Title: Synchronous correlations, Bell inequalities, and categories of nonlocal games

Date: May 24, 2017 04:00 PM

URL: http://pirsa.org/17050094

Abstract: <p>Given two sets X and Y, we consider synchronous correlations in a two-party nonlocal game with inputs X and outputs Y as a notion of generalized function between these sets (akin to a quantum graph homomorphism). We examine some structures in categories of synchronous classical, quantum, and nonsignalling strategies. We also provide analogues of Bellâ€™s inequalities for such games when Y = {0,1}. </p>

Synchronous correlations, Bell inequalities, and categories of nonlocal games

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May 24, 2017

Outline

1. Nonlocal games and synchronous correlations
2. Classical, quantum, and nonsignaling strategies
3. Bell inequalities for synchronous correlations
4. Categories of generalized functions

Essential reading

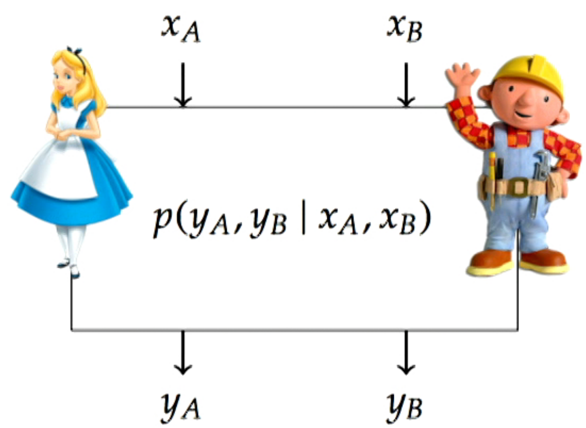
1. Abramsky, Barbosa, de Silva, and Zapata, "The quantum monad on relational structures," arXiv 1705.07310 [cs.LO].
2. Cameron, Montanaro, Newman, Severini, and Winter, "On the quantum chromatic number of a graph," *Electron. J. Combin.* **14**, 2007.
3. Mančinska and Roberson, "Quantum homomorphisms," *J. Combin. Theory B* **118**, 2016.
4. Paulsen, Severini, Stahlke, Todorov, and Winter, "Estimating quantum chromatic nubmers," *J. Funct. Analysis* **270**, 2016.

Nonlocal games

Our goal is to generalize functions $f : X \to Y$ with *nonlocal games*.

- Alice and Bob obtain inputs $(x_A, x_B) \in X^2$.
- They produce output $(y_A, y_B) \in Y^2$ (nondeterministically).

We want "synchronous" games: if they get the same input, they should produce the same output.



E.g. they preselect $f \in Y^X$.

- Alice outputs $y_A = f(x_A)$,
- Bob outputs $y_B = f(x_B)$.

In general a game is a "correlation"

$p(y_A, y_B \mid x_A, x_B)$, with a winning condition $V : X^2 \times Y^2 \to \{0, 1\}$.

E.g. "quantum graph homomorphisms" have $x_A \sim x_B$ iff $y_A \sim y_B$.

Synchronous correlations

Formally, a *synchronous* correlation satisfies:

$$p(y_A, y_B \mid x, x) = 0 \text{ whenever } y_A \neq y_B.$$

Write $\mathrm{Hom}(X, Y)$ for synchronous correlations from X to Y.

Lemma

The space of synchronous correlations is a convex set.

Example. Consider $X = \{0\}$.

- Identify Y with $Y^{\{0\}}$ via $y = f(0)$.
- A general strategy is a probability distribution on Y^2.
- A synchronous correlation has $p(y_1, y_2) = 0$ for $y_1 \neq y_2$.
- So $\mathrm{Hom}(\{0\}, Y)$ is identified with probability distributions on Y.

Example: functions as synchronous correlations

Consider the case $X = Y = \{0, 1\}$. A synchronous correlation has 4×4 (column) stochastic matrix

$$\begin{pmatrix} p(0,0|0,0) & p(0,0|0,1) & p(0,0|1,0) & p(0,0|1,1) \\ 0 & p(0,1|0,1) & p(0,1|1,0) & 0 \\ 0 & p(1,0|0,1) & p(1,0|1,0) & 0 \\ p(1,1|0,0) & p(1,1|0,1) & p(1,1|1,0) & p(1,1|1,1) \end{pmatrix}.$$

The four correlations corresponding to the four functions in Y^X are

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, R = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Here, the synchronous correlations form an eight dimensional convex polyhedron with 64 vertices (which include B, I, R, and T).

