Title: Moduli of Vacua and Categorical representations

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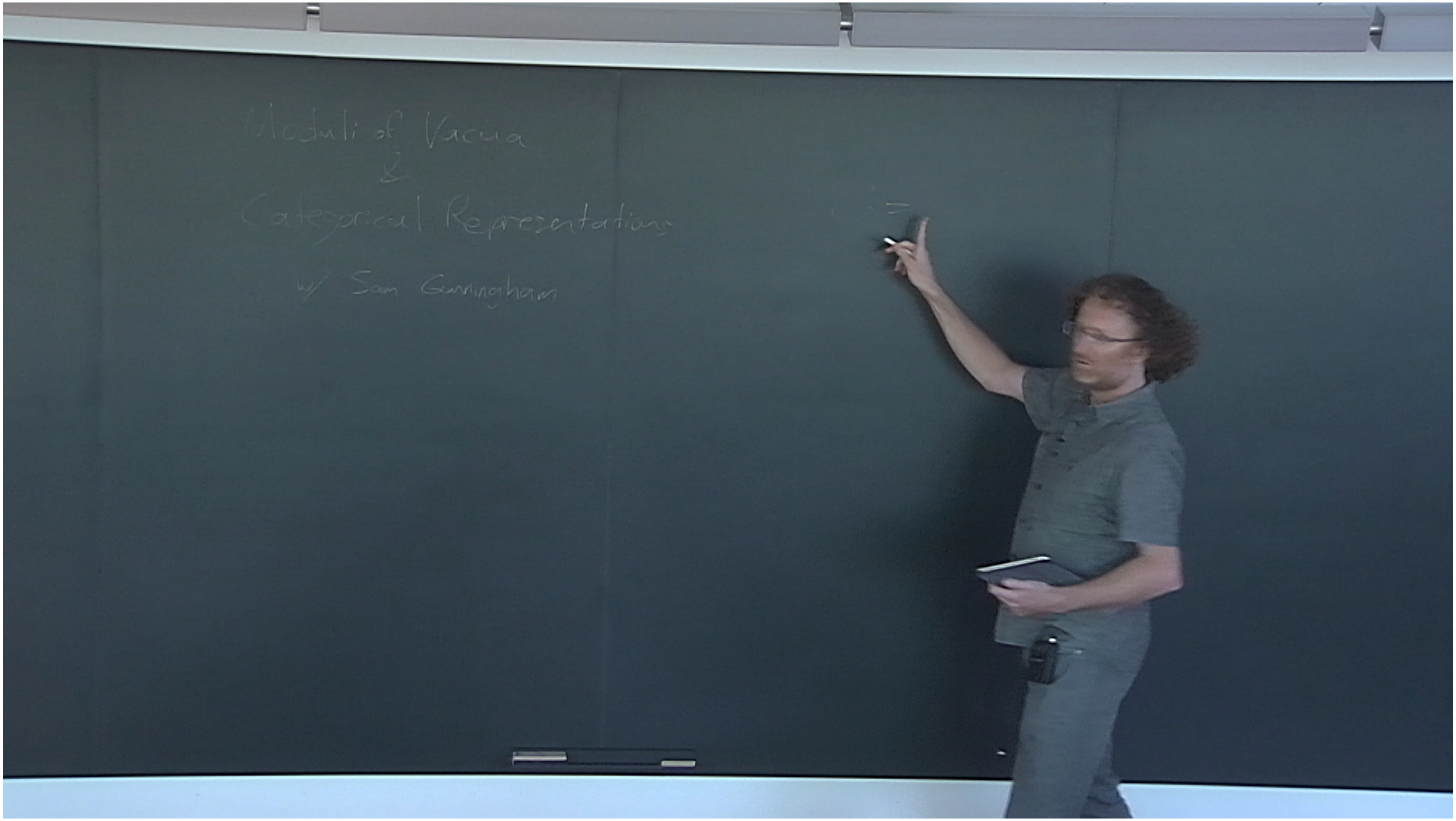
Abstract: <p>I will present some results on three-dimensional gauge theory from the point of view of extended topological field theory. In this setting a theory is specified by describing its collection of boundary conditions - in our case, a collection of categories (standing in for 2d TFTs) with a prescribed symmetry group G. We will apply ideas from Seiberg-Witten geometry to construct a new commutative algebra of symmetries for categorical representations (or line operators in the gauge theory) - a categorification of Kostant's description of the center of the enveloping algebra. (Joint with Sam Gunningham and David Nadler)</p>


Moduli of Vacua
&
Categorical Representations
w/ Sam Gunningham

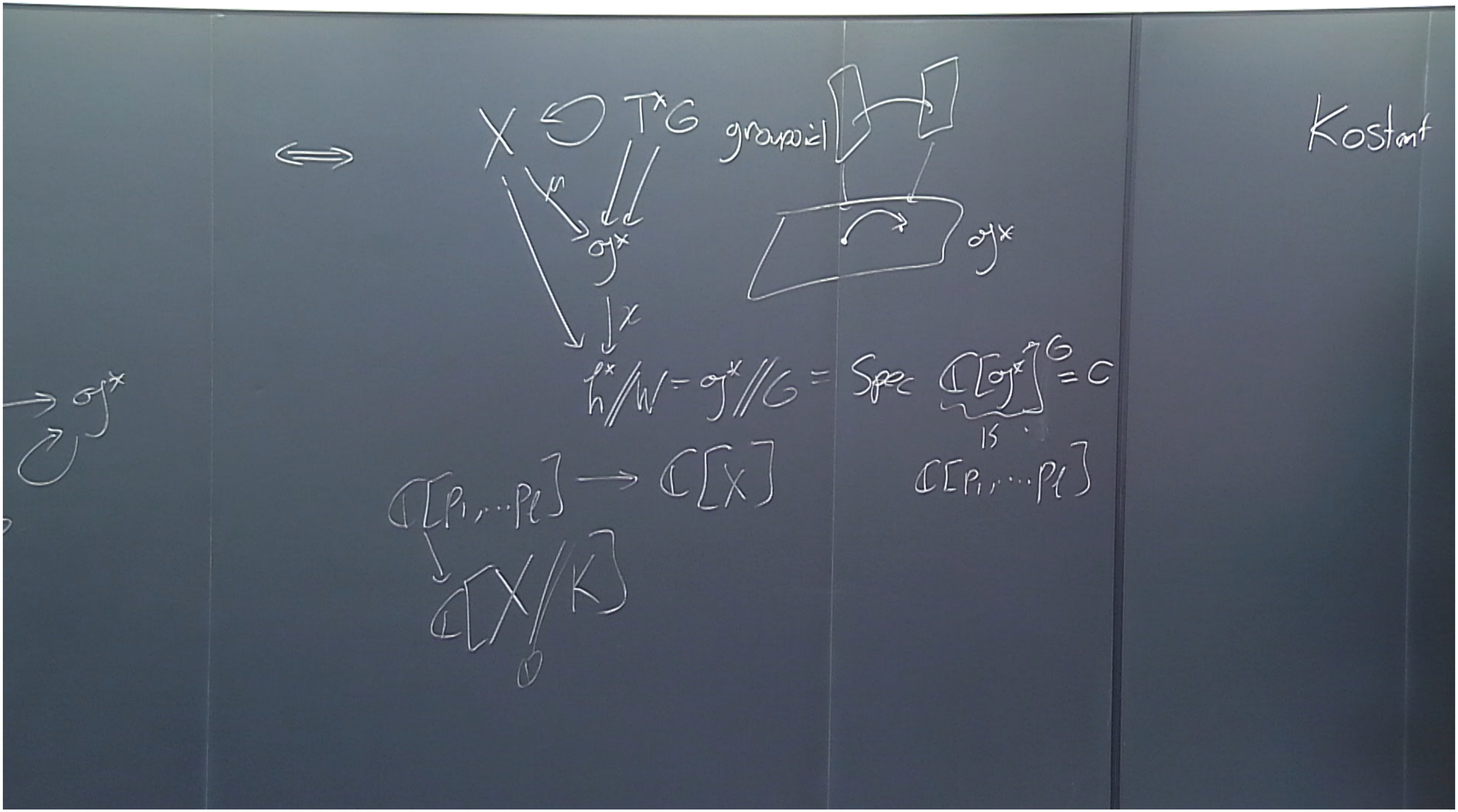
$G = G_{\mathbb{C}}$ complex reductive group

Hamiltonian G-space:

X hol. symplectic

$$X \xrightarrow{\mu} \mathfrak{g}^*$$

G


X ⟲ T*G
groupoid
μ
g*
χ
h*//W = g*//G = Spec C[g*]^G = C
C[p1, ... pl]
C[p1, ... pl] → C[X]
C[X//K]
Kostant

