Title: A Weyl-Type Theorem in Geometrized Newtonian Gravity, and How It May Bear on Shape Dynamics

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Abstract: The Weyl Theorem states that the conformal structure and the projective structure jointly suffice to fix the metric up to a global constant. This is a powerful interpretive tool in general relativity: it says in effect that if I know the paths of light rays in vacuo and I know the images of the paths of freely falling particles (i.e., the spacetime curves they follow with no preferred parametrization), then I know the metric. It has particular relevance to Shape Dynamics, where the conformal structure is taken as the object of primitive geometrical interest, and one does not generally want a preferred parametrization of timelike geodesics. The spacetimes of geometrized Newtonian gravity share many important features with the way that Shape Dynamics approaches the construction of relativistic spacetimes, in particular the fixing of a preferred foliation of spacetime by spacelike slices. Studying the way that geometrized Newtonian gravity does and does not allow one to recover a Weyl-type Theorem may, therefore, shed light on the ways that Shape Dynamics may allow one to recover the structure of relativistic spacetimes. I show that, in geometrized Newtonian gravity, the conformal structure of the spatial and temporal metrics, in conjunction with the projective structure of timelike curves, allows one to recover the full metrical structure of a geometrized Newtonian gravity only if one also fixes an affine parametreization of at least one timelike geodesic. This suggests that, in far as the analogy between geometrized Newtonian gravity and Shape Dynamics is a physically significant one, Shape Dynamics may be right not to demand preferred parametrizations of timelike geodesics from the start. The tools I develop to prove the theorem may also be applicable to problems in Shape Dynamics itself. But I make no promises.

A Weyl-Type Theorem in Classical Spacetimes, and How It May Bear on Shape Dynamics

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Weyl's Theorem in General Relativity

Classical Spacetimes

Conformal Structure on Classical Spacetime

A Weyl-Type Theorem for Classical Spacetimes

Suggestions for Shape Dynamics?

Theorem (Weyl 1918)

Given a conformal structure on a differential manifold, and a projective structure agreeing with it on (images) of its null geodesics, there is a Lorentz metric, fixed up to a constant factor, having each as its associated structure of that kind.

powerful interpretive tool...

If I know

1. the paths of light rays in vacuo
2. and the images of the paths of freely falling particles (i.e., the spacetime curves they follow with no preferred parametrization)

then I know the metric.

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Definition

A classical spacetime model *is an ordered quadruplet* $\mathcal{C} = (M,\, h^{ab},\, t_{ab}, \nabla_a)$ *such that:*

1. M *is a four-dimensional, paracompact, connected, smooth, differential manifold*
2. h^{ab} *is a smooth, symmetric tensor field on* M *of signature* $(0,\, 1,\, 1,\, 1)$
3. t_{ab} *is a smooth, symmetric tensor field on* M *of signature* $(1,\, 0,\, 0,\, 0)$
4. t_{ab} *and* h^{ab} *are orthogonal (*$t_{an}h^{nb} = 0$*)*
5. ∇_a *is a smooth derivative operator on* M*, compatible with* t_{ab} *and* h^{ab} *in the sense that* $\nabla_a t_{bc} = 0$ *and* $\nabla_a h^{bc} = 0$

(for simplicity, assume M simply connected and temporally orientable, *i.e.*, $t_{ab} = t_a t_b = \nabla_a t \nabla_b t$)

ξ^a is *timelike* if $\xi^n t_n \neq 0$, *spacelike* otherwise (but it is natural to think of $\mathbf{0}^a$ as both timelike and spacelike); h^{ab} defines the spatial metric structure in a more indirect way.

Proposition

1. *A vector ξ^a is spacelike at a point p if and only if there is a covector α_a at p such that $\xi^a = h^{an}\alpha_n$.*
2. *For all covectors α_a and β_a at a point, if $h^{an}\alpha_n = h^{an}\beta_n$, then $h^{mn}\alpha_m\alpha_n = h^{mn}\beta_m\beta_n$.*

α_a is a *representative spacelike 1-form* of ξ^a if $\xi^a = h^{an}\alpha_n$

global structure

1. the surfaces of constant t are "simultaneity slices"
2. σ^a is spacelike if and only if it is tangent to a simultaneity slice
3. there is a determinate, unambiguous temporal interval between any two events (up to constant rescaling)
4. spatial distance is defined only between simultaneous events (there is no canonical h_{ab} such that $h_{an}h^{nb} = \delta^a{}_b$, not even when restricted to simultaneity slices)

why h^{ab} has the right structure

if we knew how to assign a spatial length to timelike vectors, and so a fixed spatial separation between non-simultaneous events, then we could define a notion of absolute rest

Definition

A geometrized Newtonian gravity spacetime model *is an ordered pair* $\mathcal{N} = (\mathcal{C}, \rho)$ *such that:*

1. $\mathcal{C}$ *is a classical spacetime model*
2. ρ *is a smooth, non-negative, scalar function on* M
3. $R_{bc} = 4\pi\rho t_a t_b$ *(the "Poisson equation")*
4. $R^a{}_b{}^c{}_d = R^c{}_b{}^a{}_d$
5. $R^{ab}{}_{cd} = 0$

Proposition

A geometrized Newtonian gravity spacetime model is spatially flat ($R^{abcd} = 0$).

