Title: Spectrum of conformal gauge theories on a torus

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Abstract: <p>Many model quantum spin systems have been proposed to realize critical points or phases described by 2+1 dimensional conformal gauge theories. On a torus of size L and modular parameter Ï„, the energy levels of such gauge theories equal (1/L) times universal functions of Ï„. We compute the universal spectrum of QED3, a U(1) gauge theory with Nf two-component massless Dirac fermions, in the large-Nf limit. We also allow for a Chern-Simons term at level k, and show how the topological k-fold ground state degeneracy in the absence of fermions transforms into the universal spectrum in the presence of fermions; these computations are performed at fixed Nf/k in the large-Nf limit.</p>

Spectrum of conformal gauge theories on a torus

Alex Thomson

AT, S. Sachdev, Phys. Rev. B 95, 205128 (2017)
arXiv:1607.05279

QED$_3$: a critical spin liquid

- QED_3 is a strongly coupled conformal field theory (CFT) in $2{+}1d$ describing a U(1) gauge boson coupled to N_f two-component Dirac spinors:

$$\mathcal{L}_{\text{qed}} = -\bar{\psi}_\alpha i\gamma^\mu \left(\partial_\mu - iA_\mu\right)\psi_\alpha + \frac{1}{4e^2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2$$

It has a SU(N_f) symmetry under which the fermion flavours rotate into one another.

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  It has a SU(N_f) symmetry under which the fermion flavours rotate into one another.

- It describes a gapless, critical, U(1) spin liquid

  ⇒ Might describe:
  - the spin-1/2 kagome Heisenberg antiferromagnet
  - the J_1-J_2 antiferromagnet on the triangle lattice.

- It is also the proposed critical theory of a number of deconfined critical points.

Universality

- QED$_3$ is a conformal field theory and is therefore characterized by a number of *universal* properties
  - Depends only on IR properties, no UV dependence

- Torus spectrum may be useful to help identify the CFT in a numerical simulation.

  ⇒ Should depend only on IR cutoffs such as size and shape of torus, not the details of the lattice being simulated.

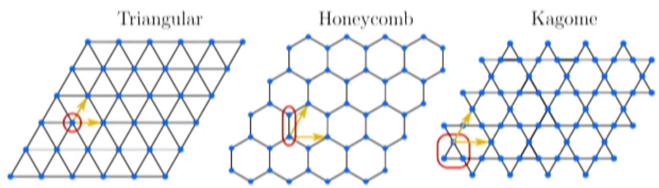
Previous work has been done for the Ising & O(N) CFTs...

Schuler et al., PRL 117, 210401 (2016)
Whitsitt, Sachdev, Phys. Rev. B 94, 085134 (2016)
Whitsitt et al., arXiv:1701.03111

Ising spectrum on torii

Critical Ising model: $H_{\mathrm{TFI}} = -\sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z - g_c \sum_i \sigma_i^x$

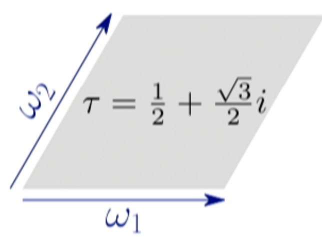
ED and QMC on multiple lattices:

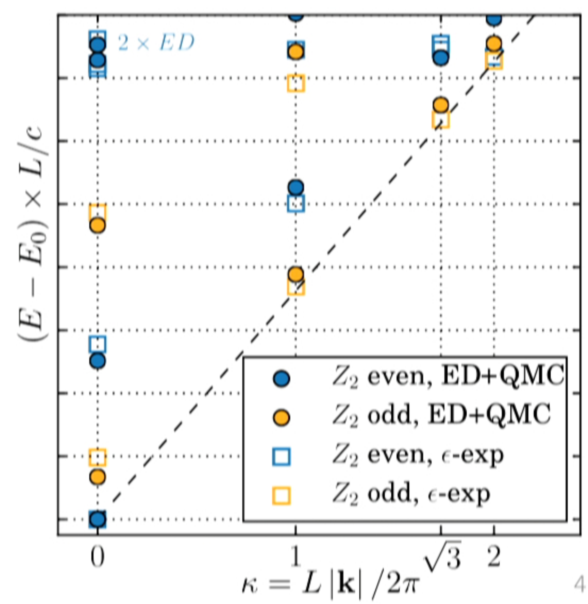


UV data

Spectrum depends only on modular parameter, τ, length L, and speed of light c.

