Title: Moduli of Vacua and Categorical representations

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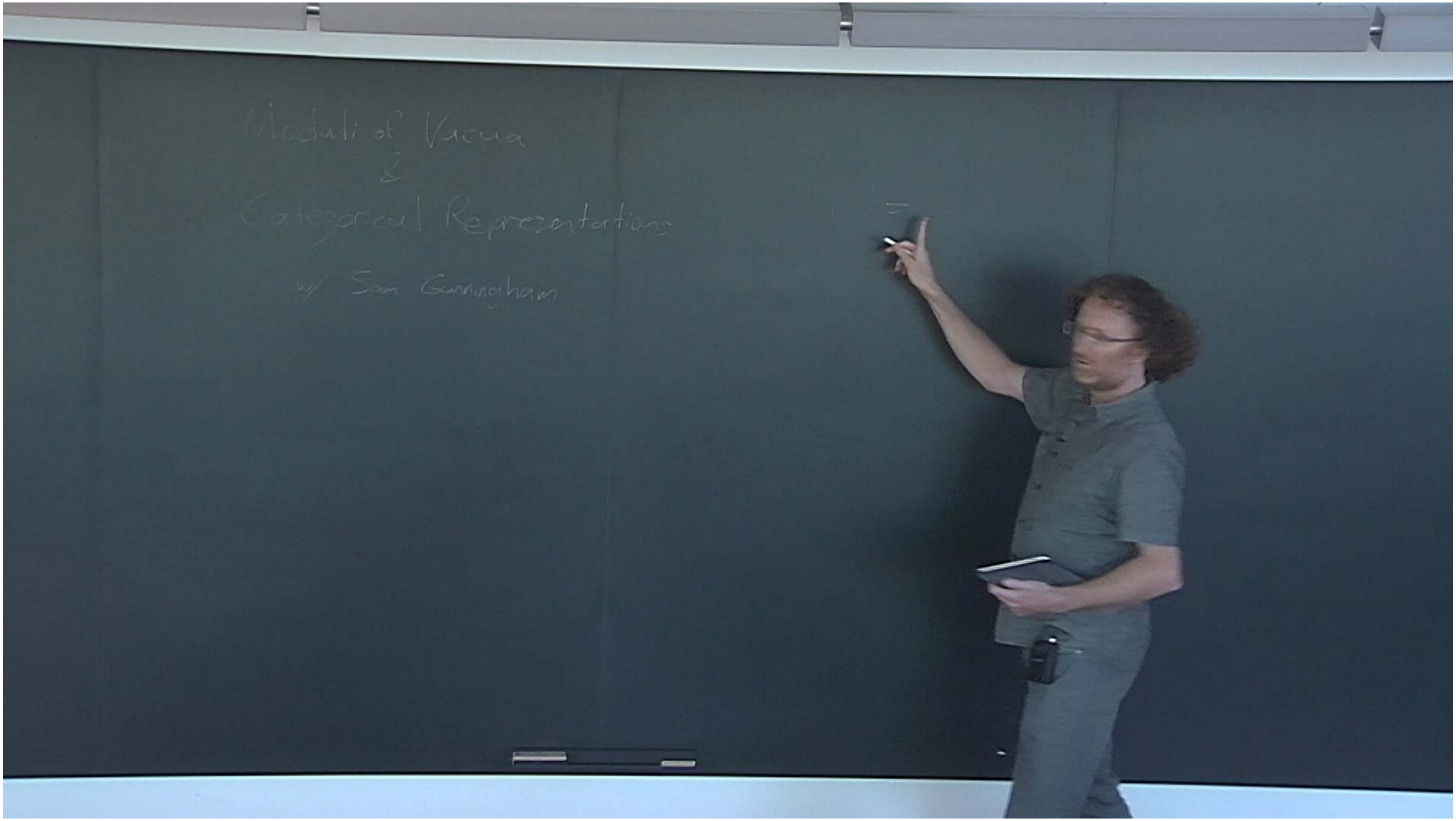
Abstract: <p>I will present some results on three-dimensional gauge theory from the point of view of extended topological field theory. In this setting a theory is specified by describing its collection of boundary conditions - in our case, a collection of categories (standing in for 2d TFTs) with a prescribed symmetry group G. We will apply ideas from Seiberg-Witten geometry to construct a new commutative algebra of symmetries for categorical representations (or line operators in the gauge theory) - a categorification of Kostant's description of the center of the enveloping algebra. (Joint with Sam Gunningham and David Nadler)</p>

