Title: Relative entropy with a twist

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Abstract: Quantum relative entropy is a measure of the indistinguishability of two quantum states in the same Hilbert space. I will discuss the relative entropy between a state with periodic boundary conditions and one with twisted boundary conditions for a free 1+1 CFT with c=1. I will also highlight the unresolved discrepancy between analytic and numeric results.

Quasi-topological quantum error correction codes

Hilary Carteret

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M. Hildred Blewett Fellowship
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www.aps.org

The Perimeter Institute, 31st May 2017

Outline

Introduction and motivation

A geometrical trick

Features of the error correction code

What might this allow us to do?

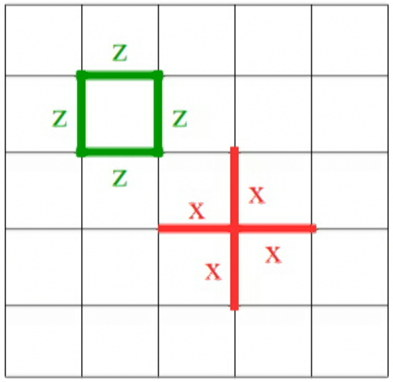
- Mimicking higher dimensional topological behaviours

Finite temperature thresholds and naturally occurring systems

- ...which can have some unexpected properties: toric bosons?!

Summary and open questions

Introduction and motivation



Topological error correction codes

Known problems with these

Dimensionality

Thermally stable in 4D (Dennis et al, JMP 2002)

Entanglement percolation transition in 5-6D (Hastings et al, PRL 2014)

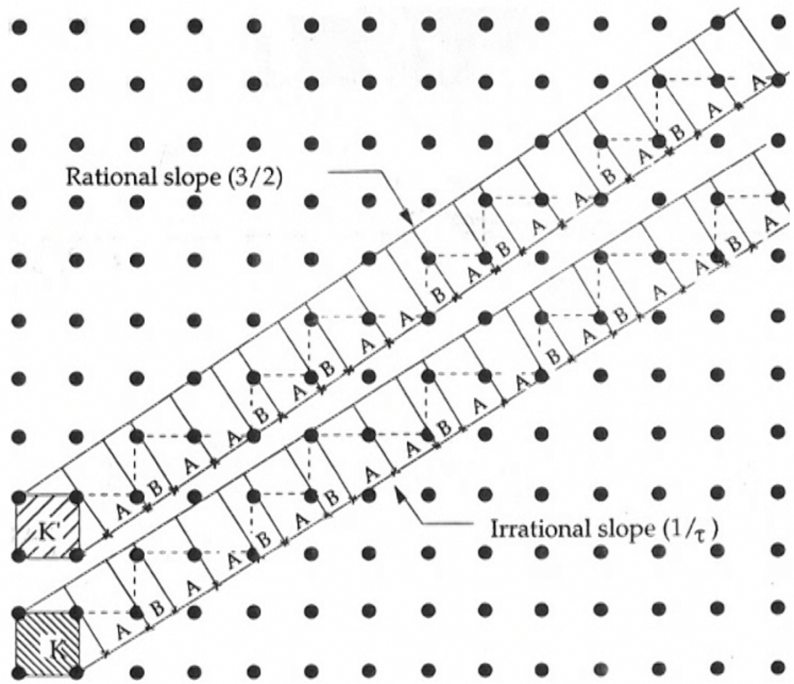
Gate errors

Beverland et al., JMP 2016

A geometrical trick, I

Note how the Quantum Monte Carlo simulations saw data collapse at about L=3, for D=5-6

Use a "cut and project" at an irrational angle.



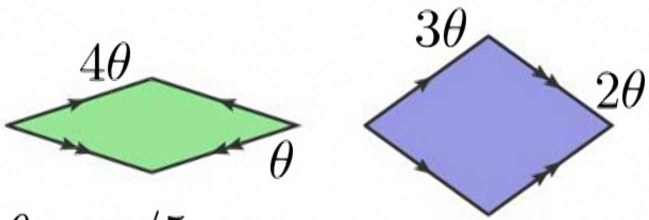
These are generally defined to be one lattice spacing wide...