IR data





Schuler, Whitsitt, Henry, Sachdev, Läuchli, PRL 117, 210401 (2016)

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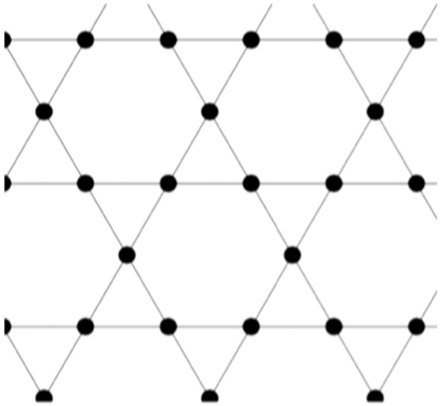
Plan

- Kagome antiferromagnet and QED_3
- Path integral approach to torus spectrum
  - QED_3 in the large-N_f limit
- QED_3 Spectrum

Kagome antiferromagnet

$$H_{\rm H} = J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$$



✧ Parton construction:

$$\mathbf{S}_i = \frac{1}{2} f_{i\alpha}^\dagger \boldsymbol{\sigma}_{\alpha\beta} f_{i\beta}, \quad 1 = \sum_\alpha f_{i\alpha}^\dagger f_{i\alpha}$$

$$\Rightarrow H_{\rm H} = -\frac{J}{2} \sum_{\langle ij \rangle} f_{i\alpha}^\dagger f_{j\alpha} f_{i\beta}^\dagger f_{j\beta}$$

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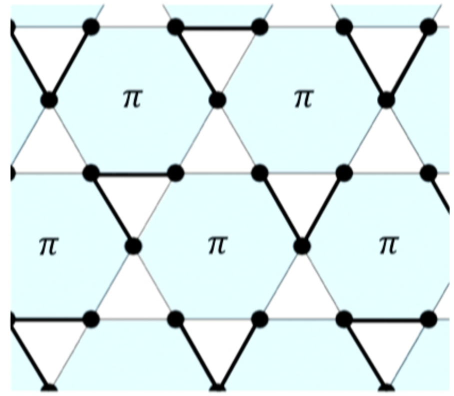
Kagome antiferromagnet

$$H_{\rm H} = J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$$



- Parton construction:

$$\mathbf{S}_i = \frac{1}{2} f^\dagger_{i\alpha} \boldsymbol{\sigma}_{\alpha\beta} f_{i\beta}, \quad 1 = \sum_\alpha f^\dagger_{i\alpha} f_{i\alpha}$$

$$\Rightarrow H_{\rm H} = -\frac{J}{2} \sum_{\langle ij \rangle} f^\dagger_{i\alpha} f_{j\alpha} f^\dagger_{i\beta} f_{j\beta}$$

- Mean field theory: decouple the 4-fermion interaction with a Hubbard-Stratonovich transformation.

$$H_{\rm MF} = -\sum_{ij} t_{ij} f^\dagger_{i\alpha} f_{j\alpha} + H.c. \qquad t_{ij} = J \left\langle f^\dagger_{i\alpha} f_{j\alpha} \right\rangle$$

- Ansatz with $\boldsymbol{\pi}$-flux per hexagon gives 2 Dirac cones per spin:

$$\mathcal{L}_{\rm MF} = -\bar{\psi}_{\sigma a} \gamma^\mu \partial_\mu \psi_{\sigma a} \qquad \underbrace{\sigma = \uparrow, \downarrow}_{\text{spin}} \quad \underbrace{a = \pm}_{\text{valley: } \pm \boldsymbol{Q}}$$

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Critical spin liquid

- Spin operators invariant under $f_{i\alpha} \to e^{i\phi_i} f_{i\alpha}$: the slave fermions carry a gauge charge.

$$\mathcal{L}_{\text{qed}} = -\bar{\psi}_\alpha i\gamma^\mu \left(\partial_\mu - iA_\mu\right)\psi_\alpha + \frac{1}{4e^2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2$$

- QED_3 with N_f=4 is strongly coupled. We work in the N_f=∞ limit.

- We can also add a Chern-Simons (CS) term to the action:

$$\mathcal{L}_{\text{CS}} = \frac{ik}{4\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho$$

This CFT can describe certain phase transitions between topological phases.

Rantner, Wen (2001); Wen (2002)
Hastings (2001); Hermele et al. (2007); Hermele et al. (2008)

Path integral definition on square torus

Partition function

$$Z = \frac{1}{\mathrm{Vol}(G)} \int DA\, D\psi\, e^{-S[A,\psi]}$$

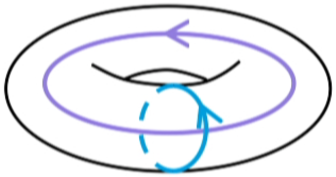
$\mathrm{Vol}(G)$ = gauge group volume

✧ Decompose gauge field as $A_\mu = a_\mu + B_\mu + \partial_\mu \phi$ $\quad \partial^2 \phi = 0$

- a_μ is the zero momentum mode: not pure gauge on the torus

$$a_\mu \to a_\mu + \frac{2\pi}{L}$$

⇒ a_μ is a periodic constant, $a_\mu \in [0, 2\pi/L)$



- B_μ is the finite-momentum gauge-fixed potential: $\partial^\mu B_\mu = 0$.

