Title: Quantum tunneling with a Lorentzian path integral

Date: Jun 01, 2017 10:10 AM

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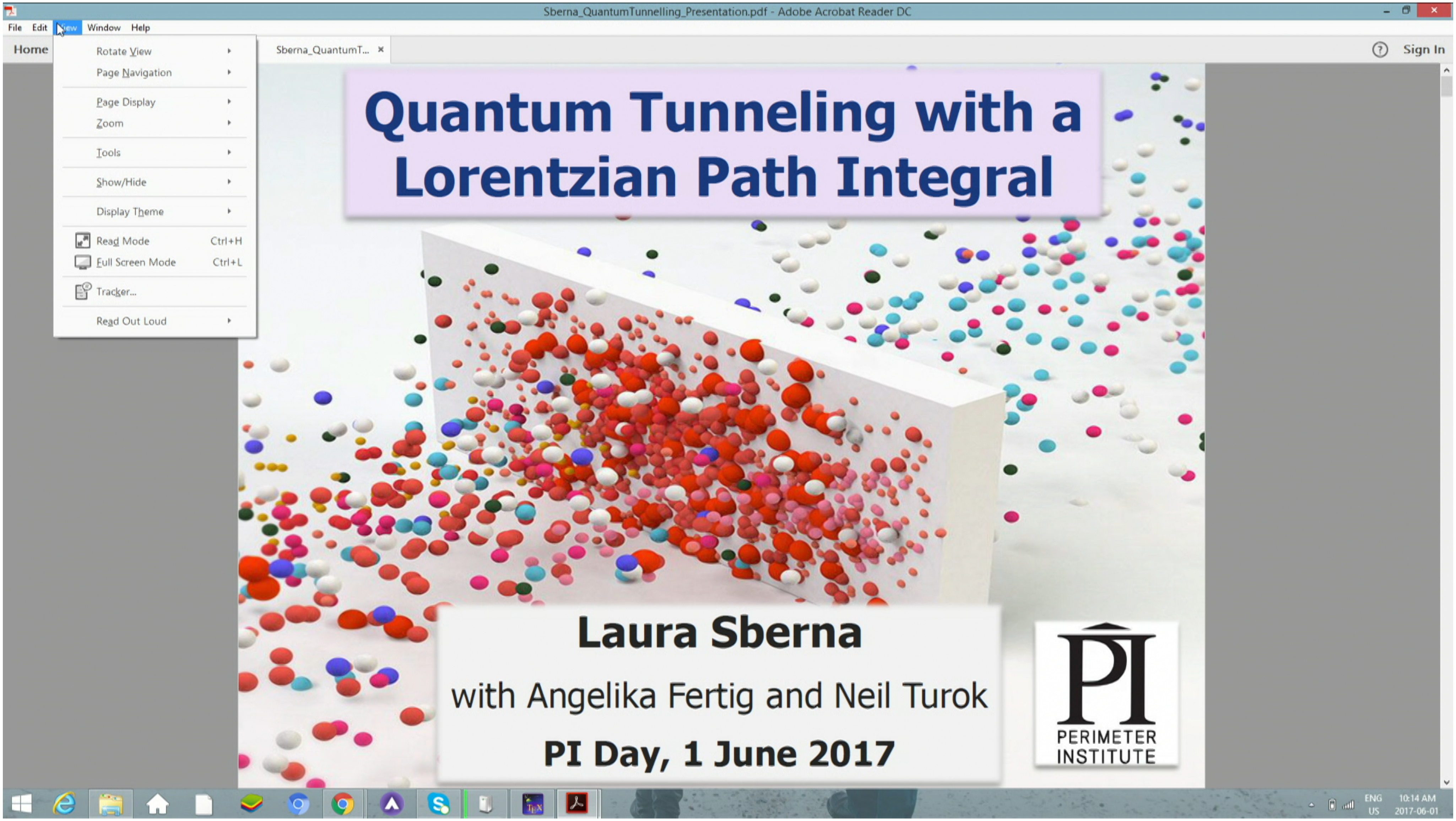
Abstract: We describe the tunneling of a quantum mechanical particle with a Lorentzian (realtime) path integral. The analysis is made concrete by application to the inverted harmonic oscillator potential, where the path integral is known exactly. We apply Picard-Lefschetz theory to the time integral of the Feynmann propagator at fixed energy, and show that the Euclidean integration contour is obtained as a Lefschetz thimble, or a sum of them, in a suitable limit. Picard-Lefschetz theory is used to make the integral manifestly convergent and is also essential for the saddle point or semiclassical approximation. The very simple example of the inverted harmonic oscillator presents many interesting mathematical features, such as the Stokes phenomenon and multiple relevant complex saddles. We also attempt to construct a more realistic picture of the tunneling process, by allowing for spreading in energy and duration.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Quantum Tunneling with a Lorentzian Path Integral
Laura Sberna
with Angelika Fertig and Neil Turok
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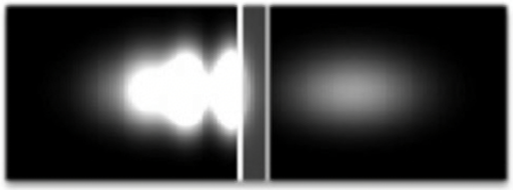
Quantum tunnelling: why?



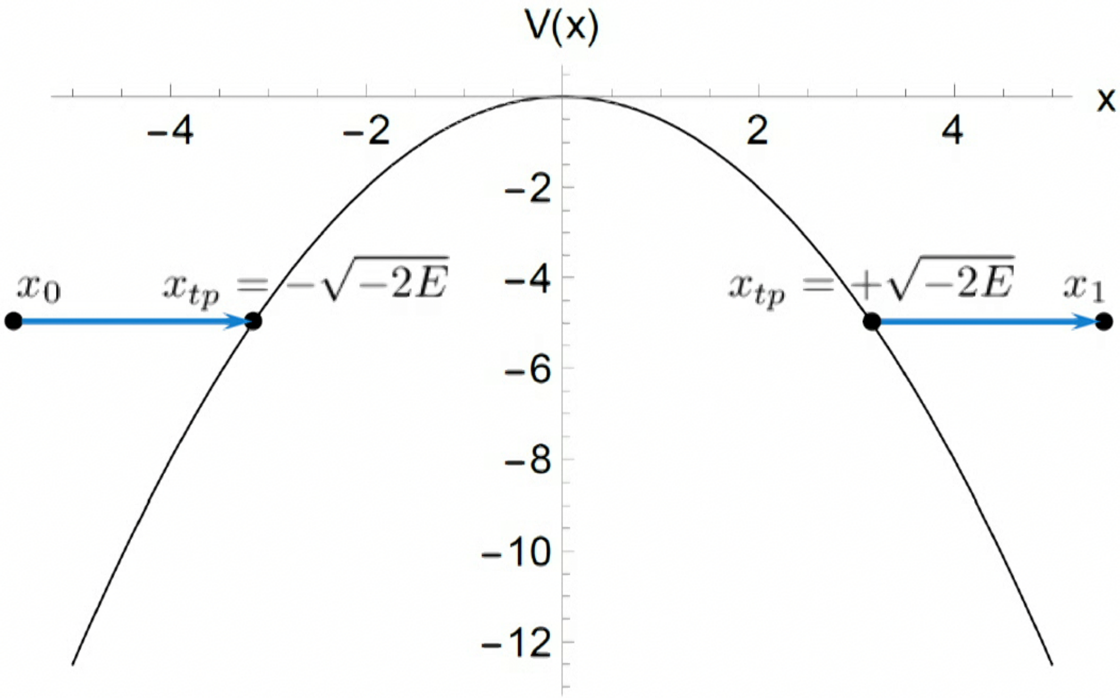
Describe process in <u>real time</u>:

