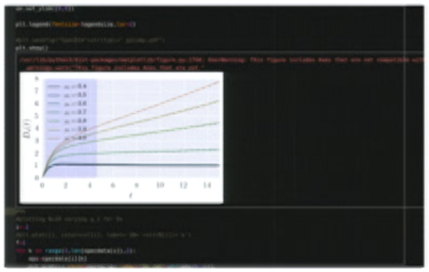
Fuzzy S^2 to exact S^2

Testing the distance measure

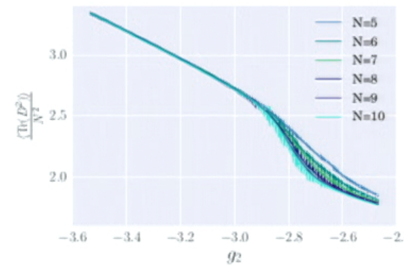


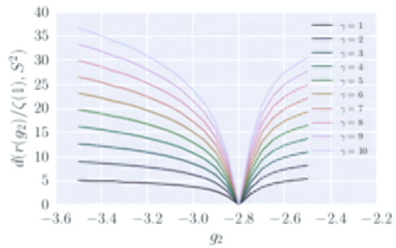
Type $(2,0)$ $N = 10$ distance between g_2

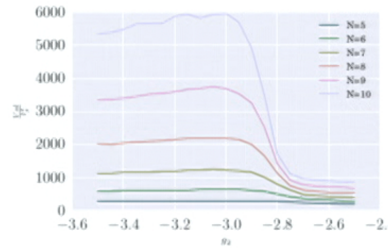
What did we learn today?

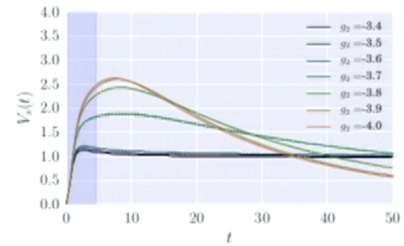


- NCG
- Fuzzy space
- The hard part: Observables

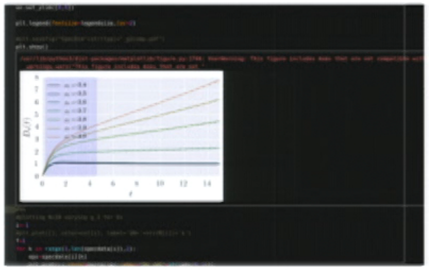








What did we learn today?



- NCG
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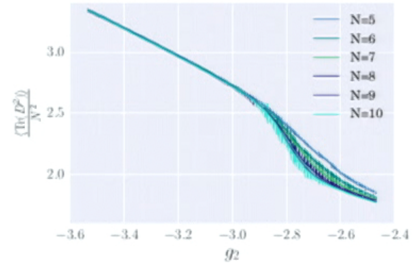

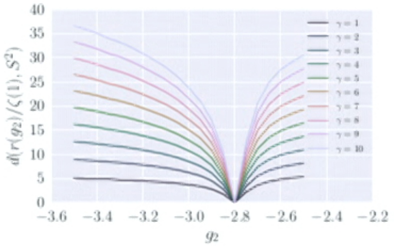

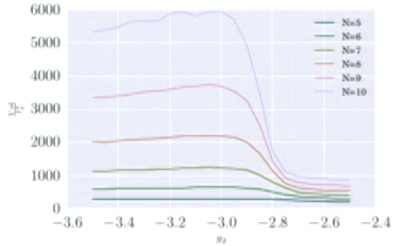

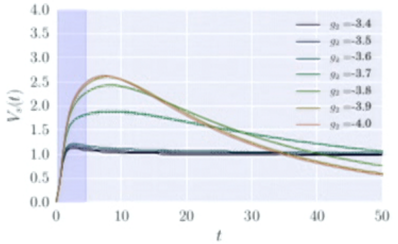

Thanks for your attention!

Any more questions?