- $\partial_\mu \phi$ parametrizes an extra gauge degree of freedom.

Klebanov et al. (2012)

Path integral normalization

$$Z = \frac{1}{\mathrm{Vol}(G)} \int DA\, D\psi\, e^{-S[A,\psi]} \qquad\qquad A_\mu = a_\mu + B_\mu + \partial_\mu \phi$$

- ✧ Measure of integration:

  $$DA = Da\, DB\, D(d\phi) = Da\, DB\, D'\phi \sqrt{\det{}'(-\nabla^2)}$$ ← Faddeev-Popov determinant

- ✧ Gauge group volume: $\mathrm{Vol}(G) = 2\pi\sqrt{\beta L^2} \int D'\phi$

$$\Rightarrow \quad Z = \frac{\beta L^2}{2\pi} \sqrt{\det{}'(-\nabla^2)} \int d^3a\, DB\, D\psi\, e^{-S[\psi,B,a]}$$

Klebanov et al. (2012)

Large-N_f QED$_3$

- Couple gauge field to N_f two-component Dirac fermions:

QED$_3$

$$S_{\rm qed}[\psi, A] = -\int d^3r \bar{\psi}_\alpha i\gamma^\mu \left(\partial_\mu - iA_\mu\right)\psi_\alpha$$

N_f even

Chern-Simons

$$S_{\rm CS}[A] = \frac{ik}{4\pi}\int d^3r \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$$

$k = 0$ or $k \sim \mathcal{O}(N_f)$

- $S_{\rm qed}$ is quadratic in the fermions, so integrate them out:

$$Z = \frac{\beta L^2}{2\pi}\sqrt{\det{}'\left(-\nabla^2\right)}\int d^3a\, DB\, e^{-S_{\rm CS}[B] + N_f \log\det\left(i\partial\!\!\!/ + a\!\!\!/ + B\!\!\!/\right)},$$

Saddle-point configurations

$$Z = \frac{\beta L^2}{2\pi}\sqrt{\det{}'\left(-\nabla^2\right)}\int d^3a\, DB\, e^{-S_{CS}[B]+N_f \log\det\left(i\partial\!\!\!/+a\!\!\!/+B\!\!\!/\right)},$$

✧ Plane: $A_\mu =$ const. are all gauge equivalent

⟹ $A_\mu=0$ is the saddle-point solution.

✧ Torus: finite zero mode a_μ exists

⟹ must minimize the free energy with respect to a_μ

✧ Rescale $B \to B/\sqrt{N_f}$ and expand in powers of N_f:

$$N_f \log\det\left(iD\!\!\!\!/\right) = \underbrace{N_f \mathrm{tr}\log\left(i\partial\!\!\!/+a\!\!\!/\right)}_{\text{free fermion free energy } F_0(a)} + \underbrace{\sqrt{N_f}\,\mathrm{tr}\left(\frac{1}{i\partial\!\!\!/+a\!\!\!/}B\!\!\!/\right)}_{\text{vanishes}} \underbrace{-\frac{1}{2}\mathrm{tr}\left(\frac{1}{i\partial\!\!\!/+a\!\!\!/}B\!\!\!/\frac{1}{i\partial\!\!\!/+a\!\!\!/}B\!\!\!/\right)}_{\text{Gauge field fluctuations Suppressed by } 1/N_f} + \cdots$$

Minimization of free energy

$$N_f \log\det(i\not{D}) = \boxed{N_f \mathrm{tr}\log(i\not{\partial} + \not{a})} - \frac{1}{2}\mathrm{tr}\left(\frac{1}{i\not{\partial}+\not{a}}\not{B}\frac{1}{i\not{\partial}+\not{a}}\not{B}\right) + \cdots$$