Theorem (The Geometrization Lemma, Trautman 1965)

Fix a geometrized Newtonian gravity spacetime model $\mathcal{N}$ *such that* ∇_a *is flat, i.e., its associated Riemann tensor* ${R^a}_{bcd} = 0$*, and a smooth scalar field* ϕ *such that* $\nabla^n \nabla_n \phi = 4\pi\rho$ *(the Poisson equation). Define another derivative operator* $\widetilde{\nabla}_a = (\nabla_a, {C^a}_{bc})$*, where* ${C^a}_{bc} = -t_{cb}\nabla^a \phi$*. Then:*

1. $(M, h^{ab}, t_{ab}, \widetilde{\nabla}_a, \rho)$ *is a geometrized Newtonian gravity spacetime model*
2. $\widetilde{\nabla}_a$ *is the unique derivative operator such that*

$$\xi^n \widetilde{\nabla}_n \xi^a = 0 \quad \textit{if and only if} \quad \xi^n \nabla_n \xi^a = -\nabla^a \phi$$

Theorem (Trautman Recovery Theorem, 1965)

Fix a geometrized Newtonian gravity spacetime model. Then there is a derivative operator $\widetilde{\nabla}_a$ and a scalar field $\tilde{\phi}$ such that

1. $\widetilde{\nabla}^n \widetilde{\nabla}_n \tilde{\phi} = 4\pi\rho$ *(the Poisson equation)*
2. $\widetilde{\nabla}^a$ *is compatible with* t_{ab} *and* h^{ab}
3. $\widetilde{\nabla}^a$ *is flat*
4. *for all timelike curves with tangent vector* ξ^a

$$\xi^n \nabla_n \xi^a = 0 \quad \textit{if and only if} \quad \xi^n \widetilde{\nabla}_n \xi^a = -\widetilde{\nabla}^a \tilde{\phi}$$

Moreover, $(\widetilde{\nabla}_a, \tilde{\phi})$ *is not unique. Any other such pair* $(\hat{\nabla}_a, \hat{\phi})$ *will satisfy the stated conditions if and only if*

1. $\nabla^a \nabla^b (\tilde{\phi} - \hat{\phi}) = 0$
2. $\hat{\nabla}_a = (\widetilde{\nabla}_a, C^a{}_{bc})$, *where* $C^a{}_{bc} = t_{bc} \nabla^a (\tilde{\phi} - \hat{\phi})$

In what follows, we restrict ourselves to classical spacetime models satisfying only

$$R^{a}{}_{b}{}^{c}{}_{d} = R^{c}{}_{b}{}^{a}{}_{d}$$

(The physical interpretation of this condition will become clearer at the end.) In particular, we do not require $R^{abcd} = 0$.

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what does “conformal structure” in a classical spacetime mean?

there are no non-trivial null cones

two spacelike vectors η^a and θ^a are *orthogonal* if $h^{mn}\alpha_m\beta_n = 0$, where α_m and β_n are any two representative spacelike 1-forms of η^a and θ^a respectively

easy to verify this is independent of representative

(This makes physical sense: we know in Newtonian physics how to determine whether two forces act at a right angle to each other.)

two spacelike vectors η^a and θ^a are *orthogonal* if $h^{mn}\alpha_m\beta_n = 0$, where α_m and β_n are any two representative spacelike 1-forms of η^a and θ^a respectively

easy to verify this is independent of representative

(This makes physical sense: we know in Newtonian physics how to determine whether two forces act at a right angle to each other.)

Two spatial metrics h^{ab} and $\tilde{h}^{ab}$ are *conformally equivalent* if they agree on orthogonality relations among all spacelike vectors.

Proposition

Two spatial metrics h^{ab} and $\tilde{h}^{ab}$ are conformally equivalent if and only if there is a non-zero scalar field Ω such that $\tilde{h}^{ab} = \Omega^2 h^{ab}$.

Definition

A Newtonian spatial conformal structure $\mathfrak{SC}$ *on a candidate Newtonian spacetime manifold* $\mathcal{M}$ *consists of:*

1. *at each point* $p \in \mathcal{M}$*, a three-dimensional vector subspace* $\mathfrak{SC}_p$ *of* T_p^*M *(the tensor space of 1-forms over* p*), smoothly varying from point to point*
2. *orthogonality relations fixed for all pairs of elements of* $\mathfrak{SC}_p$*, for all* p*, such that the distribution of three-dimensional subspaces of tangent vectors determined by the associated family of* h^{ab} *is integrable*

A spatial conformal derivative operator is the family of derivative operators each of which is compatible with some spatial metric in the conformal families. A representative of the spatial conformal structure is an ordered pair (h^{ab}, ∇_a) compatible with each other.

