Title: Introduction to Tensor Network methods - 1

Date: Jun 19, 2017 11:00 AM

URL: http://pirsa.org/17060063

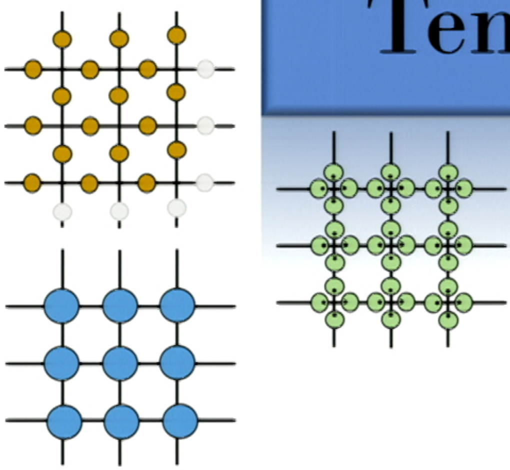
Abstract:

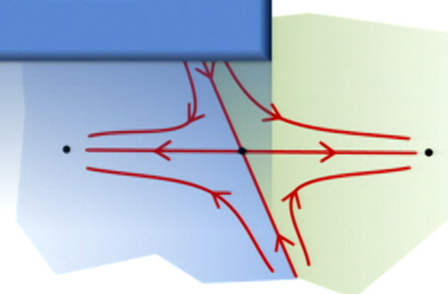
Summer School

Making Quantum Gravity Computable

Perimeter Institute, June 19th – 23rd 2017

Introduction to Tensor Networks





Guifre Vidal

PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

SIMONS FOUNDATION

Tensor Networks Initiative

NSERC

compute canada | calcul canada

MAKING QUANTUM GRAVITY COMPUTABLE

Conference Date: Monday, June 19, 2017 (All day) to Friday, June 23, 2017 (All day)
Scientific Areas: Quantum Gravity

Introduction to tensor networks

Monday - Tuesday

Guifre Vidal
Clement Delcamp

Clement Delcamp

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Introduction to tensor networks

Monday - Tuesday

Guifre Vidal
Clement Delcamp

→

Coarse-graining of spin foams

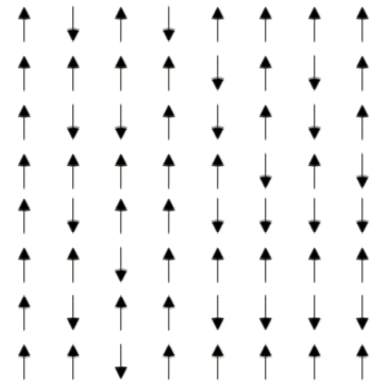
Wednesday - Thursday

Benjamin Bahr
Bianca Dittrich

Clement Delcamp

COURSE OVERVIEW

N classical spins $\sigma^{(i)} = \pm 1 \qquad i = 1, \cdots, N$



$N = 8 \times 8 = 64$ Ising spins

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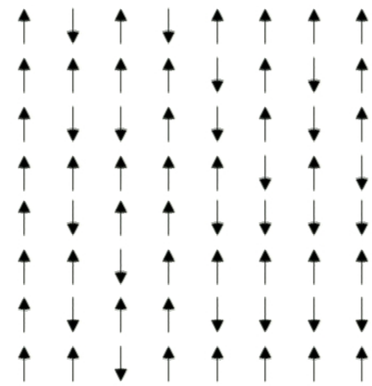
spin configuration $\{\sigma\}$ example: $\{\sigma\} = (+1, -1, -1, +1, \cdots, -1)$



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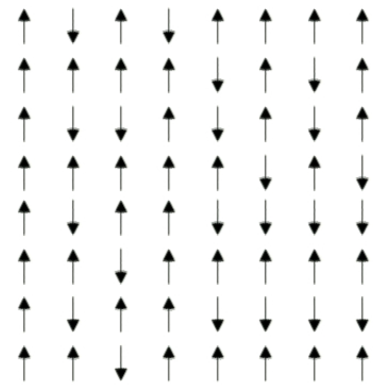

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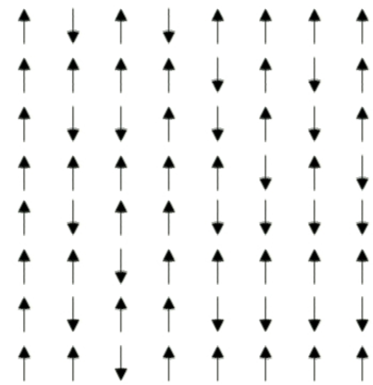
Boltzmann weight $e^{-\beta H(\{\sigma\})}$

$$Z = \sum_{\{\sigma\}} e^{-\beta H(\{\sigma\})}$$

partition function



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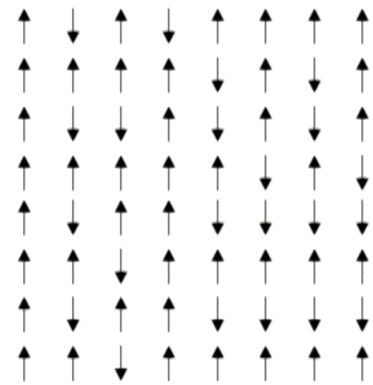

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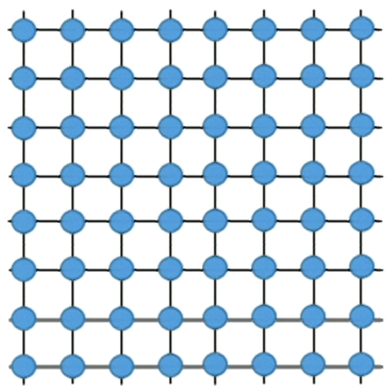

(1) tensor network for Z

(today)

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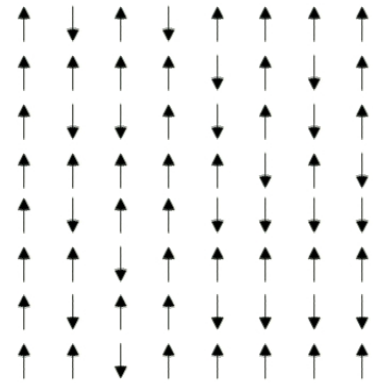

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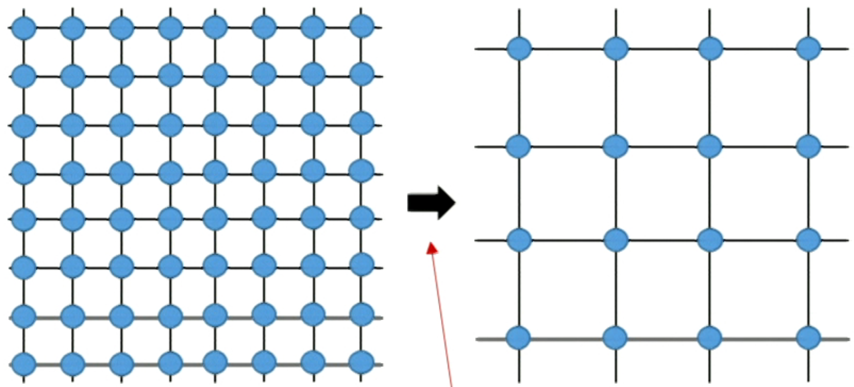

(1) tensor network for Z
(today)

(2) coarse-graining transformation
(tomorrow)

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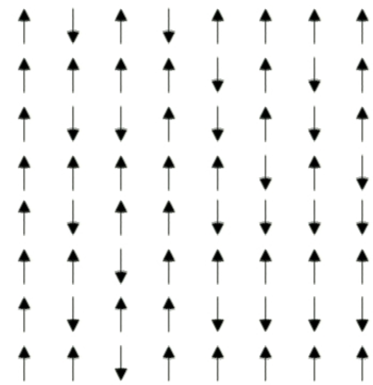

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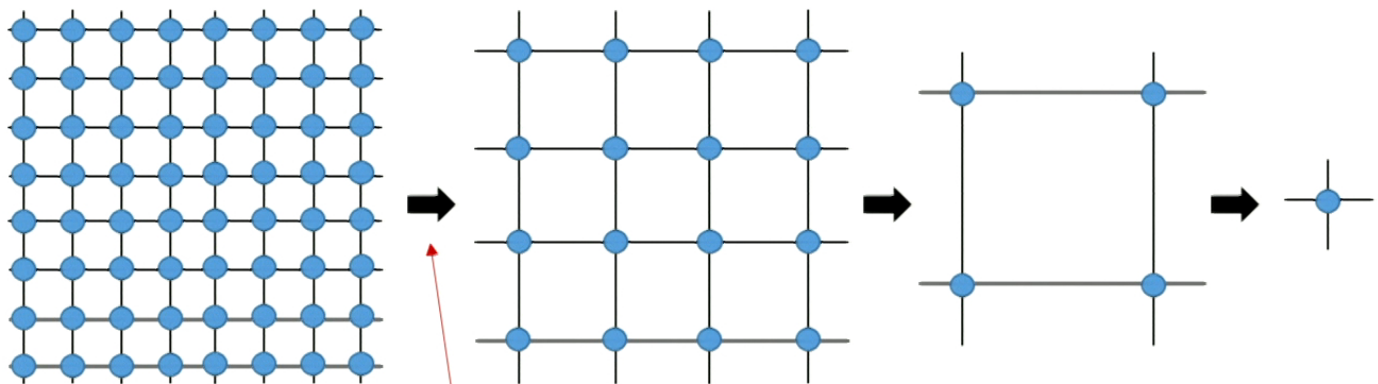

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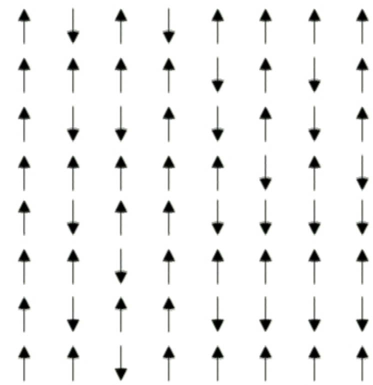

