Title: Quantum critical responses via holographic models and conformal perturbation theory

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Abstract: We investigate response functions near quantum critical points, allowing for finite temperature and a mild deformation by a relevant scalar. When the quantum critical point is described by a conformal field theory, we use conformal perturbation theory and holography to determine the two leading corrections to the scalar two-point function and to the conductivity. We build a bridge between the couplings fixed by conformal symmetry with the interaction couplings in the gravity theory. We construct a minimal holographic model that allows us to numerically obtain the response functions at all frequencies, independently confirming the corrections to the high-frequency response functions. In addition to probing the physics of the ultraviolet, the holographic model probes the physics of the infrared giving us qualitative insight into new physics scalings.

Quantum critical responses via holographic models and conformal perturbation theory

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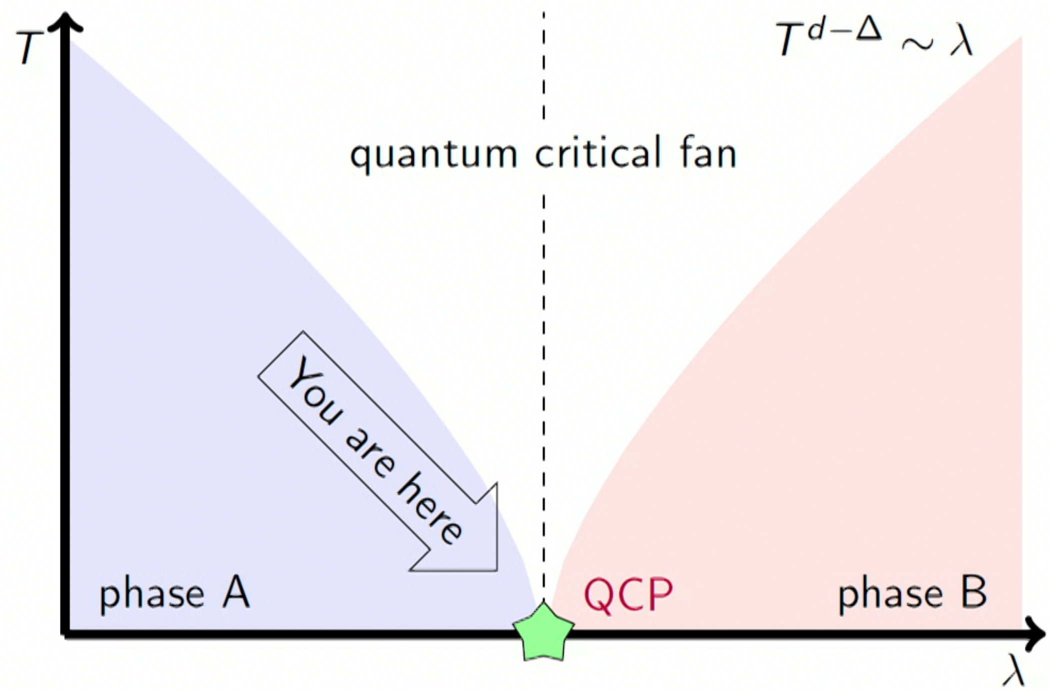
Outline

Quantum Critical System



Quantum Critical Systems

- Quantum Critical Points (QCP)s are associated with continuous phase transitions between ground states at zero temperature
- Some of the best understood instances of QCPs are described by conformal field theories (CFT)s
- A Quantum Critical Point is a many-body system with a gapless energy spectrum
- A canonical example of a CFT is the QC phase transition at zero temperature in the quantum Ising model in $1+1$ or in $2+1$ dimensions, which results from tuning the transverse magnetic field across a critical value

Field-Operator Duality

- A consequence of holography is the so-called "field-operator" duality in which for each operator in the quantum field theory, we identify a dual field in the gravitational theory

Type	**operator**	**field**	**Mass/scaling dimension**
scalar	$\mathcal{O}$	ϕ	$m^2L^2 = -\Delta(d-\Delta)$
massless spin-2	$T_{\mu\nu}$	$g_{\mu\nu}$	$m = 0,\ \Delta = d$
conserved 1-form	J_μ	A_μ	$m = 0,\ \Delta = d-1$
general p-form	$\boldsymbol{A}^{(p)}$	χ	$m^2L^2 = (\Delta - p)(d - \Delta - p)$
spin 1/2, 3/2	ψ	Ψ	$\lvert m\rvert L = \Delta - d/2$

Table: A table showing the duality between CFT operators and AdS fields with a column showing the relationship between the masses of AdS fields and the scaling dimensions for CFT operators.