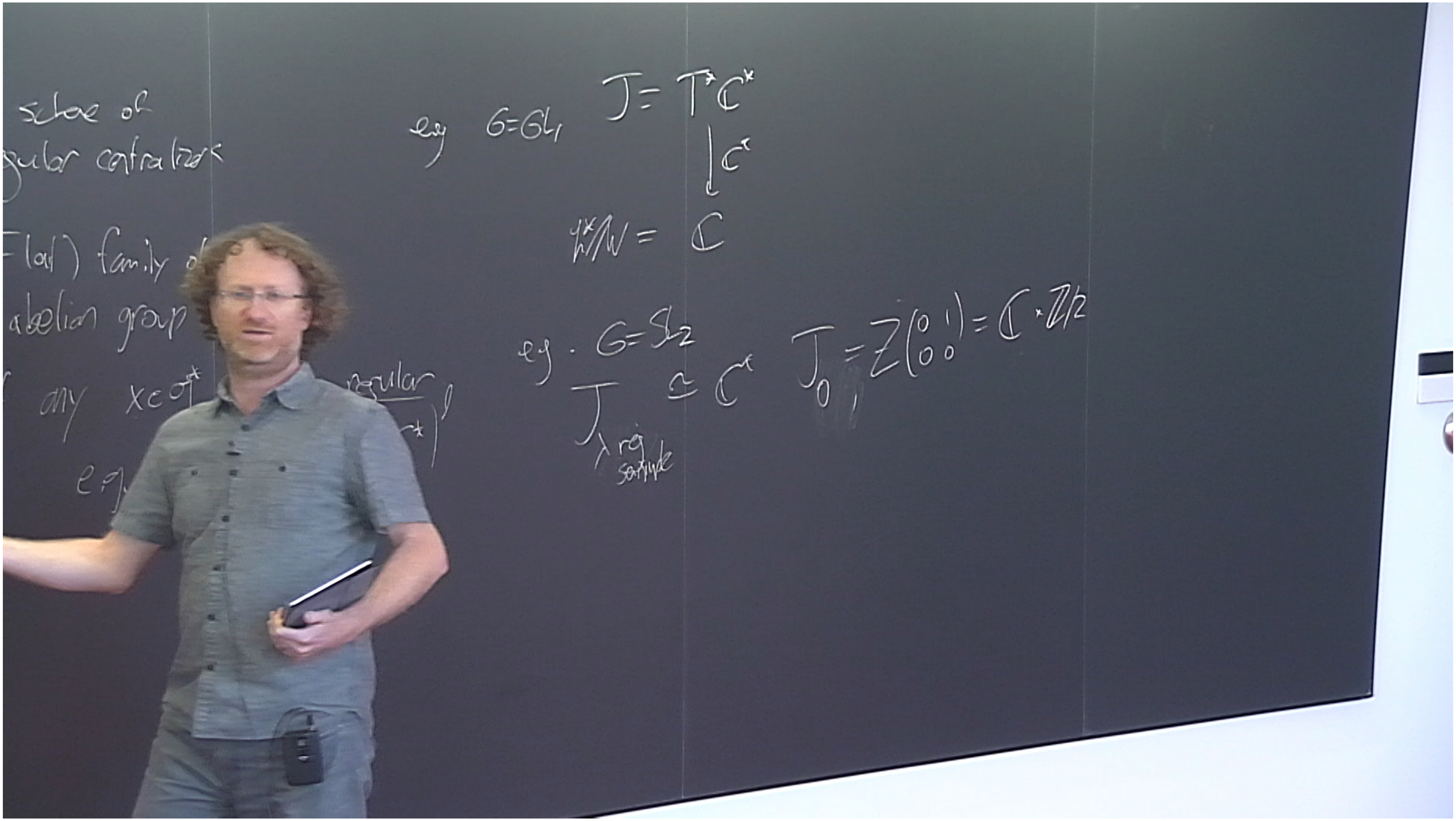
$$\begin{tikzcd} \text{Kostant: } J_\lambda \arrow[d] & \subset & J \arrow[d] \\ \lambda & \in & \mathfrak{h}^*/\!/W \end{tikzcd}$$

group scheme of regular centralizers

(Flat) family of abelian group

J_λ = centralizer of any $x \in \mathfrak{g}^*$ which is <u>regular</u>,

e.g. $x \in \mathfrak{g}^{rss}$ $J_\lambda \simeq (\mathbb{C}^*)^l$


abelian group
eg G=GL1 J = T*C*
C*
h*/W = C
eg . G=SL2
Jλ reg semisimple ≃ C*
J0 = Z(0 1 / 0 0) = C* × Z/2

Kostant $J_\lambda \subset J$ group scheme of regular centralizers

$\lambda \in \mathfrak{h}^*/W$ (Flat) family of abelian group

J_λ = centralizer of any $x \in \mathfrak{g}^*$ which is regular,

e.g. $x \in \mathfrak{g}^{rs}$ $J_\lambda \simeq (\mathbb{C}^*)^l$

J = Coulomb branch of pure 3d $\mathcal{N}=4$ SYM

e.g. $G = GL_1$ $J = T^*\mathbb{C}^*$

$$\begin{array}{c} J = T^*\mathbb{C}^* \\ \downarrow \mathbb{C}^* \\ \mathfrak{h}^*/W = \mathbb{C} \end{array}$$

e.g. $G = SL_2$

$J_{\lambda \text{ reg semisimple}} \simeq \mathbb{C}^*$ $J_0 = Z\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbb{C} \times \mathbb{Z}/2$

Ngô's Lemma: $\exists$ canonical homomorphism from regular centralizers to centralizers

$$J_{\chi(x)} \longrightarrow I_x$$

$$\chi^* J \longrightarrow I$$

$\Longleftrightarrow$

$X \circlearrowleft T^*G$ groupoid $\quad I_x = Z_G(x)$

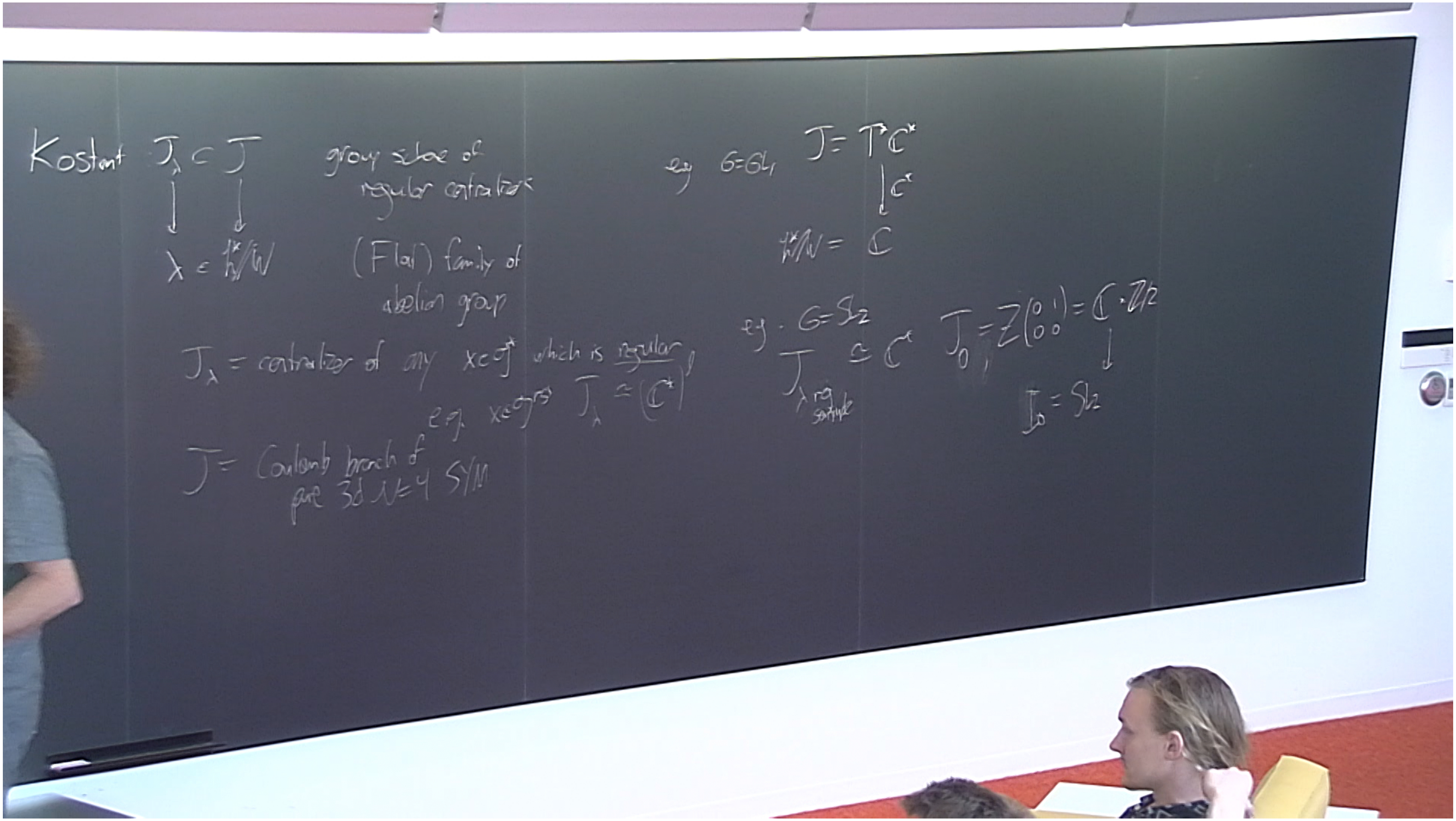
$$\begin{tikzcd} X \arrow[r, "\mu"] \arrow[dr] & \mathfrak{g}^* \ni x \arrow[d, "\chi"] \\ & \mathfrak{h}^*/\!/W = \mathfrak{g}^*/\!/G \ni \lambda \end{tikzcd}$$

$\mathfrak{g}^*$

$\mathfrak{h}^*/\!/W = \mathfrak{g}^*/\!/G = \operatorname{Spec} \underbrace{\mathbb{C}[\mathfrak{g}^*]^G}_{\cong\ \mathbb{C}[p_1, \dots p_\ell]} = \mathbb{C}$

$$\begin{tikzcd} \mathbb{C}[p_1, \dots p_\ell] \arrow[r] \arrow[d] & \mathbb{C}[X] \\ \mathbb{C}[X/\!/K] & \end{tikzcd}$$