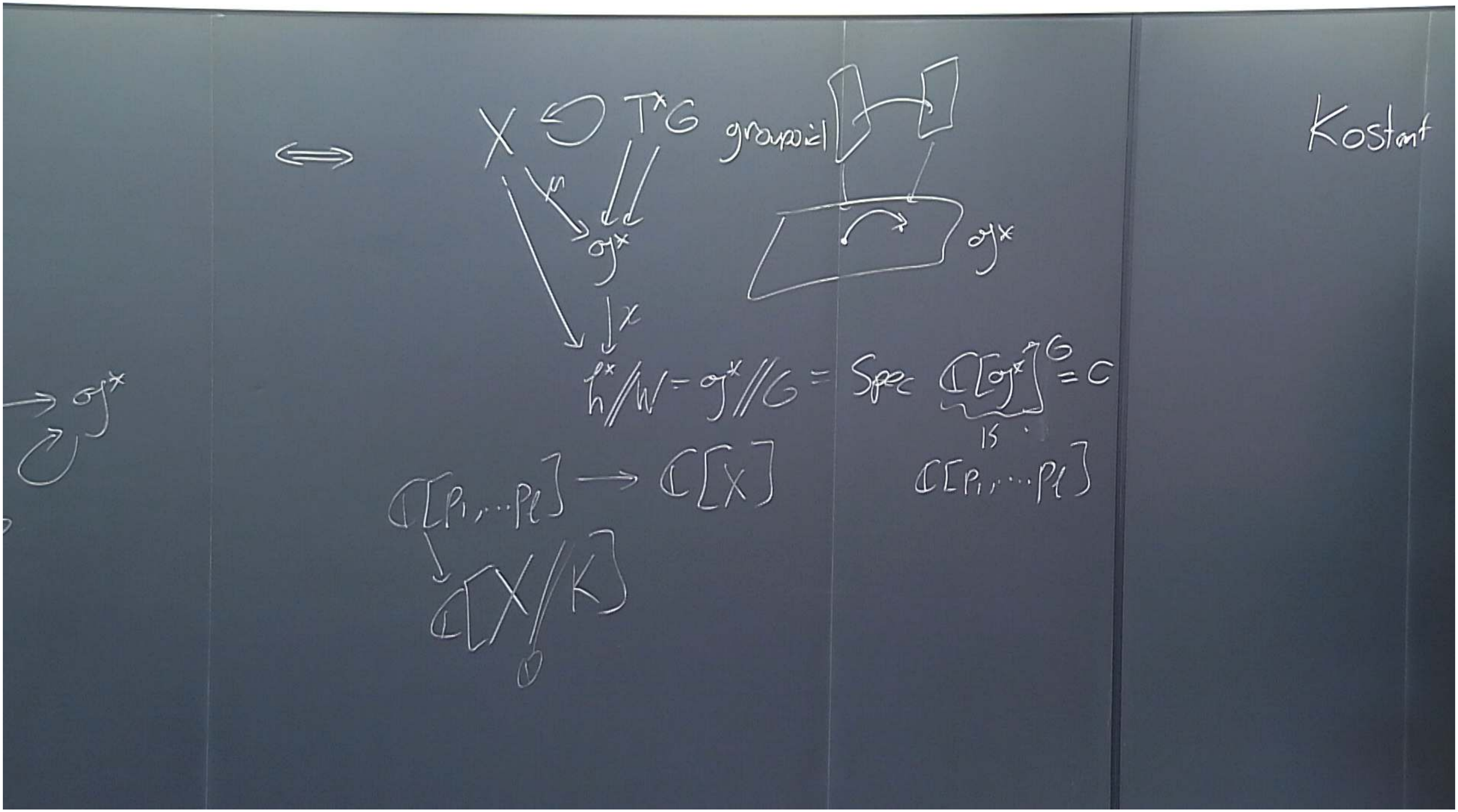

Moduli of Vacua
&
Categorical Representations
w/ Sam Gunningham

$G = G_{\mathbb{C}}$ complex reductive group

Hamiltonian G-space:

X hol. symplectic

$$\begin{array}{c} X \xrightarrow{\mu} \mathfrak{g}^* \\ \circlearrowleft \quad \circlearrowright \\ G \end{array}$$