Linear Response

- Goal: Create a formalism for computing high-frequency asymptotic behaviour for two-point functions $\langle \mathcal{O}_0 \mathcal{O}_0 \rangle$ and $\langle J_x J_x \rangle$ in the presence of a relevant $(\Delta < d)$ scalar operator $\mathcal{O}_\Delta$ deformation
- Approach: Derive the generic high-frequency expansion for two-point functions at high-temperature and finite scalar deformation
  - Expand the scalar deformation in the CFT action to find the correction due to scalar deformation
  - Expand the operator product expansion for $\mathcal{O}_0(\omega)\mathcal{O}_0(-\omega)$ and $J_x(\omega)J_x(-\omega)$ at large momenta to find the correction due to temperature
- Generic result:

$$\langle \mathcal{Y}(-\Omega)\mathcal{Y}(\Omega)\rangle = \Omega^{2\Delta_{\mathcal{Y}}-d}\left(\mathcal{C}_{\mathcal{Y}\mathcal{Y}} + b\frac{\lambda}{\Omega^{d-\Delta}} + a\frac{\langle \mathcal{O}\rangle_{\lambda=0}}{\Omega^{\Delta}} + \cdots\right) \tag{1}$$

Conductivity

- The AC conductivity at finite Euclidean frequency $\Omega = i\omega$ is given by
$$\sigma(\Omega) = -\frac{1}{\Omega}\langle J_x(\Omega) J_x(-\Omega)\rangle$$
- At non-zero temperatures the scalar one-point function $\langle \mathcal{O}_\Delta \rangle$ is non-trivial and it contributes to the conductivity
- The current-current Operator Product Expansion (OPE) has a large-frequency expansion

$$J_x(\mathbf{\Omega}) J_x(\boldsymbol{p} - \mathbf{\Omega}) = \Omega^{d-2}\left(c_{JJ}\delta^{(d)}(\boldsymbol{p}) - \frac{c_{JJO}\mathcal{O}_\Delta(\boldsymbol{p})}{\Omega^\Delta}\right) + \ldots$$

- c_{JJ} is fixed by $\langle J_x J_x \rangle_{T=0}$ and as we will soon see c_{JJO} is fixed by $\langle J_x J_x \mathcal{O}_\Delta \rangle_{T=0}$ and $\langle \mathcal{O}_\Delta \mathcal{O}_\Delta \rangle_{T=0}$

Fixing CFT Data

- C_{JJ} is fixed by taking the zero-temperature expectation value of both sides of the OPE (Recall that $\langle \mathcal{O} \rangle_{T=0} = 0$)

$$J_x(\boldsymbol{\Omega}) J_x(\boldsymbol{p} - \boldsymbol{\Omega}) = \Omega^{d-2} \left(c_{JJ}\, \delta^{(d)}(\boldsymbol{p}) - \frac{c_{JJO}\ \mathcal{O}_\Delta(\boldsymbol{p})}{\Omega^\Delta} \right)$$

- To see how we fix C_{JJO}, we can take the JJ OPE

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- To see how we fix C_{JJO}, we can take the JJ OPE
- Multiply both sides by the relevant scalar

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- To see how we fix C_{JJO}, we can take the JJ OPE
- Multiply both sides by the relevant scalar
- Take the zero-temperature expectation value of the equation

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- To see how we fix C_{JJO}, we can take the JJ OPE
- Multiply both sides by the relevant scalar
- Take the zero-temperature expectation value of the equation
- And finally, by setting $\langle \mathcal{O} \rangle_{T=0} = 0$ we find the relationship between C_{JJO}, $\langle J_x J_x \mathcal{O}_\Delta \rangle_{T=0}$ and $\langle \mathcal{O}_\Delta \mathcal{O}_\Delta \rangle_{T=0}$

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- Similarly, we can relate the correction due to the scalar deformation to CFT data
- In addition, the scalar-scalar two point function can be fixed in the same manner

Holography

- We can use the full power of holography in order to calculate these zero-temperature expectation values
- A minimal bulk action that describes a gravitational theory with non trivial $\langle JJ\mathcal{O}\rangle$ and $\langle \mathcal{O}_0\mathcal{O}_0\mathcal{O}\rangle$ is

$$S = S_0 + S_\phi + S_\psi + S_A \tag{2}$$

where

$$\begin{aligned}
S_0 &= \frac{1}{2\ell_p^{d-1}} \int d^{d+1}x\sqrt{g}\left(R + \frac{d(d-1)}{L^2}\right), \\
S_\phi &= -\frac{1}{2\ell_p^{d-1}} \int d^{d+1}x\sqrt{g}\left[(\nabla_a\phi)^2 + m^2\phi^2 - 2\alpha_C L^2\phi C_{abcd}C^{abcd}\right], \\
S_\psi &= -\frac{1}{2\ell_p^{d-1}} \int d^{d+1}x\sqrt{g}\left[(\nabla_a\psi)^2 + (m_0^2 - \alpha_\psi\phi)\psi^2\right], \\
S_A &= -\frac{1}{4g_d^2} \int d^{d+1}x\sqrt{g}(1 + \alpha_F\phi)F_{ab}F^{ab}.
\end{aligned} \tag{3}$$

Holography

- The two point functions can be calculated by taking a functional derivative of the gauge action

$$\langle J_x(\mathbf{k}) J_x(-\mathbf{k}) \rangle = -\frac{\delta^2 S_{\text{gauge}}}{\delta A_0^x(\mathbf{k}) \delta A_0^x(-\mathbf{k})}$$

$$\langle \mathcal{O}_\Delta(\mathbf{k}) \mathcal{O}_\Delta(-\mathbf{k}) \rangle = -\frac{\delta^2 S_{\text{gauge}}}{\delta \phi_0(\mathbf{k}) \delta \phi_0(-\mathbf{k})}$$