- for **experiments**
- for extensions to **time dependent** models and **gravitational effects**

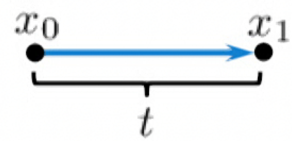
Looking for a better handle on the **Lorentzian** formulation of quantum mechanics

Inverted harmonic oscillator

$$\boxed{V(x) = -\frac{1}{2}x^2}$$



The propagator



Lorentzian / real time path integral:

$$G(x_1, x_0, t) = \int Dx\, e^{iS[x(t'); x_1, x_0, t]} \qquad \text{with} \quad x(0) = x_0, \;\; x(t) = x_1$$

Fixed-energy propagator: Fourier transform of the time propagator

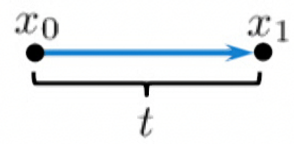
$$G_E(x_1, x_0) = \int_0^\infty dt\, G(x_1, x_0, t) e^{+iEt}$$

$$\boxed{= \int_0^\infty dN \int Dx\, e^{iS[x(\tau); x_1, x_0, N] + iEN}} \qquad t \to N\tau, \;\; \tau \in [0, 1]$$

parameter time (τ)

Schwinger time (N)

The propagator



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1. Exact

2. Oscillatory integral, need Picard-Lefschetz theory

1. Exact path integral

$$G(x_1, x_0, t) = \int Dx\, e^{iS[x(t');x_1,x_0,t]} = \frac{1}{\sqrt{2\pi i \sinh(N)}}\, e^{iS_{cl}(x_1,x_0,N)}$$

Action in parameter time:

$$S[x(t'); x_1, x_0, t] = \int_{0,\, x_0}^{1,\, x_1} d\tau\, \left[\frac{1}{2N} x'(\tau)^2 - N\, V\,(x(\tau))\right]$$

Equation of motion: Boundary conditions:

$$x''(\tau) - N^2 x(\tau) = 0, \qquad x(0) = x_0, \quad x(1) = x_1$$

Solution:

$$x_{cl}(\tau) = \frac{e^{-N\tau}\left[\left(e^{2N} - e^{2N\tau}\right) x_0 + \left(e^{N+2N\tau} - e^{N}\right) x_1\right]}{e^{2N} - 1}$$

1. Exact path integral

$$G(x_1, x_0, t) = \int Dx\, e^{iS[x(t');x_1,x_0,t]} = \frac{1}{\sqrt{2\pi i \sinh(N)}}\, e^{\boxed{iS_{cl}(x_1,x_0,N)}}$$

$$\boxed{S_{cl}(x_1, x_0, N) = \frac{1}{2}\mathrm{csch}(N)\left[\cosh(N)\left(x_0^2 + x_1^2\right) - 2x_0 x_1\right]}$$

2. Integral over N

$$G_E(x_1, x_0) = \int_0^\infty dN \frac{1}{\sqrt{2\pi i \sinh(N)}} e^{+iEN} e^{iS_{cl}(x_1,x_0,N)}$$

Cannot be performed analytically, because of how N appears in S_{cl}

Exponent:

$$\boxed{\mathcal{I}(N) = +iEN + \frac{1}{2} i \operatorname{csch}(N) \left[\cosh(N)\left(x_0^2 + x_1^2\right) - 2x_0 x_1\right]}$$

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Apply **Picard-Lefschetz** theory (Cauchy's theorem) to obtain:

- Manifest convergence
- Saddle point approximation (semi-classical expansion)

2. Integral over N

$$G_E(x_1, x_0) = \int_{\mathcal{C}_N} dN \frac{1}{\sqrt{2\pi i \sinh(N)}} e^{+iEN} e^{iS_{cl}(x_1,x_0,N)}$$

Cannot be performed analytically, but conditionally convergent

Exponent:

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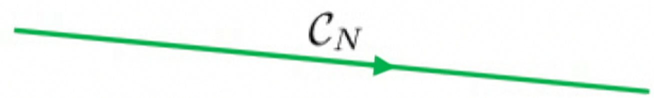
IHO: Integral over N



1. Find all the **saddle points** N_σ ;

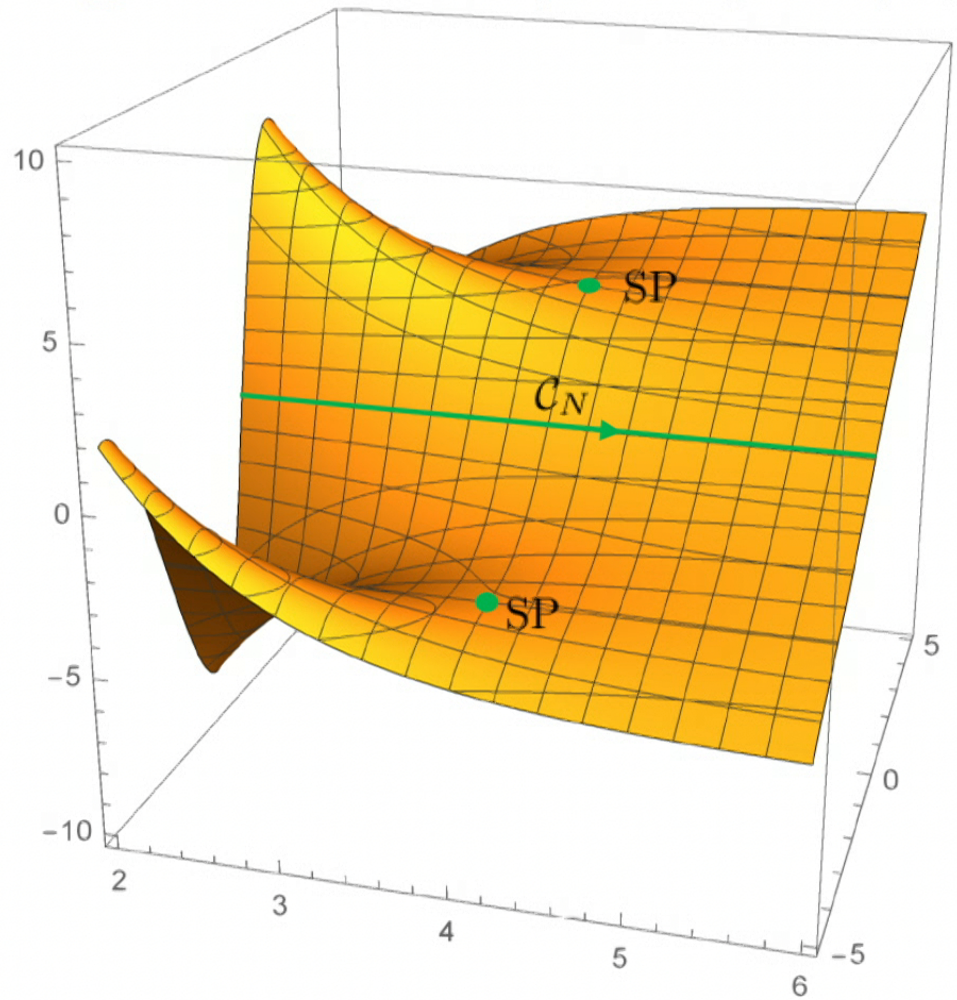
2. Distinguish the **relevant** SPs with $n_\sigma \neq 0$ from the **irrelevant** SPs;

3. Combine the **Lefschetz thimbles** (manifestly convergent contour);

$$\int_{\mathcal{C}_N} \to \sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma}$$