Hidden variables strategies

Definition

A *local hidden variables*, or simply *classical*, correlation is a conditional probability distribution that takes the form

$$p(y_A, y_B | x_A, x_B) = \sum_{\omega \in \Omega} \mu(\omega) p_A(y_A | x_A, \omega) p_B(y_B | x_B, \omega)$$

for some finite set Ω with a probability distribution μ.

We will write $\mathrm{Hom}_{\mathrm{HV}}(X, Y)$ for the synchronous hidden variables strategies.

Synchronous hidden variables strategies

Theorem

The set of synchronous local hidden variables strategies on $X \to Y$ is bijective to the set of probability distributions on Y^X. Given such a probability distribution the associated strategy is: Alice and Bob sample a function $f \in Y^X$ according the specified distribution, and upon receiving x_A, x_B they output $y_A = f(x_A)$ and $y_B = f(x_B)$.

Corollary

The extreme points of $\mathrm{Hom}_{\mathrm{HV}}(X, Y)$ can be canonically identified with Y^X.

Corollary

Every synchronous classical strategy is symmetric.

Here *symmetric* means:

$$p(y_A, y_B \mid x_A, x_B) = p(y_B, y_A \mid x_B, x_A).$$

Example: $|X| = 3$ *and* $|Y| = 2$.

Lemma

For any $p \in \mathrm{Hom}_{\mathrm{HV}}(\{0,1,2\},\{0,1\})$ *we have*

$$p(0,1|0,1) + p(0,1|1,2) + p(0,1|2,0) \leq 1.$$

Sketch of proof. Write

$$p(y_A, y_B \mid x_A, x_B) = \sum_{f \in \{0,1\}^{\{0,1,2\}}} \mu(f) \mathbb{1}_{\{y_A = F(x_A)\}} \mathbb{1}_{\{y_B = F(x_B)\}}.$$

One verifies

$$p(0,1 \mid 0,1) = \mu(0,1,0) + \mu(0,1,1),$$
$$p(0,1 \mid 1,2) = \mu(0,0,1) + \mu(1,0,1),$$
$$p(0,1 \mid 2,0) = \mu(1,0,0) + \mu(1,1,0).$$

Therefore

$$p(0,1|0,1) + p(0,1|1,2) + p(0,1|2,0) = 1 - \mu(0,0,0) - \mu(1,1,1) \leq 1.$$

Quantum strategies

Definition

A *quantum* correlation is a strategy that takes the form

$$p(y_A, y_B | x_A, x_B) = \mathrm{tr}(\rho(E^{x_A}_{y_A} \otimes F^{x_B}_{y_B})),$$

where

- ρ is a density operator on the Hilbert space $\mathfrak{H}_A \otimes \mathfrak{H}_B$, and
- for each $x \in X$ we have $\{E^x_y\}_{y \in Y}$ and $\{F^x_y\}_{y \in Y}$ are POVMs on $\mathfrak{H}_A$ and $\mathfrak{H}_B$ respectively.

We will write $\mathrm{Hom}_Q(X, Y)$ for the synchronous quantum correlations.

Synchronous quantum correlations

Lemma

In any synchronous quantum correlation, the POVMs $\{E^x_y\}_{y\in Y}$ and $\{F^x_y\}_{y\in Y}$, for $x \in X$, are projection-valued measures.

Comment.

- In general one may take $\{E^x_y\}_{y\in Y}$ and $\{F^x_y\}_{y\in Y}$ projection-valued by enlarging $\mathfrak{H}_A$ and $\mathfrak{H}_B$.
- Synchronism implies they must projection-valued already.

Proposition

Suppose $p(y_A, y_B|x_A, x_B) = \langle\psi|E^{x_A}_{y_A} \otimes F^{x_B}_{y_B}|\psi\rangle$ is any quantum correlation with projection-valued measures. If the Schmidt coefficients of $|\psi\rangle$ are all distinct then the strategy is classical.