Bulk coupling	Bulk operator	CFT correlator ($T=0$)	Observable
L^{d-1}/ℓ_P^{d-1}	R	$\langle T_{\mu\nu}\, T_{\rho\delta} \rangle$	C_T
$1/g_d^2$	$F_{ab}F^{ab}$	$\langle J_\mu\, J_\nu \rangle$	σ_∞
$m^2 L^2$	ϕ^2	$\langle \mathcal{O}\, \mathcal{O} \rangle$	Δ
α_C	$\phi\, C_{abcd} C^{abcd}$	$\langle T_{\mu\nu}\, T_{\rho\delta}\, \mathcal{O} \rangle$	$\mathcal{C}_{TT\mathcal{O}}$
α_F	$\phi\, F_{ab}F^{ab}$	$\langle J_\mu J_\nu\, \mathcal{O} \rangle$	$\mathcal{C}_{JJ\mathcal{O}}$
α_ψ	$\phi\psi^2$	$\langle \mathcal{O}_0 \mathcal{O}_0 \mathcal{O} \rangle$	$\mathcal{C}_{\mathcal{O}_0\mathcal{O}_0\mathcal{O}}$

Table: The five dimensionless parameters which characterize the bulk gravity theory and the dual correlators in the boundary CFT which they control.

Holography

- A convenient and graphical approach to doing calculations such as these, is to first find the bulk-boundary propagators $K_\Delta(z, \boldsymbol{k})$ and $G_{\mu\nu}(z, \boldsymbol{k})$ defined by source fields at the boundary

$$\phi(z, \boldsymbol{x}) = \int \frac{d^d k}{(2\pi)^d} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} K_\Delta(z, \boldsymbol{k}) \phi_0(\boldsymbol{k})$$

$$A_\mu(z, \boldsymbol{x}) = \int \frac{d^d k}{(2\pi)^d} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} G_{\mu\rho}(z, \boldsymbol{k}) A_0^\rho(\boldsymbol{k})$$

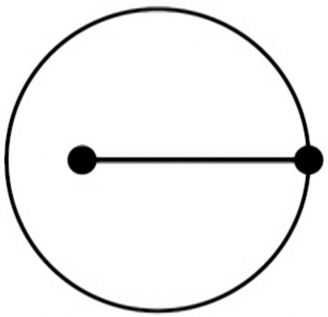

Figure: Scalar bulk-boundary propagator

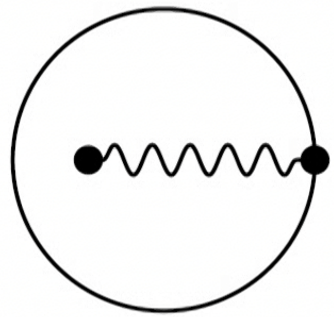

Figure: Gauge bulk-boundary propagator

Holography

- We then substitute these expansions into the action and use the equations of motion

$$S_{\text{scalar}} = -\frac{1}{2\ell_p^2} \int d^d x \sqrt{g} g^{zz} \phi \partial_z \phi \Big|_{z=\epsilon}$$

$$S_{\text{gauge}} = -\frac{1}{2g_4^2} \int d^d x \sqrt{g} g^{zz} g^{\mu\nu} A_\mu \partial_z A_\nu \Big|_{z=\epsilon}$$

- Pictorially, we can represent this calculation by Witten diagrams

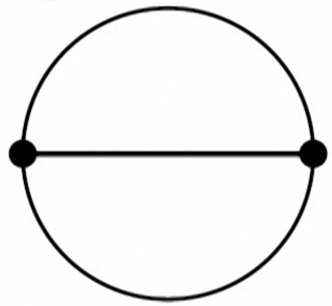

Figure: The two-point function $\langle \mathcal{O}_\Delta \mathcal{O}_\Delta \rangle_{T=0}$

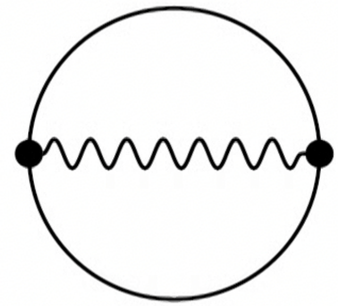

Figure: The two-point function $\langle J_x J_x \rangle_{T=0}$

Holography

- Calculating $\langle J_\mu J_\nu \mathcal{O}_\Delta \rangle$ is similar:
- We expand the action in terms of the bulk-boundary propagators and evaluate
- pictorially, this is represented by the Witten diagram shown on the right
- Providing that there is a three-point interaction involving $\phi F_{\mu\nu} F^{\mu\nu}$
- The $\langle J_\mu J_\nu \mathcal{O}_\Delta \rangle$ correlation function is calculated using both K_Δ and $G_{\mu\nu}$
- A similar calculation is done for $\langle \mathcal{O}_0 \mathcal{O}_0 \mathcal{O} \rangle$

