Title: Symmetry-protected topological phases with uniform computational power in one

Date: Jun 07, 2017 04:00 PM

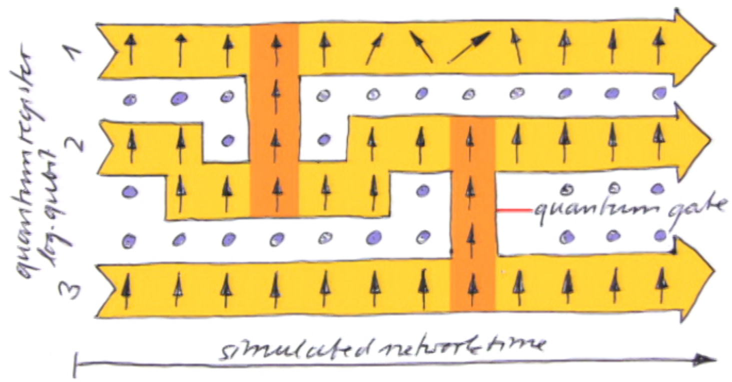
URL: http://pirsa.org/17060056

Abstract: <p>We investigate the usefulness of ground states of quantum spin chains with symmetry-protected topological order (SPTO) for measurement-based quantum computation. We show that, in spatial dimension one, if an SPTO phase supports quantum wire, then, subject to an additional symmetry condition that is satisfied in all cases so far investigated, it can also be used for quantum computation. Joint work with Dongsheng Wang, Abhishodh Prakash, Tzu-Chieh Wei and David Stephen; See arXiv:1609.07549v1</p>

Outline

1. Motivation for "computational phases of quantum matter"
2. Background review
3. Our result: computational phases of matter in 1D.
4. A question to you

Measurement-based quantum computation

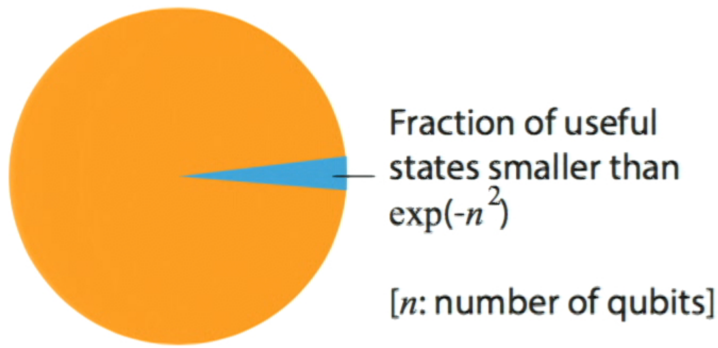


measurement of Z ($\odot$), X ($\uparrow$), $\cos\alpha\, X + \sin\alpha\, Y$ ($\nearrow$)

- Information written onto the resource state, processed and read out by one-qubit measurements only.
- Universal computational resources exist: cluster state, AKLT state.

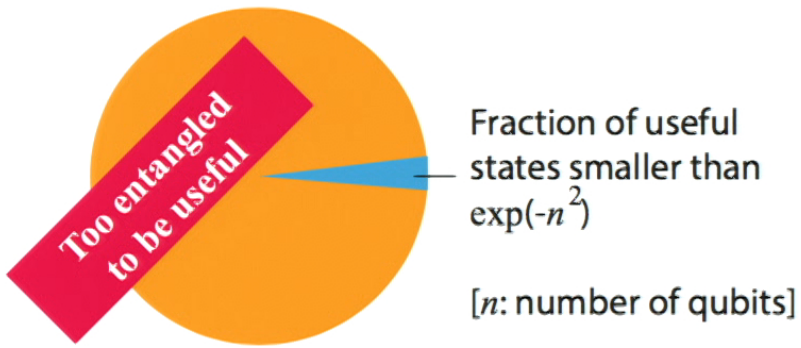

How rare are MBQC resource states
in Hilbert space?

MBQC resource states are rare



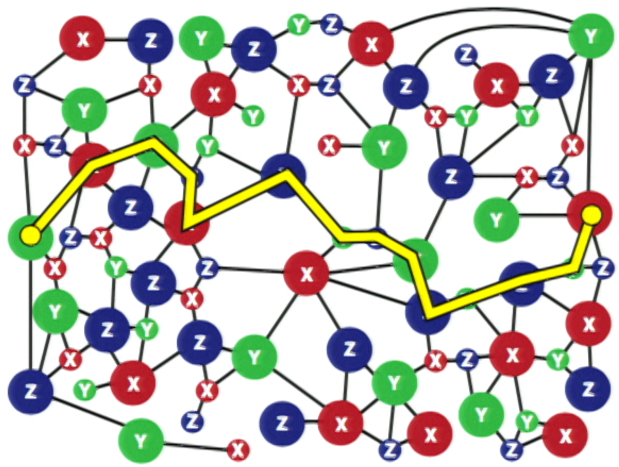
D. Gross, S.T. Flammia, J. Eisert, PRL 2009.

MBQC resource states are rare



D. Gross, S.T. Flammia, J. Eisert, PRL 2009.

What about "realistic" ground states?

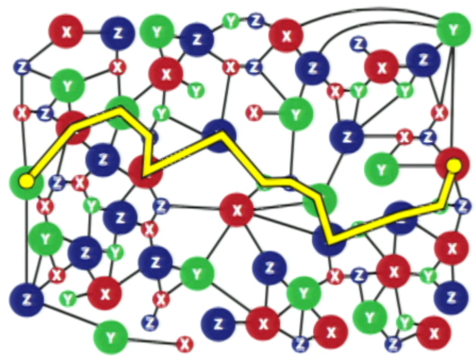


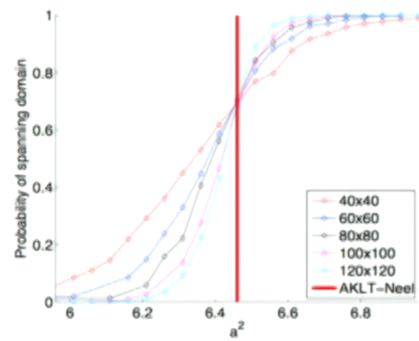
The AKLT state on a 2D honeycomb lattice is universal for MBQC.