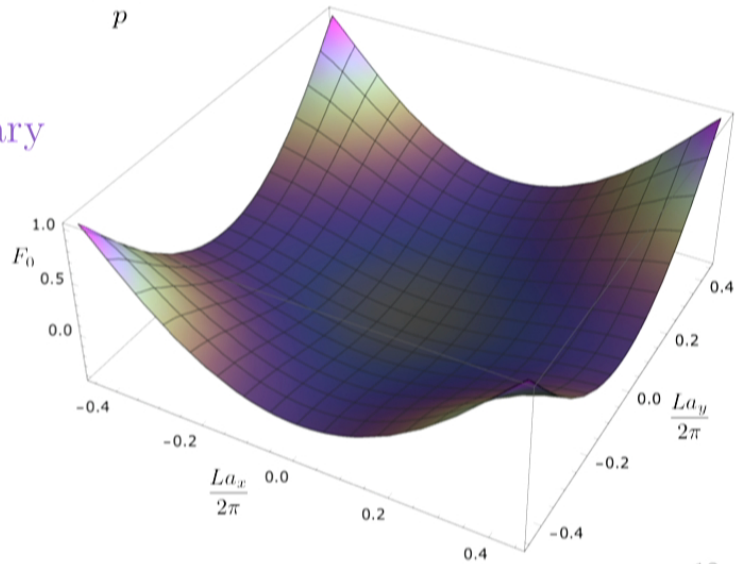
✧ The leading contribution is simply the free Dirac free energy:

$$F_0(a) = -\mathrm{tr}\log(i\not{\partial} + \not{a}) = -\sum_p \log(p+a)^2$$

✧ Find that anti-periodic boundary conditions minimize F_0:

$$\bar{a}_{x,y} = 0$$

$$p_{x,y} = \frac{2\pi}{L}\left(n_{x,y} + \frac{1}{2}\right)$$



Free Dirac-like states

Free Dirac Hamiltonian $\mathcal{H}_{\mathrm{D}} = \sum_{\mathbf{p}} |\mathbf{p}| \left[\chi^{\dagger}_{+\alpha}(\mathbf{p})\chi_{+\alpha}(\mathbf{p}) - \chi^{\dagger}_{-\alpha}(\mathbf{p})\chi_{-\alpha}(\mathbf{p}) \right]$

particles holes

✧ Constraint 1:

- $\chi_{\pm\alpha}(\mathbf{p})$ are not gauge invariant. The lowest energy states fermion states relevant to QED_3 are

$$\chi^{\dagger}_{+\alpha}(\mathbf{p}+\mathbf{q})\chi_{-\beta}(\mathbf{p})\,|0\rangle\,, \quad \chi^{\dagger}_{+\alpha}(-\mathbf{p})\chi_{-\beta}(-\mathbf{p}-\mathbf{q})\,|0\rangle$$

with energy $E_f(\mathbf{q},\mathbf{p}) = |\mathbf{q}+\mathbf{p}| + |\mathbf{p}|$.

- The free theory has $2N_f^2$ such states (except when $\mathbf{p}+\mathbf{q}=-\mathbf{p}$, in which case there are N_f^2).

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Gauge constraint

$$\chi^\dagger_{+\alpha}(\mathbf{p}+\mathbf{q})\chi_{-\beta}(\mathbf{p})\,|0\rangle\,, \quad \chi^\dagger_{+\alpha}(-\mathbf{p})\chi_{-\beta}(-\mathbf{p}-\mathbf{q})\,|0\rangle$$

✧ Constraint 2:

- The saddle-point approximation is equivalent to imposing the classical equations of motion:

$$\underbrace{\sum_\alpha \bar{\psi}_\alpha \gamma^\mu \psi_\alpha}_{J_\mu} = \frac{i}{e^2}\partial_\nu F^{\mu\nu}$$

Gauss-Ampère Law

Gauge constraint

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Gauss-Ampère Law

- All states corresponding to the poles of $\langle J_\mu(x) J_\nu(0)\rangle$ are removed. This corresponds to the removal of a single state for each momentum pair **p** & **q**.

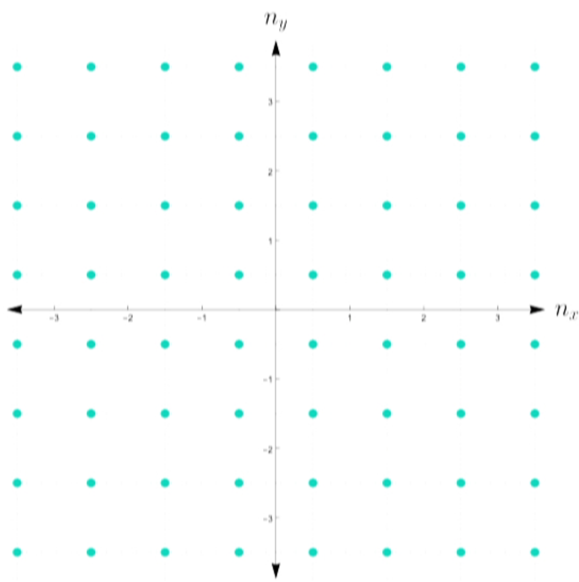
⇒ There are $2N_f^2$-1 free fermion-like states in QED_3 (except when $\mathbf{p}+\mathbf{q}=-\mathbf{p}$, in which case there are N_f^2-1).