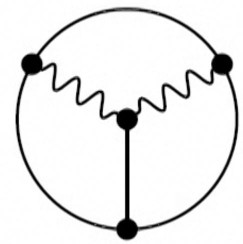

Figure: $\langle J_\mu J_\nu \mathcal{O}_\Delta \rangle$

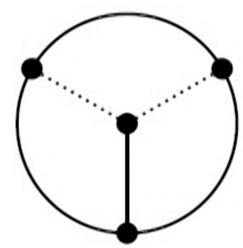

Figure: $\langle \mathcal{O}_0 \mathcal{O}_0 \mathcal{O} \rangle$

Perturbative Expansion of the Conductivity

- After calculating all of the necessary correlation functions one can now express the AC conductivity as a large-frequency expansion in terms of boundary data
- The scalar field ϕ dual to the relevant operator $\mathcal{O}_\Delta$ will have a characteristic asymptotic expansion

$$\phi(u) = \phi_0 u^{d-\Delta} + \phi_1 u^{\Delta} + O(u^{2d-\Delta})$$

- The ϕ_0 term is related to the boundary source of the scalar operator and is a representation of the scalar deformation from the CFT while the ϕ_1 term is related to the thermal scalar one-point function $\langle \mathcal{O} \rangle$

Perturbative Expansion of the Conductivity

With this expansion we see that the high-frequency corrections to the AC conductivity are

$$\sigma(\Omega) = \frac{L^{d-3}}{g_4^2}\Omega^{d-3}\Bigg(\sigma_\infty - \frac{\alpha_F \ell_p^{d-1}\lambda}{L^{d-1}\Omega^{d-\Delta}} \frac{(d-\Delta)(2-\Delta)\Psi(d, d-\Delta, d-1)}{2^{d-3}\Gamma(d/2-1)^2} - \frac{\alpha_F \ell_p^{d-1}\langle\mathcal{O}\rangle}{L^{d-1}\Omega^{\Delta}} \frac{\Delta(\Delta-d+2)\Psi(d,\Delta,d-1)}{(2\Delta-d)2^{d-3}\Gamma(d/2-1)^2}\Bigg). \tag{4}$$

where Ψ is related to the triple Bessel function integral

$$\Psi(d,\Delta,\Delta_0) = \frac{\sqrt{\pi}\Gamma\left(\frac{\Delta}{2}\right)\ \Gamma\left(\frac{\Delta+d-2\Delta_0}{2}\right)\Gamma\left(\frac{\Delta+2\Delta_0-d}{2}\right)}{4\Gamma\left(\frac{\Delta+1}{2}\right)} \tag{5}$$

A Holographic Model

- We now construct an explicit holographic model with all of the ingredients necessary to measure linear responses in the presence of a relevant scalar
- A simple metric that satisfies the background equations of motion is the planar AdS Schwarzschild metric

$$ds^2 = \frac{r_0^2}{L^2 u^2}\left(-f(u)dt^2 + dx^2 + dy^2\right) + \frac{L^2 du^2}{u^2 f(u)}$$

- r_0 sets the scale for the location of the black hole horizon
- $f(u) = 1 - u^d$
- As $u \to 0$ we approach the AdS boundary
- There is a black hole horizon located at $u = 1$
- The black hole has temperature $T = \frac{d r_0}{4\pi L^2}$
- The introduction of the black hole allows us to tune the temperature of the dual quantum field theory in order to probe the phase diagram

A Holographic Model

The solution to the scalar field equations of motion are found using Green's functions

$$\phi(u) = {}_2\mathrm{F}_1\left(\frac{\Delta}{d}, \frac{\Delta}{d}; \frac{2\Delta}{d}; u^d\right)\left(\phi_1 - \frac{d(d-1)^2(d-2)\alpha_C}{2\Delta - d} g_\Delta(u)\right) u^\Delta$$

$$- \; {}_2F_1\left(\frac{d-\Delta}{d}, \frac{d-\Delta}{d}; 2\frac{d-\Delta}{d}; u^d\right)\left(\frac{d(d-1)^2(d-2)\alpha_C}{2\Delta - d} h_\Delta(u)\right) u^{d-\Delta}$$

with

$$g_\Delta(u) = \int_0^u dy\, y^{2d-1-\Delta}\; {}_2\mathrm{F}_1\left(\frac{d-\Delta}{d}, \frac{d-\Delta}{d}; 2\frac{d-\Delta}{d}; y^d\right)$$

$$h_\Delta(u) = \int_0^u dy\, y^{d-1+\Delta}\; {}_2\mathrm{F}_1\left(\frac{\Delta}{d}, \frac{\Delta}{d}; \frac{2\Delta}{d}; y^d\right)$$

A Holographic Model

- The AC conductivity can be computed holographically via

$$\sigma(\omega) = \frac{G_{xx}(\omega)}{i\omega}$$

- where G_{xx} is the retarded Green's function for the transverse gauge field A_x. The retarded Green's function can be expressed in Fourier space by

$$G_{xx} = -\frac{1}{g_d^2}\sqrt{g}\,g^{uu}g^{xx}\frac{\partial_u A_x(u,\boldsymbol{k})}{A_x(u,\boldsymbol{k})}\bigg|_{u\to 0}$$
$$= -\frac{1}{g_d^2}\left(\frac{r_0}{L^2}\right)^{d-2}\left(\frac{L^{d-3}A'}{u^{d-3}A}\right)_{u\to 0}$$