$\mathbb{C}[X/\!/K]$ — 0


Kostant J_λ ⊂ J
group scheme of regular centralizers
λ ∈ h*/W
(Flat) family of abelian group
J_λ = centralizer of any x ∈ g* which is regular,
e.g. x ∈ g*rs J_λ ≃ (C*)^r
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e.g. G=GL1 J = T*C*
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e.g. G=SL2
J_λ reg semisimple ≃ C*
J_0 = Z(0 1 / 0 0) = C × Z/2
I_0 = SL2

Moduli of Vacua
&
Categorical Representations

w/ Sam Gunningham

Ngô's Lemma: $\exists$ canonical homomorphism from regular centralizers to centralizers

$$\begin{tikzcd} X \arrow[dr] & J \arrow[l, bend right] \arrow[d] \\ & \mathfrak{h}^*/\!/W \end{tikzcd}$$

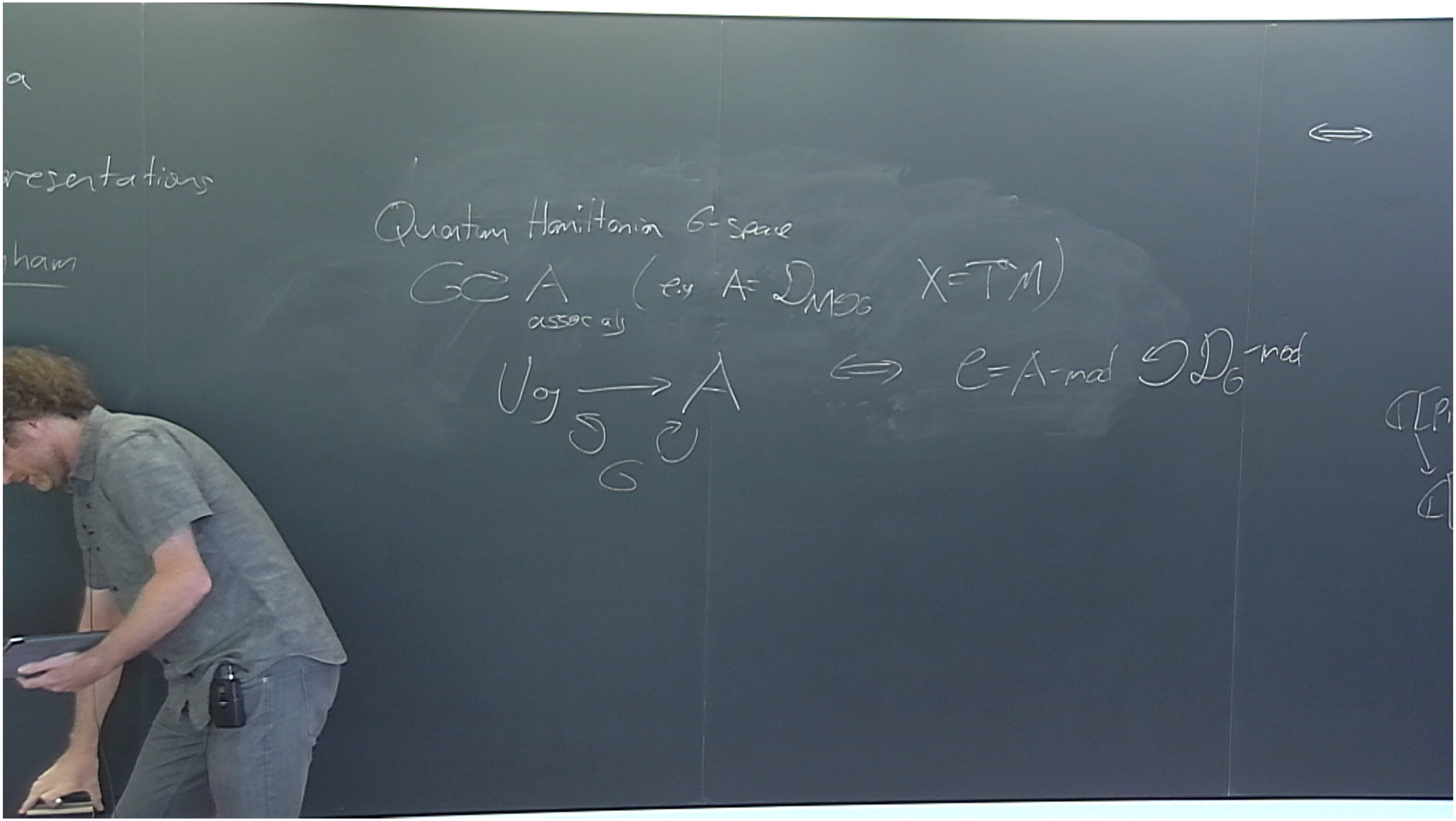
$$J_{\chi(x)} \longrightarrow I_x$$

$$\chi^* J \longrightarrow I$$

$\Longleftrightarrow$ X

$\mathbb{C}[p_1, \dots$

$\mathbb{C}[X$


a
resentations
gham
Quantum Hamiltonian G-space
G ⟳ A (e.g. A = D_{M⊗G} X = T*M)
assoc alg
Ug → A ⟺ C = A-mod ⟲ D_G-mod
G

$Z_G(\lambda)$

$\mathfrak{g}^*$

$\mathbb{C}[\mathfrak{g}^*]^G = \mathbb{C}[$

$\wr$

$p_1, \ldots p_\ell]$

$$Z(U\mathfrak{g}) = (U\mathfrak{g})^G \longrightarrow A$$

$$\wr$$

$$\mathbb{C}[\mathfrak{h}^*/\!/W] = \mathbb{C}[\Delta_1, \ldots \Delta_\ell]$$

$$\Delta_1, \ldots \Delta_\ell \in \mathcal{D}_{G/K}$$

commuting, G-invariant

Harish-Chandra system

$$M_\lambda = \{\Delta_i f = \lambda_i f\}$$

$$\lambda \in \mathfrak{h}^*/\!/W$$

a

resentations

ham

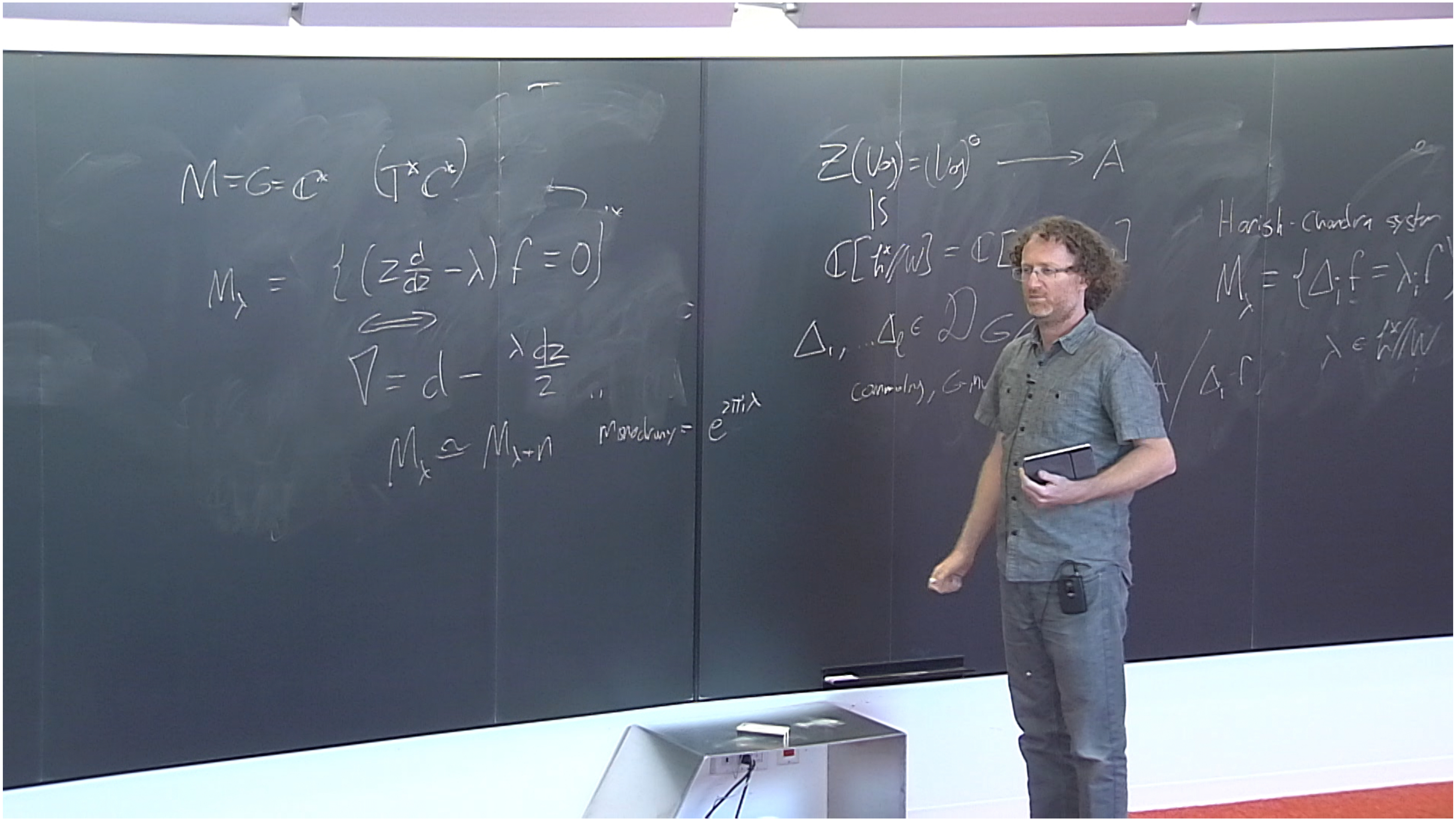
$Z \longrightarrow A$ <u>not</u> central