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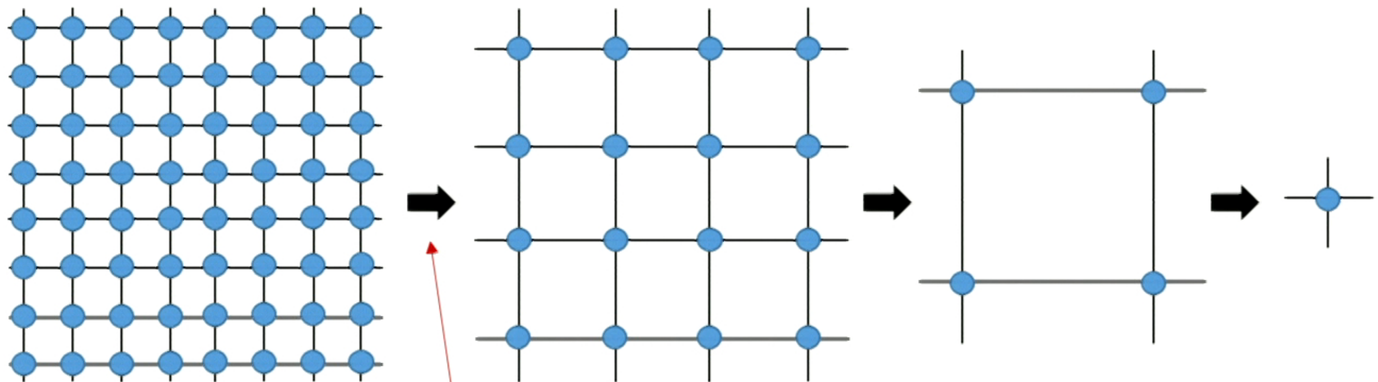
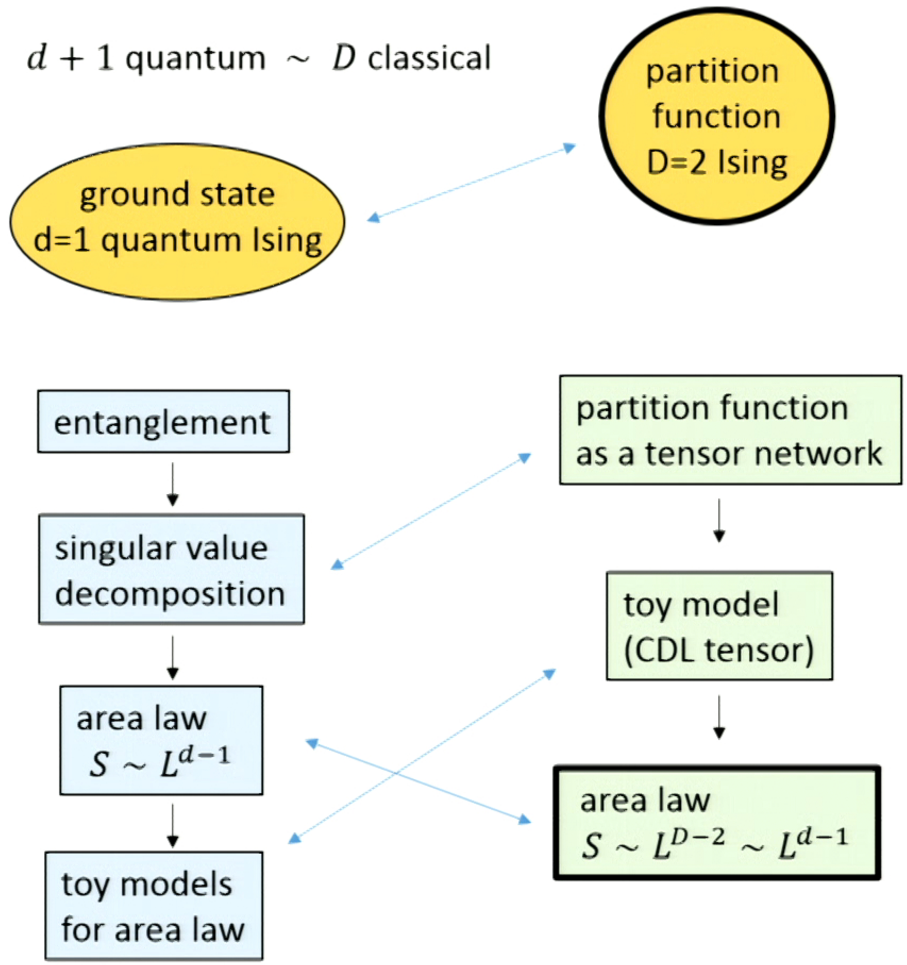


(1) tensor network for Z
(today)

(2) coarse-graining transformation
(tomorrow)

(2.a) efficient computation
(2.b) renormalization group flow

LECTURES 1 and 2



1d quantum spin chain

1 quantum spin ½

Hilbert space

$\mathbb{V} = \mathbb{C}_2$

wavefunction

$$|\Psi\rangle = \Psi_1|\uparrow\rangle + \Psi_2|\downarrow\rangle$$
$$= \Psi_1|1\rangle + \Psi_2|2\rangle$$

$\Psi_i \in \mathbb{C}$

2 complex parameters

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↑ ↓ ↑ ↓ ↑ ↑

Hilbert space

$$\mathbb{V}^{(N)} = \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \cdots \otimes \mathbb{C}_2$$

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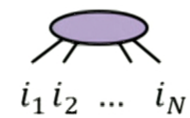

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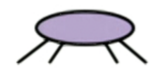

$i_1\, i_2\, \ldots\, i_N$

Quantum Hamiltonian H

example: transverse field Ising

$$H = -\sum_i \sigma_i^z \sigma_{i+1}^z + \sum_i \sigma_i^x$$

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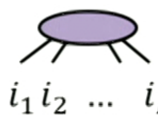

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Ground state $|\Psi\rangle$

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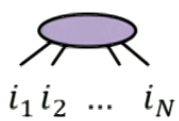

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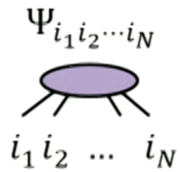
computational cost
$\exp(N)$

ground state of
local Hamiltonian



ground state of
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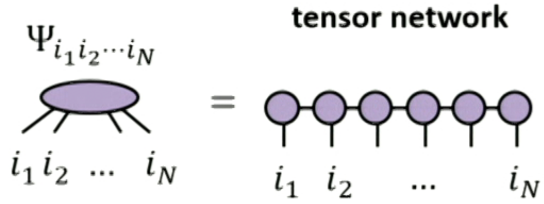

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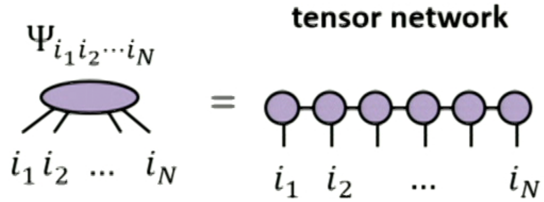

2^N

parameters

inefficient

ground state of
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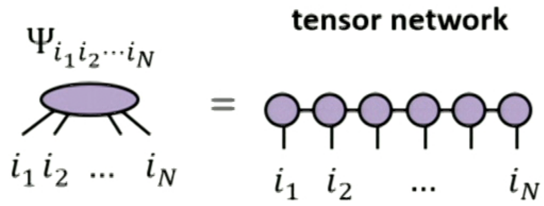

2^N
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inefficient

$O(N)$
parameters
efficient

ground state of
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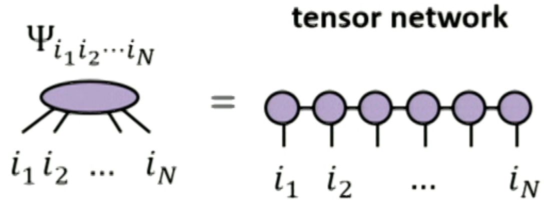

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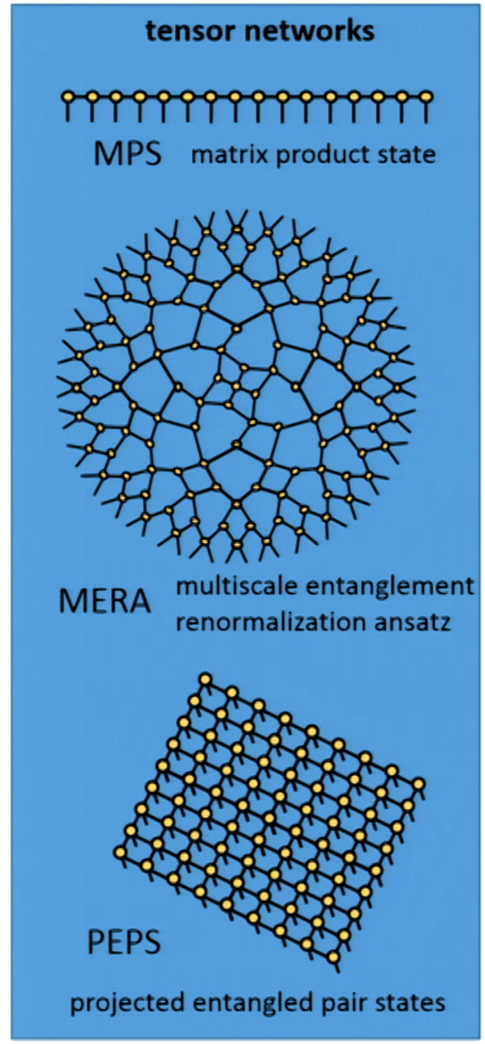
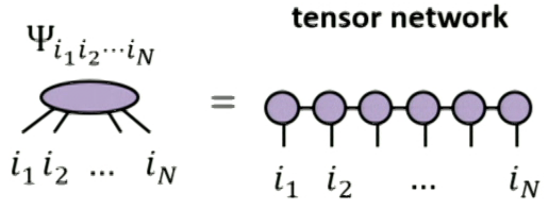


ground state of
local Hamiltonian

tensor network



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$O(N)$
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How is this possible?!

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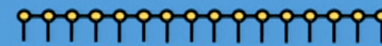
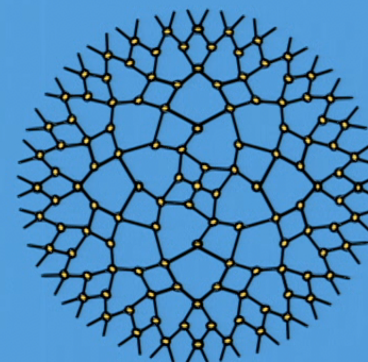
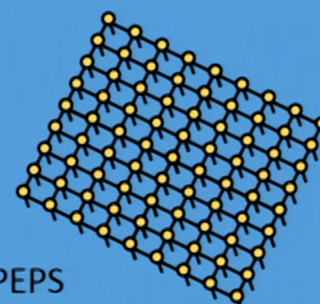
tensor networks



MPS matrix product state



MERA multiscale entanglement renormalization ansatz

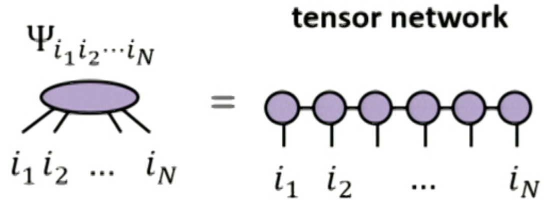


PEPS

projected entangled pair states

ground state of
local Hamiltonian

tensor network



2^N
parameters
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$O(N)$
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How is this possible?!

ground state
entanglement

AREA LAW

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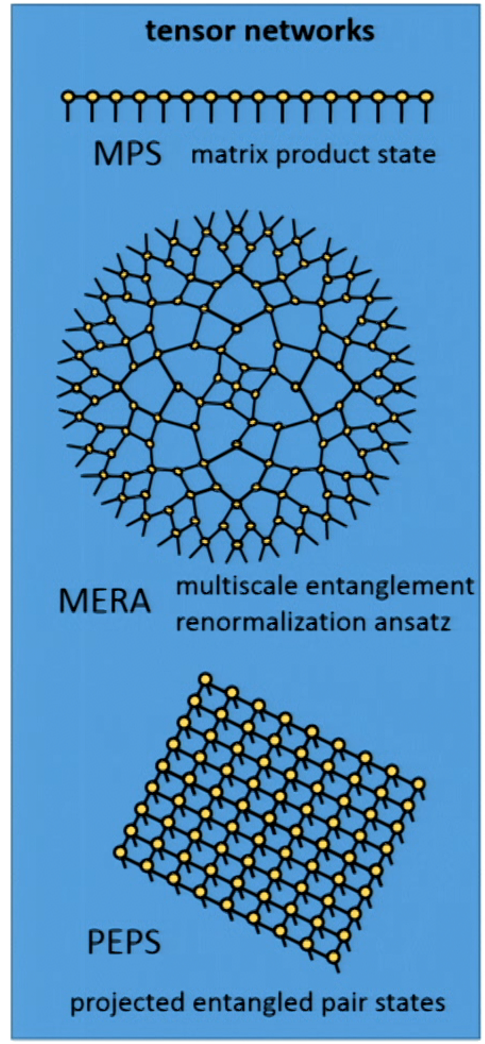

Entanglement

bipartite system

Hilbert space

$\mathbb{V}^A \otimes \mathbb{V}^B$

wavefunction

$$|\Psi\rangle = \sum_{ij} \Psi_{ij} \, |i^A\rangle \otimes |j^B\rangle$$

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def: product state

$$|\Psi\rangle = |\phi^A\rangle \otimes |\varphi^B\rangle$$

example

$$|\Psi\rangle = |1^A\rangle \otimes |1^B\rangle$$
$$= |1^A 1^B\rangle$$

Entanglement

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def: entangled state $$|\Psi\rangle \neq |\phi^A\rangle \otimes |\varphi^B\rangle$$

example $|\Psi\rangle = |1^A\rangle \otimes |1^B\rangle + |2^A\rangle \otimes |2^B\rangle$
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Entanglement

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Question: $|\Psi\rangle = |1^A 1^B\rangle + |1^A 2^B\rangle + |2^A 1^B\rangle + |2^A 2^B\rangle$ entangled?

Entanglement

bipartite system

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Question: $|\Psi\rangle = |1^A 1^B\rangle + |1^A 2^B\rangle + |2^A 1^B\rangle + |2^A 2^B\rangle$ entangled?