Numerical Approach

$$0 = A_x'' + \left(\frac{X'}{X} + \frac{f'}{f} - \frac{d-3}{u}\right) A_x' + \left(\frac{d\,\omega}{4\pi T}\right)^2 \frac{A_x}{f^2}$$

where $X = 1 + \alpha_F \phi$

- Integrate from horizon
- The gauge field $A_x(u)$ has an oscillatory divergence
- Define $A_x(u) = f(u)^b F(u)$ where $b = -\frac{i\omega}{4\pi T}$
- Infalling-boundary conditions
- $F(u)$ is regular everywhere

A Holographic Model

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A Holographic Model

- In order to calculate the conductivity we need to know the gauge field A is affected by this scalar deformation
- We look at the gauge equations of motion

$$\nabla_a \left[(1 + \alpha_F \phi) F^{ab} \right] = 0$$

- There is little hope in finding an exact solution to this equation of motion, so we will have to be satisfied with a numerical solution

Numerical Approach

$$0 = A_x'' + \left(\frac{X'}{X} + \frac{f'}{f} - \frac{d-3}{u}\right) A_x' + \left(\frac{d\,\omega}{4\pi T}\right)^2 \frac{A_x}{f^2}$$

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A Holographic Model

Recall

$$\sigma(\omega) = \frac{T^{d-2}L^{d-3}}{i\omega g_4^2}\left(\frac{4\pi}{d}\right)^{d-2}\left(\frac{\partial_u A_x}{u^{d-3}A_x}\right)_{u\to 0}$$

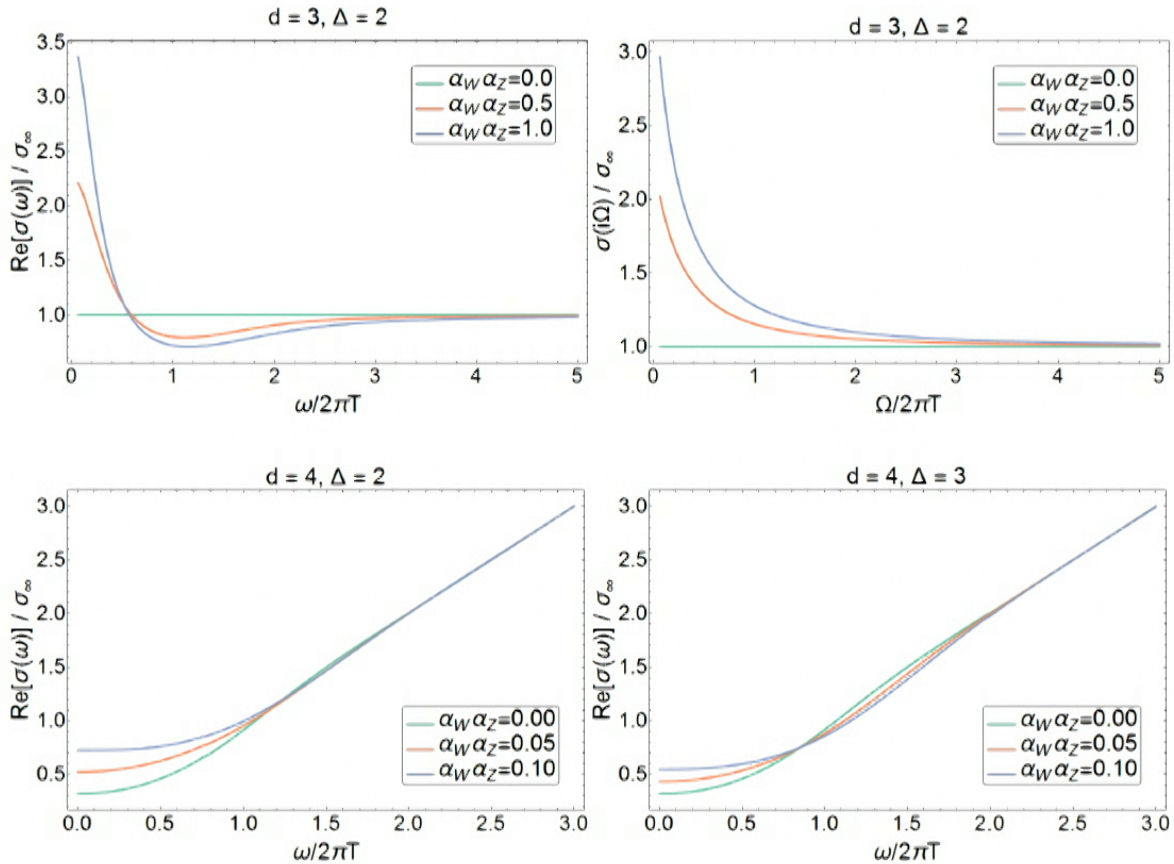
- With the near-boundary behavior of the gauge field A_x known we can extract the conductivity
- The gauge field will have two integration constants near the boundary
- We can extract these numerically from the integral solutions