"Spec A" doesn't live over $\mathrm{Spec}\, Z = \mathfrak{h}^*/W$

e.g. "quantum fibers" $M_\lambda \simeq M_{\lambda'}$

$\lambda \neq \lambda'$

$\Longleftrightarrow$



$M = G = \mathbb{C}^\times \quad (T^*\mathbb{C}^\times)$
$M_\lambda = \{(z\frac{d}{dz} - \lambda)f = 0\}$
$\nabla = d - \lambda\frac{dz}{z}$
$M_\lambda \simeq M_{\lambda+n}$
Monodromy = $e^{2\pi i\lambda}$
$Z(U\mathfrak{g}) = (U\mathfrak{g})^G \longrightarrow A$
$\mathbb{C}[\mathfrak{h}^*/W] = \mathbb{C}[$
$\Delta_1, \ldots \Delta_\ell \in \mathcal{D}G$
commuting, G-in
Harish-Chandra system
$M_\lambda = \{\Delta_i f = \lambda_i f\}$
$\lambda \in \mathfrak{h}^*/W$

$M = G = \mathbb{C}^\times \quad (T^*\mathbb{C}^\times)$

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$\Longleftrightarrow$

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Monodromy $= e^{2\pi i \lambda}$

$M_\lambda \simeq M_{\lambda+n}$

Moduli is $\mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^\times$

$Z(U\mathfrak{g}) = (U\mathfrak{g})^G \longrightarrow A$

$\wr$

$\mathbb{C}[\mathfrak{h}^*/W] = \mathbb{C}[\Delta_1, \dots \Delta_\ell]$

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Harish-Chandra system

$M_\lambda = \{ \Delta_i f = \lambda_i f \}$

$A/\Delta_i - r$

$\lambda \in \mathfrak{h}^*/W$

ie $\mathcal{D}$

Theorem (BZ-Gunningham)

Any quantum Hamiltonian G-space is linear over $\mathfrak{h}^*/\!/W^{aff} \simeq H^\vee/\!/W$

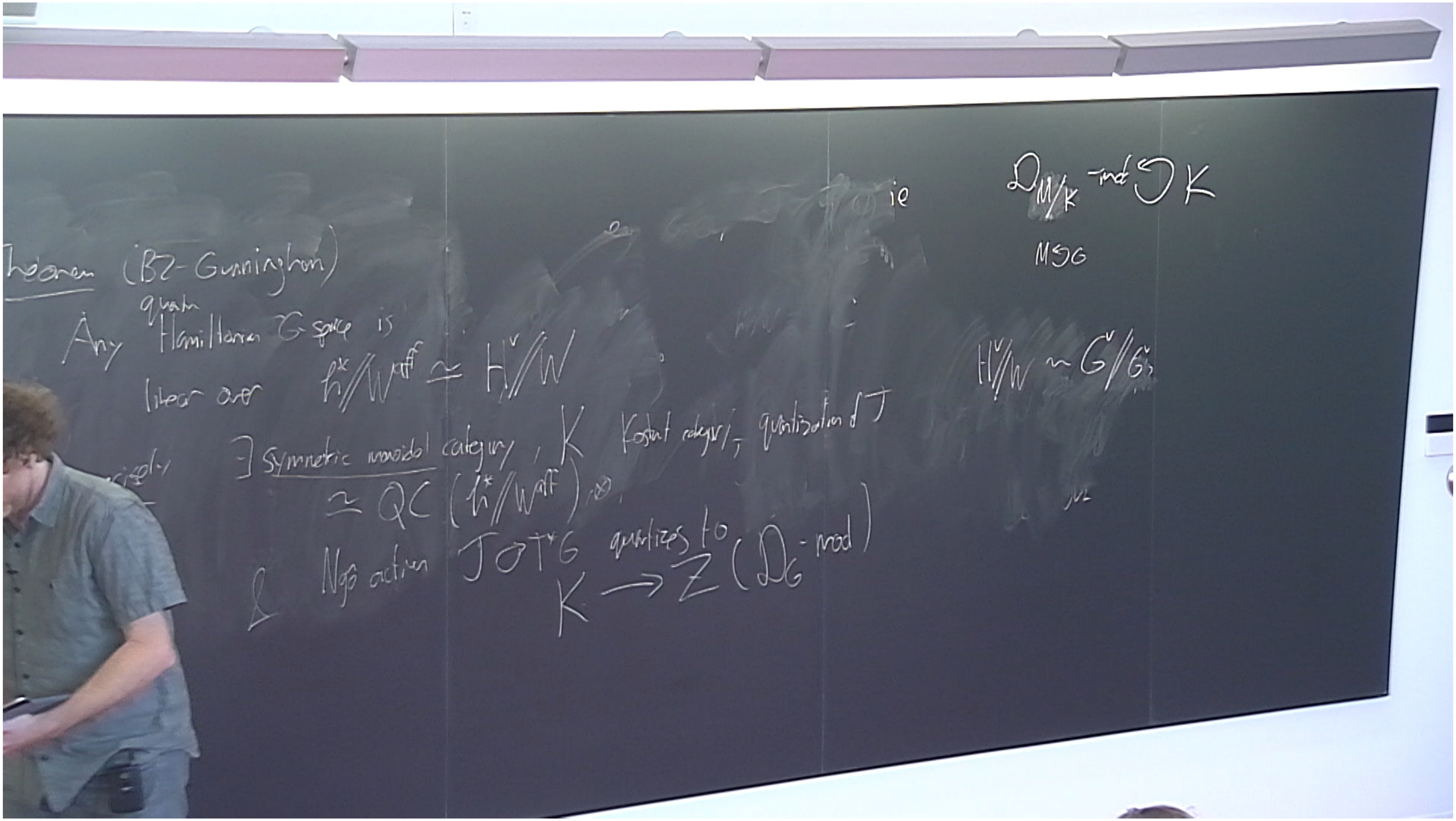
More precisely $\exists$ symmetric monoidal category, K Kostant category, quantization of J

$\simeq QC(\mathfrak{h}^*/\!/W^{aff})^{\otimes}$

& $N\mathfrak{g}$ action $J \circlearrowright T^*G$ quantizes to

$K \longrightarrow Z(\mathcal{D}_G\text{-mod})$

G-space is
Symmetric monoidal category
Ngô action
MSG
ie


Theorem (BZ-Gunningham)
Any quantum Hamiltonian G-space is
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Kostant category, quantization of J
≃ QC
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K → Z (D_G - mod)
ie
D_{M/K} -mod ⟲ K
M⟲G

$$K = H_*^{G\times S^1}(\mathrm{Gr}_{G^\vee})\text{-mod}$$

$=$ line operators in 3d $\mathcal{N}=4$ in Ω-background

$=$ quantized sheaves on J

$=$ Whittaker Hecke category

$Z \longrightarrow A$ not central

"Spec A" doesn't live over Spec $Z = \mathfrak{h}^*/W$

e.g. "quantum fibers" $M_\lambda \simeq M_{\lambda'}$, $\lambda \neq \lambda'$

$\mathcal{D}(N_\psi \backslash G / N_\psi)$

$Z =$ Whittaker reduction of $U\mathfrak{g}$

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= modules for affine nil-Hecke algebra