(but they don't have to be!)

Maciá, Rep. Prog. Phys. 2005

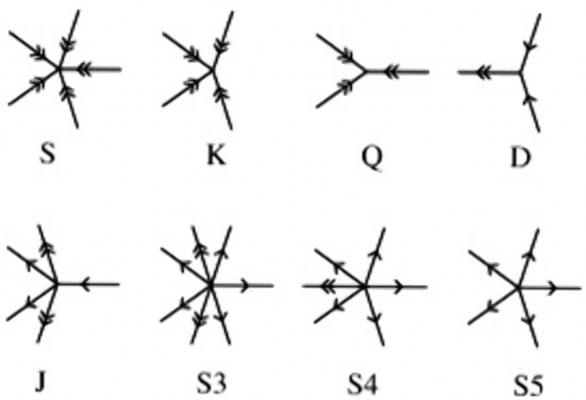
A geometrical trick, II

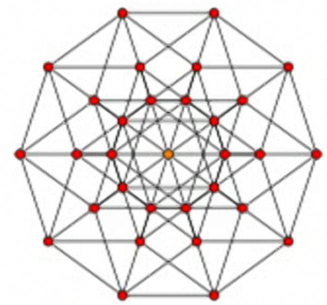
If we starting from a 5D hypercubic lattice
and project to 2D, we get a Penrose tiling:

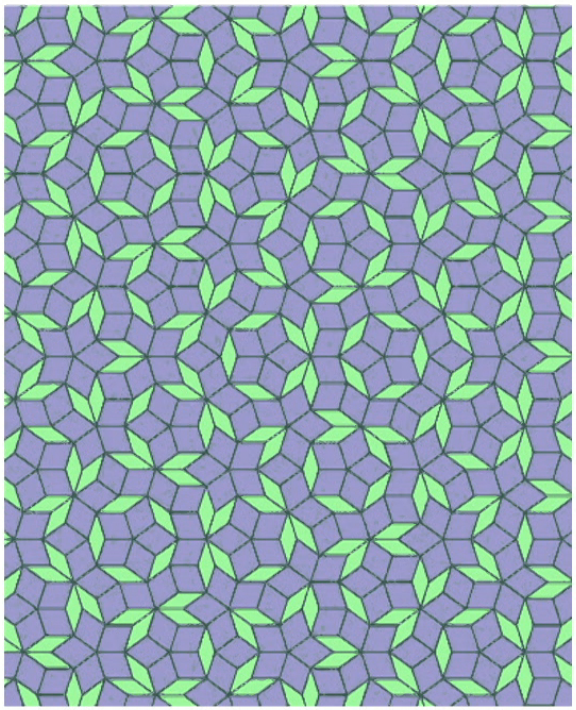


$\theta = \pi/5$

Allowed vertices:







http://www.ams.org/samplings/feature-column/fcarc-penrose

Choosing the code I

Will assume (for now) that we need to start with either a 5D simple hypercubic lattice, or a 4D face-centred hypercubic lattice.

(We are not guaranteed to get a Penrose tiling just because we started out with 5 dimensions and then projected down to 2 at the correct angle!)

There are several candidates. Look at more than one, as some Have potential problems...

1. The 4D toric code from (Dennis et al., JMP 2002)

The qubits are located at the centres of the faces of a hypercubic lattice. (However the corners are empty, which could complicate things later...)

Pros: low generator weight (6 and 6) and has a thermal stability proof in 4D (but that may not survive the projection step)

Cons: doesn't occupy the corners in the A_4 lattice

Choosing the code II

2. Some 5D toric code (e.g. 5D X-star, 4D Z-plaquette)

   Pros: occupies all the edges in the 5D hypercubic lattice

   Cons: ridiculously high generator weight:
   X-star: 10, Z-plaquette: 32
   (may lose some of that at the projection step, but see later)
   Don't know if it's thermally stable at finite T.