A. Miyake, Ann. Phys. 2011

T.-C. Wei, I. Affleck and R. Raussendorf, PRL 2011.

What about phases?





Phase transition in MBQC power coincides with physical phase transition AKLT-order to Neel order.

H. Niggemann, A. Klümper, and J. Zittartz, Z. Phys. 1997.

A.S. Darmawan, G.K. Brennen, and S.D. Bartlett, NJP 2012.

What about symmetry?

MBQC-AQC hybrid:

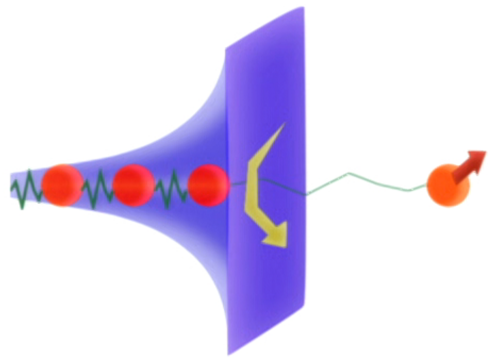
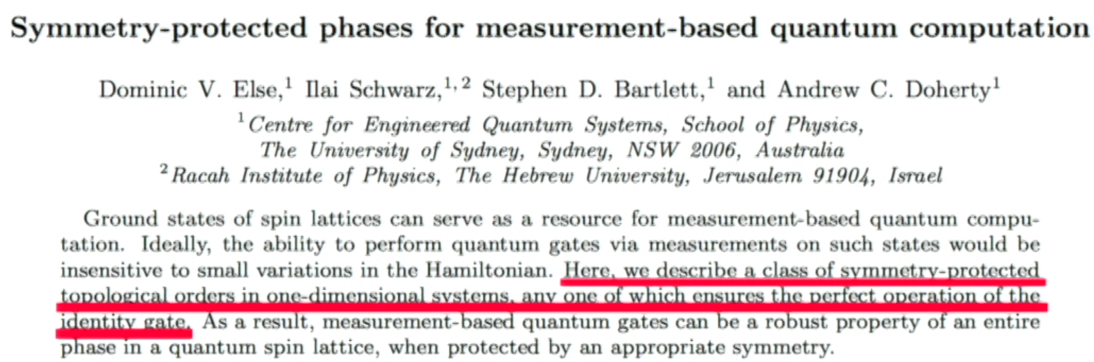


FIG. 1: Quantum computation is processed on the edge state which plays the role of a "holographic screen," while its computational capability relies on the symmetry-protected topological entanglement from the bulk.

A. Miyake, PRL 2010.

What about symmetry?


Symmetry-protected phases for measurement-based quantum computation

Dominic V. Else,[1] Ilai Schwarz,[1,2] Stephen D. Bartlett,[1] and Andrew C. Doherty[1]
[1] Centre for Engineered Quantum Systems, School of Physics, The University of Sydney, Sydney, NSW 2006, Australia
[2] Racah Institute of Physics, The Hebrew University, Jerusalem 91904, Israel

Ground states of spin lattices can serve as a resource for measurement-based quantum computation. Ideally, the ability to perform quantum gates via measurements on such states would be insensitive to small variations in the Hamiltonian. Here, we describe a class of symmetry-protected topological orders in one-dimensional systems, any one of which ensures the perfect operation of the identity gate. As a result, measurement-based quantum gates can be a robust property of an entire phase in a quantum spin lattice, when protected by an appropriate symmetry.

This gives wire. *Can we have universal quantum computation?*

D.V. Else,1 I. Schwarz, S.D. Bartlett, and A.C. Doherty, PRL 2012.

Part II:

Review of background material

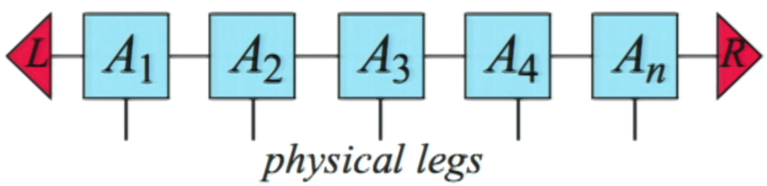
Matrix-product states [MPS]

... are states of the form:

$$|\Phi\rangle = \sum_{i_1,..,i_n} \underbrace{\langle R|A[i_1]A[i_2]..A[i_n]|L\rangle}_{\text{expansion coefficient}} |i_1, i_2, .., i_n\rangle$$

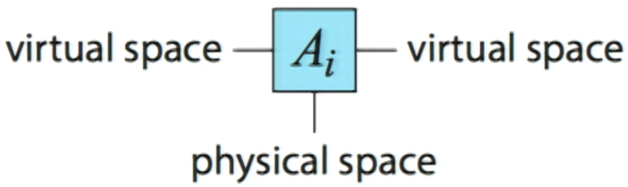
with the $A[i_k]$ are $D \times D$-matrices, and $i_k = 1, .., d$.

- d is the physical dimension
- D is the *bond dimension*.