Synchronous quantum correlations, cont'd

Theorem

Every synchronous quantum correlation is convex combination of ones with maximally entangled pure states. In particular, if

$$p(y_A, y_B \mid x_A, x_B) = \mathrm{tr}(\rho(E^{x_A}_{y_A} \otimes F^{x_B}_{y_B}))$$

is extremal, then we may take $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle$ maximally entangled.

Sketch of proof. Take $\rho = |\psi\rangle\langle\psi|$ with σ_j the Schmidt coefficients of $|\psi\rangle$.

- Then $\mathrm{tr}_B(|\psi\rangle\langle\psi|) = \sum_{j=1}^r \sigma_j \Pi_j^A$.
- This decomposes $\mathfrak{H}_A = \mathfrak{H}_0^A \oplus \bigoplus_{j=1}^r \mathrm{im}(\Pi_j^A)$.
- Similarly for $\mathfrak{H}_B$.

We can use this to write

$$p(y_A, y_B|x_A, x_B) = \sum_{j=1}^r (\ell_j\sigma_j)\langle\psi_j|E^x_{y,j} \otimes F^x_{y,j}|\psi_j\rangle$$

where $|\psi\rangle_j$ is maximally entangled on $\mathrm{im}(\Pi_j^A) \otimes \mathrm{im}(\Pi_j^B)$

Example

Returning to the example $X = Y = \{0,1\}$, consider the quantum correlation $p(y_A, y_B | x_A, x_B) = \langle\psi|(E^{x_A}_{y_A} \otimes E^{x_B}_{y_B})|\psi\rangle$ on $\mathbb{C}^2$ where:

- $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$,
- $E^0_0 = |0\rangle\langle 0|$ and $E^0_1 = |1\rangle\langle 1|$, and
- $E^1_0 = |\phi_0\rangle\langle\phi_0|$ and $E^1_1 = |\phi_0^\perp\rangle\langle\phi_0^\perp|$ where

$$|\phi_0\rangle = \cos\theta|0\rangle + e^{i\varphi}\sin\theta|1\rangle, \text{ and } |\phi_0^\perp\rangle = -\sin\theta|0\rangle + e^{i\varphi}\cos\theta|1\rangle.$$

The associated stochastic matrix for this strategy is

$$\frac{1}{2}\begin{pmatrix} 1 & \cos^2\theta & \cos^2\theta & 1 \\ 0 & \sin^2\theta & \sin^2\theta & 0 \\ 0 & \sin^2\theta & \sin^2\theta & 0 \\ 1 & \cos^2\theta & \cos^2\theta & 1 \end{pmatrix} = \frac{\cos^2\theta}{2}(B+T) + \frac{\sin^2\theta}{2}(I+R).$$

So we see this correlation is not extremal, and in fact is classical.

Further reductions

Theorem

Let X, Y be finite sets, and $\mathfrak{H}$ a d-dimensional Hilbert space. Suppose for each $x \in X$, we have a projection-valued measure $\{F^x_y\}_{y \in Y}$. Then

$$p(y_A, y_B | x_A, x_B) = \frac{1}{d} \mathrm{tr}(F^{x_A}_{y_A} F^{x_B}_{y_B})$$

defines a synchronous quantum correlation. Moreover, every synchronous quantum correlation with maximally entangled pure state has this form.

Corollary

Synchronous quantum correlations are symmetric.

General nonsignaling strategies

Definition

A correlation is *nonsignaling* if it satisfies:

$$\sum_{y_B} p(y_A, y_B | x_A, x_B) = \sum_{y_B} p(y_A, y_B | x_A, x'_B) \text{ for all } y_A, x_A, x_B, x'_B, \text{ and}$$
$$\sum_{y_A} p(y_A, y_B | x_A, x_B) = \sum_{y_A} p(y_A, y_B | x'_A, x_B) \text{ for all } y_B, x_A, x'_A, x_B.$$

We will write $\mathrm{Hom}_{\mathrm{NS}}(X, Y)$ for the synchronous nonsignaling correlations.