$= (|1^A\rangle + |2^A\rangle) \otimes (|1^B\rangle + |2^B\rangle)$

Entanglement

bipartite system

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Question: $|\Psi\rangle = |1^A 1^B\rangle + |1^A 2^B\rangle + |2^A 1^B\rangle + |2^A 2^B\rangle$ entangled?

$= (|1^A\rangle + |2^A\rangle) \otimes (|1^B\rangle + |2^B\rangle)$

Question: $|\Psi\rangle = |1^A 1^B\rangle + |1^A 2^B\rangle + |2^A 1^B\rangle - |2^A 2^B\rangle$ entangled?

matrix

$$M$$

singular value decomposition

$$M = USV^\dagger$$

$$U^\dagger U = I \quad V^\dagger V = I$$

$$S = \begin{pmatrix} s_1 & & & \\ & s_2 & & \\ & & \ddots & \\ & & & s_d \end{pmatrix}$$

$s_\alpha \geq 0$ singular values

matrix

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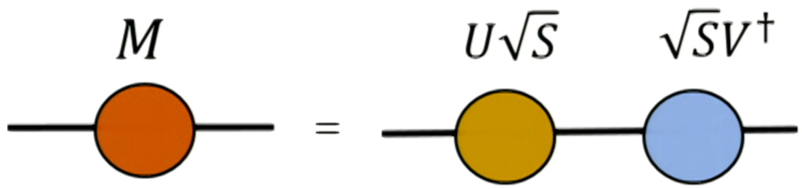

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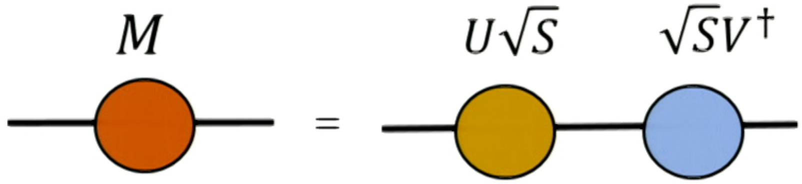

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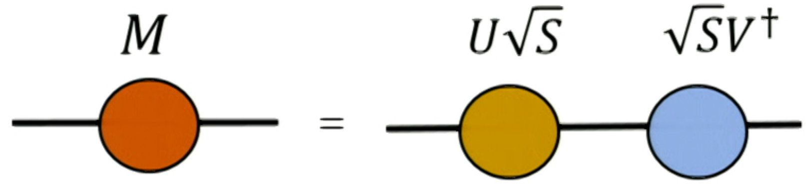

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Schmidt values
(=singular values)

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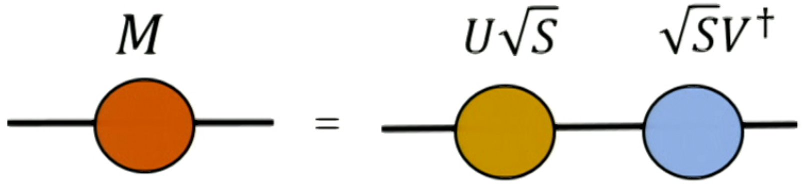

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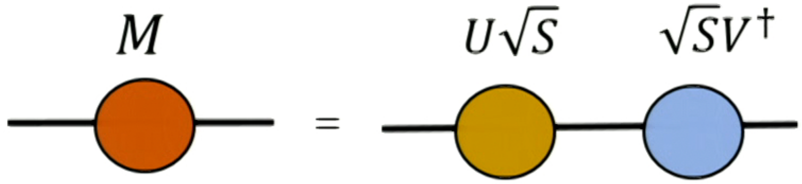

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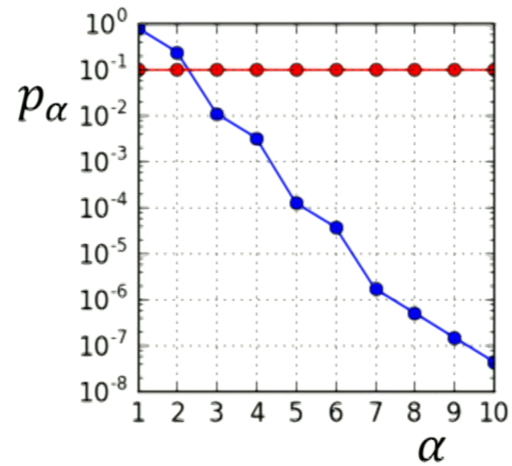

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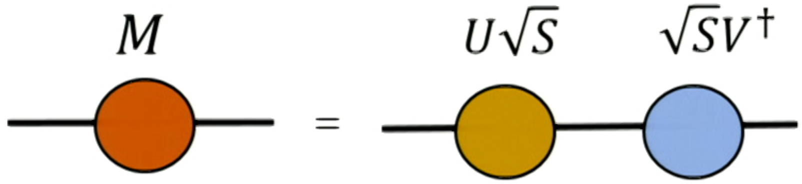

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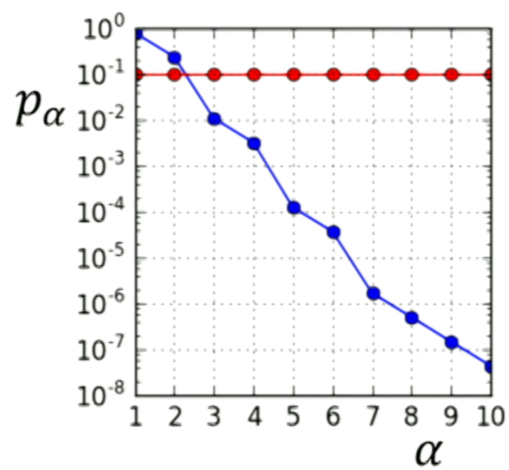

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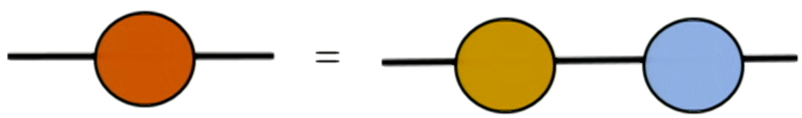

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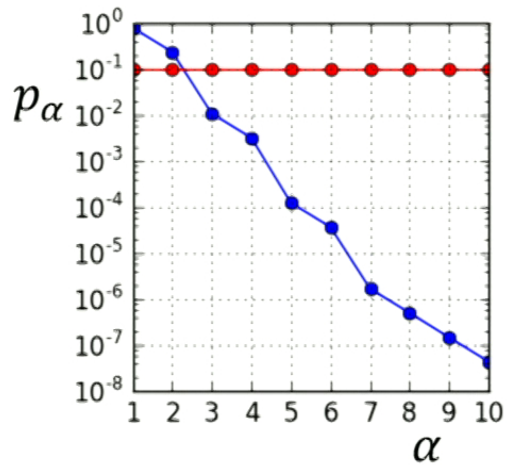

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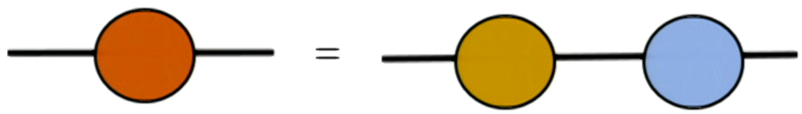

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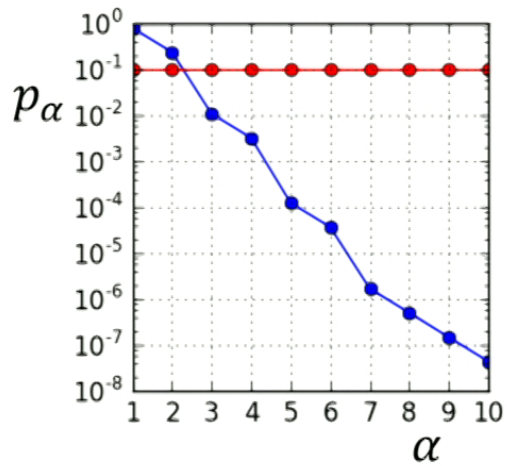

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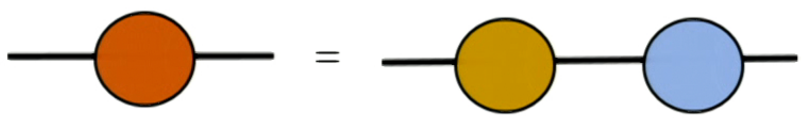
M $\quad$ $U\sqrt{S}$ $\quad$ $\sqrt{S}V^{\dagger}$



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Schmidt values
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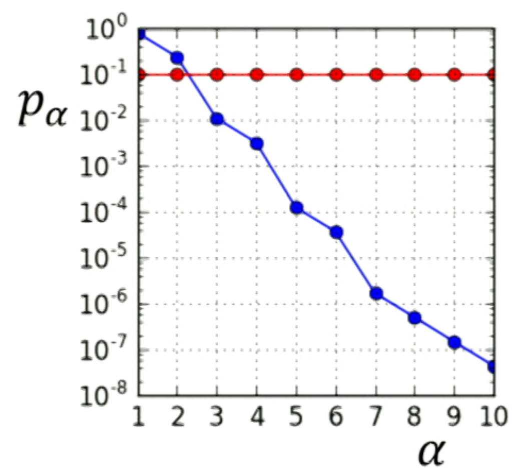

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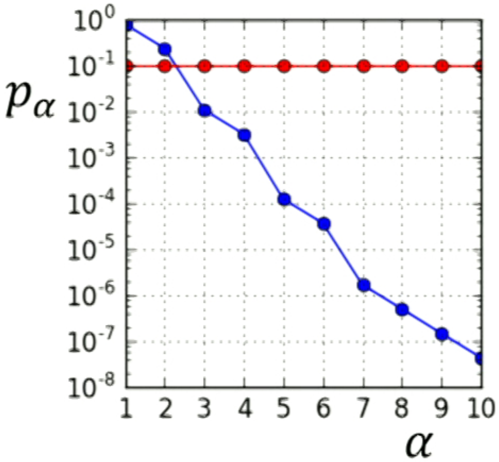
Truncation

Schmidt decomposition

$$|\Psi\rangle = \sum_{\alpha=1}^{10} \sqrt{p_\alpha}\, |\phi_\alpha^A\rangle \otimes |\varphi_\alpha^B\rangle$$

examples of singular values:



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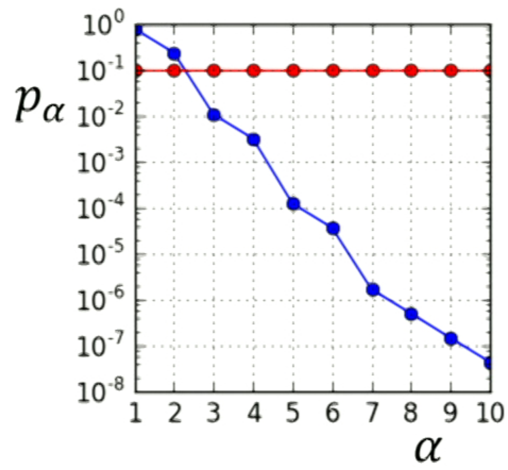
examples of singular values:



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approximation

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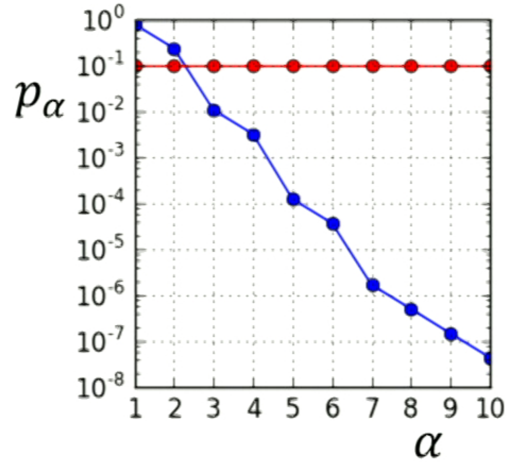