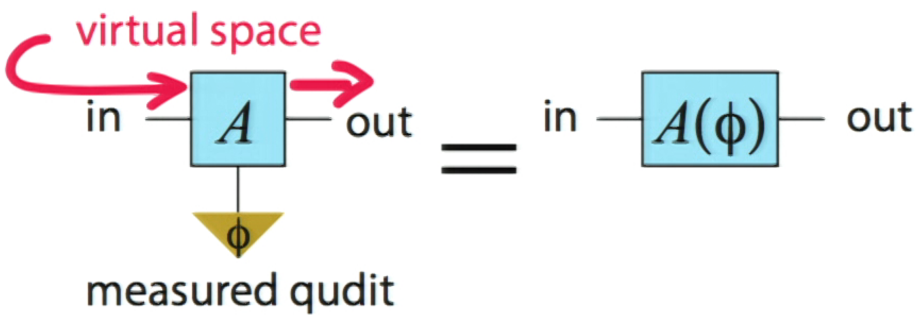


Matrix-product states [MPS]

- Ground states of 1D gapped phases are described by MPS.
- MBQC resource states are described by MPS.
- Advantage: MPS tensors are local objects.



MBQC with MPS



The circuit equivalent of MBQC lives on the virtual space.
The $A(\phi)$ are the gates.

Q: For which post-measurement states ϕ is $A(\phi)$ unitary?

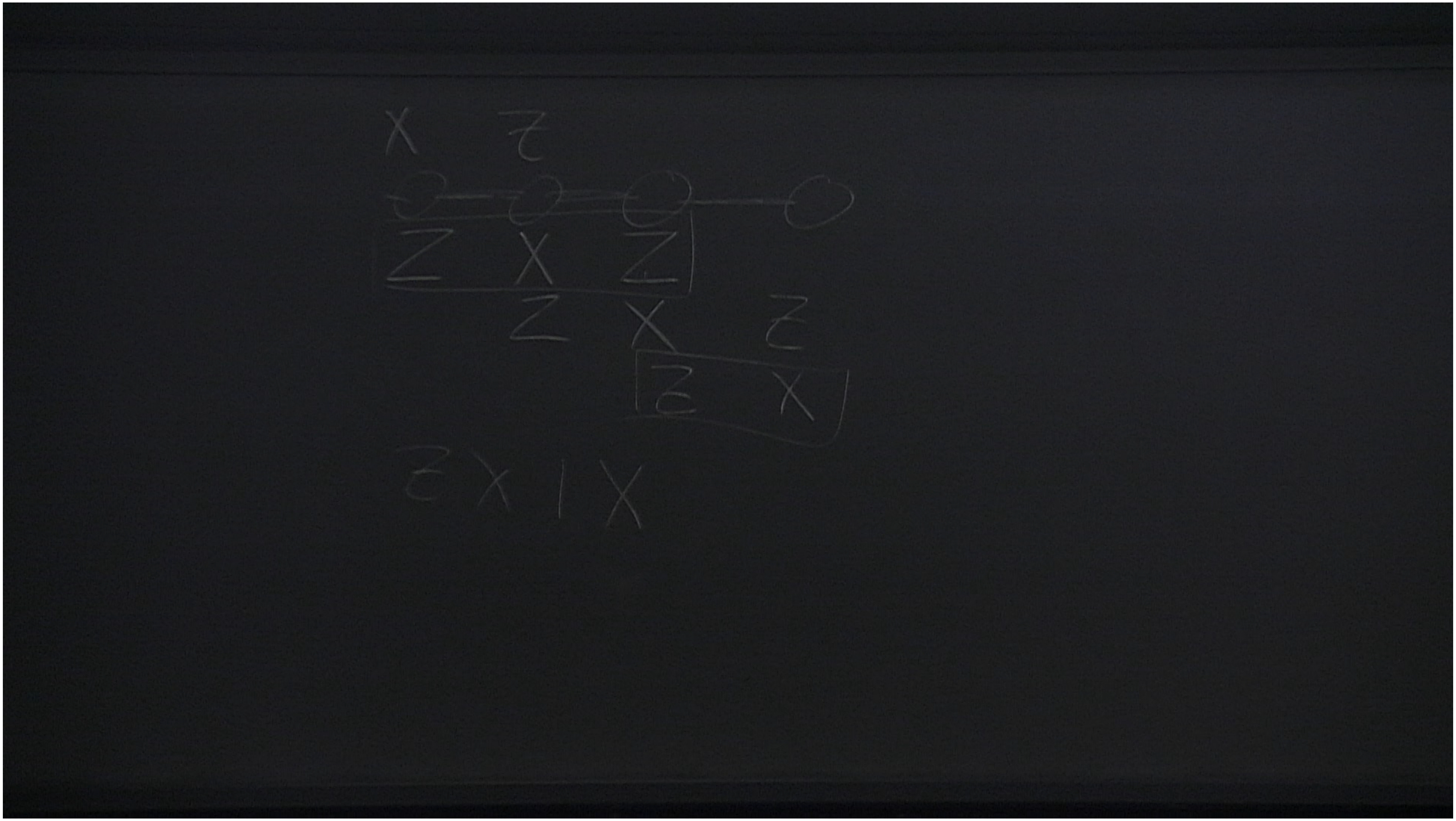
D. Gross, J. Eisert, PRL 2007.

Example: 1D cluster states are MPS

In the eigenbasis of local σ_x, the 2-blocked tensor $A(\pm,\pm)$ is

$$\begin{aligned} A(+,+) &= I, & A(+,-) &= \sigma_x, \\ A(-,+) &= \sigma_z, & A(-,-) &= \sigma_y. \end{aligned}$$

- Measurement in the σ_x-basis gives wire on the virtual space.
- Unitary gates and logical measurement in other bases.


X Z
Z X Z
Z X Z
Z X
ZX | X

Example: 1D cluster states are MPS

In the eigenbasis of local σ_x, the 2-blocked tensor $A(\pm,\pm)$ is

$$\begin{aligned} A(+,+) &= I, & A(+,-) &= \sigma_x, \\ A(-,+) &= \sigma_z, & A(-,-) &= \sigma_y. \end{aligned}$$

- Measurement in the σ_x-basis gives wire on the virtual space.
- Unitary gates and logical measurement in other bases.