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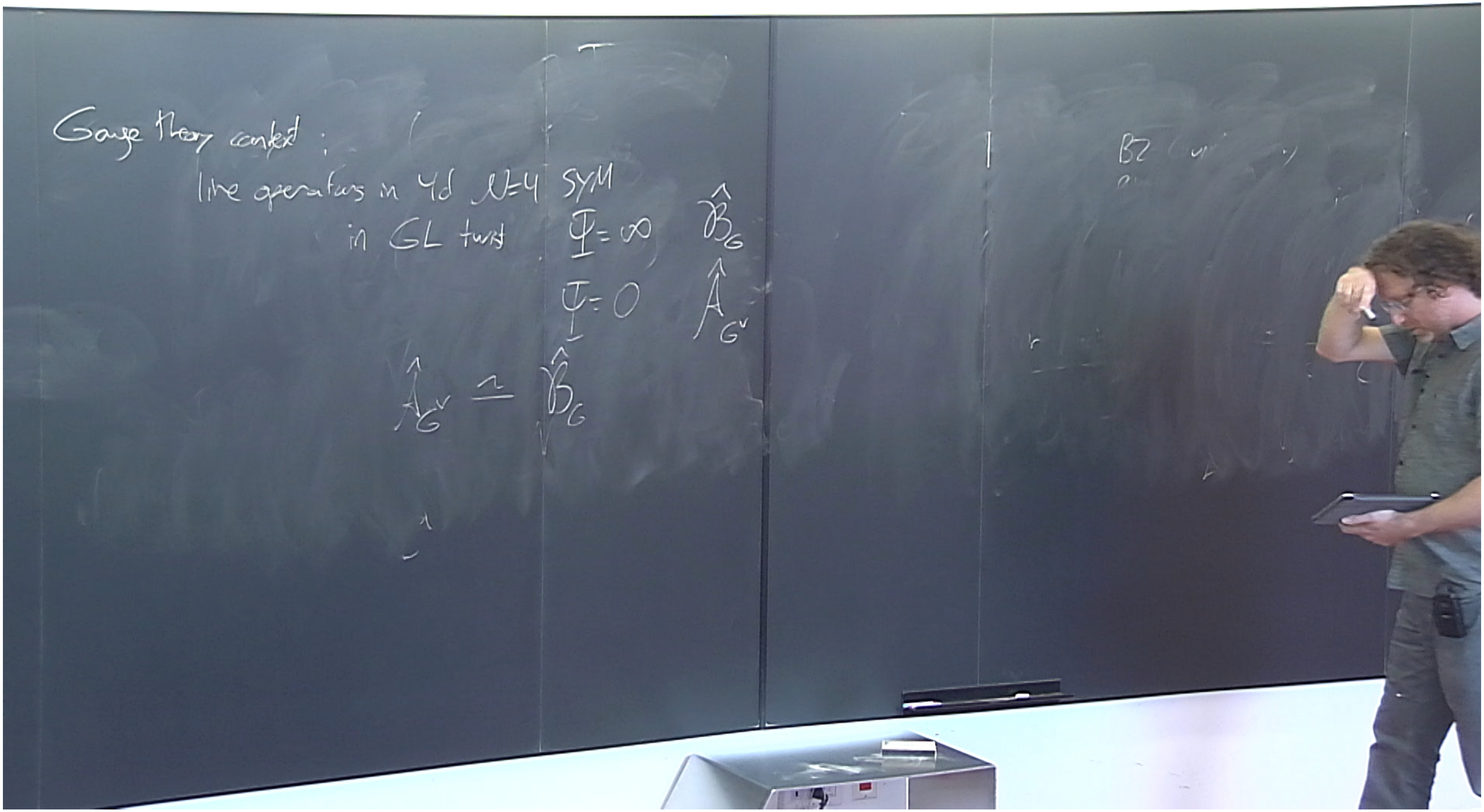
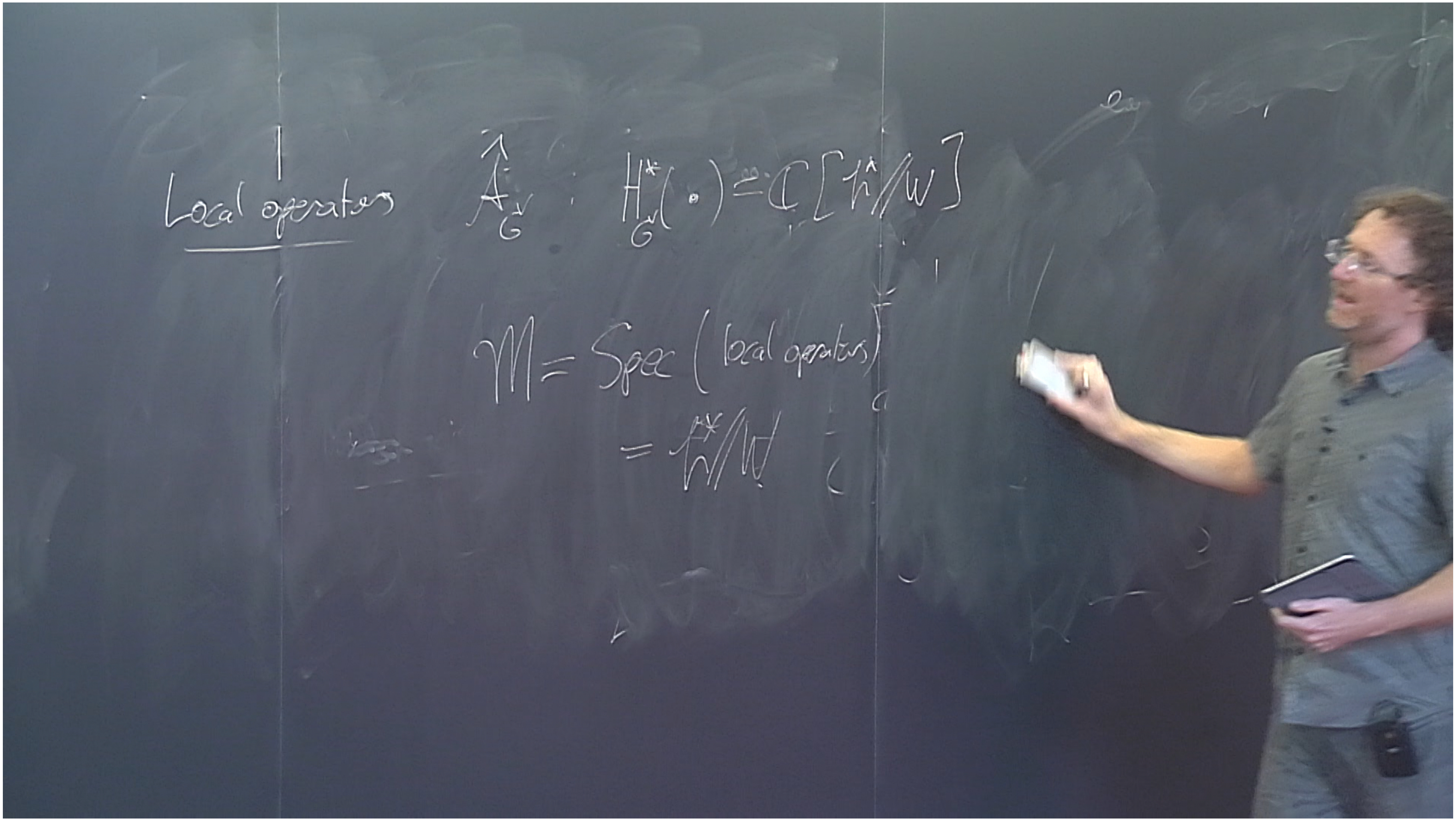
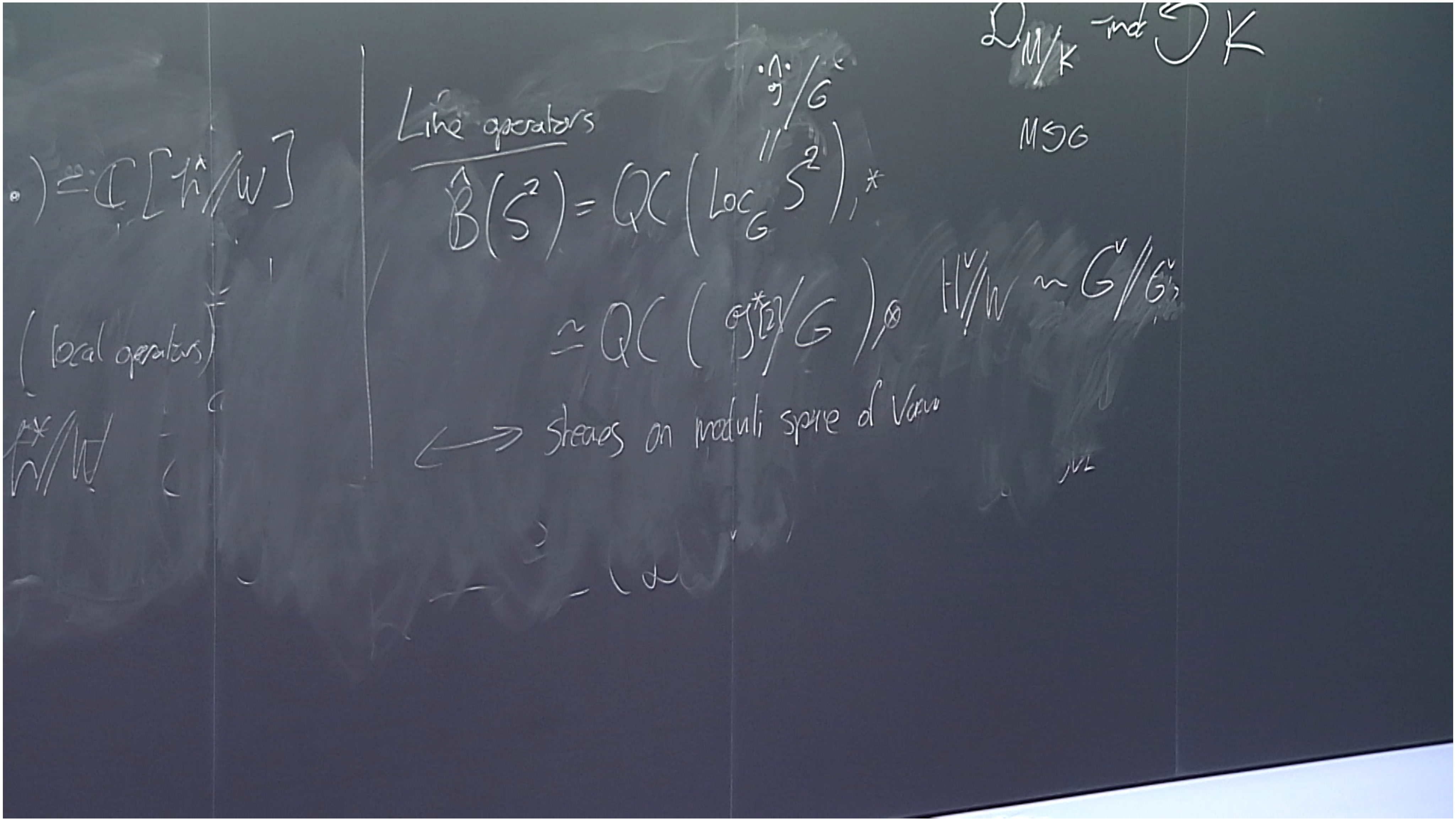
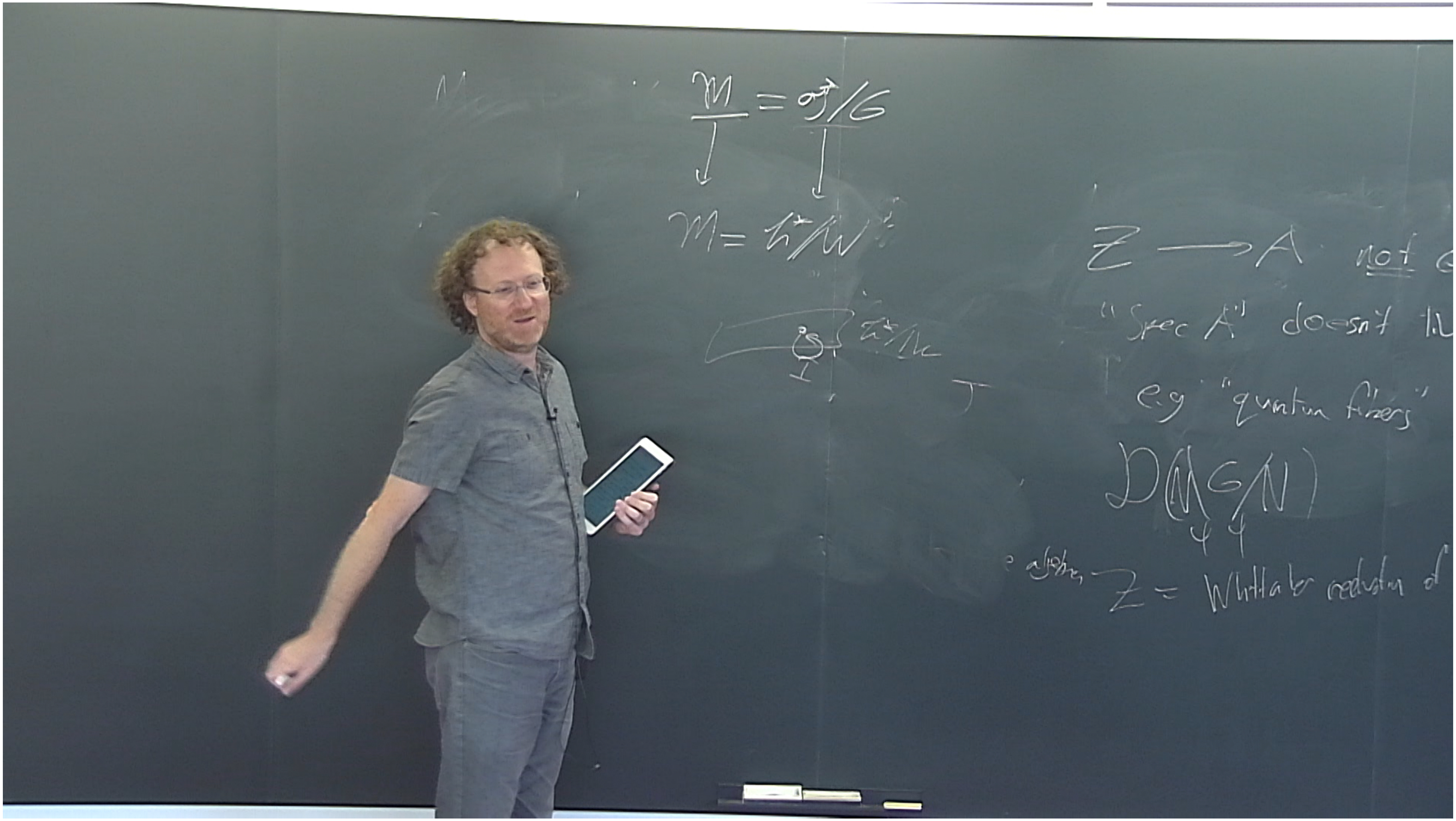