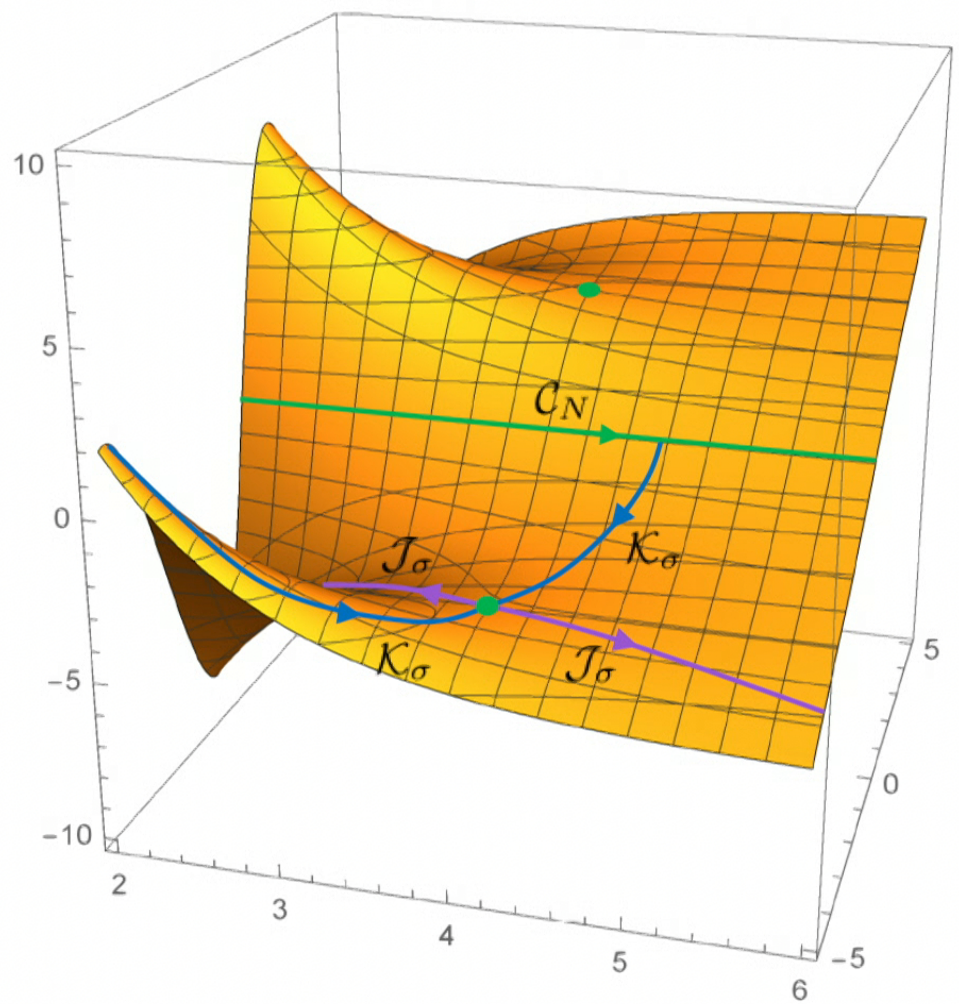
4. **Expand** in $\hbar$ around each SP.

IHO: Integral over N complex SPs given by $\mathcal{I}_{,N}(N_\sigma)=0$



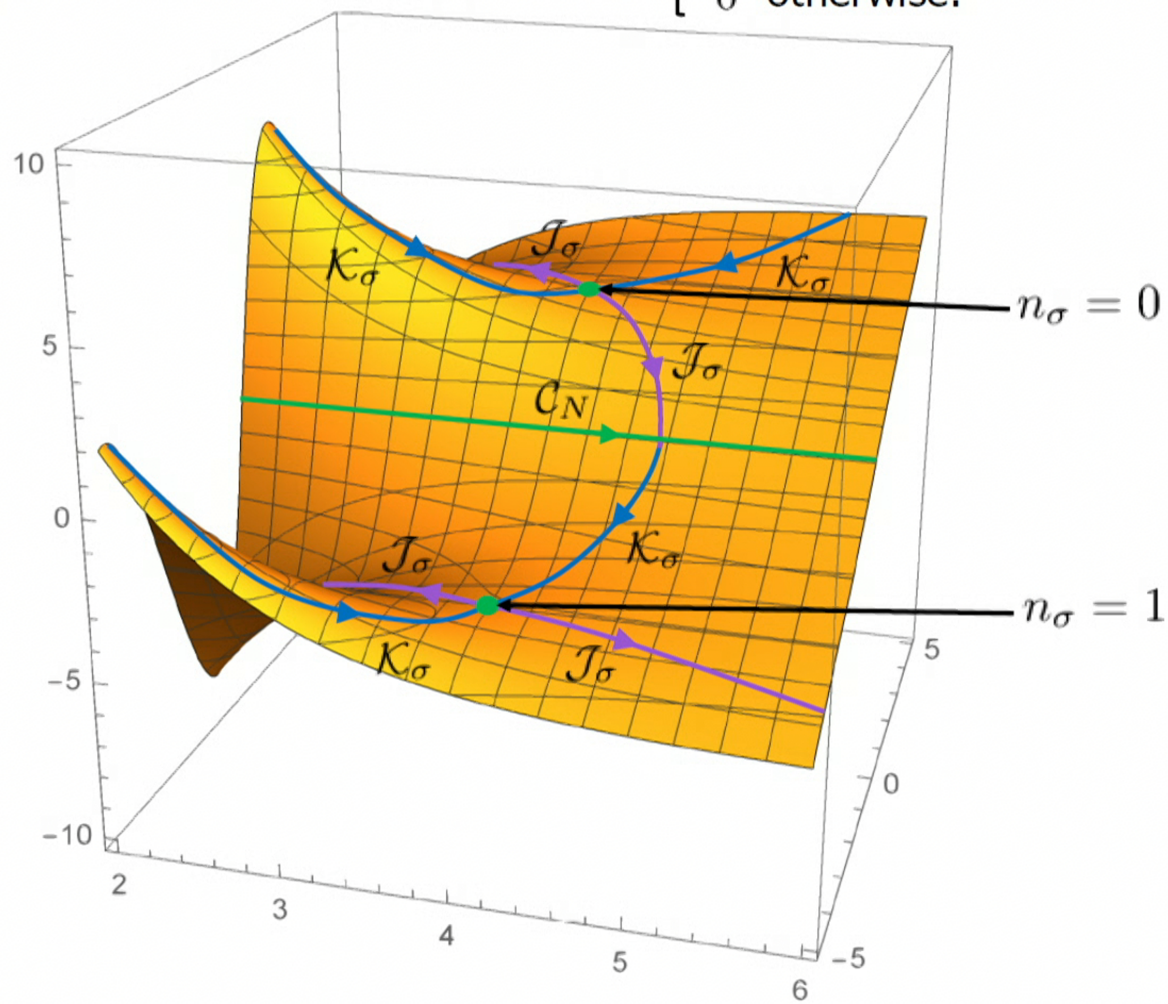
Lefschetz thimble $\mathcal{J}_\sigma$ = steepest **descent** contour through SP