Cluster states and symmetry

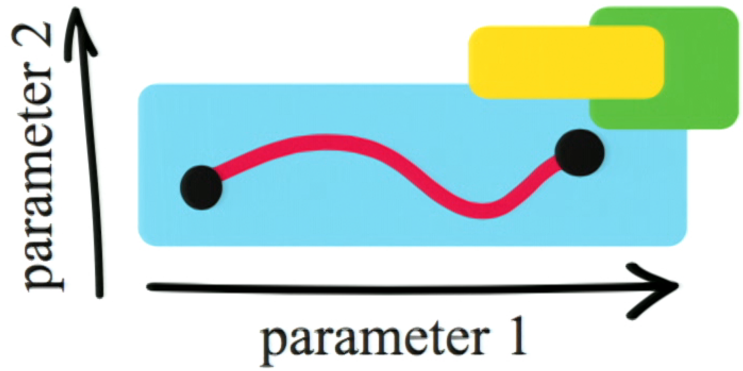
1D cluster states have an on-site $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetry.

$$..\, I \quad X \quad I \quad X \quad I \quad X\, ..$$

$$..\, X \quad I \quad X \quad I \quad X \quad I\, ..$$

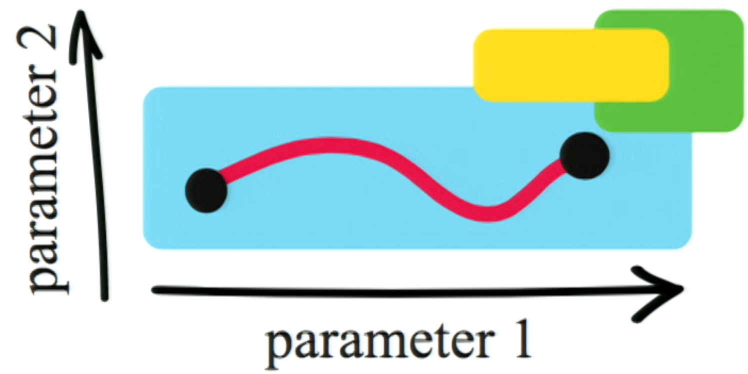
Symmetry-protected topological order

Definition of SPTO phases:



We consider ground states of Hamiltonians that are invariant under a symmetry group G.

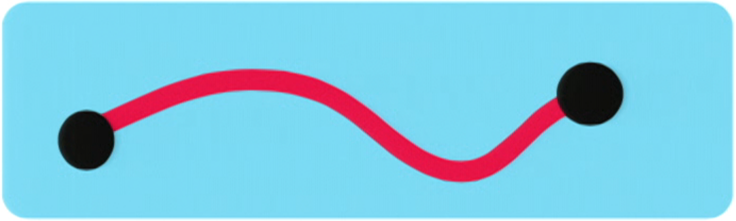
Symmetry-protected topological order



Two points in parameter space lie in the same SPTO phase iff they can be connected by a path of Hamiltonians such that

1. At every point on the path, the corresponding Hamiltonian is invariant under G.
2. Along the path the energy gap never closes.

Topological characterization of SPT phases



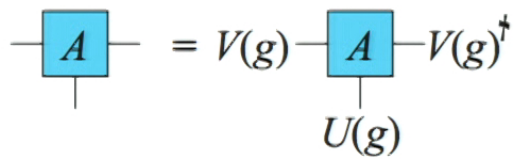
Fact: In spatial dimension 1, SPT phases are characterized by the symmetry group G and the cohomology class $[\omega] \in H^2(G)$.

What is the cocycle ω?

X. Chen, Z.C. Gu, and X.G. Wen, PRB 2011.
N. Schuch, D. Perez-Garcia, I. Cirac, Phys. RRB 2011.

Topological characterization of SPT phases

The MPS tensors A satisfy the symmetry constraint, $\forall g \in G$,



U is a unitary representation, V a *projective* representation of G,

$$V(gh) = \omega(g,h)V(g)V(h),$$

for some function $\omega : G \times G \longrightarrow U(1)$.

X. Chen, Z.C. Gu, and X.G. Wen, PRB 2011.

N. Schuch, D. Perez-Garcia, I. Cirac, Phys. RRB 2011.

Topological characterization of SPT phases

$$V(gh) = \omega(g,h)V(g)V(h).$$

The function $\omega : G \times G \longrightarrow U(1)$ is subject to a constraint and an identification.

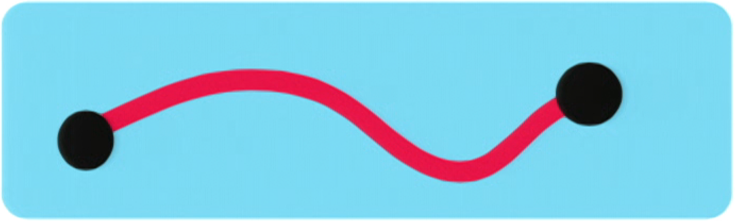
- The **constraint** comes from $\big(V(g)V(h)\big)V(k) = V(g)\big(V(h)V(k)\big)$.
- The **identification** comes from equivalence under rephasing,

  $$V(g) \mapsto \chi(g)V(g),$$

  where χ is some phase factor $\chi : G \longrightarrow U(1)$.

$\boxed{\text{This makes } [\omega] \text{ an element in the cohomology group } H^2(G).}$

Main conjecture & our result



Conjecture: The computational power of resource states for MBQC is *uniform* across symmetry-protected topological phases.

Our result: The conjecture holds in spatial dimension 1. Available logical gates, state preparations and measurements are determined by G, $[\omega]$.