$$d = 3; \qquad A_x(u) \to A_0 + A_1 u + O(u^2)$$

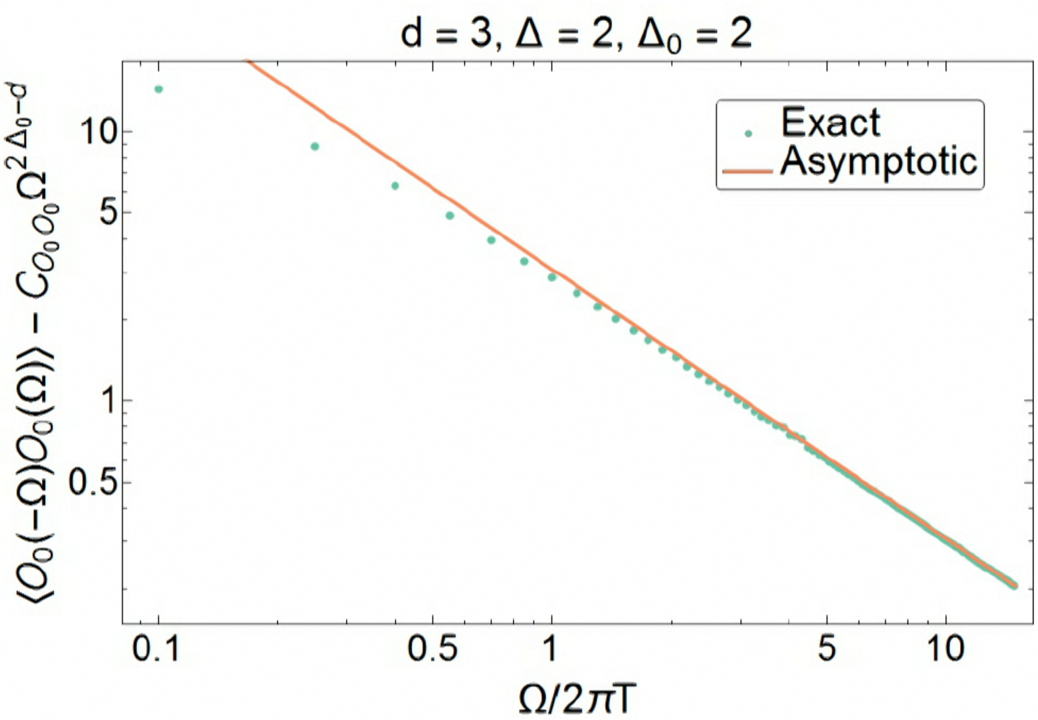
$$d = 4; \qquad A_x(u) \to A_0 + A_2 u^2 + 8b^2 A_0 u^2 \log(\Lambda u) + O(u^3)$$

$$d = 5; \qquad A_x(u) \to A_0 - \frac{b^2 d^2 A_0}{2} u^2 + A_3 u^3 + O(u^4)\,.$$