$\mathcal{K}_\sigma$:
steepest
ascent
through
SP

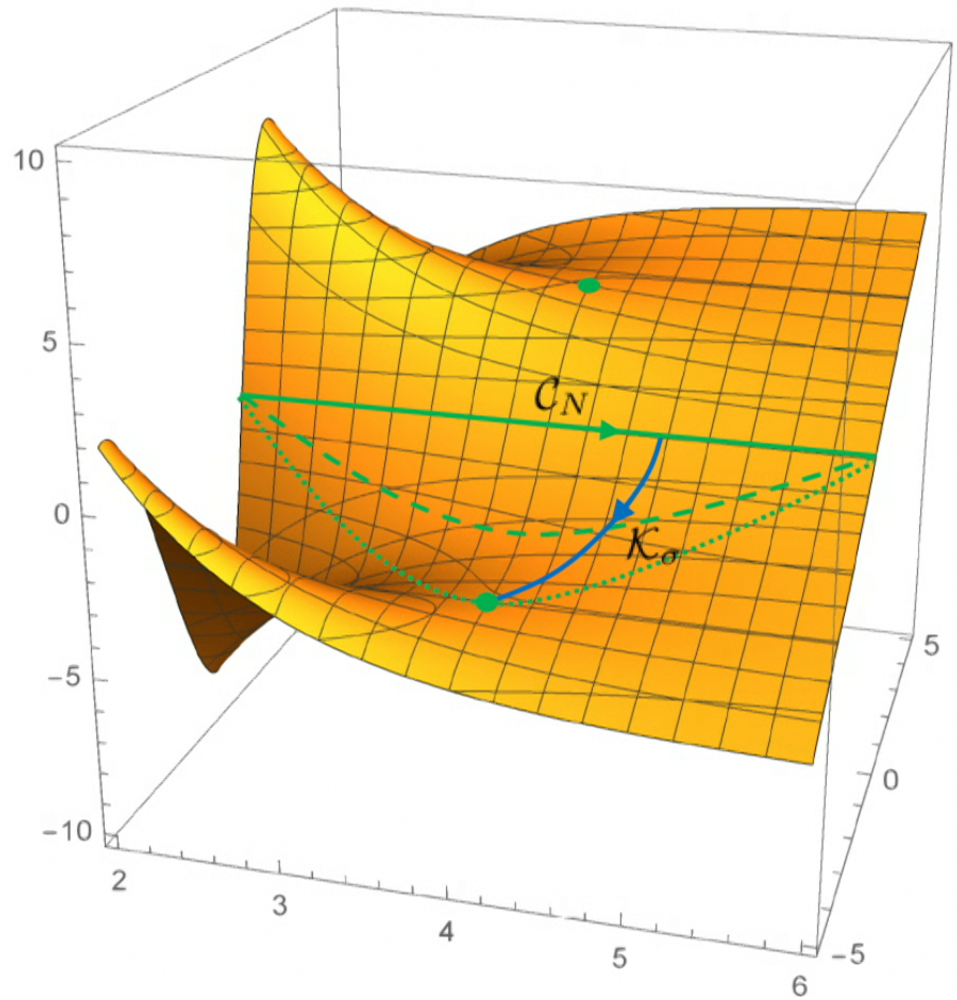


Intersection number

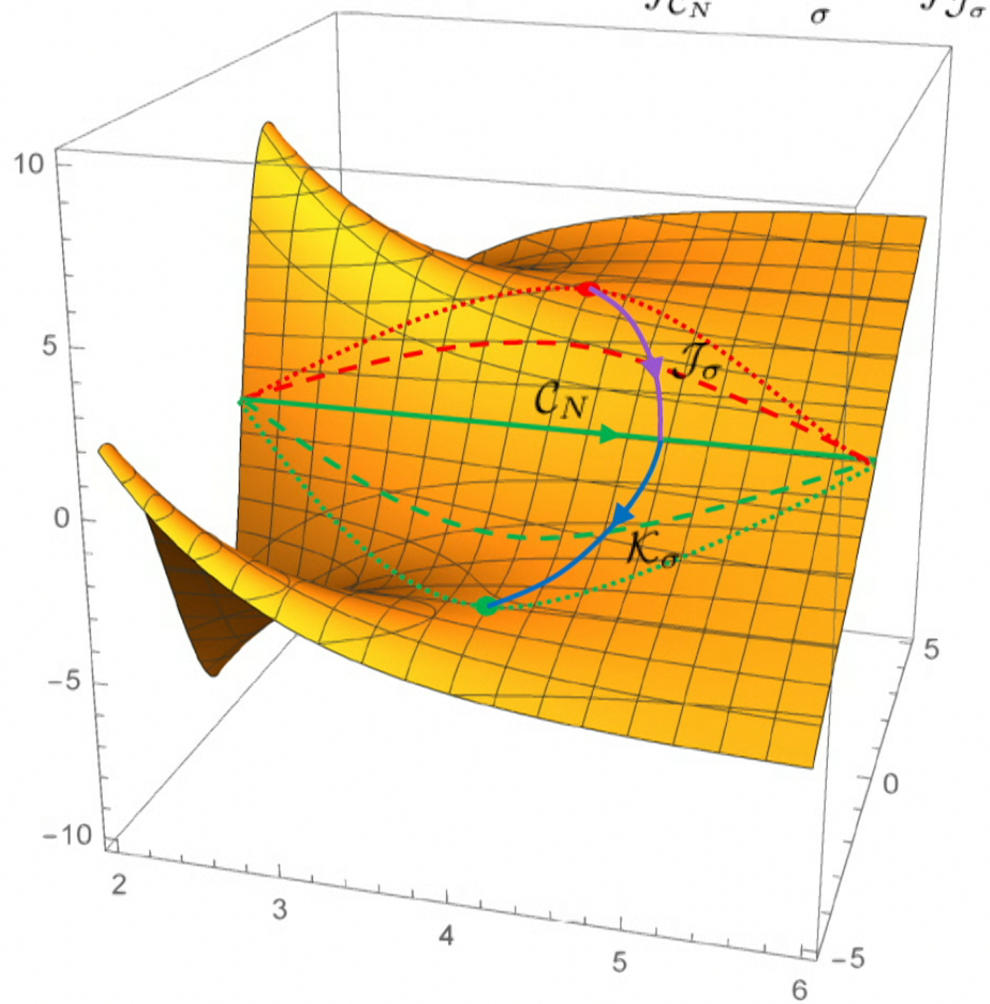
$$n_\sigma = \mathrm{Int}\{\mathcal{K}_\sigma, \mathcal{C}_N\} \begin{cases} 1 & \text{for } \textbf{\textit{relevant}} \text{ SPs,} \\ 0 & \text{otherwise.} \end{cases}$$



Can smoothly "slide" the contour from $\mathcal{C}_N$ onto the thimble $\mathcal{J}_\sigma$

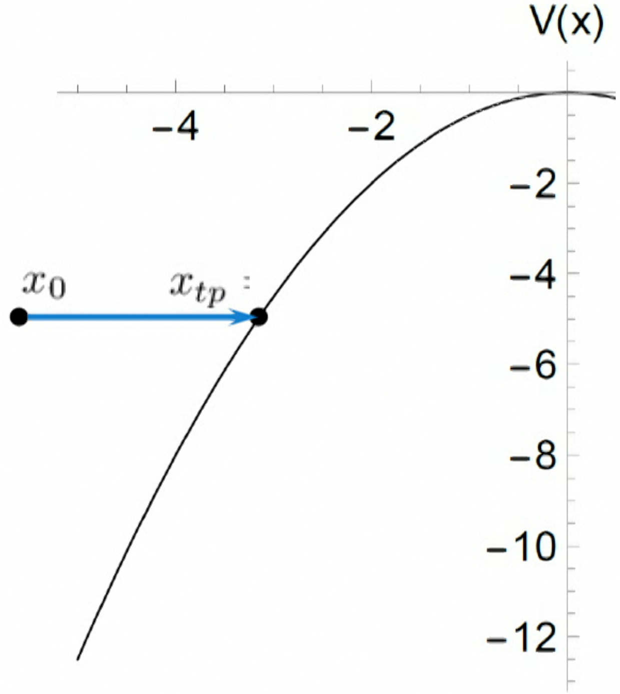


Picard-Lefschetz theory: $\int_{\mathcal{C}_N} \to \sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma}$