Gauge theory context:
line operators in 4d N=4 SYM
in GL twist
Ψ = ∞
Ψ = 0
Â_{G^v} ≃ B̂_G


Local operators
$\hat{A}_{G^\vee}$: $H^*_{G^\vee}(\cdot) \cong \mathbb{C}[\hat{t}/\!/W]$
$\mathcal{M}$ = Spec (local operators)
$= \hat{t}^*/\!/W$


Line operators
$\hat{B}(S^2) = QC(Loc_G S^2)$, *
$\simeq QC(\mathfrak{g}^*[2]/G)$, ⊗
$H^\vee//W \sim G^\vee//G^\vee$
⟷ sheaves on moduli space of Vacua
MSG
(local operators)
$\simeq \mathbb{C}[\hat{\mathfrak{h}}//W]$


Z ⟶ A not
"Spec A" doesn't
e.g. "quantum fibers"
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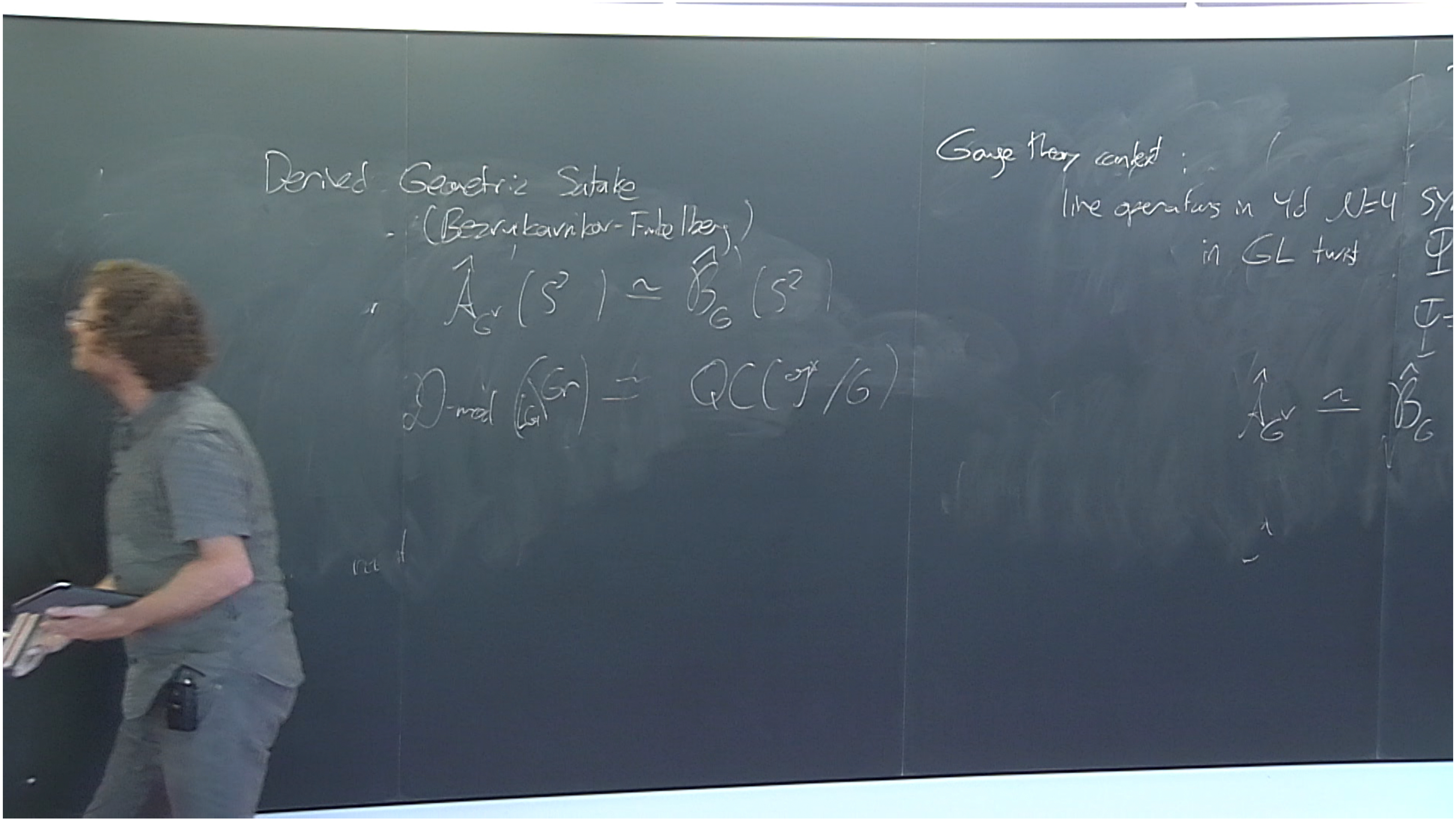


Derived Geometric Satake
(Bezrukavnikov-Finkelberg)
$\hat{A}_{G^\vee}(S^2) \simeq \hat{B}_G(S^2)$
$\mathcal{D}\text{-mod}_{(G_\mathcal{O})}(Gr) \simeq QC(\mathfrak{g}^*/G)$
Gauge theory context:
line operators in 4d $\mathcal{N}=4$ SY
in GL twist
$\hat{A}_{G^\vee} \simeq \hat{B}_G$

$$\begin{array}{ccc} \mathcal{M} = \mathfrak{g}^*/G & \longleftrightarrow & T^*G \\ \downarrow & & \downarrow \\ M = \mathfrak{h}^*/W & & \mathfrak{g}^* \end{array}$$

$QC(\mathfrak{g}^*/G)$

Satzger?