Stepping stone

Theorem 1 [*]. Consider a symmetry-protected phase characterized by a finite Abelian group and a maximally non-commutative cohomology class $[\omega]$. Then, for every MPS in this phase there exists a basis w.r.t. which the MPS tensor A has the decomposition

$$A_i = (B_i)_{\text{junk}} \otimes (C_i)_{\text{logical}}.$$

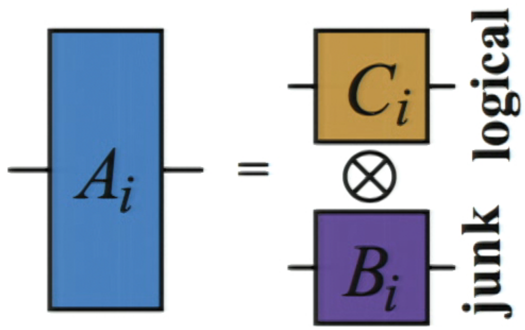
Therein, the operators C_i are elements of a finite group, and are constant throughout the phase.

Physical implication:

Can realize quantum wire on the logical subsystem.

*: D. Else *et al.*, PRL 2012.

Obstacle to quantum computation



There exists a basis in which this factorization holds

Obstacle:

For other measurement bases,
logical and junk subsystem become entangled.

Part III:

Computational phases of quantum matter in 1D

Page 27 of 52

Result

Theorem 2 [*] Consider a symmetry-protected phase of a group G with the properties

(i) the ground state is unique,
(ii) there is a wire basis, and
(iii) for all C_i exists a $g \in G$ such that $C_i \otimes I = V(g)$.

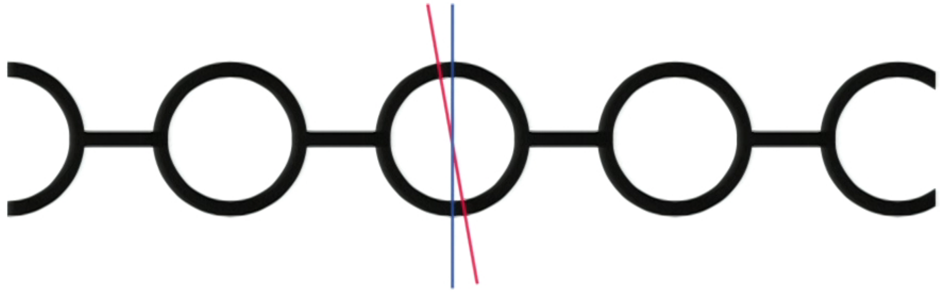
Then, this SPTO phase has the uniform computational power to execute MBQC simulations of measuring the logical observables

$$\mathcal{O} = \left\{ \frac{C_i^{-1}C_j + C_j^{-1}C_i}{2}, \frac{C_i^{-1}C_j - C_j^{-1}C_i}{2}, \ \forall i, j \right\}$$

and of the Lie group unitary gates generated by $\mathcal{O}$.

*: R.Raussendorf, D.S. Wang, A. Prakash, T.-C. Wei, D.T. Stephen, arXiv:1609.
!: Also see for 1 phase: Miller and Miyake, PRL 2016.

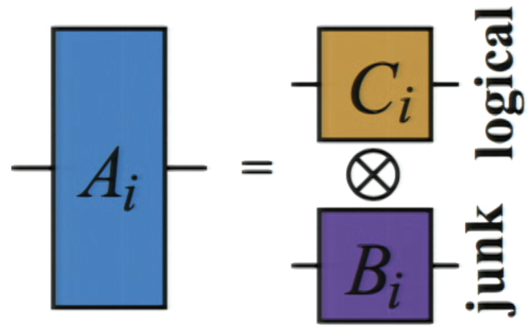
Task



Deviate from protected basis
without losing control

Computational primitives

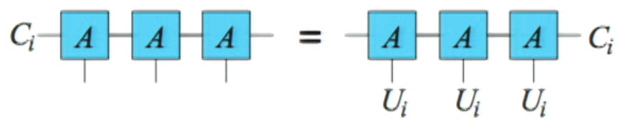
1. Oblivious wire: drives junk subsystem towards a fixed point state
2. Unitary operations with small rotation angle
3. Measurement



Primitive 1: oblivious wire

1. Measure a given qudit in the protected basis.

2. Propagate byproduct C_i through the chain, using the symmetry relation.



3. Forget the outcome i.

Primitive 1: oblivious wire

- This procedure implements the channel $\mathcal{L}$ on the junk system
$$\mathcal{L}(\cdot) = \sum_i B_i(\cdot)B_i^\dagger.$$
- If the ground state is unique then $\mathcal{L}$ has a unique fixed point ρ_{fix}.

This generates reproducible conditions
on the junk subsystem.

Primitive 2: small-angle unitary

Procedure:

1. Given a wire basis $\mathcal{B}_W = \{|i\rangle\}$, measure in the basis $\mathcal{B}'$

$$\begin{aligned} |1'\rangle &= |1\rangle + d\alpha|2\rangle, \\ |2'\rangle &= |2\rangle - d\alpha|1\rangle. \end{aligned}$$

   and $|3'\rangle = |3\rangle$ etc.

2. Propagate the byproduct C_i as before.

3. Apply (several rounds of) oblivious wire.

Primitive 2: small-angle unitary

Given the outcome $1'$, the result of this procedure on an input state $\sigma \otimes \rho_{\text{fix}}$ is, to linear order in $d\alpha$,

$$\sigma \otimes \rho_{\text{fix}} \;\longrightarrow\; \nu_{11}\, \sigma \otimes \rho_{\text{fix}} + \frac{d\alpha}{2}[\nu_{21}C - \nu_{21}^{*}C^{\dagger}, \sigma] \otimes \rho_{\text{fix}} + \\ + \frac{d\alpha}{2}\{\nu_{21}C + \nu_{21}^{*}C^{\dagger}, \sigma\} \otimes \rho_{\text{fix}} \quad ,$$

where $C := C_1^{-1}C_2$ and the ν_{ij} are given by $\lim_{n\to\infty} \mathcal{L}^n B_i \rho_{\text{fix}} B_j^{\dagger} = \nu_{ij}\rho_{\text{fix}}$.