Example: nonsignaling correlations when $|X| = |Y| = 2$

For $X = Y = \{0,1\}$, the synchronous nonsignaling correlations have:

$$\begin{aligned}
0 + p(0,0|0,0) &= p(0,0|0,1) + p(0,1|0,1),\\
p(0,0|1,0) + p(0,1|1,0) &= p(0,0|1,1) + 0,\\
0 + p(1,1|0,0) &= p(1,0|0,1) + p(1,1|0,1),\\
p(1,0|1,0) + p(1,1|1,0) &= p(1,1|1,1) + 0,\\
0 + p(0,0|0,0) &= p(0,0|1,0) + p(1,0|1,0),\\
p(0,0|0,1) + p(1,0|0,1) &= p(0,0|1,1) + 0,\\
0 + p(1,1|0,0) &= p(0,1|1,0) + p(1,1|1,0),\\
p(0,1|0,1) + p(1,1|0,1) &= p(1,1|1,1) + 0.
\end{aligned}$$

With $p(y_A, y_B|x_A, x_B) \geq 0$ and $\sum_{y_A,y_B} p(y_A, y_B|x_A, x_B) = 1$, we get a four dimensional polytope with six extreme points. Four of these vertices are familiar: B, I, R, and T. The two additional ones are PR-boxes

$$PR_0 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \text{ and } PR_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

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3. **Bell inequalities for synchronous correlations**
4. Categories of generalized functions

Nonsignaling correlations with two-point domain

Lemma

Let $u = u(y_A, y_B)$ and $v = v(y_A, y_B)$ be probabilities on Y^2 with:

- $\sum_{y'} u(y, y') = \sum_{y'} v(y', y) =: \theta(y)$ *and*
- $\sum_{y'} u(y', y) = \sum_{y'} v(y, y') =: \phi(y)$.

Define

$$p(y_A, y_B|0,0) = \mathbb{1}_{\{y_A = y_B\}}\theta(y_A)$$
$$p(y_A, y_B|0,1) = u(y_A, y_B),$$
$$p(y_A, y_B|1,0) = v(y_A, y_B),$$
$$p(y_A, y_B|1,1) = \mathbb{1}_{\{y_A = y_B\}}\phi(y_A).$$

Then p is a synchronous nonsignaling correlation. Moreover, every synchronous nonsignaling correlation from $\{0,1\}$ to Y arises this way.

The proof is straightforward.

Hidden variables correlations with two-point domain

Lemma

Let $u = u(y_A, y_B)$ be any probability distribution on Y^2 and set $\theta(y) = \sum_{y'} u(y, y')$ and $\phi(y) = \sum_{y'} u(y', y)$. Define

$$p(y_A, y_B|0,0) = \mathbb{1}_{\{y_A = y_B\}}\theta(y_A)$$
$$p(y_A, y_B|0,1) = u(y_A, y_B),$$
$$p(y_A, y_B|1,0) = u(y_B, y_A),$$
$$p(y_A, y_B|1,1) = \mathbb{1}_{\{y_A = y_B\}}\phi(y_A).$$

Then p is a synchronous classical correlation. Moreover, every synchronous classical correlation. from $\{0,1\}$ to Y arises this way.

Sketch of proof. The probability distribution on $Y^{\{0,1\}}$ is

$$\mu(f) = u(f(0), f(1)).$$

No Bell inequalities with two-point domains

Corollary

A synchronous nonsignaling correlation from $\{0,1\}$ to a set Y is classical if and only if it is symmetric.

Proof. If p is a symmetric synchronous nonsignaling correlation, then the u and v in the above lemma satisfy $v(y_A, y_B) = u(y_B, y_A)$, and so from this latter lemma p is classical. Conversely, every synchronous classical correlation is symmetric and nonsignaling.

Corollary

Every synchronous quantum correlation from $\{0,1\}$ to a finite set Y is classical.

Proof. Extremal synchronous quantum correlations are symmetric, and hence classical by the previous corollary.

Nonsignaling correlations with two-point range

Lemma

Suppose $|X| \geq 2$ *and let* $w = w(x_A, x_B)$ *be a nonnegative function on* X^2 *such that for every* $x_A, x_B \in X$:

1. $w(x_A, x_B) \leq w(x_A, x_A)$,
2. $w(x_A, x_B) \leq w(x_B, x_B)$, *and*
3. $w(x_A, x_A) + w(x_B, x_B) \leq 1 + w(x_A, x_B)$.