X ⟲ T*G
groupoid
g*
χ
h*//W = g*//G = Spec C[g*]^G = C
C[p1, ... pl]
C[p1, ... pl] → C[X]
C[X//K]
Kostant

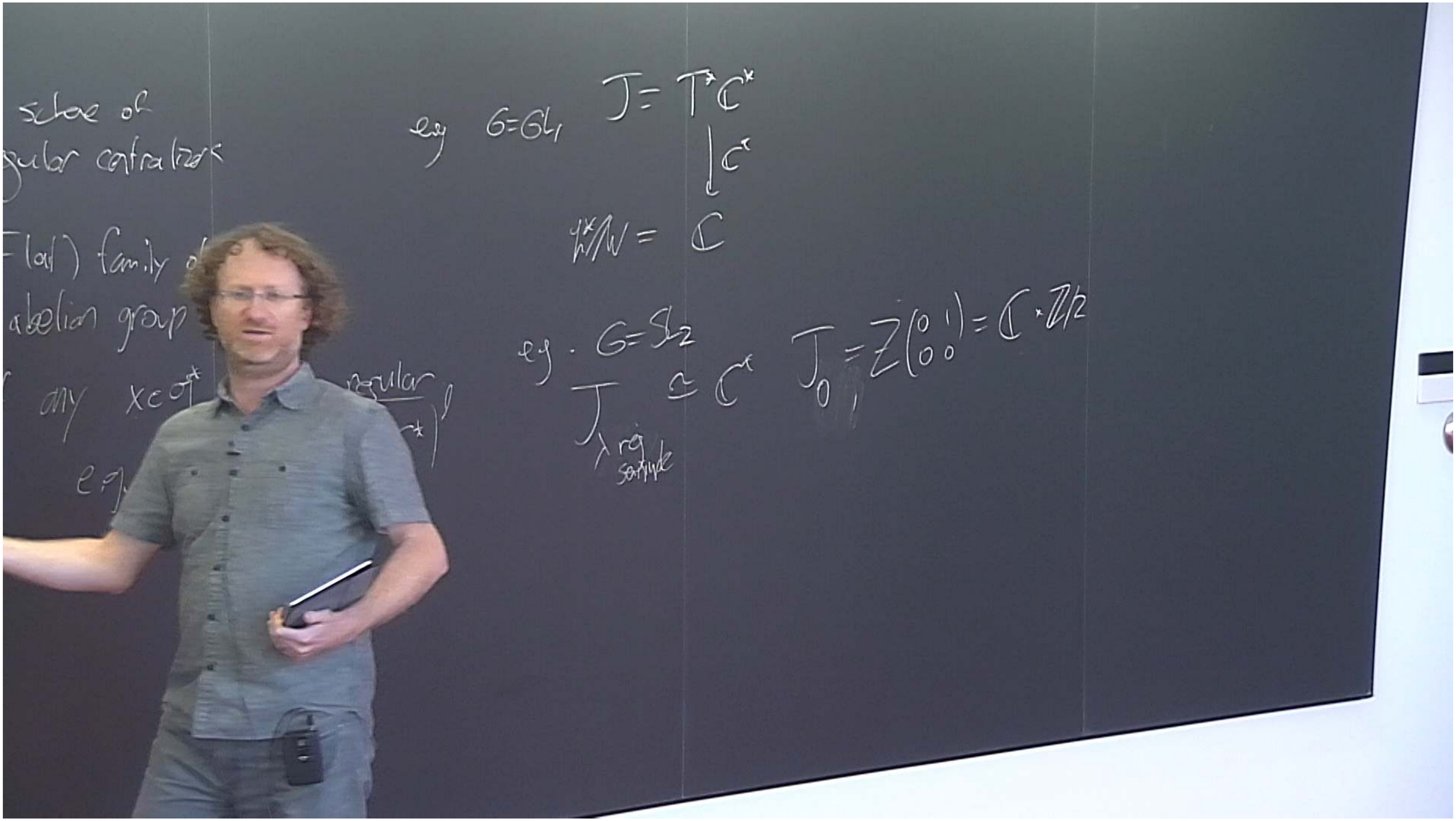
$$
\text{Kostant: }\begin{array}{ccc} J_\lambda & \subset & J \\ \big\downarrow & & \big\downarrow \\ \lambda & \in & \mathfrak{h}^*/W \end{array}
$$