Fermion energies

✧ Fermion energies: $E_f(\mathbf{q},\mathbf{p}) = |\mathbf{q}+\mathbf{p}| + |\mathbf{p}|$.

$$\mathbf{q} = \frac{2\pi}{L}\left(m_x + \frac{1}{2}, m_y + \frac{1}{2}\right), \quad \mathbf{p} = \frac{2\pi}{L}\left(n_x + \frac{1}{2}, n_y + \frac{1}{2}\right),$$

$$E_f(\mathbf{q},\mathbf{p}) = \frac{2\pi}{L}\left[\sqrt{(n_x + m_x + 1)^2 + (n_x + m_y + 1)^2} + \sqrt{(n_x + 1/2)^2 + (n_y + 1/2)^2}\right]$$



$\bar{\mathbf{q}} = (0,0)$	
$\bar{E}_f$	d_f
1.414214	$4N_f^2 - 2$
3.162278	$8N_f^2 - 4$
4.242640	$4N_f^2 - 2$

$$\bar{\mathbf{q}} = L\mathbf{q}/2\pi,$$
$$\bar{E}_f = LE_f/2\pi$$

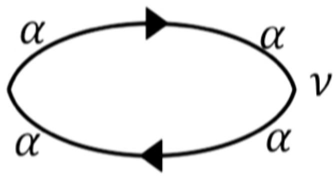
Photon contribution

⇨ the spectrum is non-extensive in N_f , so $\mathcal{O}(1)$ corrections must be included as well.

$$N_f \log\det\left(i\not{D}\right) = N_f \mathrm{tr}\log\left(i\not{\partial} + \not{a}\right) - \boxed{\frac{1}{2}\mathrm{tr}\left(\frac{1}{i\not{\partial}+\not{a}}\not{B}\frac{1}{i\not{\partial}+\not{a}}\not{B}\right)} + \cdots$$

✧ The only contribution at next-to-leading order is from the photon:

$$\boxed{S_{\mathrm{eff}}[B] = \frac{1}{2}\sum_q B_\mu(-q)\Pi^{\mu\nu}(q)B_\nu(q),} \qquad \Pi^{\mu\nu} = \Pi_f^{\mu\nu} + \Pi_{\mathrm{CS}}^{\mu\nu}$$

$\Pi_f^{\mu\nu}(q) = \mu$ 

= quadratic term in determinant expansion.

$$\Pi_{\mathrm{CS}}^{\mu\nu}(q) = \frac{1}{2\pi\lambda}\epsilon^{\mu\nu\rho}q_\rho$$

$$\lambda = \frac{N_f}{k}$$

Photon states

$$S_{\text{eff}}[B] = \frac{N_f}{2}\sum_q B_\mu(-q)\Pi^{\mu\nu}(q)B_\nu(q), \quad \Pi^{\mu\nu}(q) = \Pi_f^{\mu\nu}(q) + \Pi_{\text{CS}}^{\mu\nu}(q)$$

✧ When $k{=}0$, this is simply the expansion:

Spectrum from path integral

Harmonic Oscillator $H = \frac{1}{2}p^2 + \frac{1}{2}\Omega^2 q^2$

$$Z = \sum_{n=0}^{\infty} e^{-\beta\left(n+\frac{1}{2}\right)\Omega}$$

✧ Path integral:

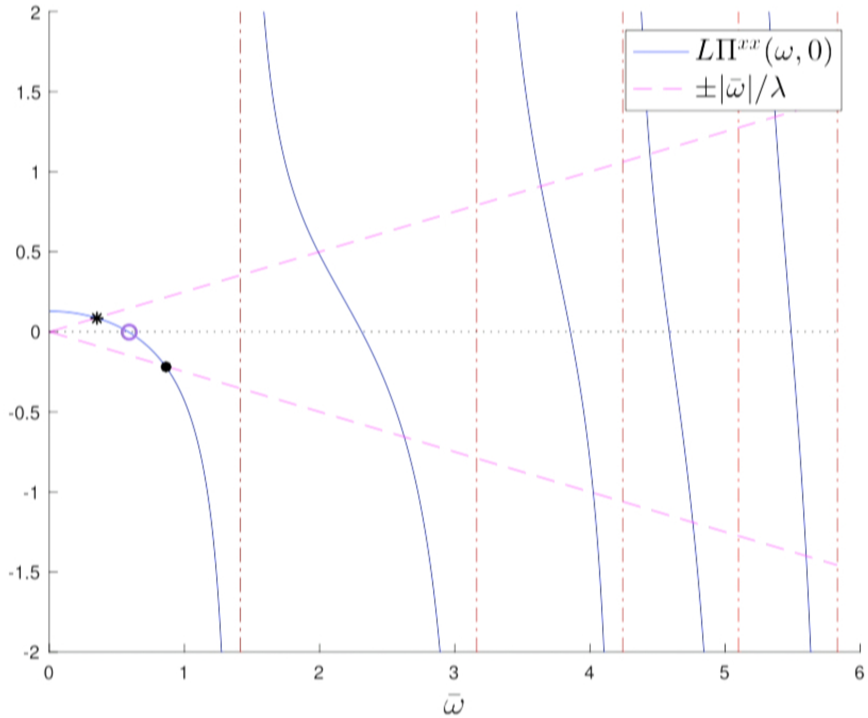
$$Z = \int Dq \exp\left[-\frac{1}{2}\int_0^\beta d\tau\, q(\tau)\left(-\partial_\tau^2 + \Omega^2\right) q(\tau)\right] = \left[\det\left(\frac{-\partial_\tau^2 + \Omega^2}{2\pi}\right)\right]^{-1/2}$$

$$\Rightarrow \quad F = -\frac{1}{\beta}\log Z = \frac{1}{2\beta}\log\det(-\partial_\tau^2 + \Omega^2)$$

$$= \frac{1}{2\beta}\sum_{\omega_n}\log\left[\omega_n^2 + \Omega^2\right] = -\frac{1}{\beta}\log\left(\frac{e^{-\beta\Omega/2}}{1 - e^{-\beta\Omega}}\right)$$

Photon states in QED_3

✧ Argument of log for $\mathbf{q}=0$: $\mathcal{F}(\omega, \mathbf{q}=0) = -\frac{\Pi_f^{xx}\Pi_f^{yy}}{\omega^2} + \frac{1}{4\pi^2\lambda}$



Plot of $\Pi_f^{xx}(\omega,\mathbf{q}=0)$ and $\pm|\omega|/\lambda$ for $\lambda=N_f/k=4$.

Photon states in QED_3

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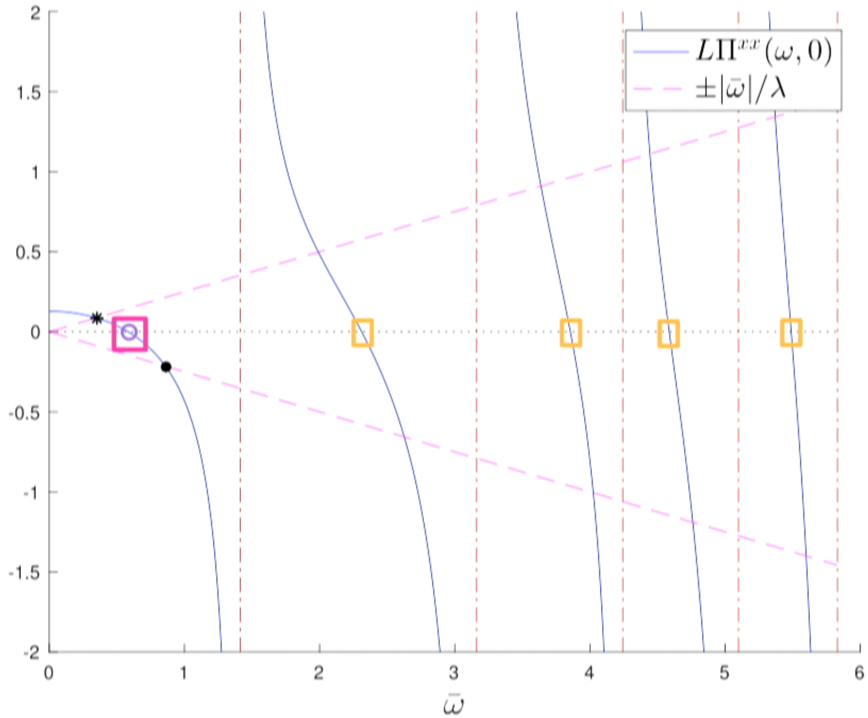

Plot of $\Pi_f^{xx}(\omega,\mathbf{q}=0)$ and $\pm|\omega|/\lambda$ for $\lambda=N_f/k=4$.