3. Some "A_4" code (that I just made up)

   Z-plaquettes are 3-cubes with qubits at the centre of all the faces *and* at all the corners (weight = 6+8 = 14)

   X-stars are 4 orthogonal intersecting 2x2 planes with qubits at the centre of all the faces *and* one at the centre, but none on the edges of the planes (weight = 4x4+1 = 17)

   Pros: occupies all the nodes in the A_4 lattice

   Cons: high generator weight; don't know about thresholds

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June 1, 2017

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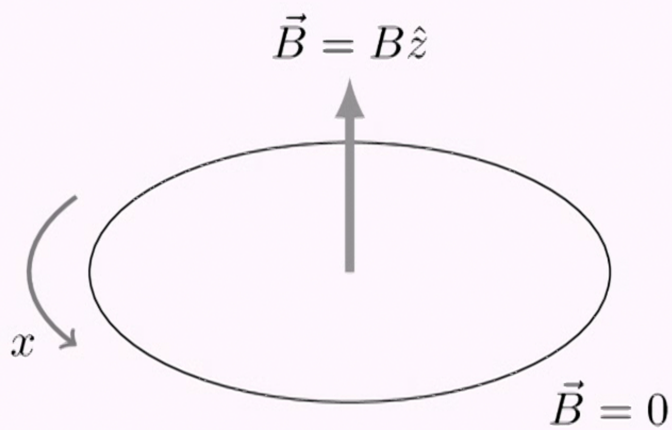
Twisted BCs

Periodic field $\phi_{_{PBC}}(x+L) = \phi_{_{PBC}}(x)$ subject to a constant gauge field $A_\mu(x) = A_x$.



Twisted BCs

We eliminate A_x with a field redefinition

$$\phi(x) = e^{-iq\int dx' A_x}\phi_{_{PBC}}(x)$$

The PBC translate to

$$e^{iqA_x(x+L)}\phi(x+L) = e^{iqA_x x}\phi(x)$$

so the new field has twisted BCs

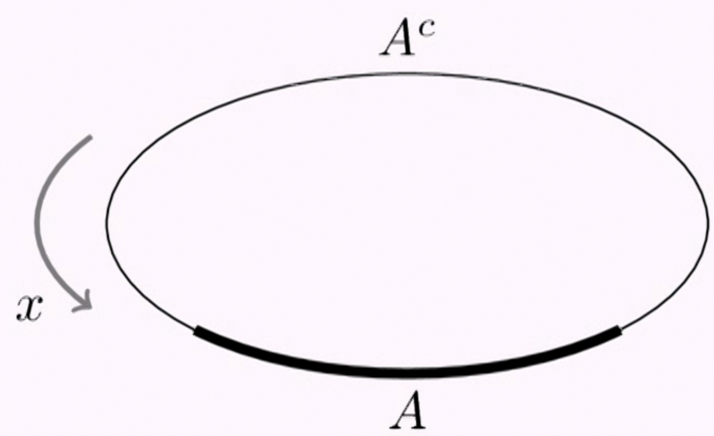
$$\phi(x+L) = e^{-iqA_x L}\phi(x) = e^{i\theta}\phi(x)$$

Q: How does the entanglement entropy depend on a twist θ?

Entanglement entropy

EE is defined as

$$S(\rho_A) = -\mathrm{Tr}\left(\rho_A \log \rho_A\right) = \lim_{\alpha\to 1} \frac{1}{1-\alpha}\mathrm{Tr}\left(\rho_A^\alpha\right)$$



Real time method

Write density matrix as

$$\rho_A = \frac{1}{\mathcal{N}} e^{-H_{\rho_A}} \qquad H_{\rho_A} = \sum_k \epsilon_k a_k^\dagger a_k$$

So the EE is

$$S(\rho_A) = -\mathrm{Tr}\, \rho_A \left(-\sum_k \epsilon_k a_k^\dagger a_k + \log \mathcal{N} \right)$$

From Bose-Einstein statistics, $\langle a_k^\dagger a_k \rangle = (e^{\epsilon_k} - 1)^{-1}$ and $\mathcal{N} = \prod_k (1 - e^{-\epsilon_k})^{-1}$