group scheme of regular centralizers

(Flat) family of abelian group

J_λ = centralizer of any $x \in \mathfrak{g}^*$ which is <u>regular</u>,

e.g. $x \in \mathfrak{g}^{rs}$ $J_\lambda \simeq (\mathbb{C}^*)^l$


scheme of
regular centralizers
abelian group
any x ∈ g*
regular
e.g. G = GL1 J = T*C*
C*
h*/W = C
e.g. G = SL2
J λ reg semisimple ≃ C*
J0 = Z(0 1 / 0 0) = C × Z/2

Kostant $J_\lambda \subset J$ group scheme of regular centralizers

$$\begin{array}{ccc} J_\lambda & \subset & J \\ \downarrow & & \downarrow \\ \lambda & \in & \mathfrak{h}^*/W \end{array}$$

(Flat) family of abelian group

J_λ = centralizer of any $x \in \mathfrak{g}^*$ which is regular, e.g. $x \in \mathfrak{g}^{rs}$ $J_\lambda \simeq (\mathbb{C}^*)^l$

J = Coulomb branch of pure 3d $\mathcal{N}=4$ SYM

e.g. $G = GL_1$ $J = T^*\mathbb{C}^*$

$$\begin{array}{c} J = T^*\mathbb{C}^* \\ \downarrow \mathbb{C}^* \\ \mathfrak{h}^*/W = \mathbb{C} \end{array}$$

e.g. $G = SL_2$ $J_{\lambda \text{ reg semisimple}} \simeq \mathbb{C}^*$ $J_0 = Z\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbb{C} \times \mathbb{Z}/2$

Ngô's Lemma: $\exists$ canonical homomorphism from regular centralizers to centralizers

$$J_{\chi(x)} \longrightarrow I_x$$

$$\chi^* J \longrightarrow I$$

$\Longleftrightarrow$ $X \circlearrowleft T^*G$ groupoid $I_x = Z_G(x)$

$\mathfrak{g}^* \ni x$

$\mathfrak{h}^*/\!/W = \mathfrak{g}^*/\!/G = \operatorname{Spec} \mathbb{C}[\mathfrak{g}^*]^G = \mathbb{C}[p_1, \dots p_l]$