$\wr$ Morita

$QC(T^*G)$

Derived Geometric Satake

(Bezrukavnikov-Finkelberg)

$\hat{A}_{G^\vee}(S^2) \simeq \hat{B}_G(S$

$\mathcal{D}\text{-mod}(\mathcal{L}G)^{Gr} \simeq QC($

(S^2)

$\mathbb{C}(\mathfrak{g}^* / G)$

Omega background

$\mathbb{R}^4$

line operator

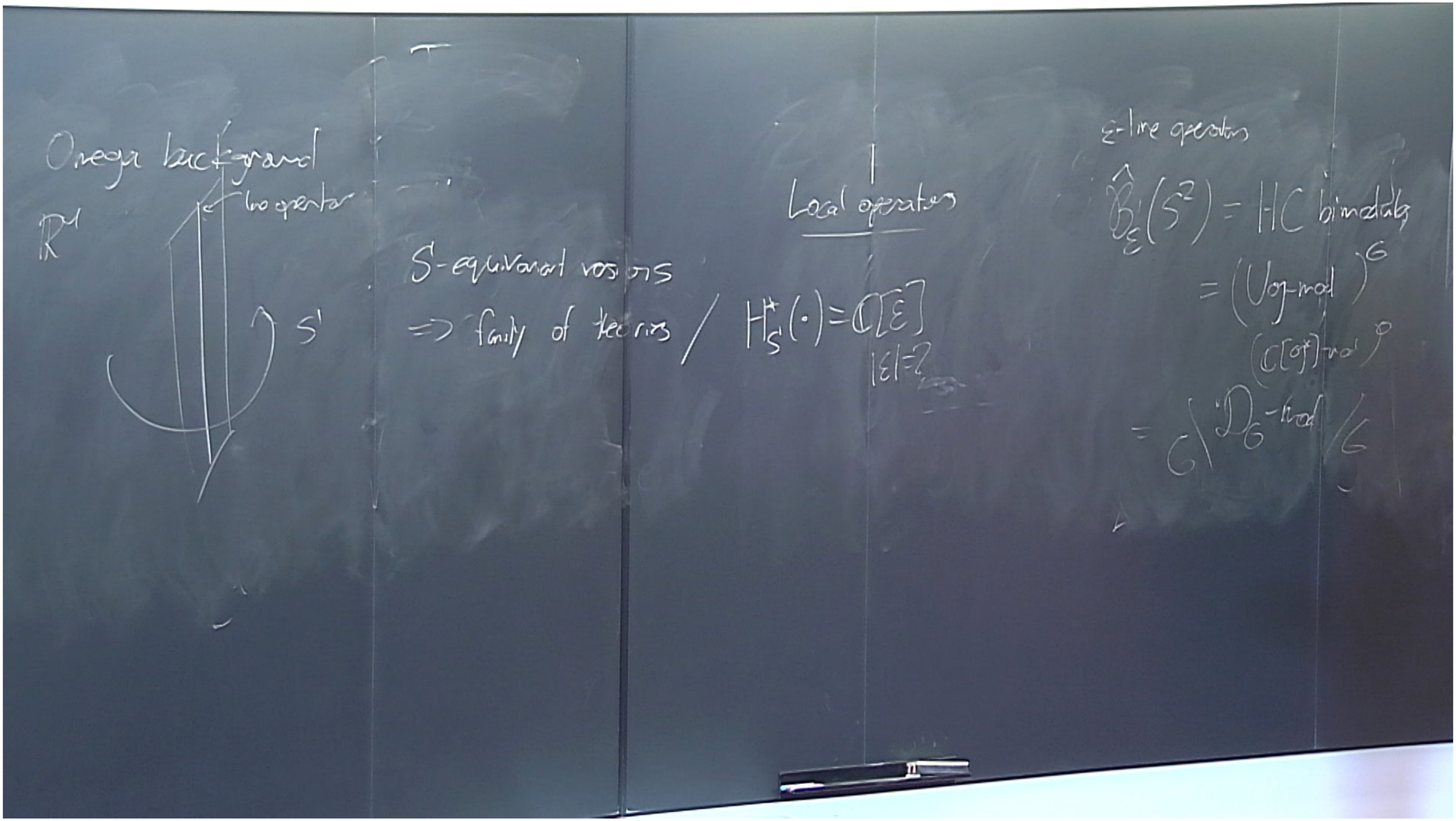
S^1

S-equivariant versions

$\Rightarrow$ family of theories / $H^*_{S^1}(\cdot) = \mathbb{C}[\varepsilon]$

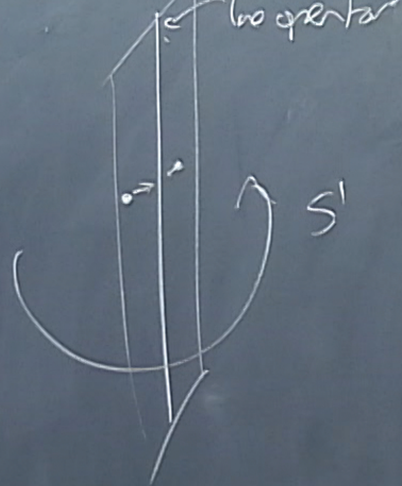
$|\varepsilon| = 2$

Local operators


Omega background
$\mathbb{R}^4$
line operator
S^1
S^1-equivariant versions
=> family of theories / $H^*_{S^1}(\cdot) = \mathbb{C}[\varepsilon]$
$|\varepsilon| = 2$
Local operators
ε-line operators
$\hat{\mathcal{B}}_\varepsilon(S^2)$ = HC bimodules
$= (U\mathfrak{g}\text{-mod})^G$
$(\mathbb{C}[\mathfrak{g}^*]\text{-mod})^G$
$= G \backslash \mathcal{D}_G\text{-mod} / G$