Define

$$
\begin{aligned}
p(0,0 \mid x_A, x_B) &= 1 + w(x_A, x_B) - w(x_A, x_A) - w(x_B, x_B) \\
p(0,1 \mid x_A, x_B) &= w(x_B, x_B) - w(x_A, x_B), \\
p(1,0 \mid x_A, x_B) &= w(x_A, x_A) - w(x_A, x_B), \\
p(1,1 \mid x_A, x_B) &= w(x_A, x_B).
\end{aligned}
$$

Then p *is a synchronous nonsignaling correlation. Moreover, every synchronous nonsignaling correlation from* X *to* $\{0, 1\}$ *arises this way.*

Bell inequalities for $|X| = 3$ and $|Y| = 2$

The nonsignaling polytope

Consider nonsignaling correlations from $\{0, 1, 2\}$ to $\{0, 1\}$.

- The lemma shows there are 9 essential parameters, $w_0, \ldots, w_8$, where
$$p(1, 1 \mid y_A, y_B) = w_{3 * y_A + y_B}.$$
- The conditions in the lemma form 24 linear inequalities
$$\begin{array}{lll lll lll lll}
0 & \leq & w_1, & w_1 & \leq & w_0, & w_1 & \leq & w_4, & w_0 + w_4 & \leq & 1 + w_1, \\
0 & \leq & w_2, & w_2 & \leq & w_0, & w_7 & \leq & w_4, & w_0 + w_4 & \leq & 1 + w_3, \\
0 & \leq & w_3, & w_3 & \leq & w_4, & w_2 & \leq & w_8, & w_0 + w_8 & \leq & 1 + w_2, \\
0 & \leq & w_5, & w_5 & \leq & w_4, & w_5 & \leq & w_8, & w_0 + w_8 & \leq & 1 + w_6, \\
0 & \leq & w_6, & w_6 & \leq & w_8, & w_3 & \leq & w_0, & w_4 + w_8 & \leq & 1 + w_5, \\
0 & \leq & w_7, & w_7 & \leq & w_8, & w_6 & \leq & w_0, & w_4 + w_8 & \leq & 1 + w_7.
\end{array}$$
- $0 \leq w_0, w_4, w_9$ are implicitly true.

These inequalities define a polytope in $\mathbb{R}^9$, which has 80 vertices. Any point in this polytope describes a synchronous nonsignaling strategy.

Bell inequalities for $|X| = 3$ and $|Y| = 2$

The hidden variables polytope

The hidden variables polytope is:

- the convex hull of the 8 functions in $\{0,1\}^{\{0,1,2\}}$;
- a 6 dimensional polytope with these 8 functions as vertices;
- lives in the space defined by

$$w_1 = w_3, \qquad w_2 = w_6, \qquad w_5 = w_7.$$

These are exactly expressing the symmetry of classical strategies.
These equations reduces nonsignaling conditions to 12 inequalities.
However there are four additional inequalities that are required:

$$\begin{aligned} J_0 = w_0 - w_3 + w_4 - w_6 - w_7 + w_8 &\leq 1, \\ J_1 = w_0 - w_3 - w_6 + w_7 &\geq 0, \\ J_2 = -w_3 + w_4 + w_6 - w_7 &\geq 0, \\ J_3 = w_3 - w_6 - w_7 + w_8 &\geq 0. \end{aligned}$$

Note: first of these, $J_0 \leq 1$ is precisely the inequality we saw before.

Nonsignaling violations for $|X| = 3$ and $|Y| = 2$

Proposition

No synchronous nonsignaling correlation can violate more than one of the inequalities $J_0 \leq 1$ and $J_1, J_2, J_3 \geq 0$. The greatest possible violation among these strategies is $J_0^{max} = \frac{3}{2}$, and $J_1^{min} = J_2^{min} = J_3^{min} = -\frac{1}{2}$.

Sketch of proof. Enumerate the vertices of the nonsignaling polytope.

- Eight of these have

$$(1 - J_0, J_1, J_2, J_3) = \left(-\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right).$$

- Similarly eight each with $(1 - J_0, J_1, J_2, J_3)$ being

$$\left(\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right), \left(\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}\right), \text{ and } \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}\right).$$

- The other 48 do not violate any of the inequalities.