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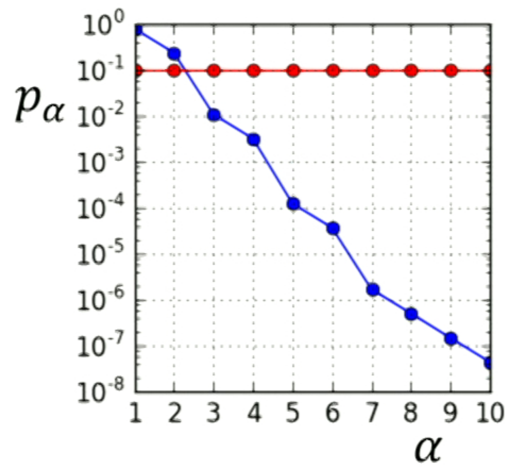

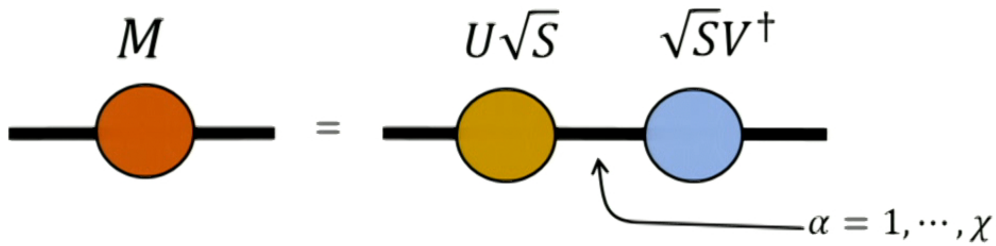

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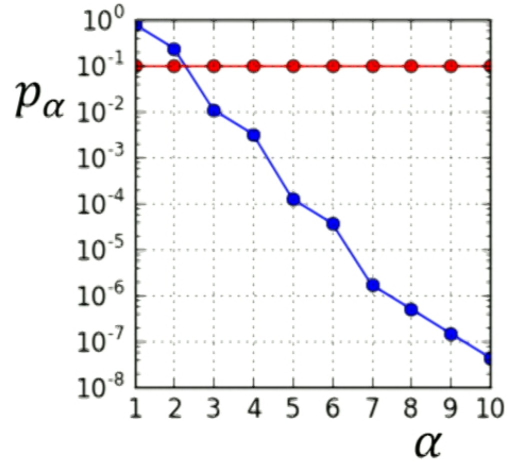
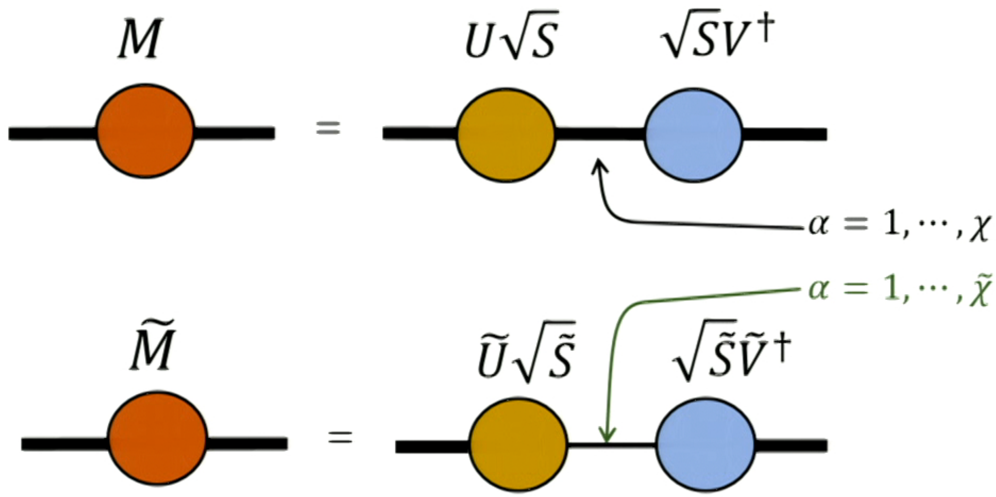
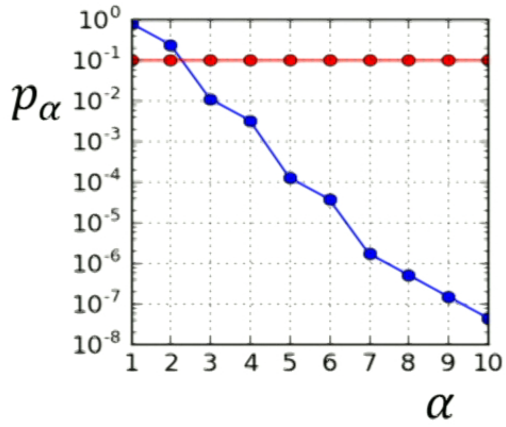




entanglement entropy as a measure of entanglement

(squared)
singular values $\{p_\alpha\}$

examples of singular values:



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$$S(\{p_\alpha\}) \equiv -\sum_\alpha p_\alpha \log_2(p_\alpha)$$

entanglement entropy

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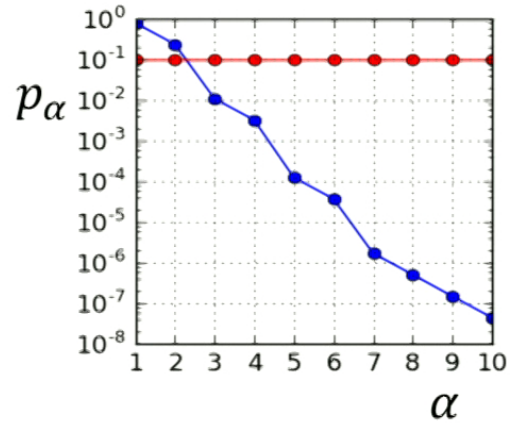

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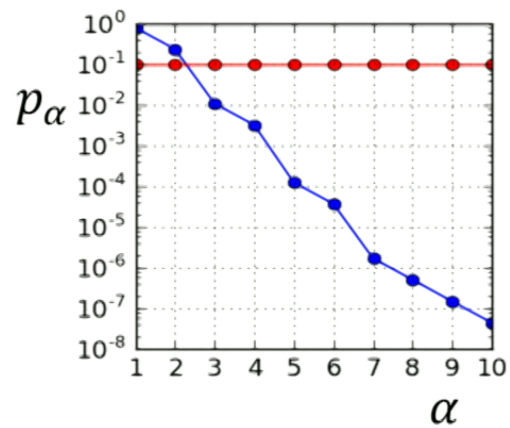

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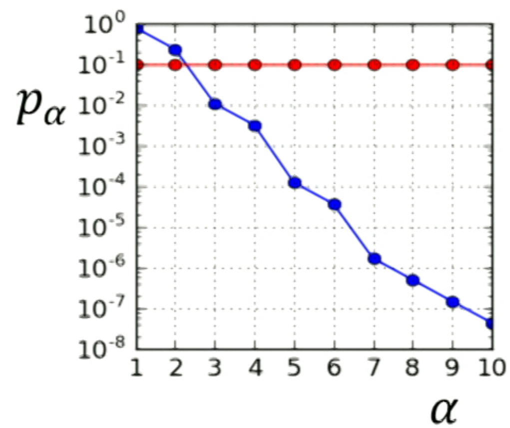

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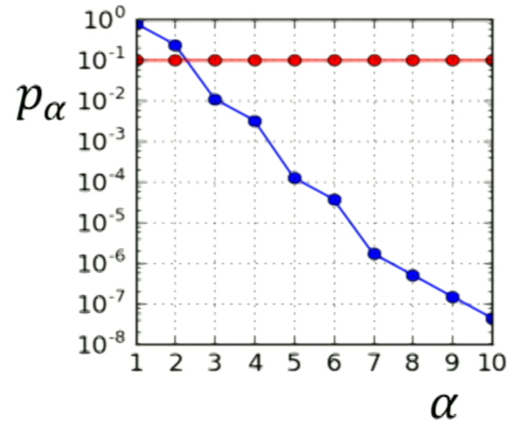

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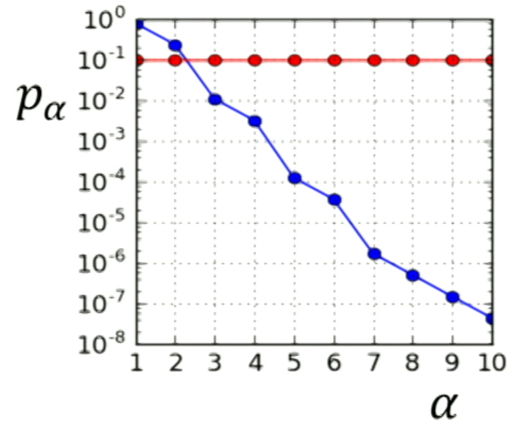

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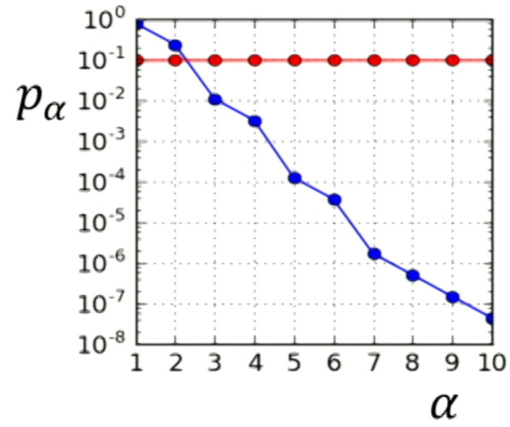

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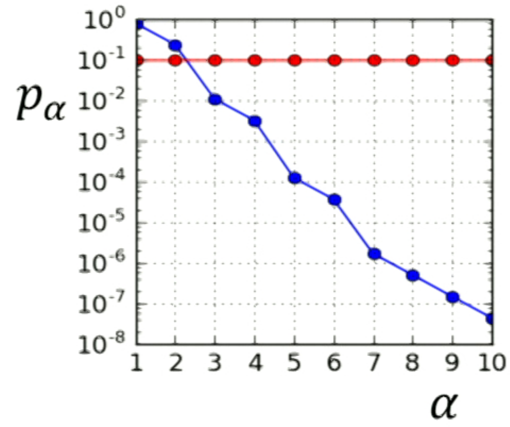

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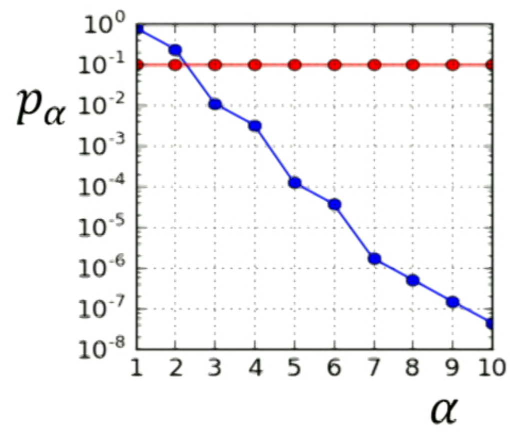

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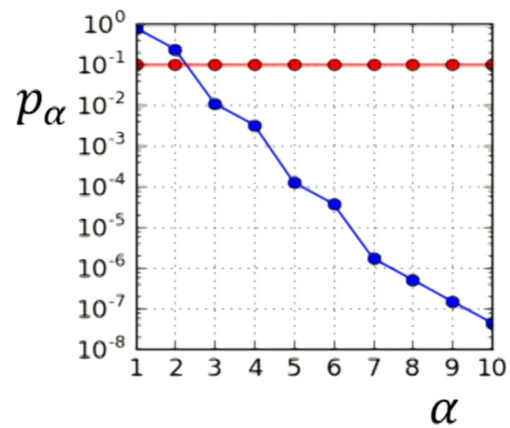

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entanglement entropy as a measure of entanglement

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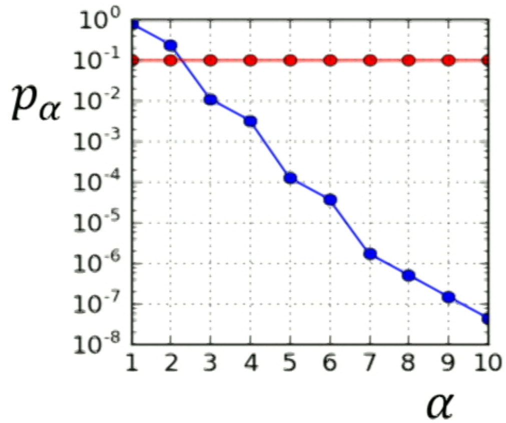