Conductivity



Comparisons



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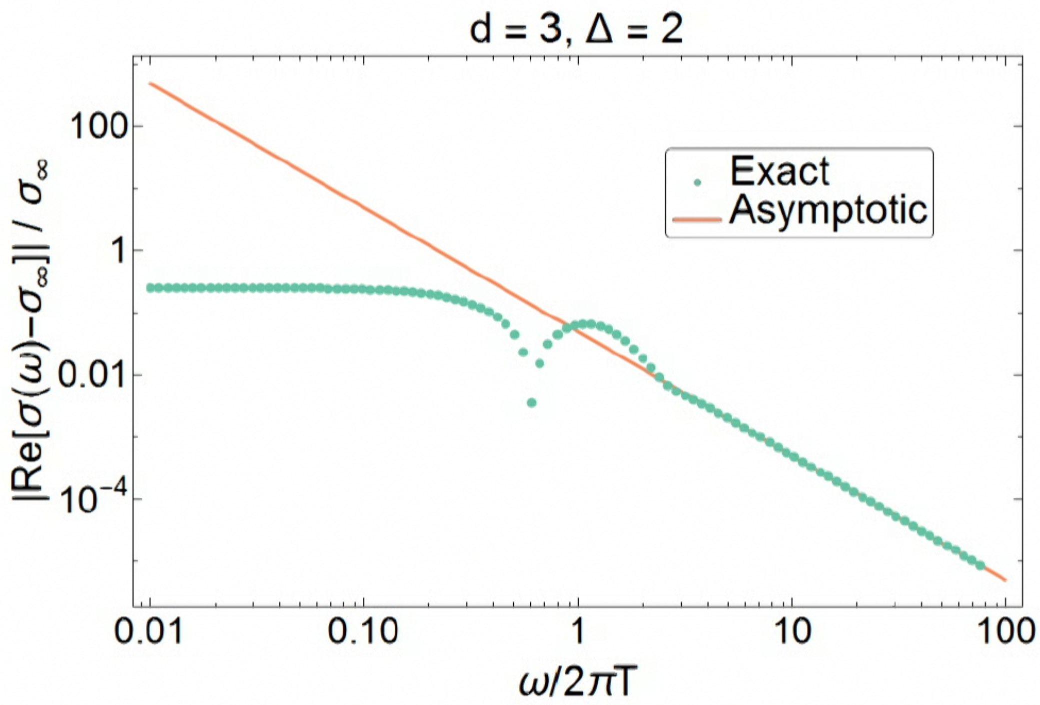
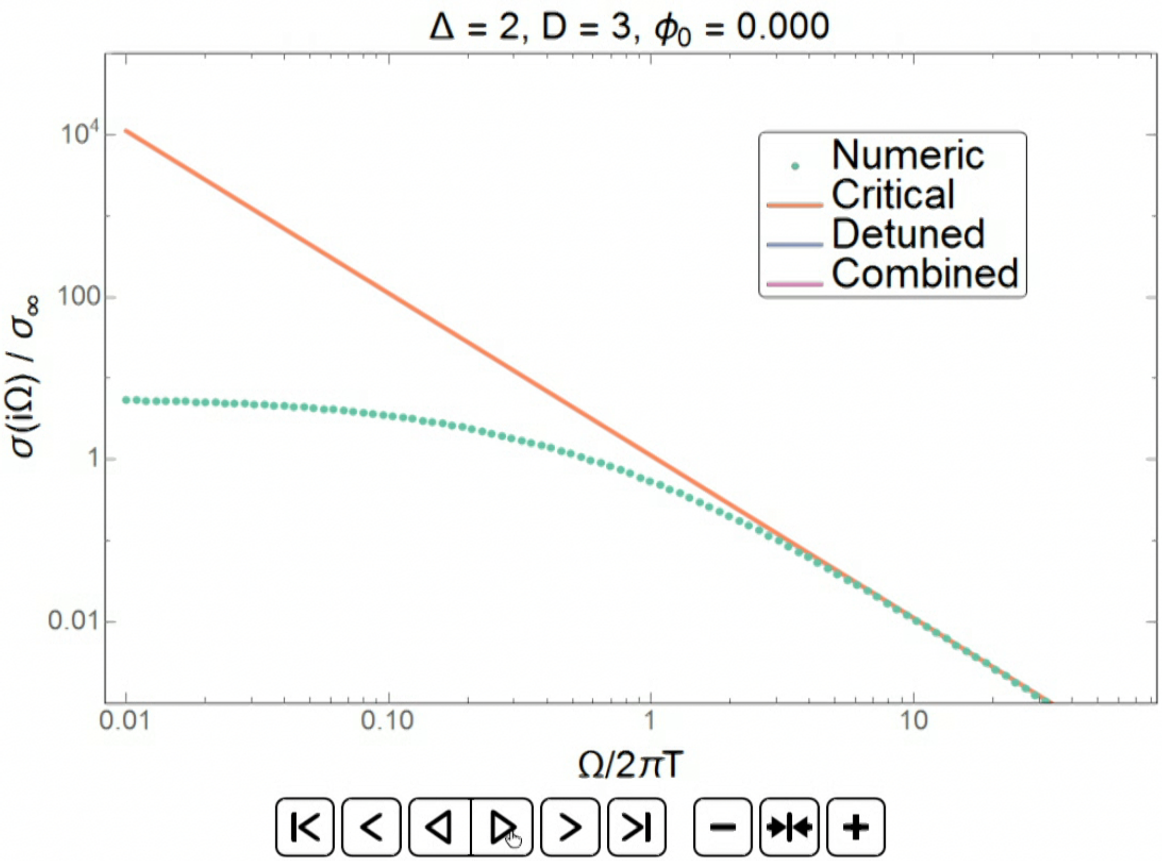


Deformation Animation



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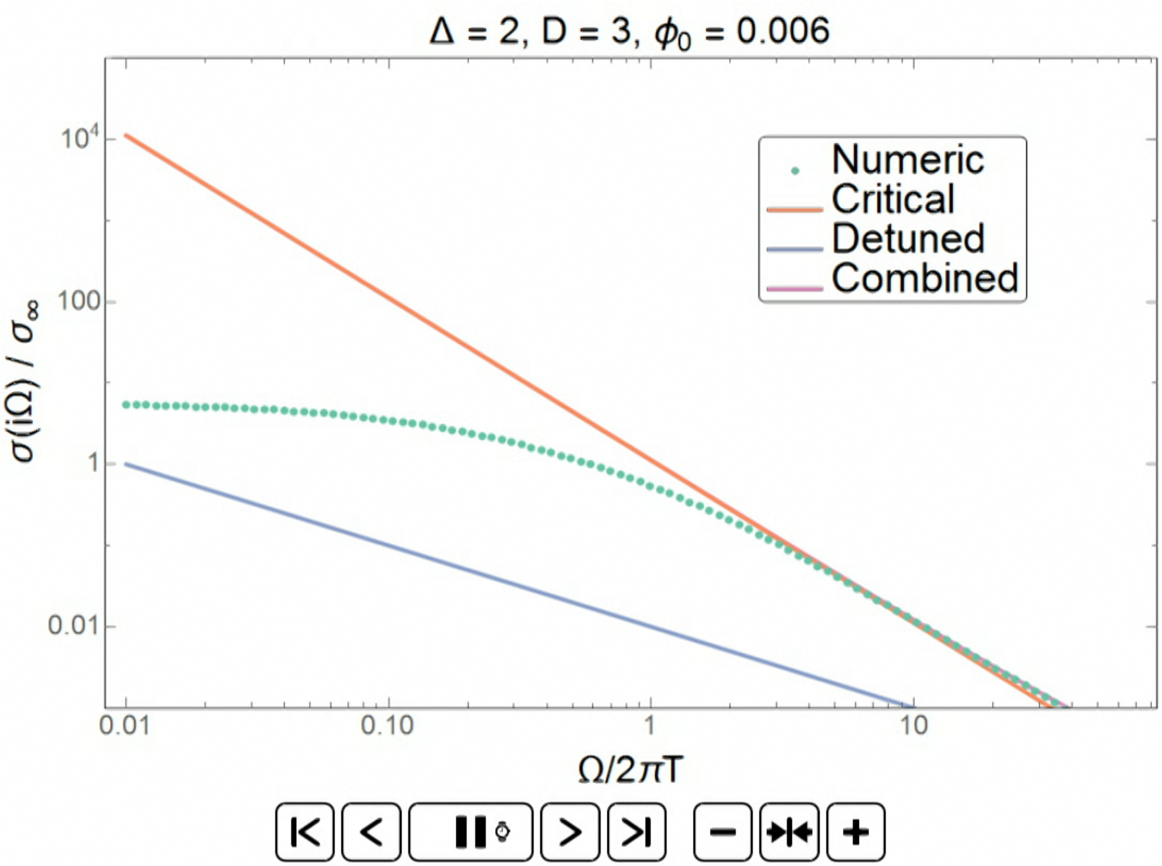