Real time method continued

For Hamiltonians for the form

$$\mathcal{H} = \frac{1}{2}\sum_i \pi_i^2 + \frac{1}{2}\sum_{i,j} \phi_i K_{ij} \phi_j$$

we can directly compute the eigenvalues ϵ_k of H_{ρ_A} from correlation functions

$$X_{ij} = \langle \phi_i \phi_j \rangle = \frac{1}{2}\left(K^{-\frac{1}{2}}\right)_{ij}$$

$$P_{ij} = \langle \pi_i \pi_j \rangle = \frac{1}{2}\left(K^{\frac{1}{2}}\right)_{ij}$$

The effect of twisted boundary conditions is to change $K_{1,L}, K_{L,1}$

Entanglement entropy

Restrict the correlation matricies X, P to the region A and define $C_A = \sqrt{X_A P_A}$

The eigenvalues λ_k of C_A are related to ϵ_k of H_{ρ_A}

$$\lambda_k = \frac{1}{2} \coth \frac{\epsilon_k}{2}$$

The EE in terms of λ_k is

$$S(\rho_A) = \sum_k \left[\left(\lambda_k + \frac{1}{2} \right) \ln \left(\lambda_k - \frac{1}{2} \right) - \left(\lambda_k - \frac{1}{2} \right) \ln \left(\lambda_k - \frac{1}{2} \right) \right]$$

Real time method continued

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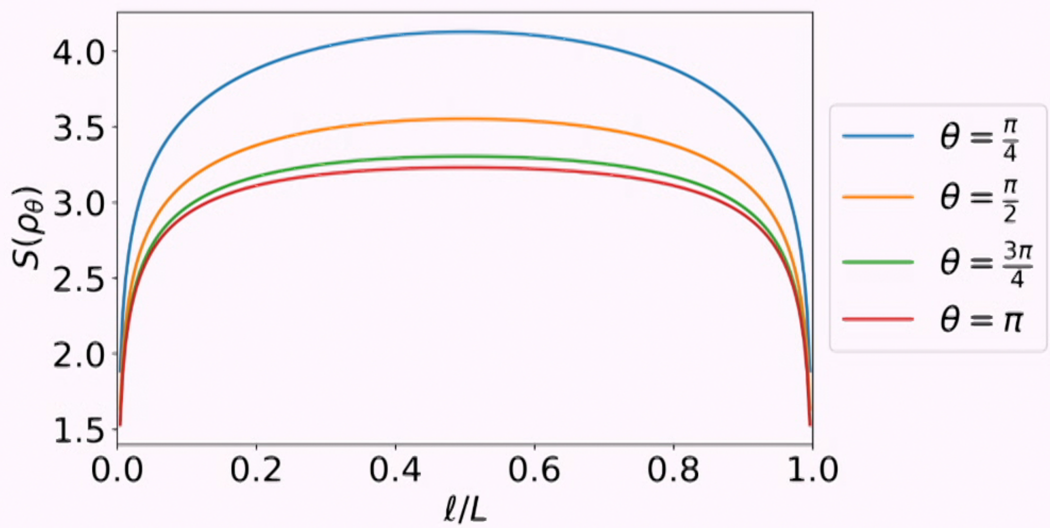
Results: Entanglement entropy