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product state $S = 0$ (no entanglement)

example 2: $|\Psi\rangle = \frac{1}{\sqrt{2}}(|1^A 1^B\rangle + |2^A 2^B\rangle)$ $\quad p_\alpha = \left(\frac{1}{2}, \frac{1}{2}\right)$ $\quad S = -2 \times \frac{1}{2}\log_2 \frac{1}{2}$

$$= -\log_2 \frac{1}{2} = \log_2 2 = 1$$

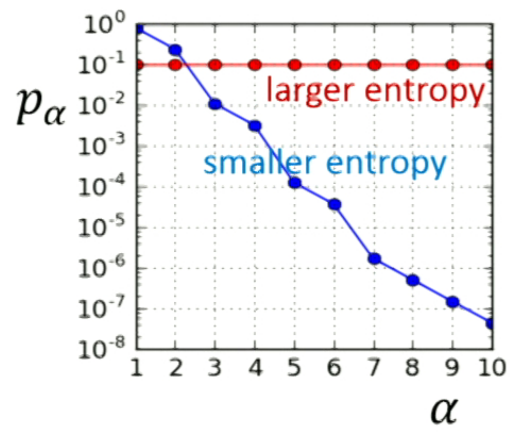
example 3: $|\Psi\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d} |i^A i^B\rangle$ $\quad p_\alpha = \left(\frac{1}{d}, \frac{1}{d}, \cdots, \frac{1}{d}\right)$ $\quad S = -d \times \frac{1}{d}\log_2 \frac{1}{d}$

QUESTION!

$$= -\log_2 \frac{1}{d} = \log_2 d$$

entanglement entropy as a measure of entanglement

examples of singular values:



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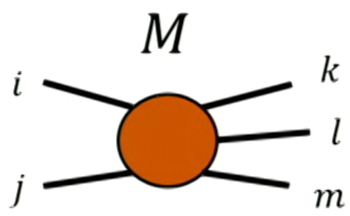
QUESTION!

tensor SVD

tensor

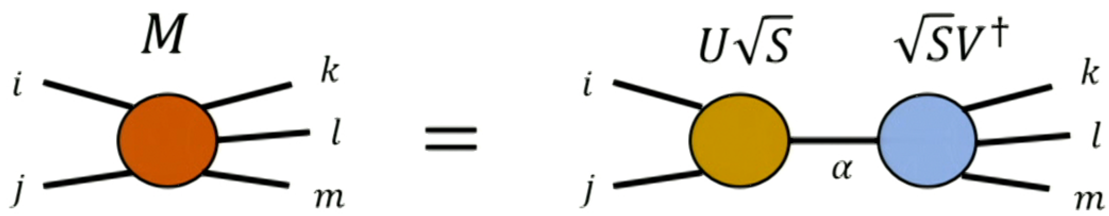
$$M_{ijklm}$$



tensor SVD

tensor — singular value decomposition

$$M_{ijklm} \qquad M_{(ij)(klm)} = \sum_{\alpha} U_{(ij)\alpha} S_{\alpha\alpha} \left(V^{\dagger}\right)_{\alpha(klm)}$$



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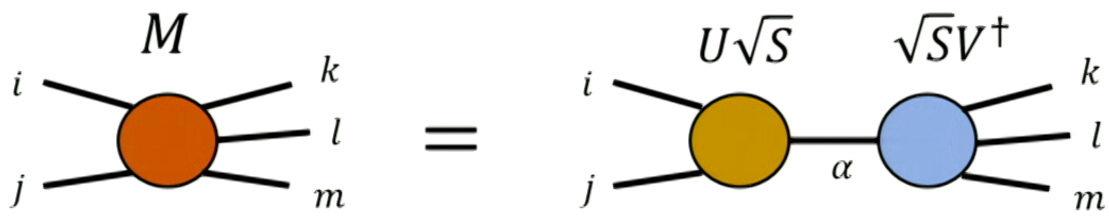

wavefunction

$$|\Psi\rangle = \sum_{ijklm} \Psi_{ijklm} \, |i^A j^B k^C l^D m^E\rangle$$

Schmidt decomposition

$$|\Psi\rangle = \sum_{\alpha} \sqrt{p_\alpha} \, |\phi_\alpha^{AB}\rangle \otimes |\varphi_\alpha^{CDE}\rangle$$

$$\sqrt{p_\alpha} \geq 0$$

Schmidt values
(=singular values)

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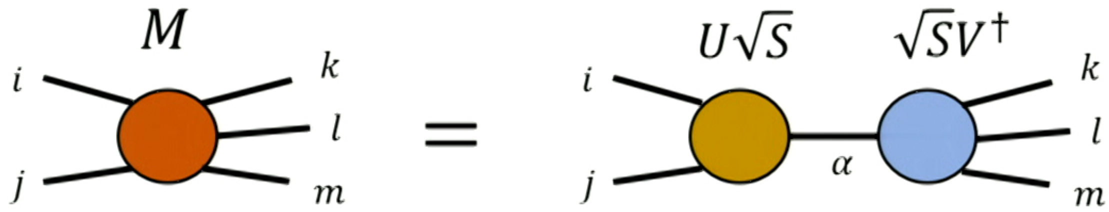

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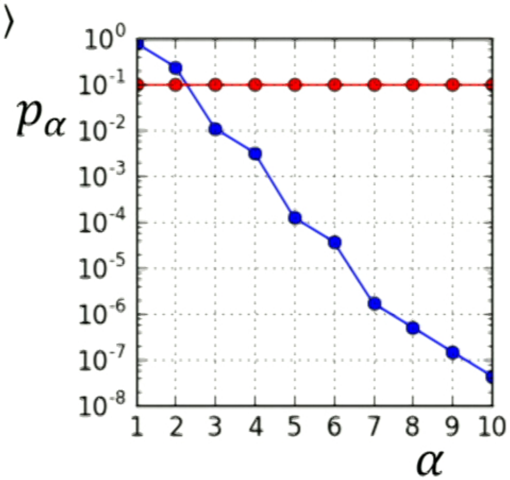
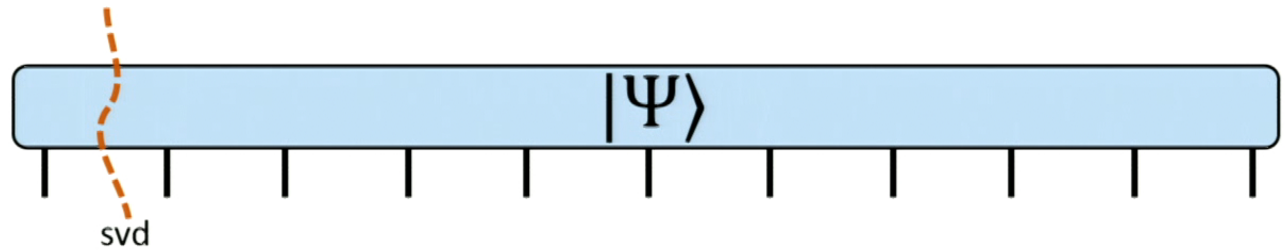
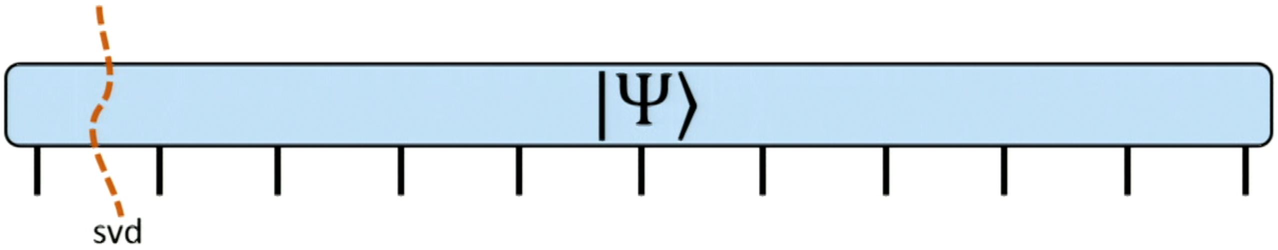



|Ψ⟩
svd



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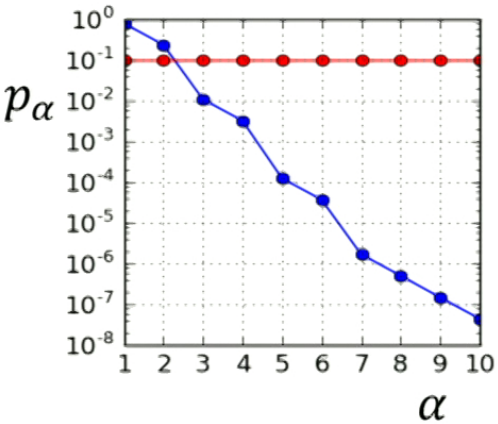

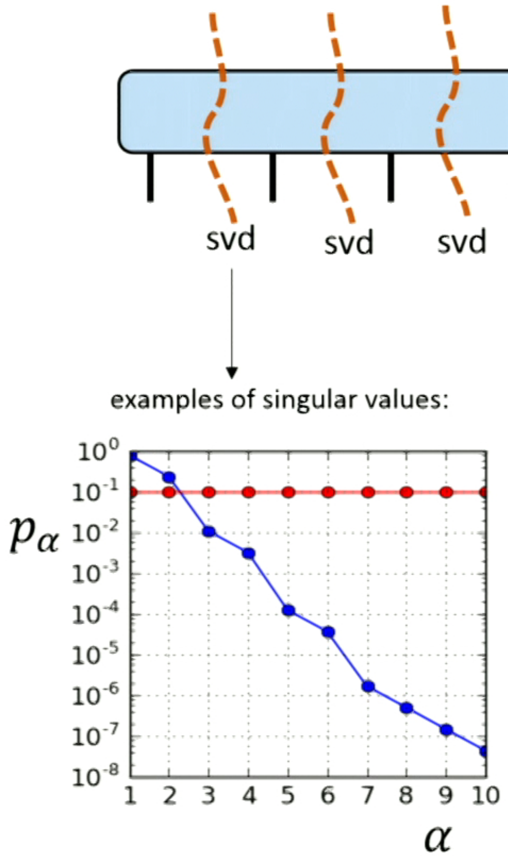
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p_α
10^0
10^{-1}
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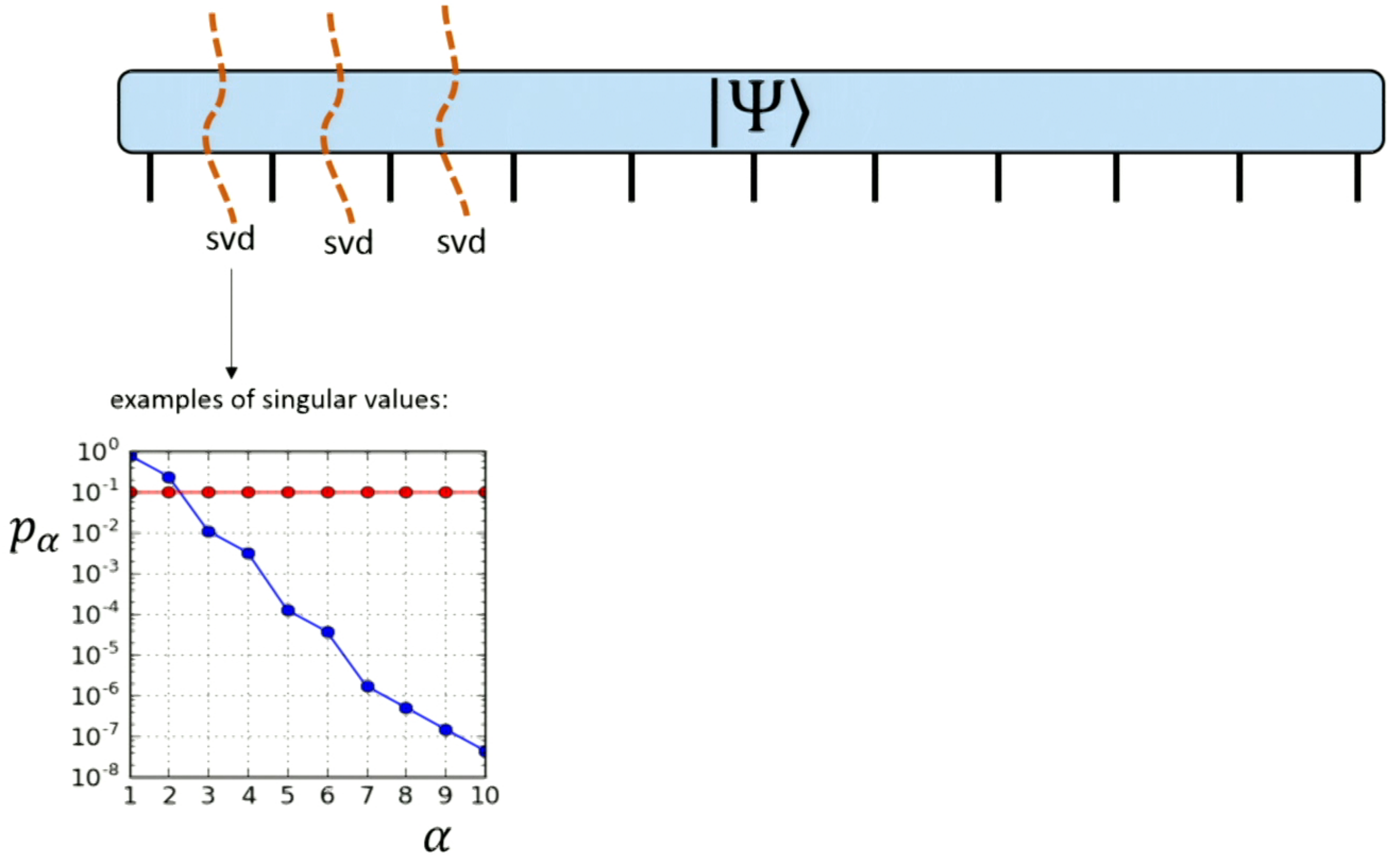
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
1 2 3 4 5 6 7 8 9 10
α