Conclude one J_j is negative implies the others are nonnegative.

Quantum violations for $|X| = 3$ and $|Y| = 2$

Theorem

On $\mathfrak{H}_A = \mathfrak{H}_B = \mathbb{C}^2$ every synchronous quantum correlation satisfies $J_0 \leq \frac{9}{8}$ and $J_1, J_2, J_3 \geq -\frac{1}{8}$. Moreover these bounds are sharp, with each of $J_0 = \frac{9}{8}$, $J_1 = -\frac{1}{8}$, $J_2 = -\frac{1}{8}$, and $J_3 = -\frac{1}{8}$ realized by a unique (up to symmetry) extremal synchronous quantum correlation.

E.g. the three measurement settings that achieve $J_0 = \frac{9}{8}$ have

$$F_1^0 = |1\rangle\langle 1|,\ F_1^1 = |\phi_1\rangle\langle\phi_1|,\ \text{and } F_1^2 = |\phi_2\rangle\langle\phi_2|$$

where

$$|\phi_1\rangle = -\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle \text{ and } |\phi_2\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle.$$

This correlation has $J_1 = J_2 = J_3 = \frac{3}{8}$.

Outline

Categories

Here we wish to extend the category FinSet of finite sets using larger Hom-sets. We have already used the notation:

- $\mathrm{Hom}_{\mathrm{HV}}(X, Y)$ for the synchronous hidden variables strategies,
- $\mathrm{Hom}_{\mathrm{Q}}(X, Y)$ for the synchronous quantum correlations,
- $\mathrm{Hom}_{\mathrm{NS}}(X, Y)$ for the synchronous nonsignaling strategies.
- $\mathrm{Hom}(X, Y)$ for all synchronous correlations.

The composition rule is the obvious one.

- This extends of composition of functions.
- Each Hom-set is closed under composition.
- Corresponds to multiplication of associated stochastic matrices.
- The identity id_X is the identity function in all cases.

Denote these $\mathsf{FinSet}_{\mathrm{HV}}$, $\mathsf{FinSet}_{\mathrm{Q}}$, $\mathsf{FinSet}_{\mathrm{NS}}$, etc., for these categories.

Notions of injective nonlocal games

In a general category C a morphism $f \in \mathrm{Hom}^{\mathsf{C}}(A, B)$ is:

- a *section* if there exists a $g \in \mathrm{Hom}^{\mathsf{C}}(B, A)$ with $g \circ f = \mathrm{id}_A$,
- a *monomorphism* if whenever $g, h \in \mathrm{Hom}^{\mathsf{C}}(Z, A)$ satisfy $f \circ g = f \circ h$ then $g = h$.

Theorem

In FinSet_{NS}, the sections are precisely the deterministic strategies corresponding to a one-to-one functions.

Corollary

In each of FinSet_{HV} and FinSet_{Q}, the sections are precisely the deterministic strategies corresponding to a one-to-one functions.

Proposition

In each of FinSet_{HV}, FinSet_{Q}, and FinSet_{NS}, the monomorphisms are precisely those strategies whose stochastic maps have zero right nullspace.

Notions of surjective nonlocal games

In a general category C a morphism $f \in \mathrm{Hom}^{\mathsf{C}}(A, B)$ is:

- a *retract* if there exists a $g \in \mathrm{Hom}^{\mathsf{C}}(B, A)$ with $f \circ g = \mathrm{id}_A$,
- an *epimorphism* if whenever $g, h \in \mathrm{Hom}^{\mathsf{C}}(Z, A)$ satisfy $g \circ f = h \circ f$ then $g = h$.

Proposition

In $\mathsf{FinSet}_{\mathrm{NS}}$, epimorphisms are precisely those strategies whose stochastic matrices have zero left nullspace.

- The proof of this proposition relies on our characterization of $\mathrm{Hom}_{\mathrm{NS}}(X, \{0, 1\})$, which does not naturally extend to FinSet_{HV} and FinSet_{Q}.
- Nonetheless in FinSet_{HV}: an *isomorphism* is a deterministic bijective functions, and a *bimorphisms* is synchronous nonsignaling correlations whose stochastic matrix is nonsingular.