Commutator term: unitary rotation – good.

Anti-commutator term: non-unitary stretching – bad.

Primitive 2: small-angle unitary

Outcome $1'$:

$$\sigma \otimes \rho_{\text{fix}} \longrightarrow \begin{array}{l} \nu_{11}\,\sigma \otimes \rho_{\text{fix}} + \frac{d\alpha}{2}[\nu_{21}C - \nu_{21}^{*}C^{\dagger}, \sigma] \otimes \rho_{\text{fix}} + \\ + \frac{d\alpha}{2}\{\nu_{21}C + \nu_{21}^{*}C^{\dagger}, \sigma\} \otimes \rho_{\text{fix}} \end{array},$$

Outcome $2'$:

$$\sigma \otimes \rho_{\text{fix}} \longrightarrow \begin{array}{l} \nu_{22}\,\sigma \otimes \rho_{\text{fix}} + \frac{d\alpha}{2}[\nu_{21}C - \nu_{21}^{*}C^{\dagger}, \sigma] \otimes \rho_{\text{fix}} + \\ - \frac{d\alpha}{2}\{\nu_{21}C + \nu_{21}^{*}C^{\dagger}, \sigma\} \otimes \rho_{\text{fix}} \end{array},$$

Probabilistically add both branches (forget outcome):

$$\sigma \otimes \rho_{\text{fix}} \longrightarrow (\nu_{11} + \nu_{22})\sigma \otimes \rho_{\text{fix}} + d\alpha\,[\nu_{21}C - \nu_{21}^{*}C^{\dagger}, \sigma] \otimes \rho_{\text{fix}}$$

Primitive 2: small-angle unitary

This implements, with probability $\nu_{11} + \nu_{22}$, a heralded unitary

$$U(d\alpha) = \exp\left(id\alpha \frac{\nu_{21}C - \nu_{21}^* C^\dagger}{i(\nu_{11} + \nu_{22})}\right).$$

Recall $C := C_1^{-1}C_2$ is from the algebraic part of A; ν_{ij} are complex numbers describing the fixed point state of the junk system, $\lim_{n\to\infty} \mathcal{L}^n B_i \rho_{\mathsf{fix}} B_j^\dagger = \nu_{ij}\rho_{\mathsf{fix}}$.

Primitive 2: finite-angle unitary

- Chop up a rotation about a finite angle ϕ into N rotations about an angle ϕ/N.

- Error per individual rotation is $O(\phi^2/N^2)$ [second order in $d\alpha$]

- Total error is $O(\phi^2/N)$. Hence for a total error ϵ need

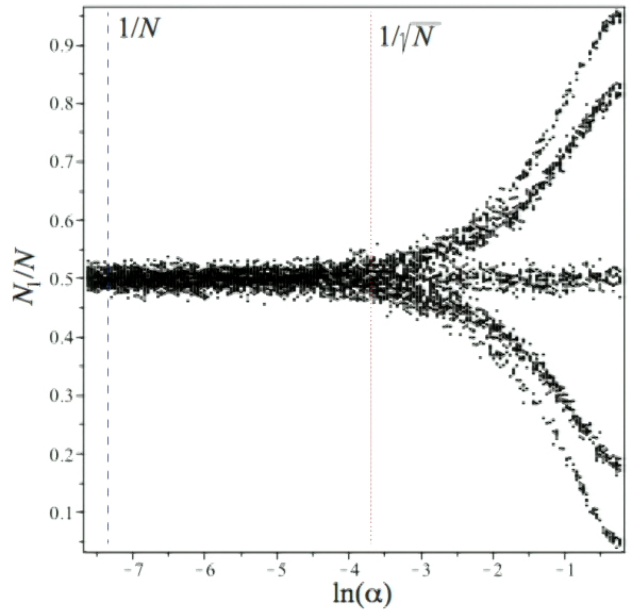
$$N = O(\phi^2/\epsilon)$$

small-angle rotations.

Primitive 3: measurement

- Second-order corrections in $d\alpha$ (or α) indeed violate unitarity
- Can use this fact to implement measurement
- Project to eigenstates of C

Unitary-to-measurement changeover



Total error for unitary $\epsilon = O(\alpha^2 N)$. $N = 1600$ in this plot.

Result

Theorem 2 [*] Consider a symmetry-protected phase of a group G with the properties