Omega background

$\mathbb{R}^4$



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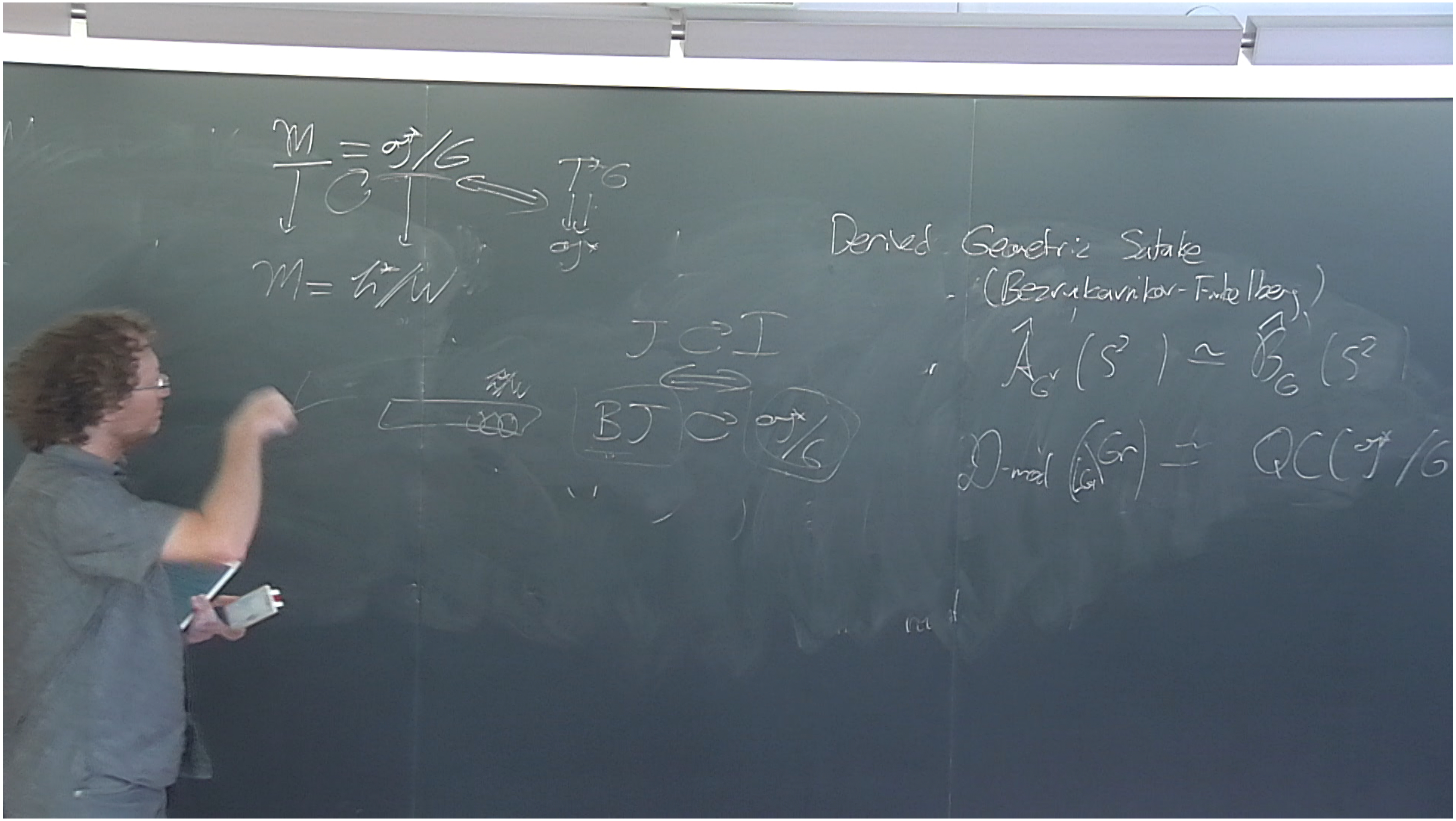
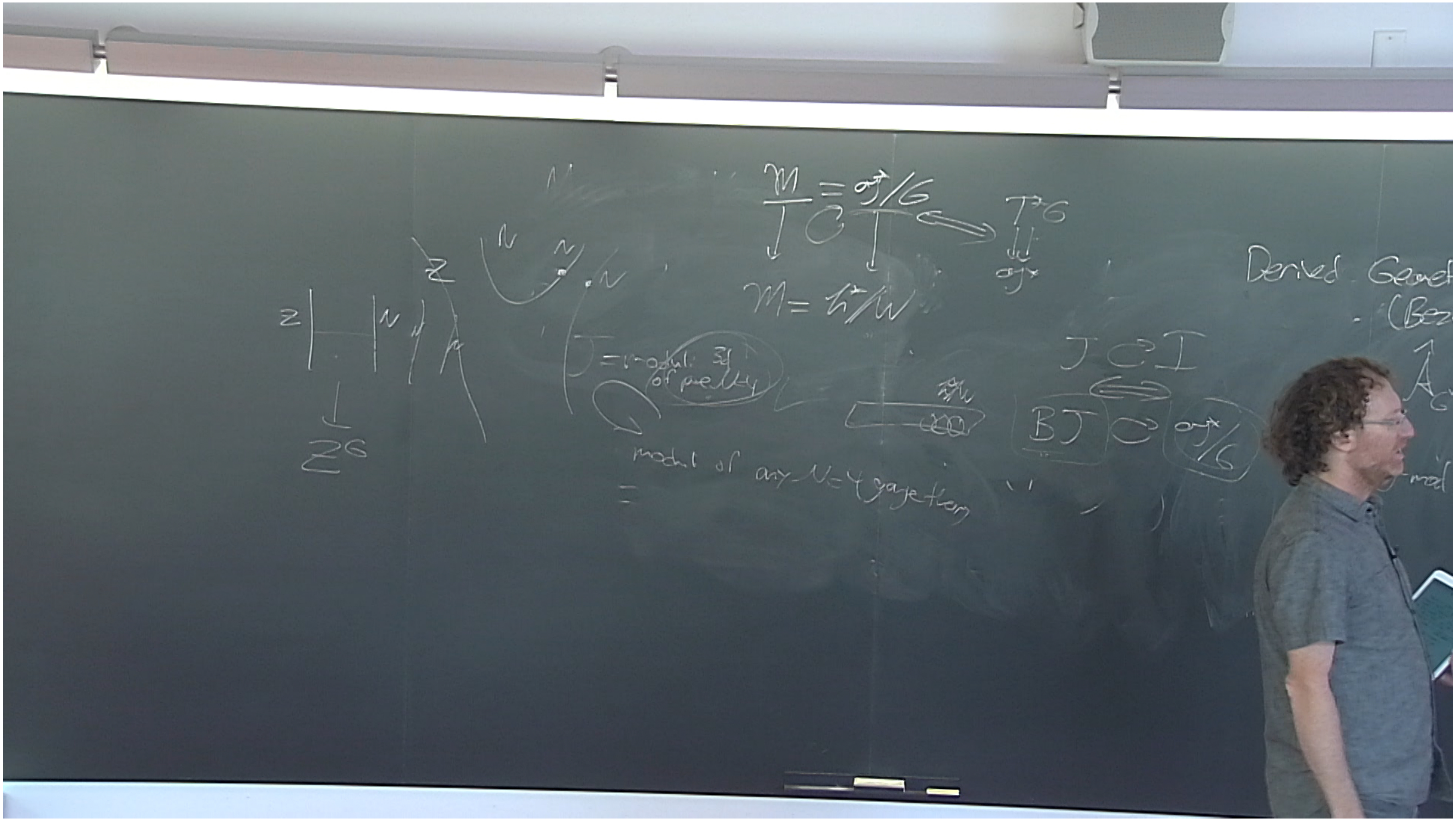
Local operators: $Z(U\mathfrak{g})$

$\simeq \mathbb{C}[\mathfrak{h}^*/W]$


Derived Geometric Satake
(Bezrukavnikov-Finkelberg)
BT
D-mod
QC

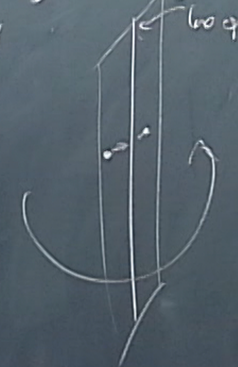

Derived
BJ
modul of any N=4 gauge theory

Derived G

$$QC(BJ) = \mathrm{Rep}\, J$$
$$= H_*^G(Gr)\text{-comod}$$
$$\sim \mathrm{Loc}^G(Gr)$$
$$= \mathrm{Loc}([S^2, BG])$$

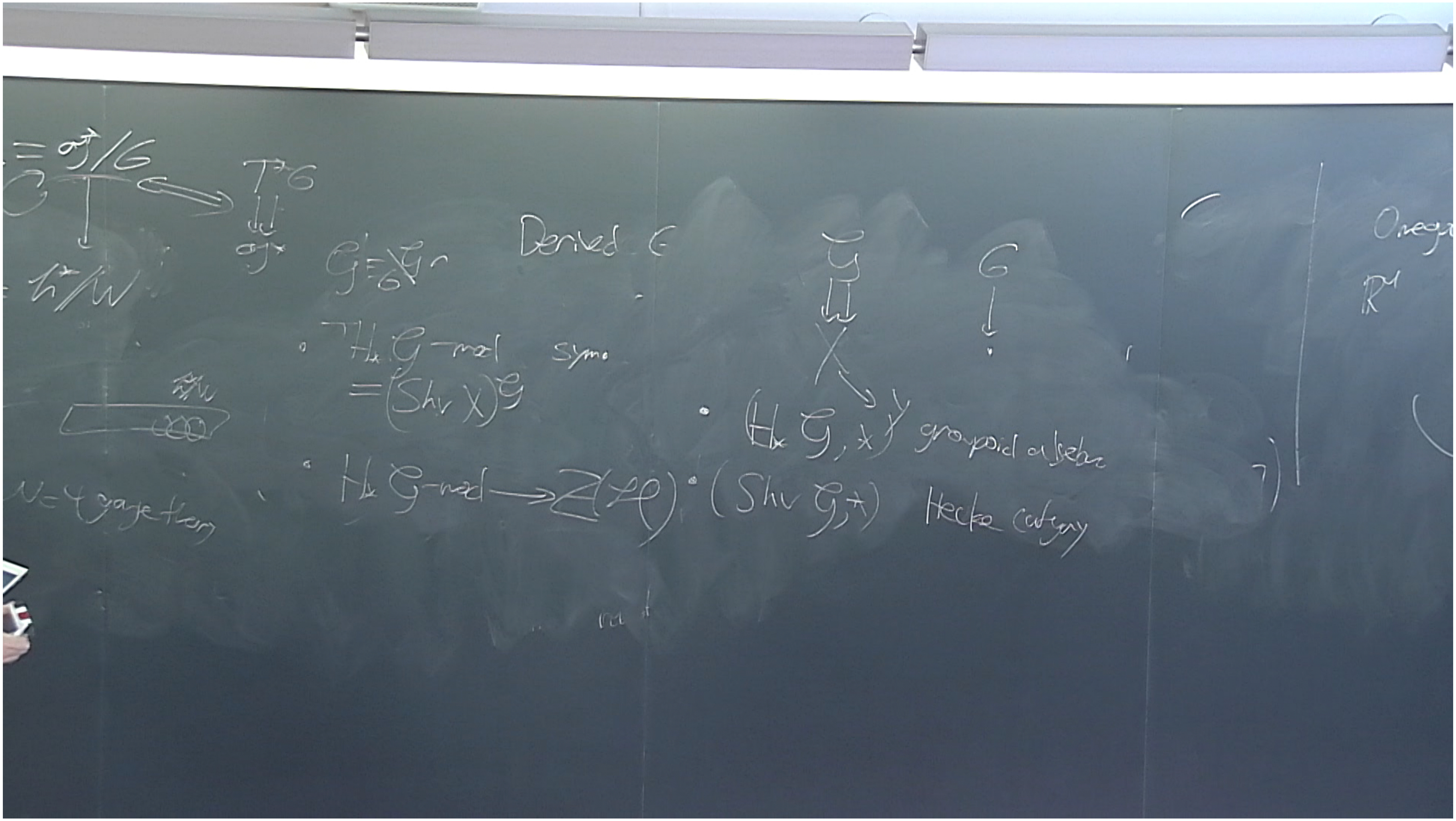
Omega background

$\mathbb{R}^4$



S-equivariant vac

$\Rightarrow$ family of the


$= \mathfrak{g}^*/G$
T^*G
$\mathfrak{g}^*$
$\mathfrak{h}^*/W$
Derived G
$\mathcal{G} = G \backslash \mathcal{G}r$
$H_* \mathcal{G}$-mod symp.
$= (\mathrm{Shv}\, X)^{\mathcal{G}}$
$(H_* \mathcal{G}, *)$ groupoid algebra
$H_* \mathcal{G}$-mod $\longrightarrow \mathcal{Z}(\mathcal{L}\mathcal{C})$
$(\mathrm{Shv}\, \mathcal{G}, *)$ Hecke category
$\mathcal{N}=4$ gauge theory
$\mathbb{R}^4$