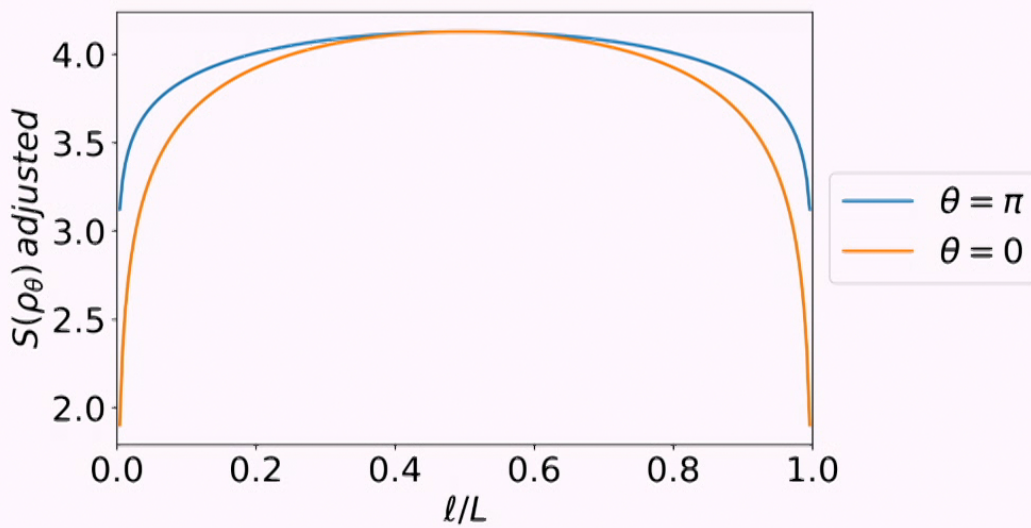
Results: EE

- Entanglement grows as $\theta \to 0$
- Reasonable since $S(\rho_0) \sim -\log m$



Results

- Different functional form
- $S(\rho_A) = \frac{c}{3} \ln\left(\frac{L}{\pi} \sin \frac{\pi \ell}{L}\right) + (\text{non-universal}) \quad \text{only for } \theta = 0$



Analytic formula

In 2017 Shiba (arxiv:1701.00688) derived the EE for these twists,

$$S^{\alpha}(\rho_{\theta}) \sim \sum_{k=1}^{\alpha-1} \ln\langle \sigma_{k/n}(0)\sigma_{1-k/n}(x)\sigma_{\theta/2\pi}(1)\sigma_{1-\theta/2\pi}(\infty)\rangle$$

$$\sim \sum_{k=1}^{n-1} \ln \frac{\kappa^2 |x|^{-2\frac{k}{n}(1-\frac{k}{n})}|1-x|^{-2\frac{k}{n}(1-\nu)}}{A+\bar{A}+B}$$

with $|x| = 2|\sin\frac{\pi\ell}{L}|$ and

$$A = \frac{\Gamma(1-\nu)\Gamma(k/n)}{\Gamma(1+k/n-\nu)}\ {}_2F_1(1-\nu, k/n, 1, x)\ {}_2F_1(1-\nu, k/n, 1+k/n-\nu, 1-\bar{x})$$

$$B = \frac{\pi\sin\pi(k/n-\nu)}{\sin(\pi k/n)\sin(\pi\nu)}\ {}_2F_1(1-\nu, k/n, 1, x)\ {}_2F_1(1-\nu, k/n, 1, \bar{x})$$

Q: How is $\rho_A(\theta)$ different from $\rho_A(0)$?

Background: Relative Entropy

Given two reduced density matrices ρ and σ on a region A, the relative entropy is defined as

$$S(\rho||\sigma) = \mathrm{Tr}\left(\rho \log \rho - \rho \log \sigma\right)$$

This has useful properties such as

- Non-negative, $S(\rho||\sigma) \geq 0$
- Monotonically decreases under CPTP operations
- (Monotonically increases under increasing volume of A)
- Jointly convex function of ρ, σ

Modified real time approach

Use

$$\rho_A = \frac{1}{\mathcal{N}_\rho} e^{-\sum_k \epsilon_k^\rho b_k^\dagger b_k}\,, \qquad \sigma_A = \frac{1}{\mathcal{N}_\sigma} e^{-\sum_k \epsilon_k^\sigma a_k^\dagger a_k}$$

so the second term in the relative entropy becomes

$$\mathrm{Tr}\left(\rho_A \log \sigma_A\right) \sim \mathrm{Tr}\left(\rho_A \sum_k \epsilon_k^\sigma a_k^\dagger a_k\right) + \mathrm{const}$$

So we need to compute $\mathrm{Tr}\left(\rho_A\, a_k^\dagger a_k\right)$

Transformation

Take a linear combination

$$a_j^\dagger = M_{ij} b_j^\dagger + N_{ij} b_j$$

$$a_j = N_{ji} b_j^\dagger + M_{ji} b_j$$

expanding $\mathrm{Tr}\left(\rho_A\, a_i^\dagger a_j\right)$ gives

$$M_{ik} M_{jl} \mathrm{Tr}\left(\rho_A b_k^\dagger b_l\right) + N_{ij} N_{kl} \mathrm{Tr}\left(\rho_A b_k b_l^\dagger\right) + 0 + 0$$

Transformation ...