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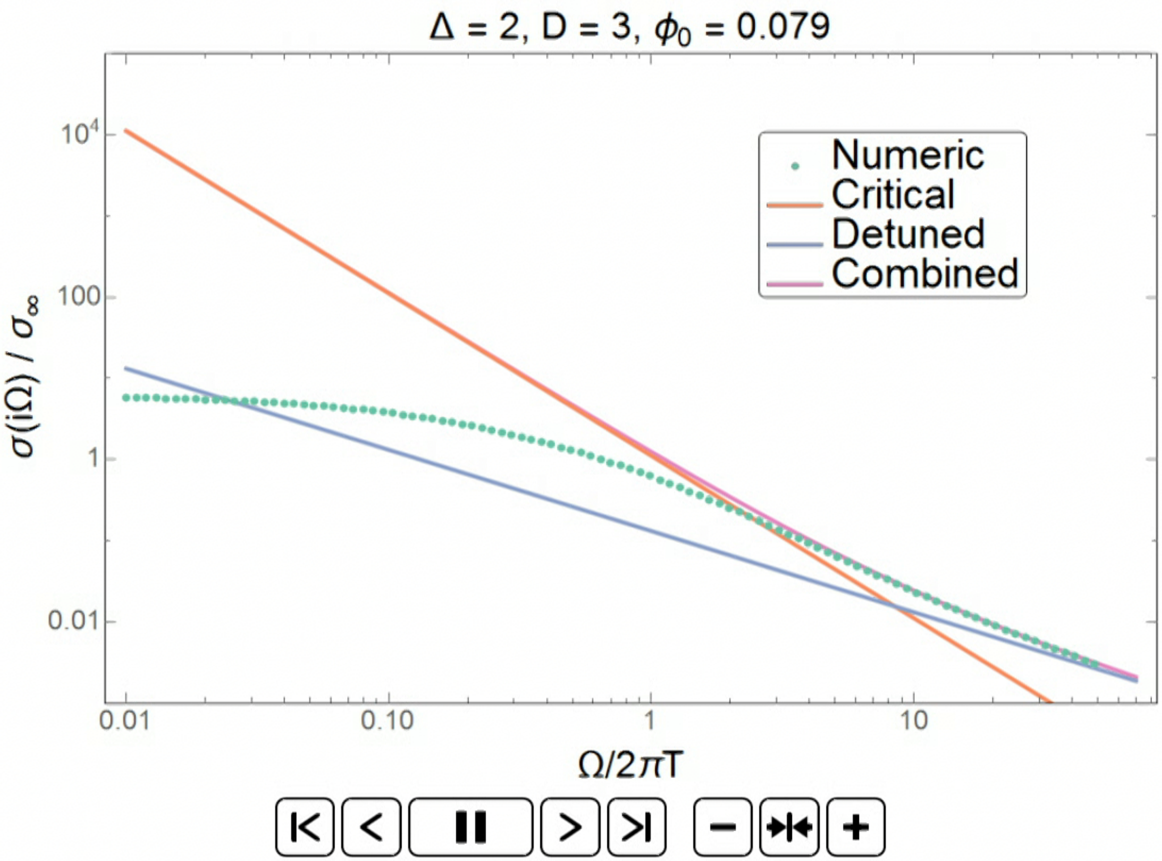

Conclusions

- We can calculate the high temperature linear responses for quantum field theories being deformed from a critical point by a relevant scalar operator
- By exploring conformal perturbation theory and operator product expansions we can predict the high-frequency corrections to linear response functions which we can verify using an explicit holographic model
- The holographic model presented can go on and make full-frequency predictions, offering some insight that can point towards discovering new universal qualities in CFTs