$\mathbb{C}[p_1, \dots p_l] \longrightarrow \mathbb{C}[X]$

$\mathbb{C}[X/K]$

$$\begin{array}{ccc} \text{Kostant} \quad J_\lambda & \subset & J \\ \downarrow & & \downarrow \\ \lambda & \in & \mathfrak{h}^*/W \end{array}$$

group scheme of regular centralizers

(Flat) family of abelian group

J_λ = centralizer of any $x \in \mathfrak{g}^*$ which is regular, e.g. $x \in \mathfrak{g}^{rs}$ $J_\lambda \simeq (\mathbb{C}^*)^r$

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Moduli of Vacua
&
Categorical Representations

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Ngô's Lemma: $\exists$ canonical homomorphism from regular centralizers to centralizers

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$$\begin{tikzcd} X \arrow[dr] & J \arrow[l, bend right] \arrow[d] \\ & \mathfrak{h}^*/\!/W \end{tikzcd}$$

$\Longleftrightarrow$

$\mathbb{C}[p_1, \ldots$

$\mathbb{C}[X$

a

resentations

ham

Quantum Hamiltonian G-space

$G \circlearrowright \underset{\text{assoc alg}}{A}$ (e.g. $A = \mathcal{D}_{M \circlearrowleft G}$ $X = T^*M$)

$\underset{\circlearrowright}{U\mathfrak{g}} \longrightarrow \underset{\circlearrowright}{A} \quad \Longleftrightarrow \quad \mathcal{C} = A\text{-mod} \circlearrowleft \mathcal{D}_G\text{-mod}$