For a free theory, you can already write ϕ, π in terms of creation-annihilation operators

$$\phi_j = \alpha_{ij}\left(a_i^\dagger + a_i\right) = \beta_{ij}\left(b_i^\dagger + b_i\right)$$

$$\pi_j = \frac{i}{2}(\alpha_{ji})^{-1}\left(a_i^\dagger - a_i\right) = \frac{i}{2}(\beta_{ji})^{-1}\left(b_i^\dagger - b_i\right)$$

Solving for $a^\dagger, a$ in terms of $b^\dagger, b$ give the M, N matrices

$$M_{jk} = \frac{1}{2}(\alpha_{ji})^{-1}\beta_{ik} + \frac{1}{2}\alpha_{ji}(\beta_{ik})^{-1}$$

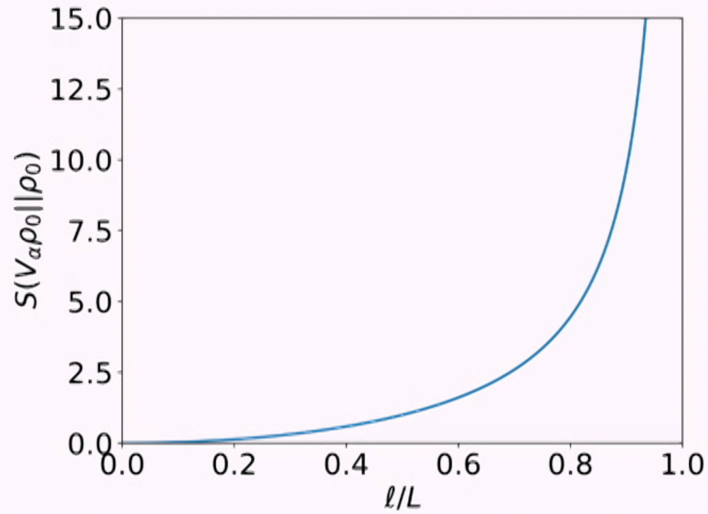
$$N_{jk} = \frac{1}{2}(\alpha_{ji})^{-1}\beta_{ik} - \frac{1}{2}\alpha_{ji}(\beta_{ik})^{-1}$$

The α, β matrices can be found from X_A, P_A.

Results: Relative entropy

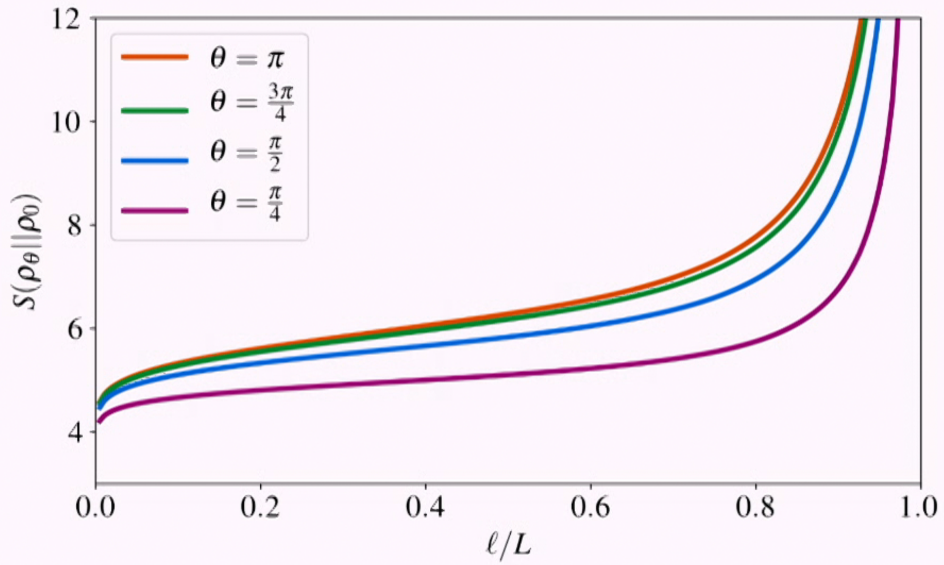
Aside: what to expect

Relative entropy for an state excited by a vertex operator $V_\alpha = e^{i\alpha\phi}$ (Lashkari, arxiv:1404.3216)