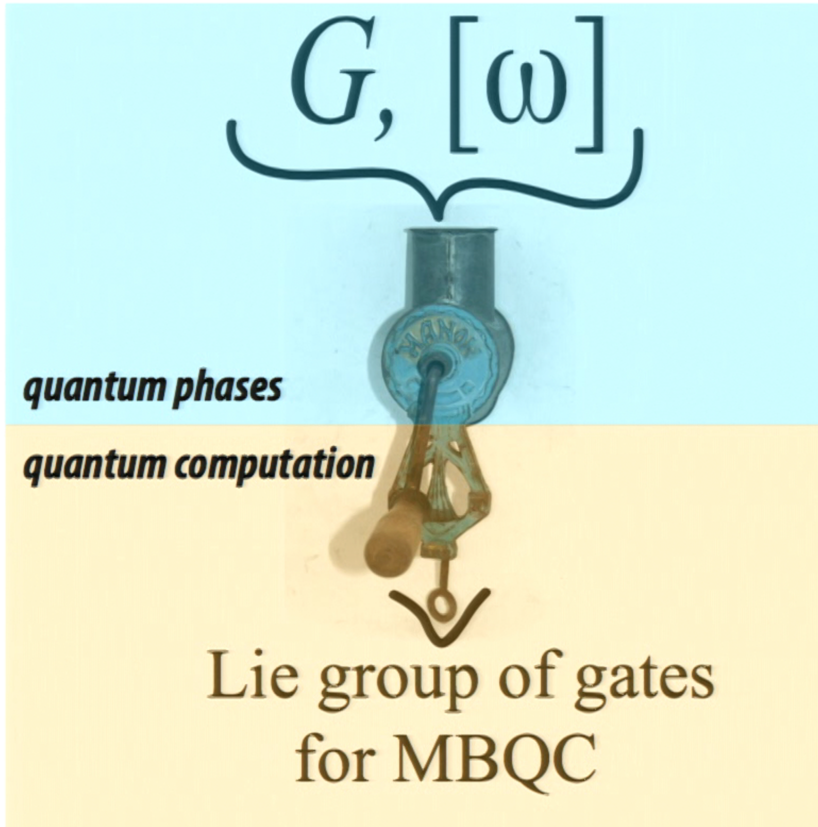
(i) the ground state is unique,
(ii) there is a wire basis, and
(iii) for all C_i exists a $g \in G$ such that $C_i \otimes I = V(g)$.

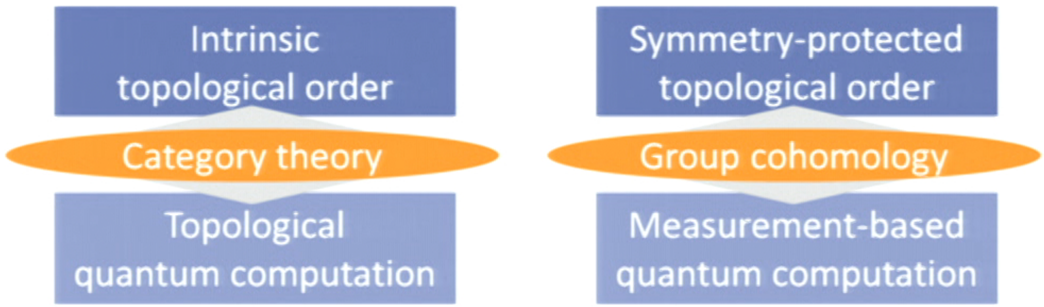
Then, this SPTO phase has the uniform computational power to execute MBQC simulations of measuring the logical observables

$$\mathcal{O} = \left\{ \frac{C_i^{-1}C_j + C_j^{-1}C_i}{2}, \frac{C_i^{-1}C_j - C_j^{-1}C_i}{2}, \ \forall i, j \right\}$$

and of the Lie group unitary gates generated by $\mathcal{O}$.

*: R.Raussendorf, D.S. Wang, A. Prakash, T.-C. Wei, D.T. Stephen, arXiv:1609.


$G, [\omega]$
quantum phases
quantum computation
Lie group of gates
for MBQC



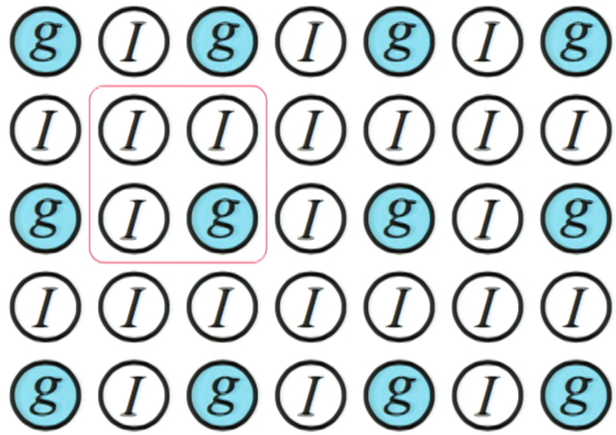
MBQC relates to symmetry-protected topological order like topological QC relates to topological order.

Summary and outlook

- Computational power for MBQC is uniform across 1D-SPT phases.
- Algorithm converts the topological characterization $G, [\omega]$ of an SPT phase into the corresponding MBQC scheme
- Goal: Reproduce the above in spatial dimension 2 (and higher).

arXiv:1609.07549
arXiv:1611.08053

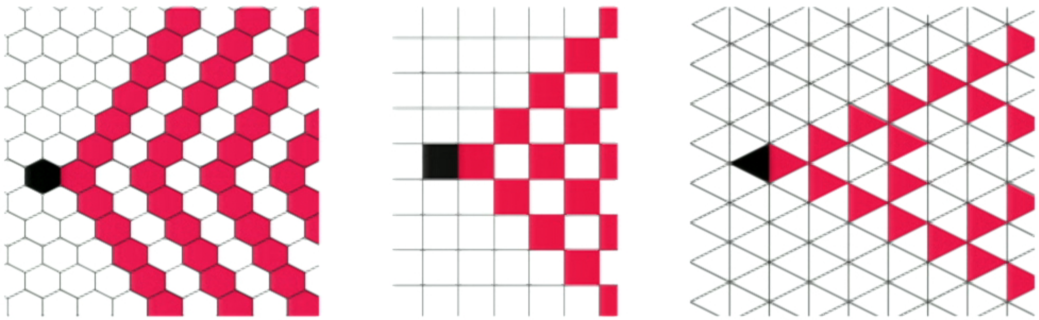
Question: How to implement the symmetry?



... each symmetry $g \in G$ acts in a local-global *translation-invariant* fashion.

This approach seems to marginally fit in 1D, but not in higher dimension.