1. Saddle Points for $x_0 = -x_1$: $\mathcal{I}_{,N}(N_\sigma) = 0$

$$N_{sp} = \boxed{\pm 2T_{cl}} \pm \boxed{i(2n+1)\pi}, \quad n \in \mathbb{N}$$



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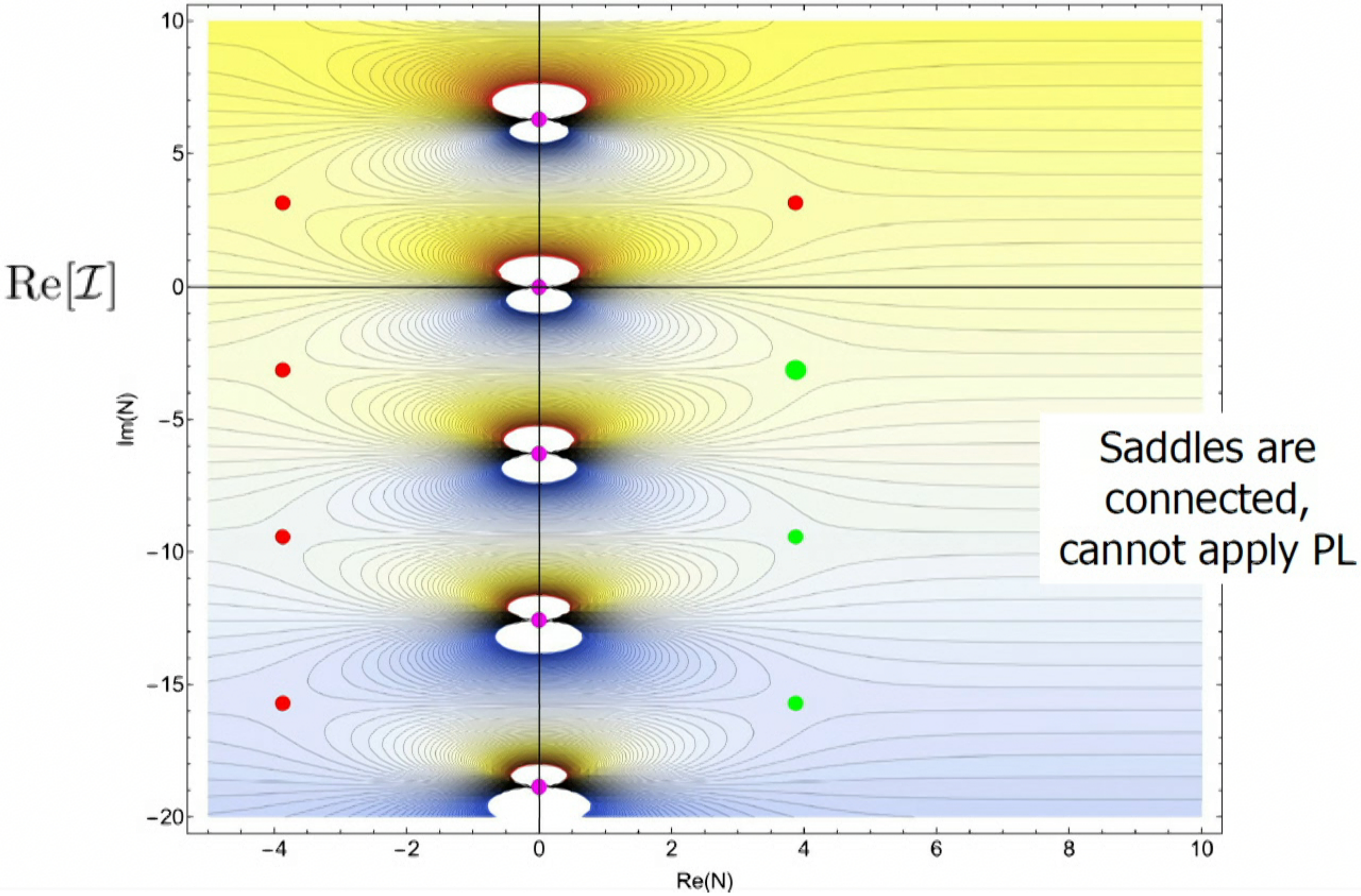
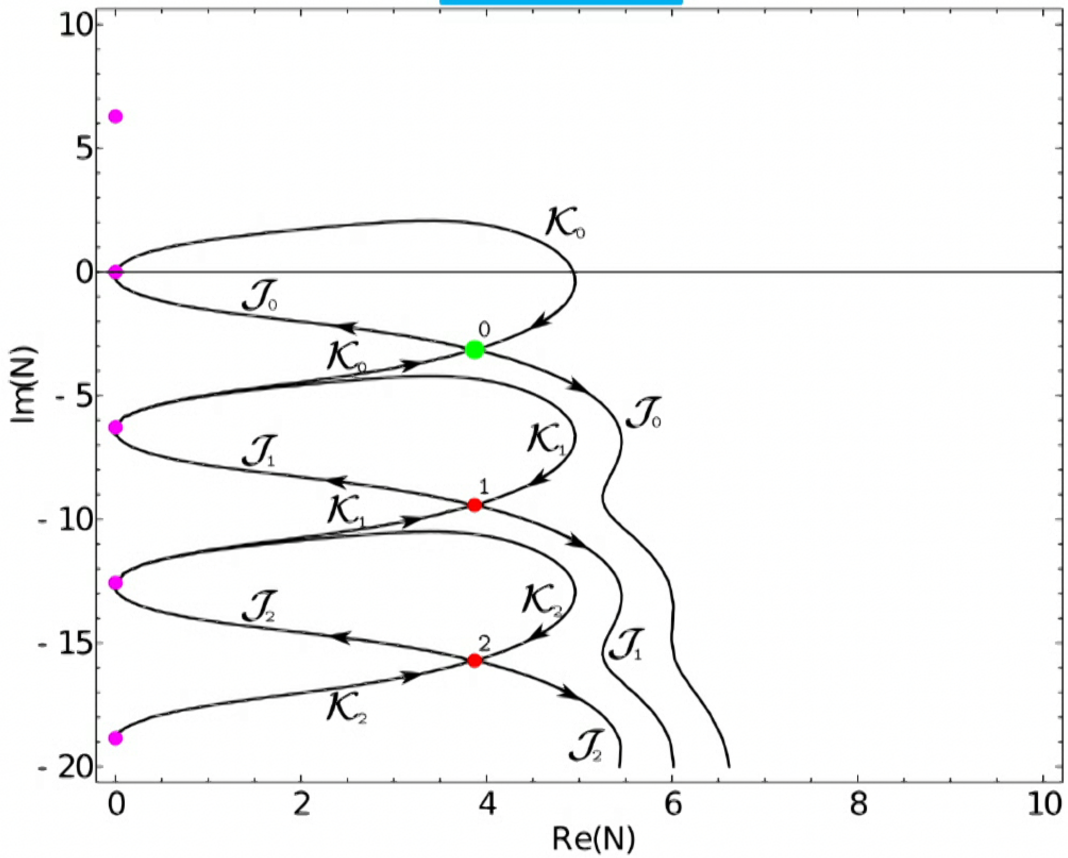