✧ Π_f^{xx} diverges at E_f :

$$\Pi_f^{xx}(\omega,0) \sim \frac{\Pi_\gamma\left(\omega^2-\omega_\gamma^2\right)}{\Pi_{\mathbf{p}}\left(\omega^2-E_f(0,\mathbf{p})^2\right)}$$

⇨ These diverges effectively remove the states created by J^μ

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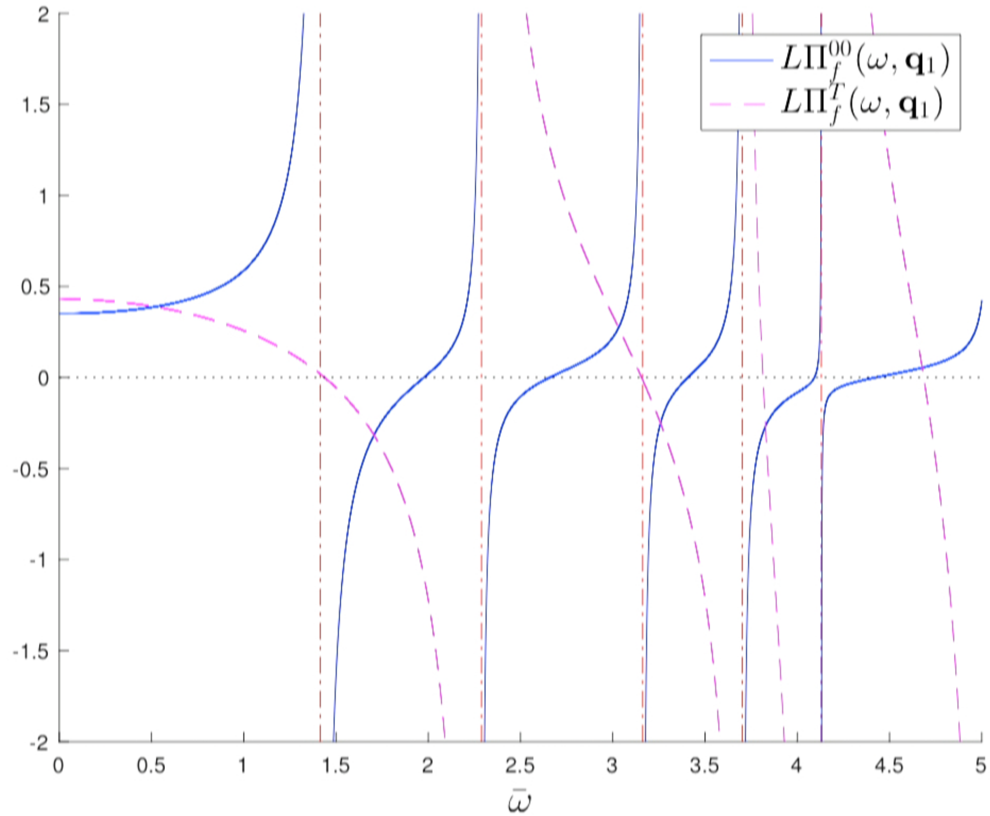
Finite momentum: $\mathbf{q}_1 = 2\pi(1,0)/L$ $(k=0)$