- $S(V_\alpha \rho_0 || \rho_0) \to 0$ as $\ell/L \to 0$
- $S(V_\alpha \rho_0 || \rho_0) \to \infty$ as $\ell/L \to 1$

Relative Entropy

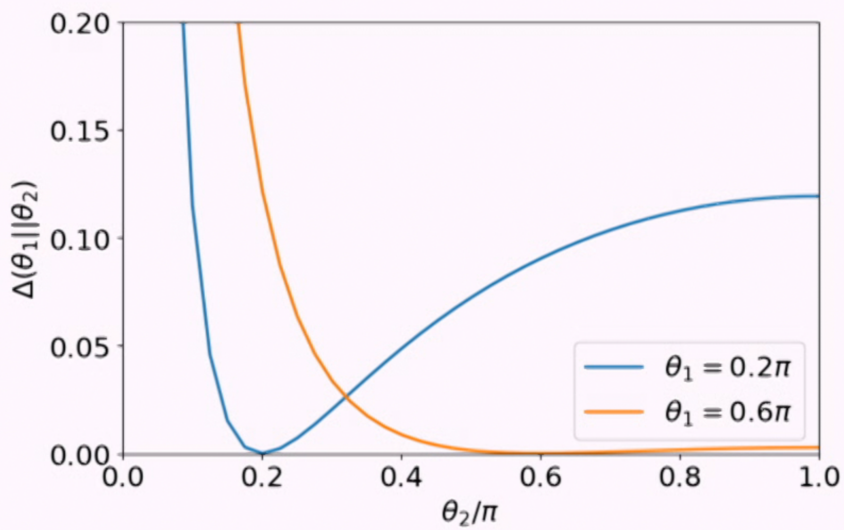


Surprising features

- Non-zero $\Delta(\theta, 0) \equiv S(\rho_\theta||\rho_0)\big|_{\ell=1}$ as $\ell/L \to 0$
- Gap diverges with mass $\Delta(\theta, 0) \sim -\log(m)$
- Inflection point

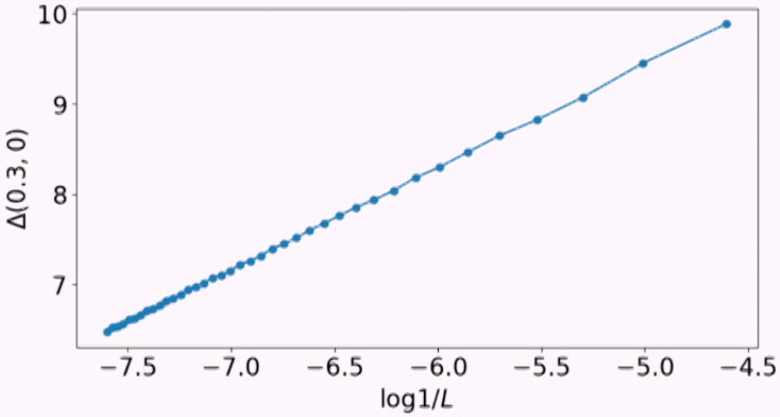
More than just a $\theta = 0$ problem

- Between two angles, θ_1, θ_2 there is still a gap
- Relative entropy vanishes if $\theta_1 = \theta_2$



Finite-size scaling

How does $\Delta(\theta, 0)$ scale with L?



Without a proper finite-size scaling prediction it's ambiguous is the gap vanishes or not

Final Remarks

- Entanglement entropy depends non-trivially on θ
- Gap at $S(\rho_\theta||\rho_0)\big|_{\ell=1}$ means ρ_θ depends on some information about the BCs
- Can't expand in θ as $\rho_\theta = \rho_0 + \lambda\delta\rho$
- Unknown finite-size scaling of relative entropy for twisted BCs

Thanks for listening!