IHO: 2. Relevant SPs :

the **Stokes phenomenon** $\hbar \to \hbar\, e^{i\alpha}$

$\boxed{\alpha \to 0^+}$



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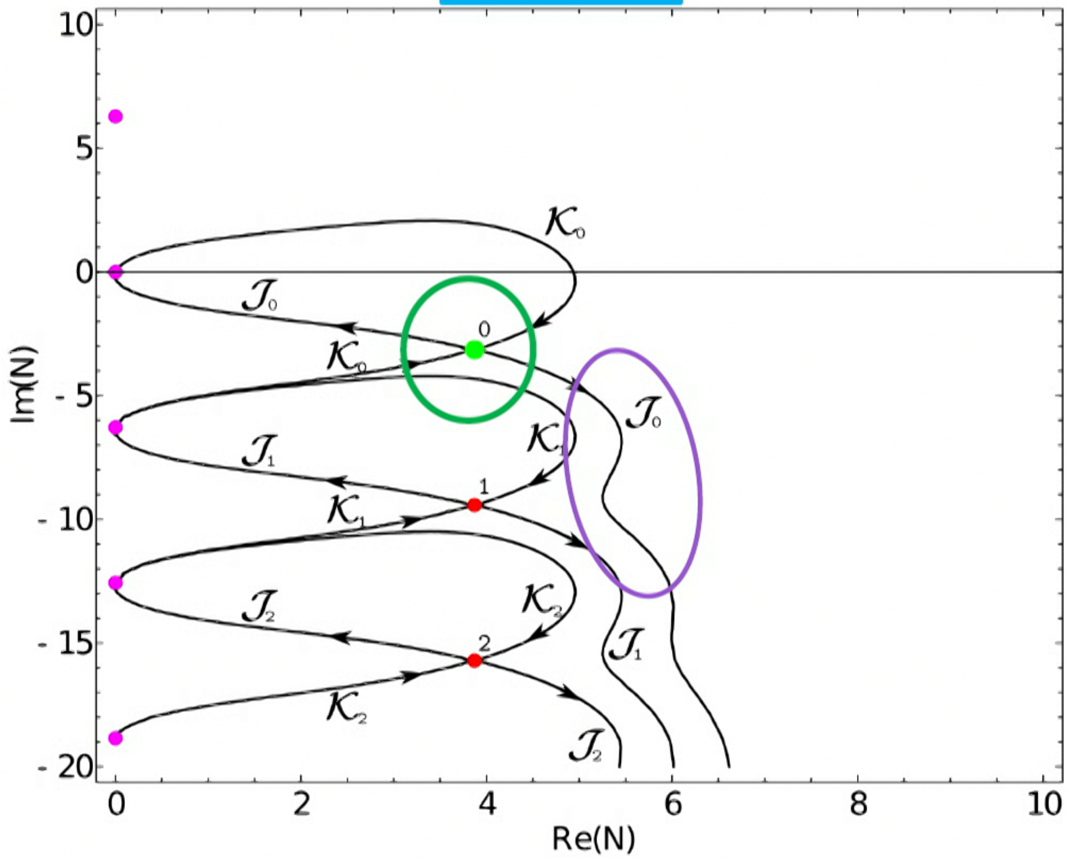

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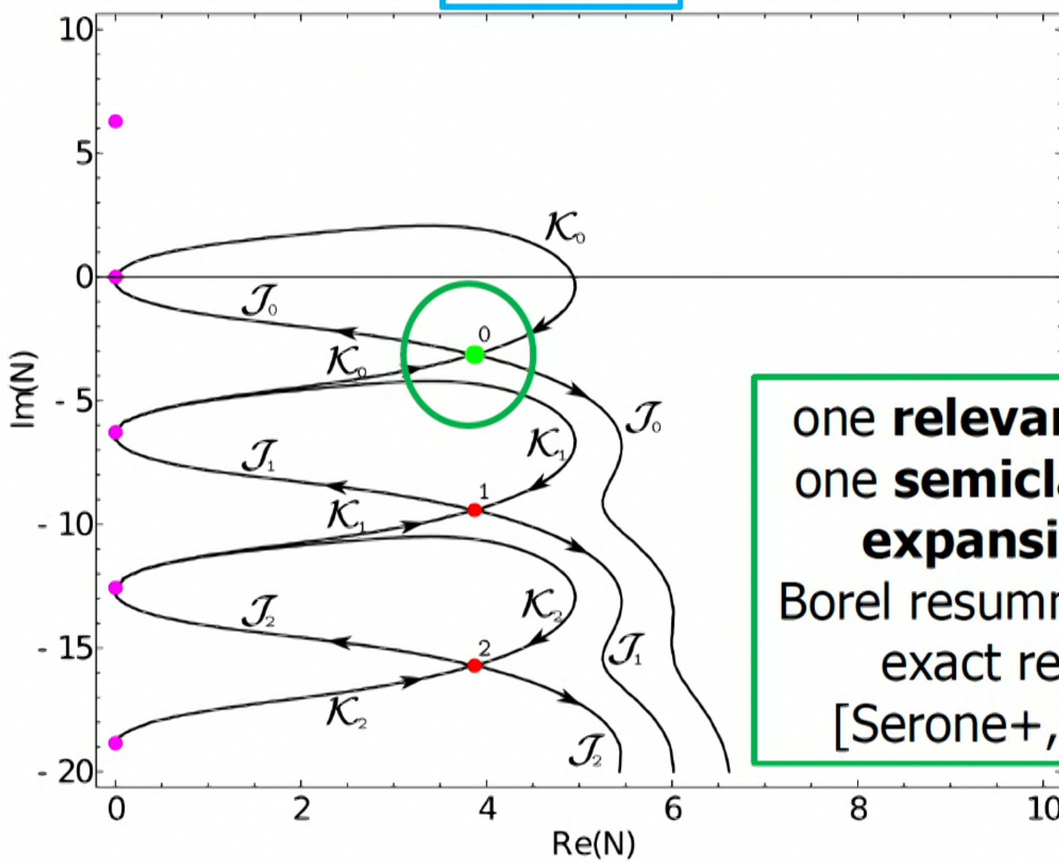
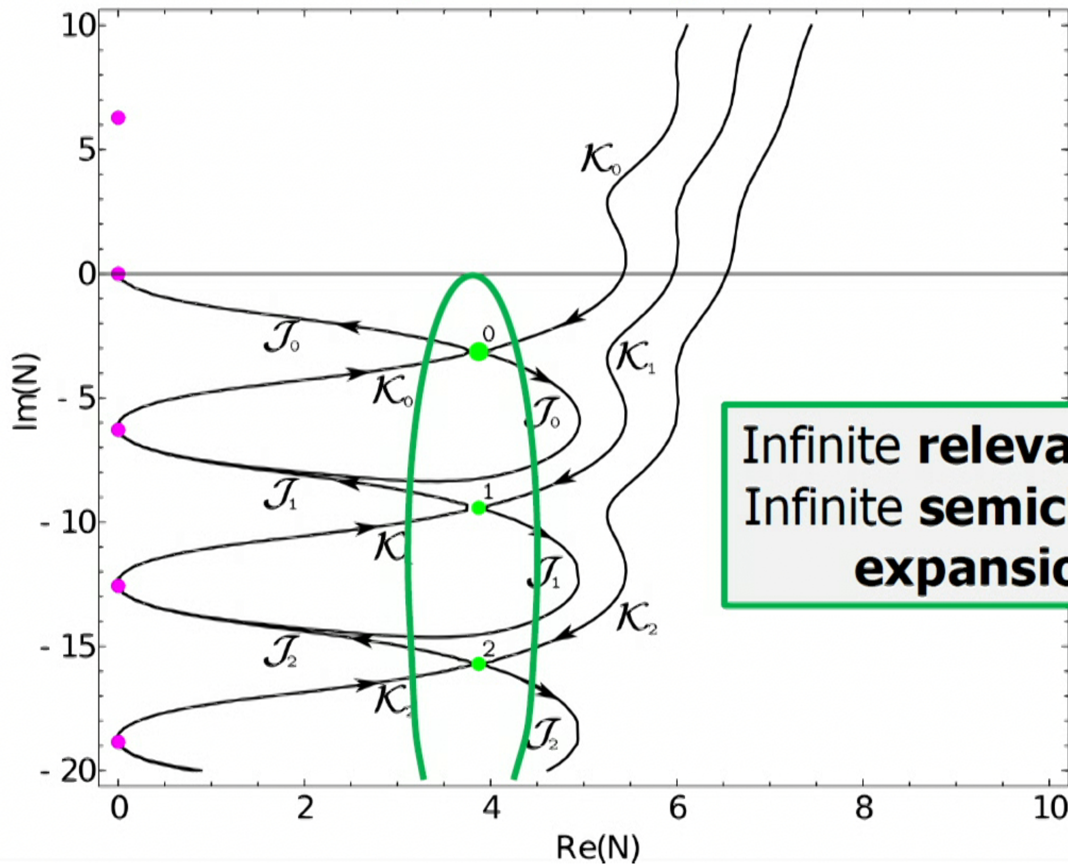