MBQC forward cones: discrete electrodynamics



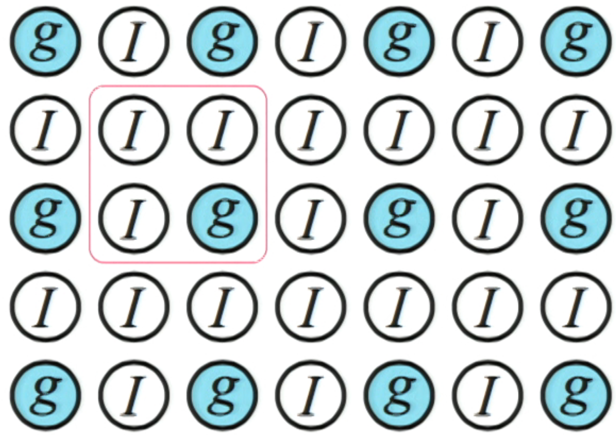
$\Box\psi = \delta_{r,\tau}$

$\Box\psi = \delta_{r,\tau}$

$\Box\psi + \psi = \delta_{r,\tau}$

$$\Box := \frac{d^2}{d^2x} + \frac{d^2}{d^2t} \mod 2.$$

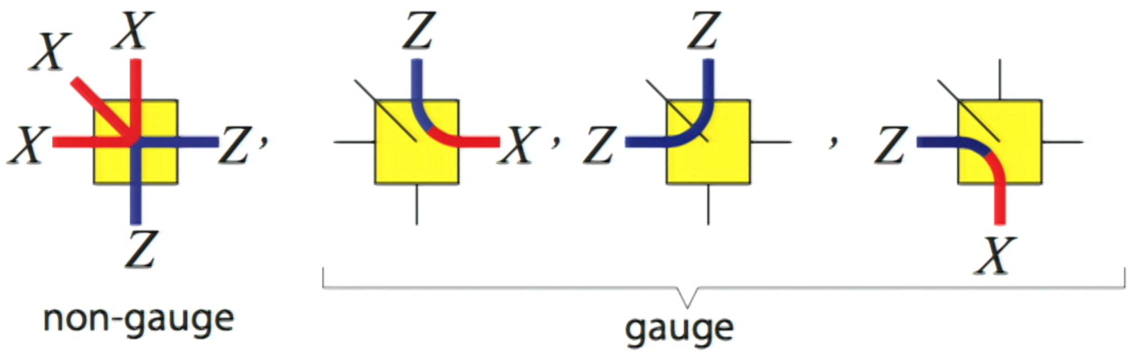
Question: How to implement the symmetry?



... each symmetry $g \in G$ acts in a local-global *translation-invariant* fashion.

This approach seems to marginally fit in 1D, but not in higher dimension.

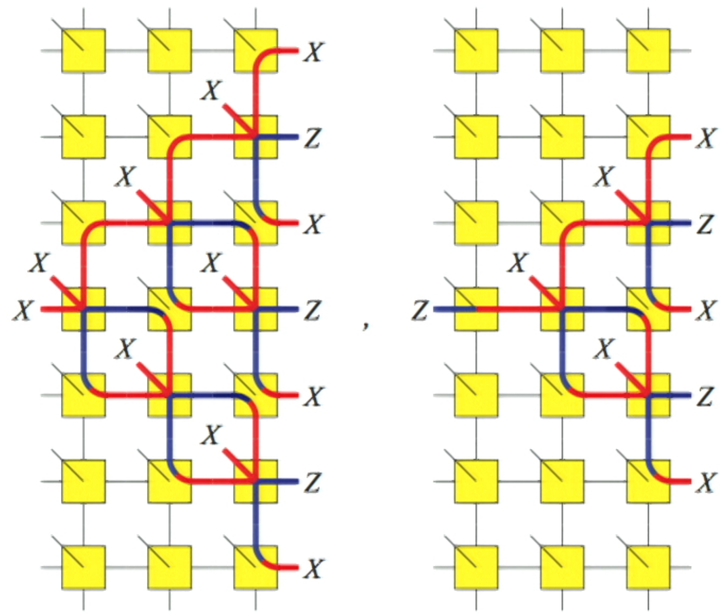
A $\mathbb{Z}_2^{\times 4}$-symmetry



These symmetries describe what matters about 2D cluster states.

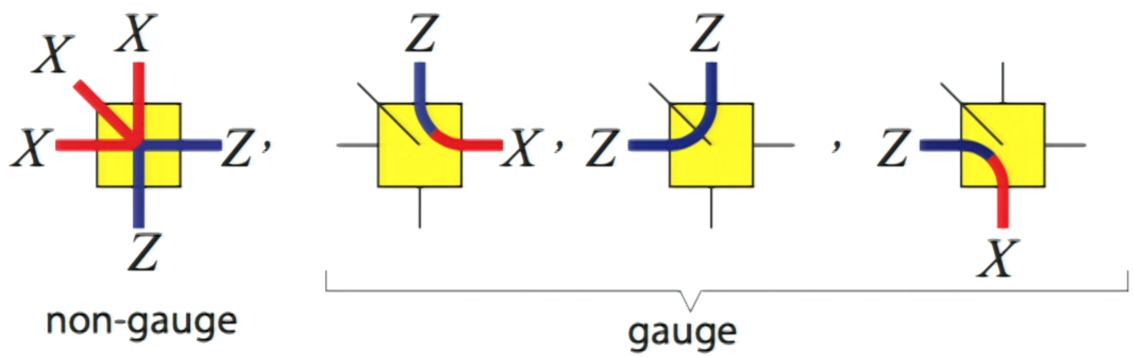
We want to regard those as the fundamental symmetries

Symmetry



$\mathbb{Z}_2^{\times 2n}$ *symmetry emerges via Lego*

Symmetry



But the question is:

Is there a natural physical phase throughout which these symmetries persist?