$$\mathbf{q}_1 = \frac{2\pi}{L}(1,0)$$

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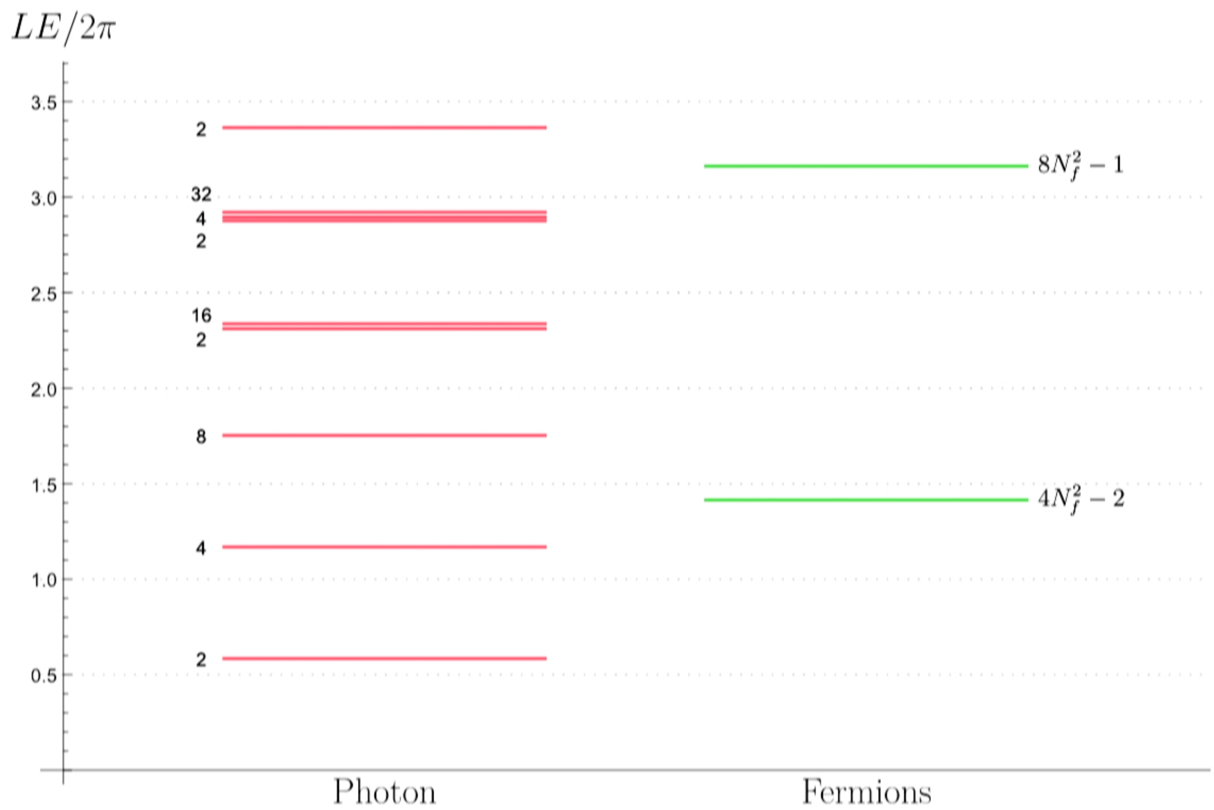
Comparison with fermion energies, $\mathbf{q}=0$



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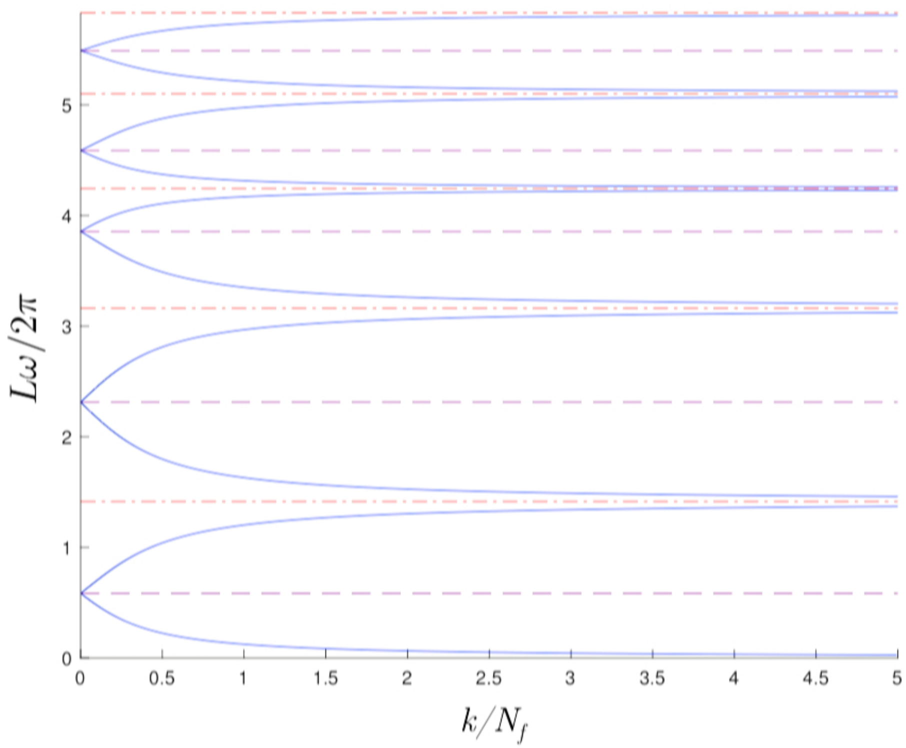
Photon modes as a function of k



Plot of energy modes for $\mathbf{q}=0$ as a function of $1/\lambda = k/N_f$.

- For $k/N_f \to 0$, the modes approach the QED_3 values.
- As $k/N_f \to \infty$, $\omega_0 \to 0$ and all higher energy modes approach the free theory energy $E_f = 2|\mathbf{p}|$.

⇒ The (infinite) ground state degeneracy is 'restored'.

Thank you

AT, S. Sachdev, Phys. Rev. B 95, 205128 (2017)
arXiv:1607.05279