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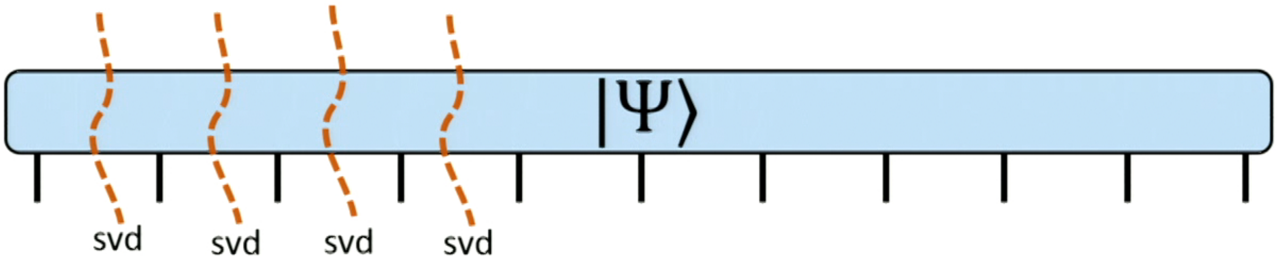
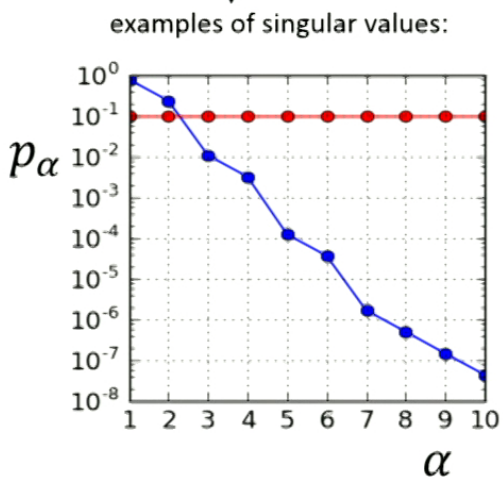
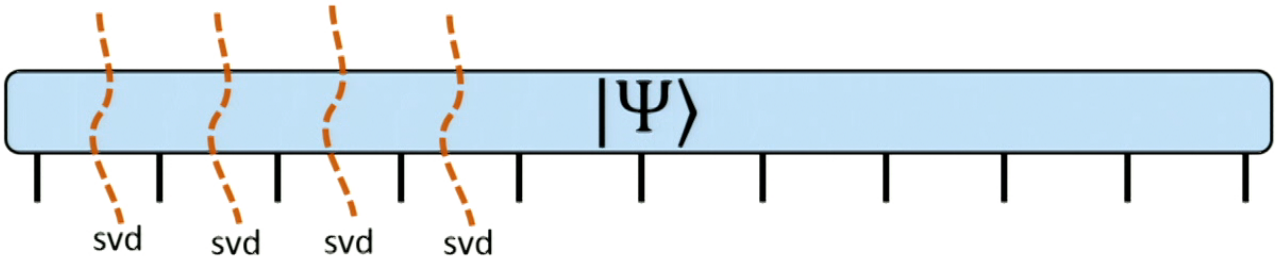




for each *entanglement cut*,

(squared)
singular values $\{p_\alpha\}$



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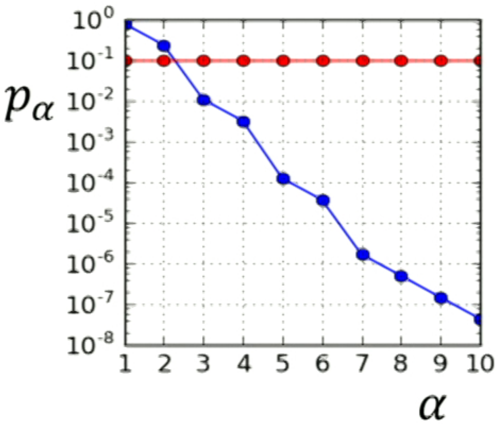

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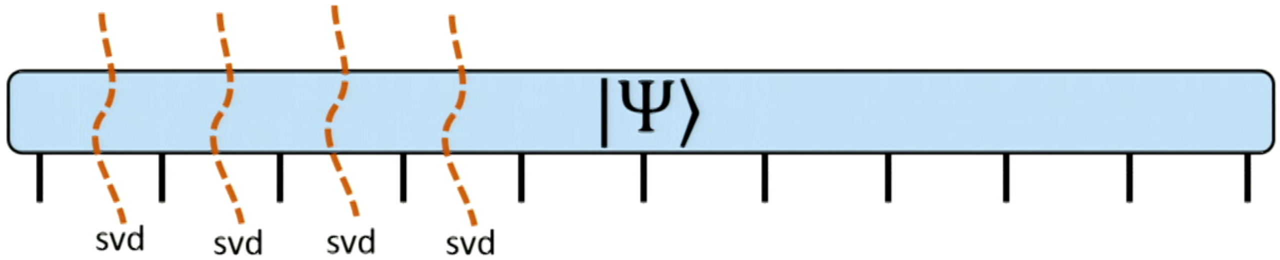

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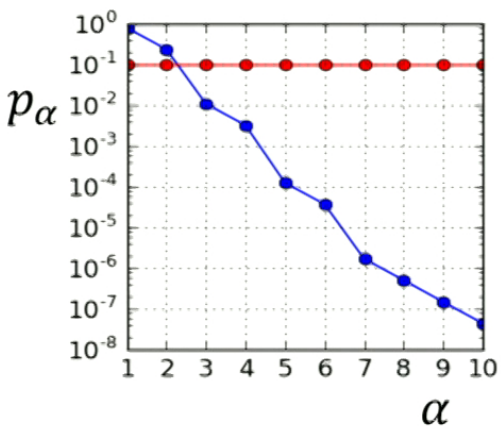

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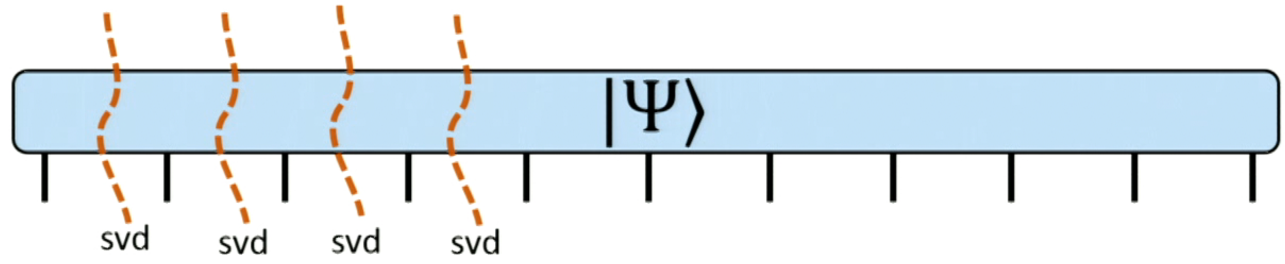
entanglement entropy



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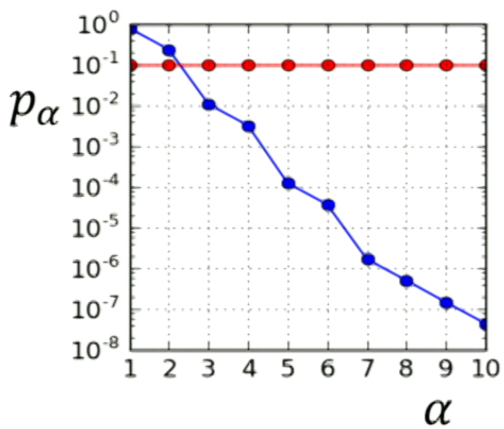

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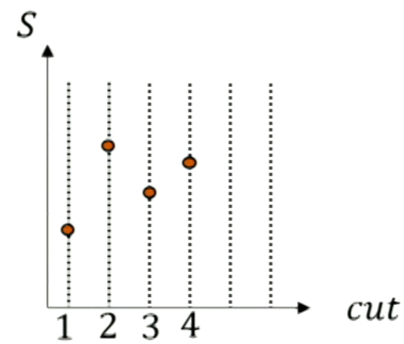

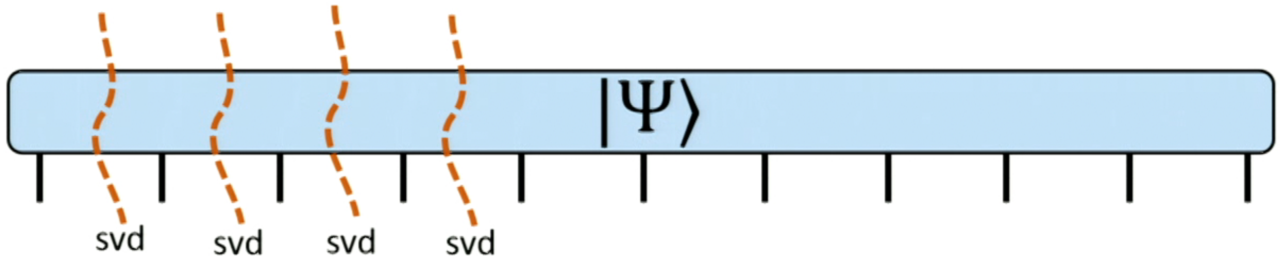

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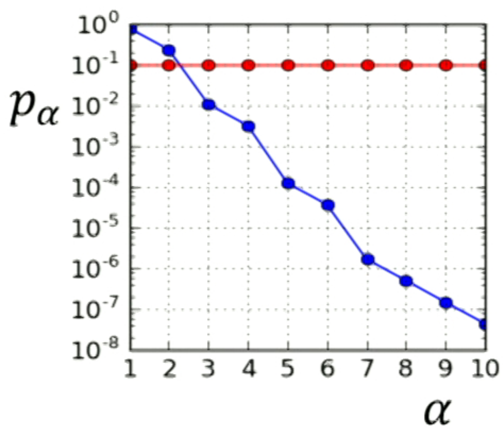