Proposition

Being spatially twist-free for a vector field $(\nabla^{[a}\xi^{b]} = 0)$ *is invariant with respect to spatial conformal structure.*

Definition

A Newtonian temporal conformal structure $\mathfrak{TC}$ *on a candidate Newtonian spacetime manifold* $\mathcal{M}$ *consists of a four-dimensional open, convex set of tangent vectors* $\mathfrak{TC}_p$ *at each point* $p \in \mathcal{M}$ *and a fixing of hyperbolic angles among all pairs of vectors in each* $\mathfrak{TC}_p$*, all such that:*

1. *the associated family of* t_{ab} *(fixed up to multiplication by a positive function) are all smooth, and of the form* $t_{ab} = t_a t_b$
2. *the distribution of three-dimensional vector subspaces of the tangent planes determined by a representative 1-form* t_a *is integrable*

A temporal conformal derivative operator is the family of derivative operators each of which is compatible with some temporal metric in the conformal families. A representative of the temporal conformal structure is an ordered pair (t_{ab}, ∇_a) compatible with each other.

Definition

A Newtonian conformal structure $\mathfrak{C}$ *on a candidate Newtonian spacetime manifold* M *consists of a spatial and a temporal conformal structure, compatible in the sense that they agree on their associated distribution of three-dimensional vector subspaces, i.e., on the leaves of the induced foliations.*

It follows that every h^{ab} in a Newtonian conformal structure is compatible with every t_{ab} in it, *i.e.*, $h^{an}t_{nb} = 0$. To fix a Newtonian spatial conformal structure fixes the entire conformal structure, as there is only one possible Newtonian temporal conformal structure compatible with it.

A conformal derivative operator is the family of derivative operators each of which is compatible with some pair of spatial and temporal metrics in the conformal families. A conformal representative is a triplet $(h^{ab},\, t_{ab},\ \nabla_a)$ all compatible with each other.

For a Newtonian conformal structure to be a physically meaningful analogue of conformal structure in general relativity, it should allow one to distinguish a preferred family of images of "null" curves, the *conformal spacelike geodesics*.

Proposition

Let ξ^a and η^a be two spacelike vector-fields everywhere orthogonal to each other and with vanishing Lie bracket. Then their integral curves are projective geodesics of the conformal derivative operator, i.e., for any representative ∇_a, $\xi^n \nabla_n \xi^a = \lambda \xi^a$, for some non-negative λ (and similarly for η^a). Every projective geodesic of the conformal derivative operator is part of such a family.

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Definition

A projective structure and a Newtonian conformal structure are compatible *with each other if the conformal spacelike geodesics determined by the Newtonian conformal structure are also projective geodesics.*

Lemma

A Newtonian spatial conformal structure and a compatible projective structure suffice to fix the temporal and spatial metrics up to spatially constant factors, and the derivative operator up to a difference tensor $C^a{}_{bc} = \lambda \delta^a{}_{(b} t_{c)}$, where λ is spatially constant.

Corollary

A Newtonian spatial conformal structure and a compatible projective structure suffice to fix a proper affine parametrization for the spacelike conformal/projective geodesics, i.e., they determine affine structures on every simultaneity slice.

the closest thing to Weyl's Theorem

Proposition

Given a Newtonian spatial conformal structure and a compatible projective structure, and ξ^a a spatially twist-free timelike vector-field of constant length on each simultaneity slice, then there exists a unique conformal representative $(h^{ab}, t_{ab}, \nabla_a)$ such that ξ^a is unit with respect to t_{ab} and geodesic with respect to ∇_a if and only if $R^a{}_b{}^c{}_d = R^c{}_b{}^a{}_d$.

(that ξ^a be of constant length on each simultaneity slice makes sense, since t_a is fixed up to a spatially constant factor)

the interpretation

1. To know what forces are orthogonal to each other and to know the images of the paths of freely falling bodies fixes the spatial derivative operator, the total derivative operator up to factors of t_a, and the spatial and temporal metrics up to spatially constant factors.
2. To fix then one non-rotating family of standard clocks all "traveling at the same speed" is to fix the entire geometry of the classical spacetime.
3. $R^a{}_b{}^c{}_d = R^c{}_b{}^a{}_d$ is an "integrability" condition that allows one to set a standard of non-rotational inertial motion.

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1. reformulate Newtonian Shape Dynamics results in geometrized picture, where "conformal structure" can be made precise and geometrically intuitive: insights into relation of Newtonian picture to relativity?
2. given special status accorded embedded 3-manifolds in reconstruction of general relativity, use geometric structure like this one, with h^{ab} "encoding" shift, t_a lapse? $\Rightarrow$ automatically get non-privileged temporal and spatial scales