one **relevant** SP =
one **semiclassical expansion**=
Borel resummable to exact result
[Serone+,2017]

IHO: 2. Relevant SPs :

the **Stokes phenomenon** $\hbar \to \hbar\, e^{i\alpha}$

$$\boxed{\alpha \to 0^-}$$



Infinite **relevant** SP = Infinite **semiclassical expansions**

IHO: 4. Semi-classical expansion in $\hbar$

Angle at which the thimble meets the SP

The energy propagator:

Depending on **Stokes edge**: 0 or ∞

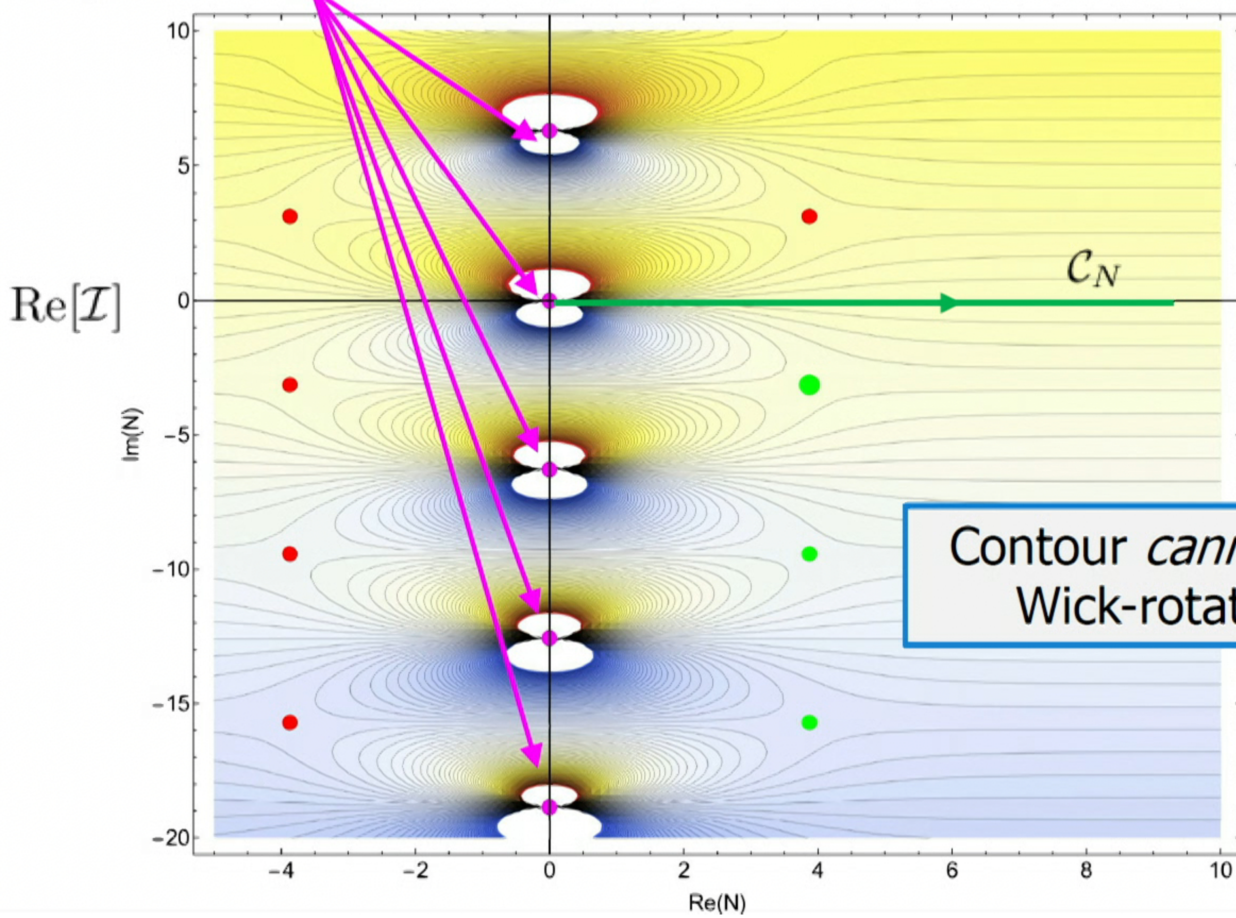
$$\boxed{G_E(x_1,x_0) = \frac{e^{i\left(2E\,T_{cl}/\hbar + x_0^2/\hbar \tanh(T_{cl})\right)}}{2\sqrt{x_0^2 \tanh^2(T_{cl})}} e^{i\theta(\hbar)} \sum_{n=0}^{\infty} e^{-|E|\pi(1+2n)/\hbar}(1+O(\hbar))}$$

Contribution from **dominant** SP = **WKB** result:

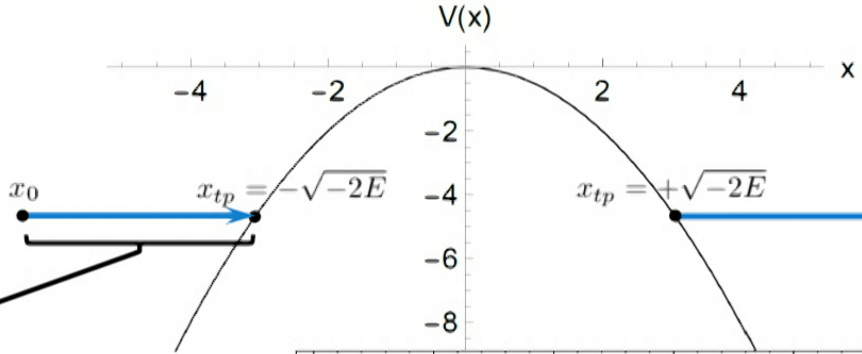
$$\boxed{G_E(x_1,x_0) \simeq e^{-|E|\pi}}$$

Relation to the Euclidean Path Integral

Singularities on the imaginary axis! $N_{\rm sing} = \pm ik\pi, \text{ with } k \in \mathbb{N}$