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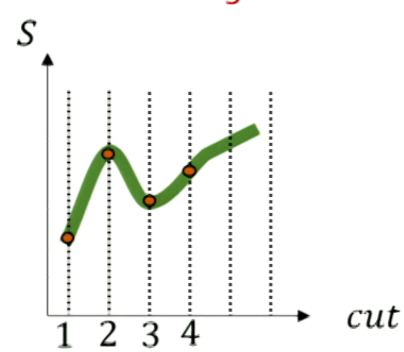

N quantum spins ½

Hilbert space $\mathbb{V}^{(N)} = \mathbb{C}_2 \otimes \mathbb{C}_2 \otimes \cdots \otimes \mathbb{C}_2$

$$|\Psi\rangle = \sum_{i_1,\cdots,i_N} \Psi_{i_1 i_2 \cdots i_N} |i_1 i_2 \cdots i_N\rangle$$

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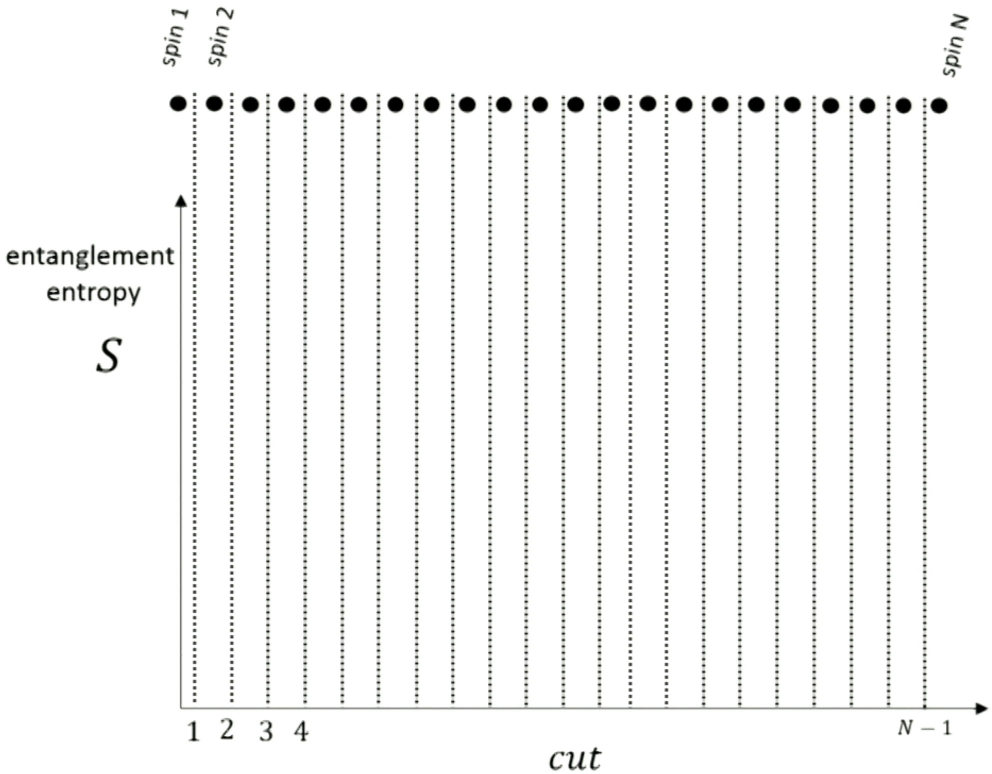

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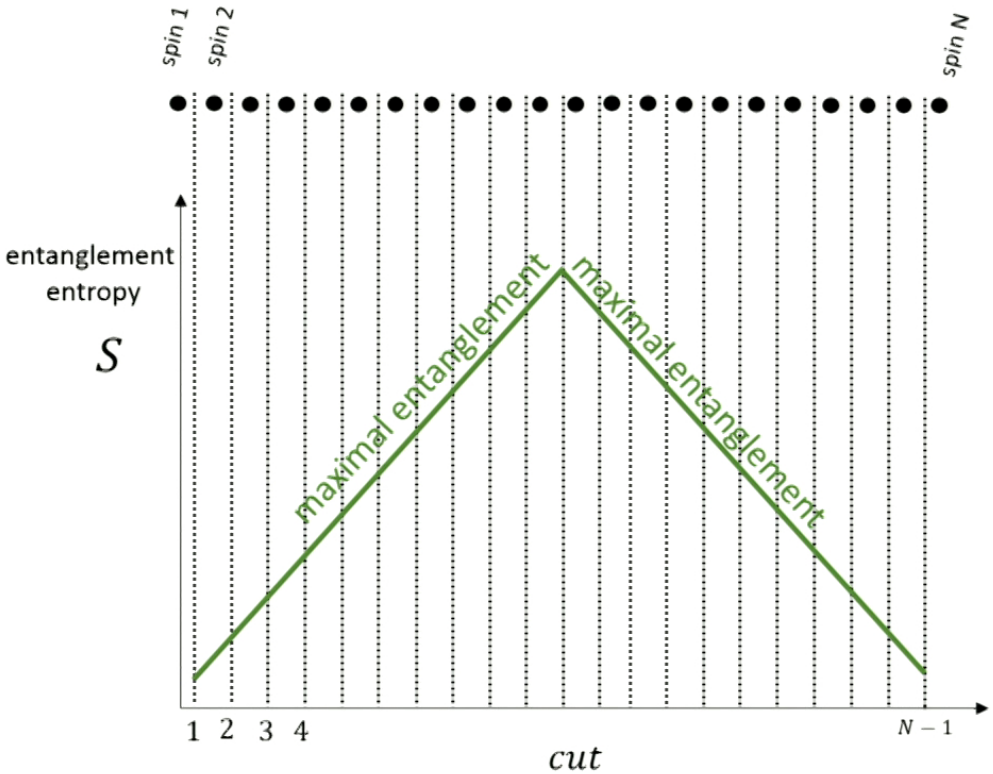

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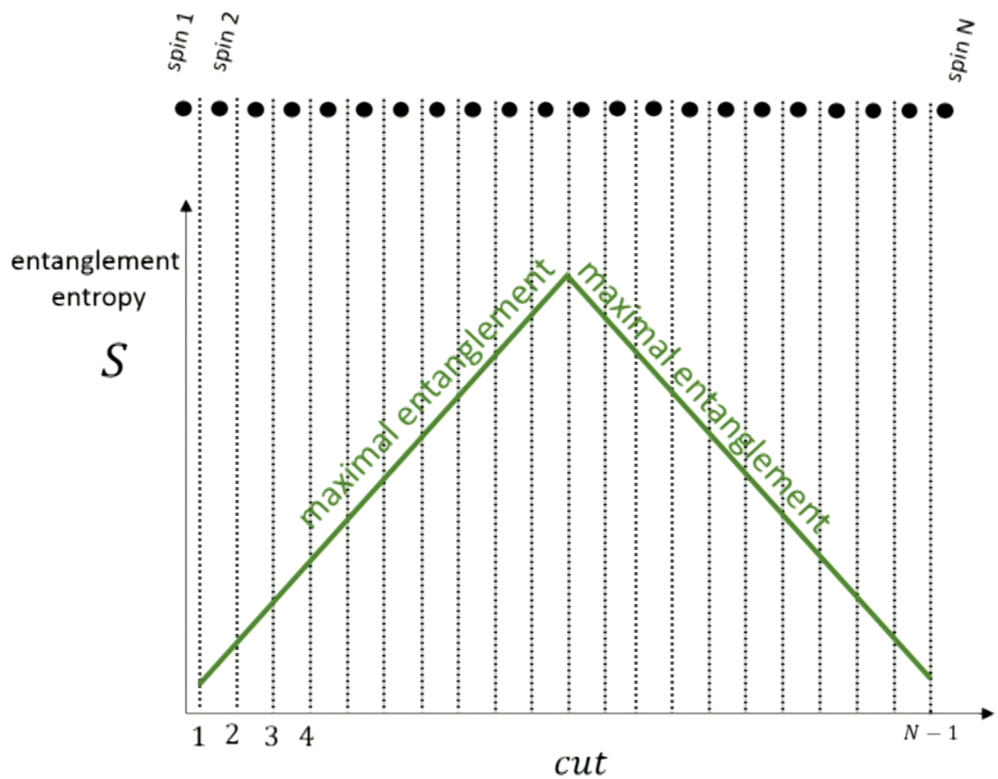

generic state

2^N random complex coefficients $\Psi_{i_1 i_2 \cdots i_N}$

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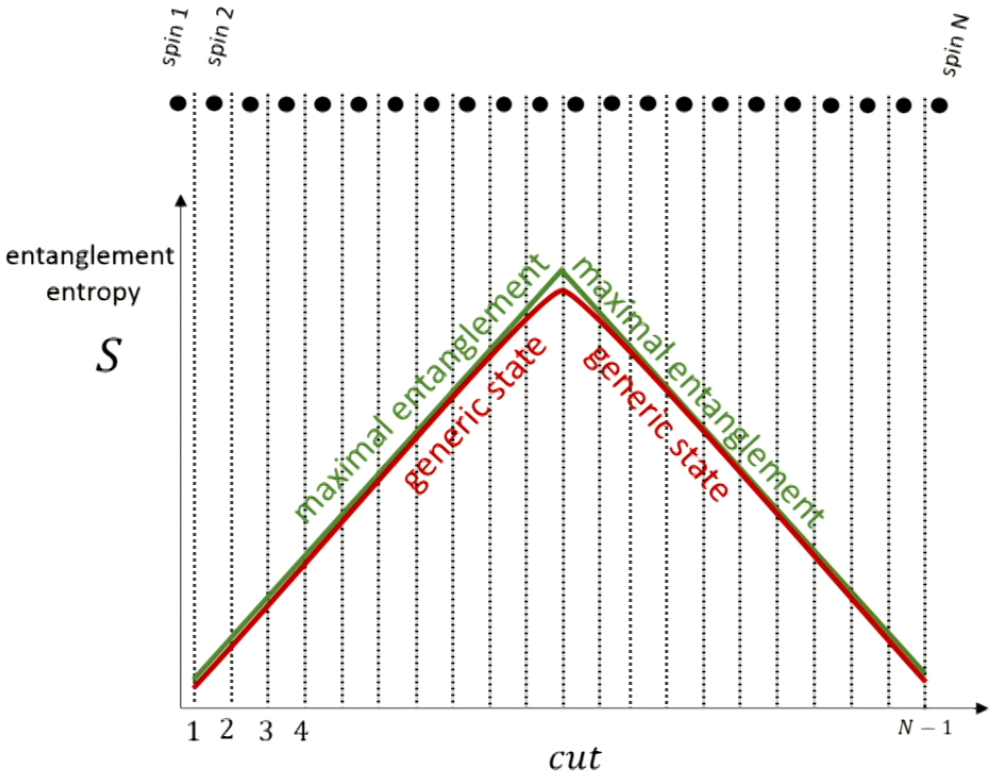

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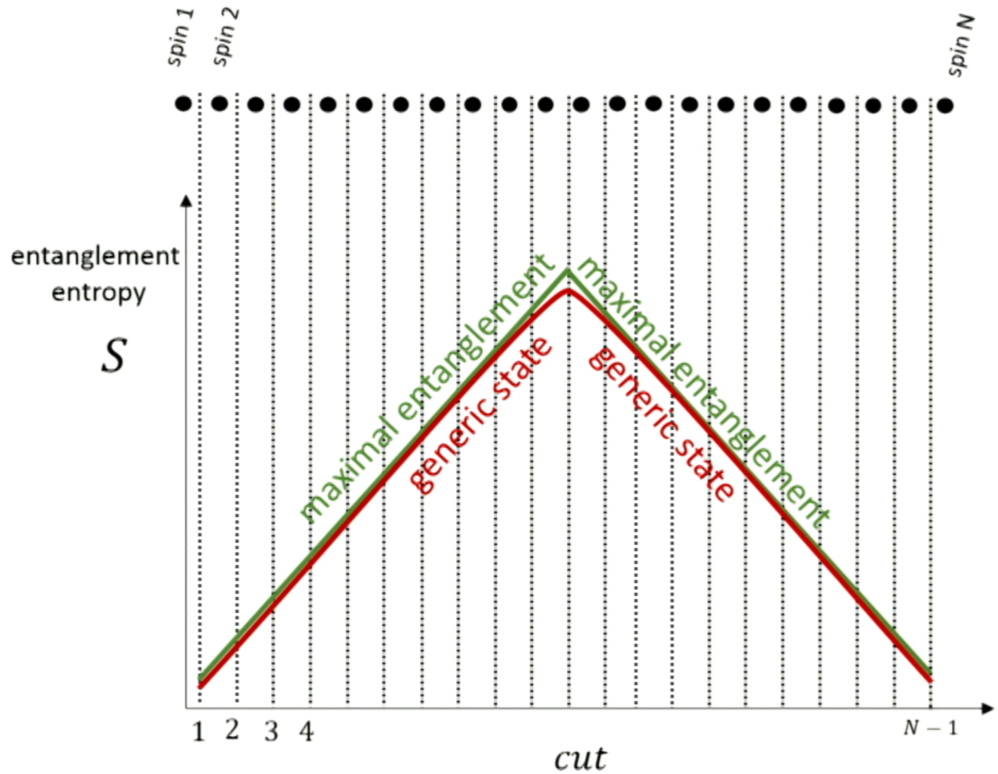

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ground state of local Hamiltonian

$H|\Psi\rangle = E_0|\Psi\rangle$

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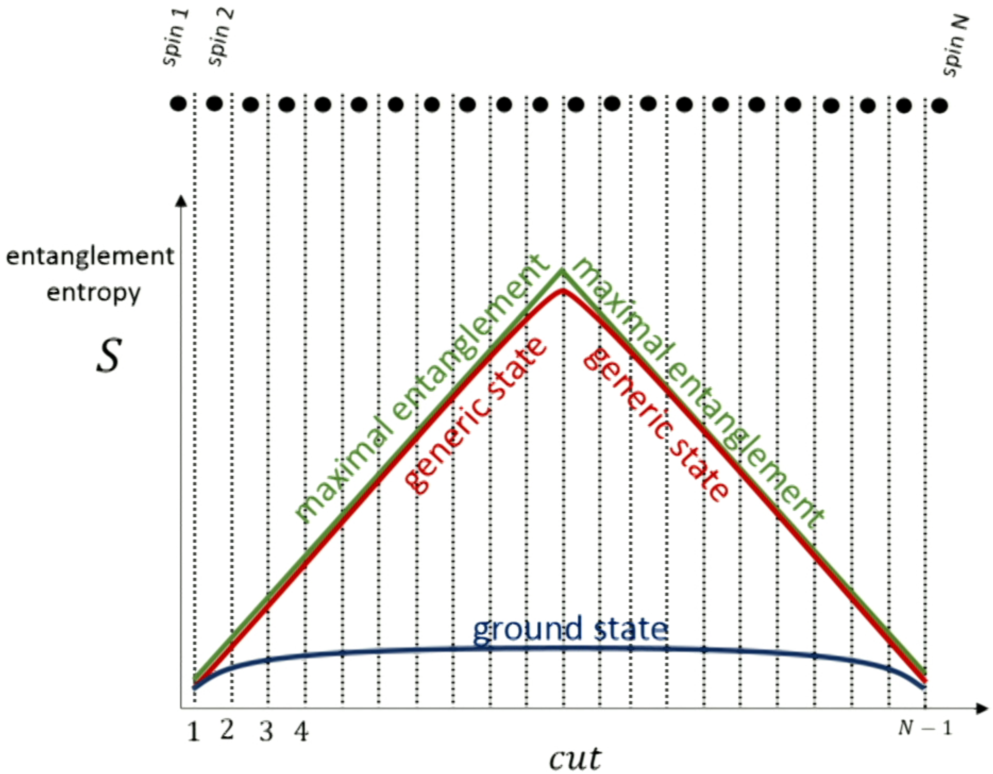

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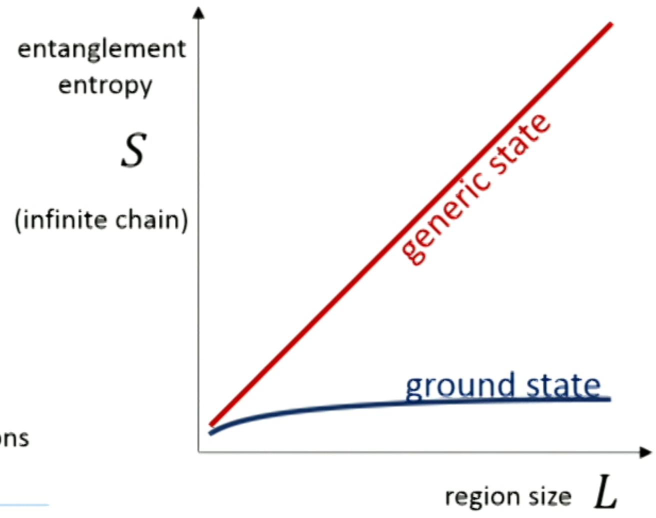
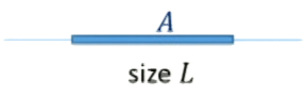
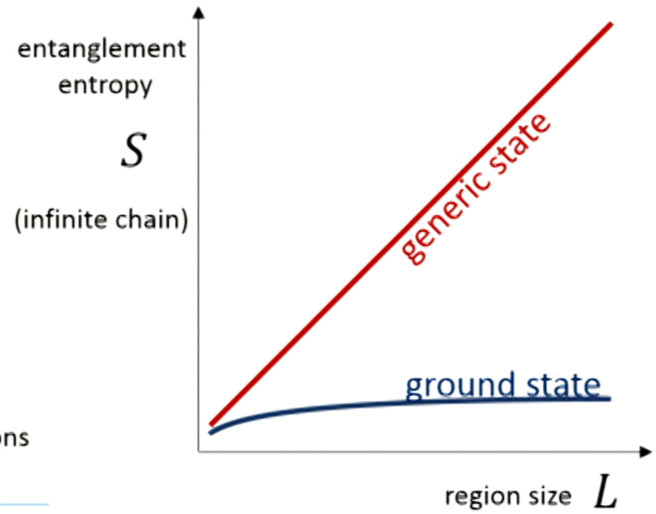
Area law



$d = 1$ space dimensions



Area law



generic state

$S(A) \sim L$

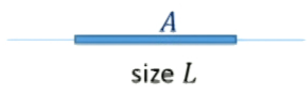
volume law

ground state

$S(A) \sim L^0$

area law

$d = 1$ space dimensions

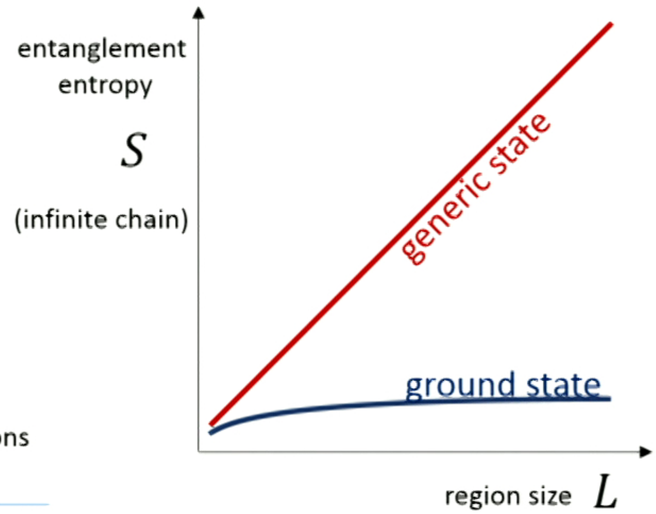


ground state

$S(A) \sim |\partial A| \sim L^{d-1}$

area law

Area law



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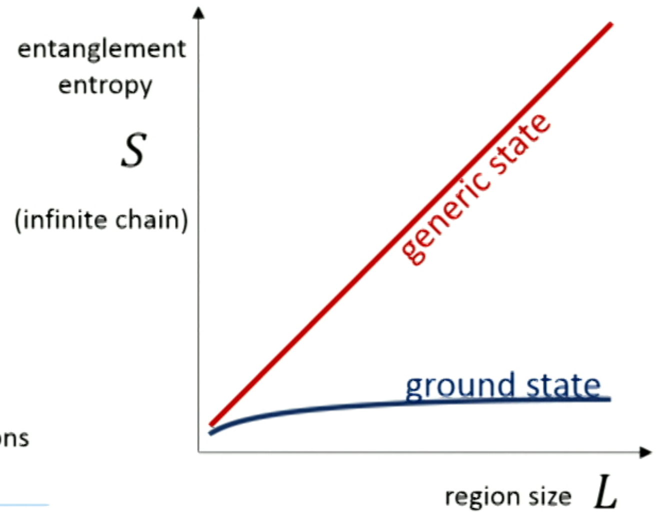


generic state

$S(A) \sim L$

volume law

ground state

$S(A) \sim L^0$

area law

$d = 1$ space dimensions

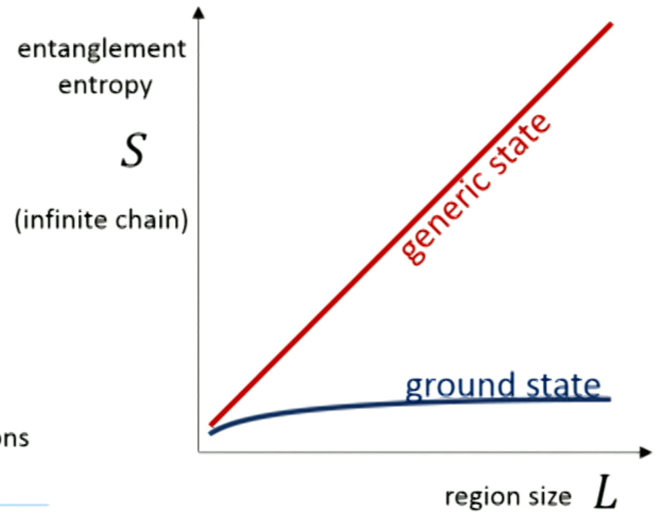
region A of linear size L

ground state

$$S(A) \sim |\partial A| \sim L^{d-1}$$

area law

Area law



generic state

$$S(A) \sim L$$

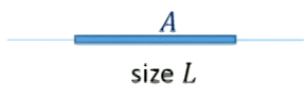
volume law

ground state

$$S(A) \sim L^0$$

area law

$d = 1$ space dimensions



region A of linear size L

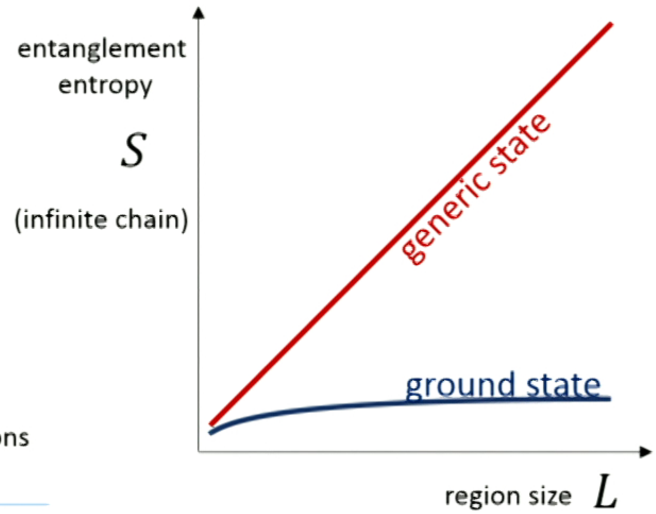
size of boundary of region A

ground state

$$S(A) \sim |\partial A| \sim L^{d-1}$$

area law

Area law



generic state

$S(A) \sim L$

volume law

ground state

$S(A) \sim L^0$

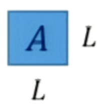
area law

$d = 1$ space dimensions

region A of linear size L

size of boundary of region A

$d = 2$ space dimensions



ground state $S(A) \sim L^1$

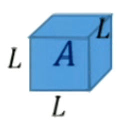
generic state $S(A) \sim L^2$

ground state

$S(A) \sim |\partial A| \sim L^{d-1}$

area law

$d = 3$ space dimensions



ground state $S(A) \sim L^2$

generic state $S(A) \sim L^3$

generic state

$S(A) \sim |A| \sim L^d$

volume law