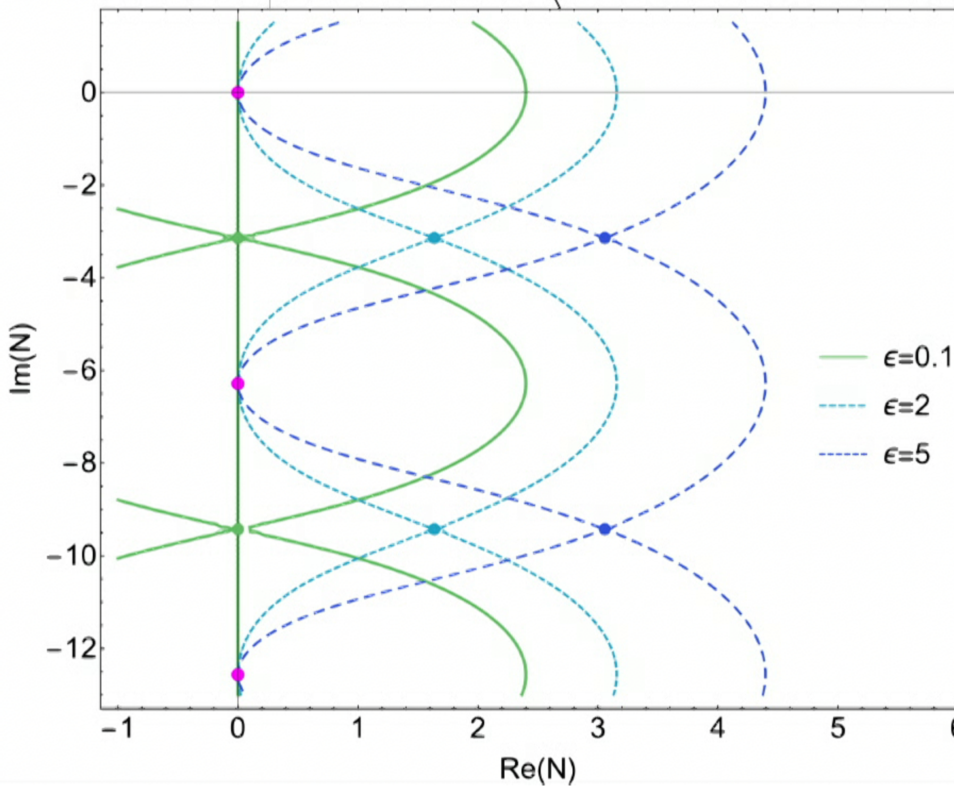
Contour *cannot* be Wick-rotated.



$\epsilon = x_0 - x_{tp}$

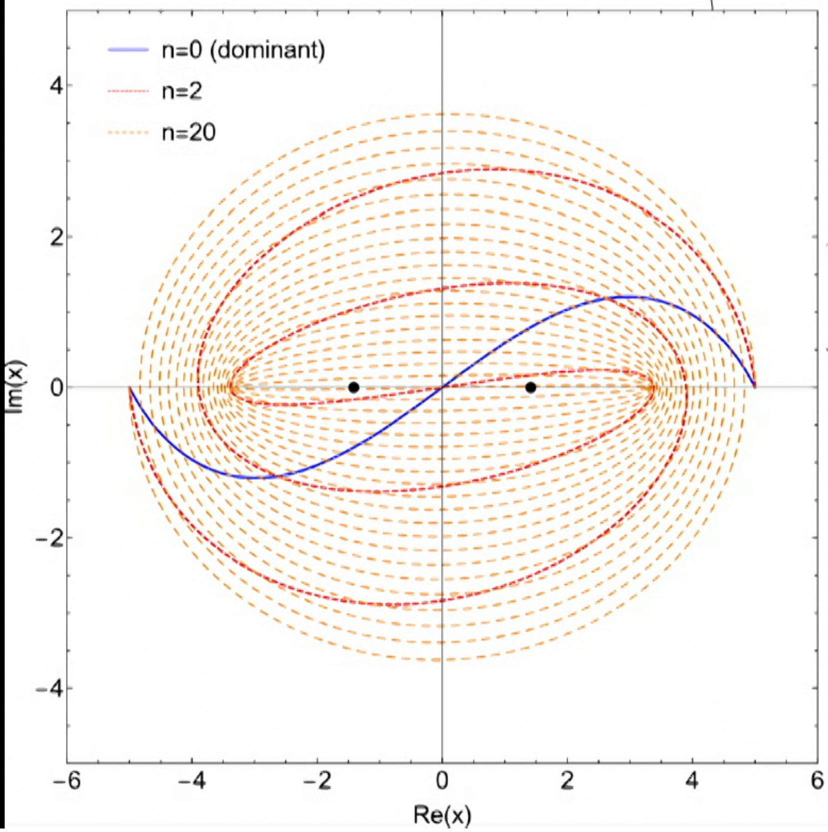
In the *now* well-defined limit $\epsilon \to 0$:

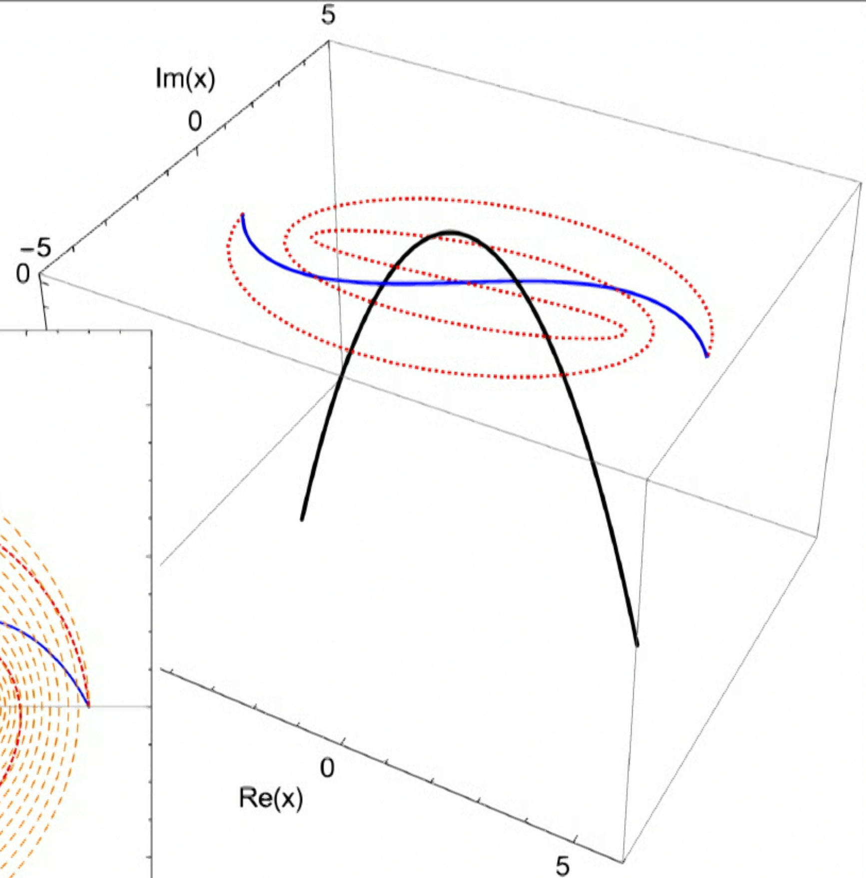
$\mathrm{Re}[N_{sp}] = \pm 2T_{cl} \to 0$



Classical Solution

in parameter time





Take-home points

We can describe **quantum tunneling** with a **Lorentzian Path Integral.**

The energy propagator has **complex-time saddle points**.

Thanks to **Picard-Lefschetz theory**, the integral is made manifestly convergent and we can use the SP approximation.

Euclidean path integral can be recovered **as a limiting case** in the Lorentzian approach.

On going: more physical picture (**energy average**), an application to cosmology (**membrane bubble**).


Thank you!