G

$\Longleftrightarrow$

$\mathbb{C}[P$

$\mathbb{C}[$

$Z_G(\lambda)$

$\mathfrak{g}^*$

$[\mathfrak{g}^*]^G = \mathbb{C}$

$\cong$

$[p_1, \ldots p_\ell]$

$$Z(U\mathfrak{g}) = (U\mathfrak{g})^G \longrightarrow A$$

$$\cong$$

$$\mathbb{C}[\mathfrak{h}^*/\!/W] = \mathbb{C}[\Delta_1, \ldots \Delta_\ell]$$

$$\Delta_1, \ldots \Delta_\ell \in \mathcal{D}\, G/K$$

commuting, G-invariant

Harish-Chandra system

$$M_\lambda = \{\Delta_i f = \lambda_i f\}$$

$$\lambda \in \mathfrak{h}^*/\!/W$$

a

representations

ham

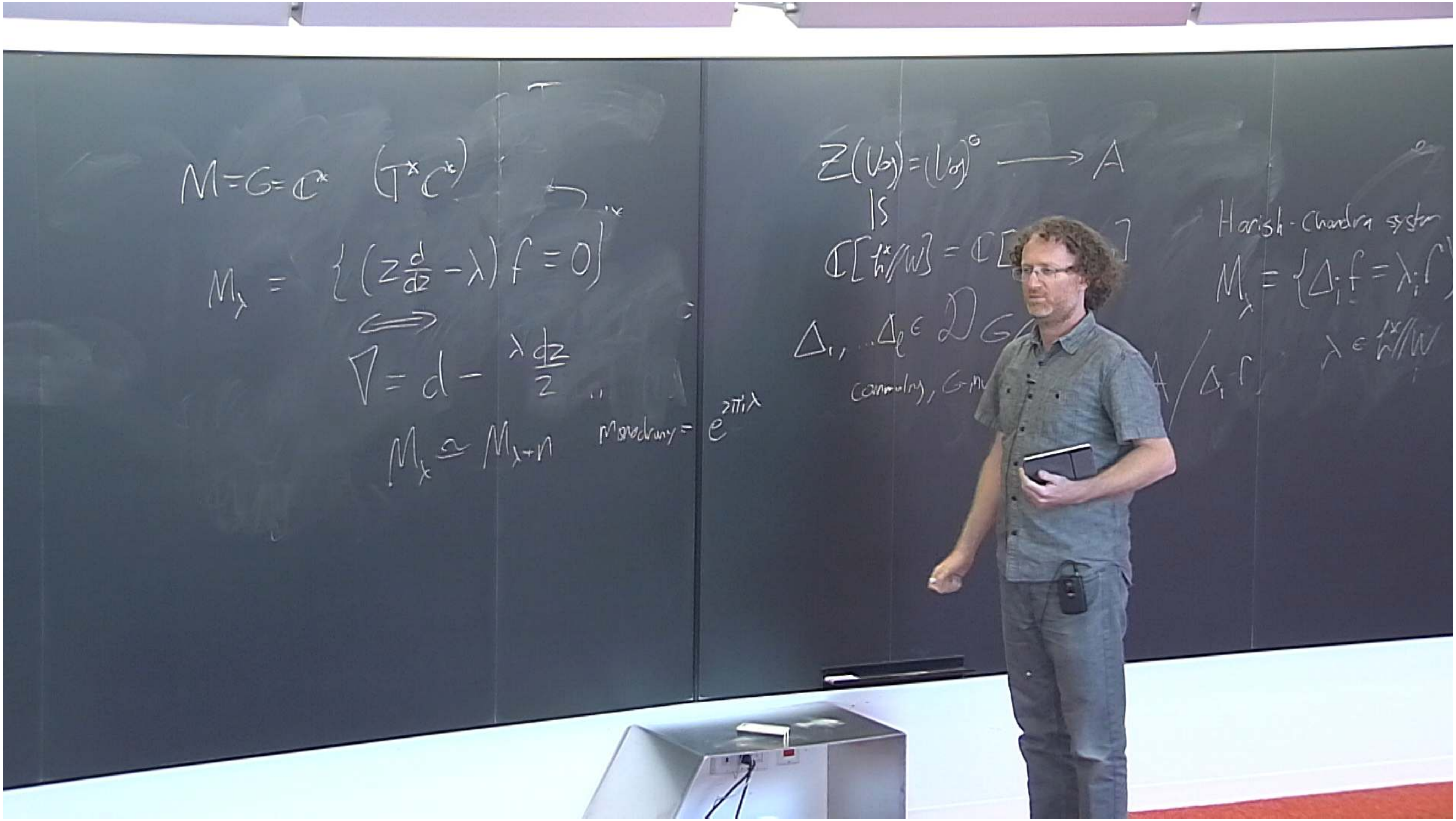
$Z \longrightarrow A$ not central

"Spec A" doesn't live over Spec $Z = \mathfrak{h}^* /\!/ W$

e.g. "quantum fibers" $M_\lambda \simeq M_{\lambda'}$

$\lambda \neq \lambda'$

$\Longleftrightarrow$


Harish-Chandra system
Monodromy =
commuting, G-inv

$$M = G = \mathbb{C}^* \quad (T^*\mathbb{C}^*)$$

$$M_\lambda = \left\{ \left(z\tfrac{d}{dz} - \lambda\right) f = 0 \right\}$$

$$\Longleftrightarrow$$

$$\nabla = d - \lambda \frac{dz}{z}$$

$$M_\lambda \simeq M_{\lambda+n} \qquad \text{Monodromy} = e^{2\pi i\lambda}$$

Moduli is $\mathbb{C}/\mathbb{Z} \cong \mathbb{C}^*$

$$Z(U\mathfrak{g}) = (U\mathfrak{g})^G \longrightarrow A$$

$$\wr\!\wr$$

$$\mathbb{C}[\mathfrak{h}^*/W] = \mathbb{C}[\Delta_1, \dots \Delta_\ell]$$

$$\Delta_1, \dots \Delta_\ell \in \mathcal{D}_{G/K}$$

commuting, G-invariant

Harish-Chandra system

$$M_\lambda = \{\Delta_i f = \lambda_i f\}$$

$$\lambda \in \mathfrak{h}^*/W$$

$$A/\Delta_i f$$

ie $\mathcal{D}$

Theorem (BZ-Gunningham)

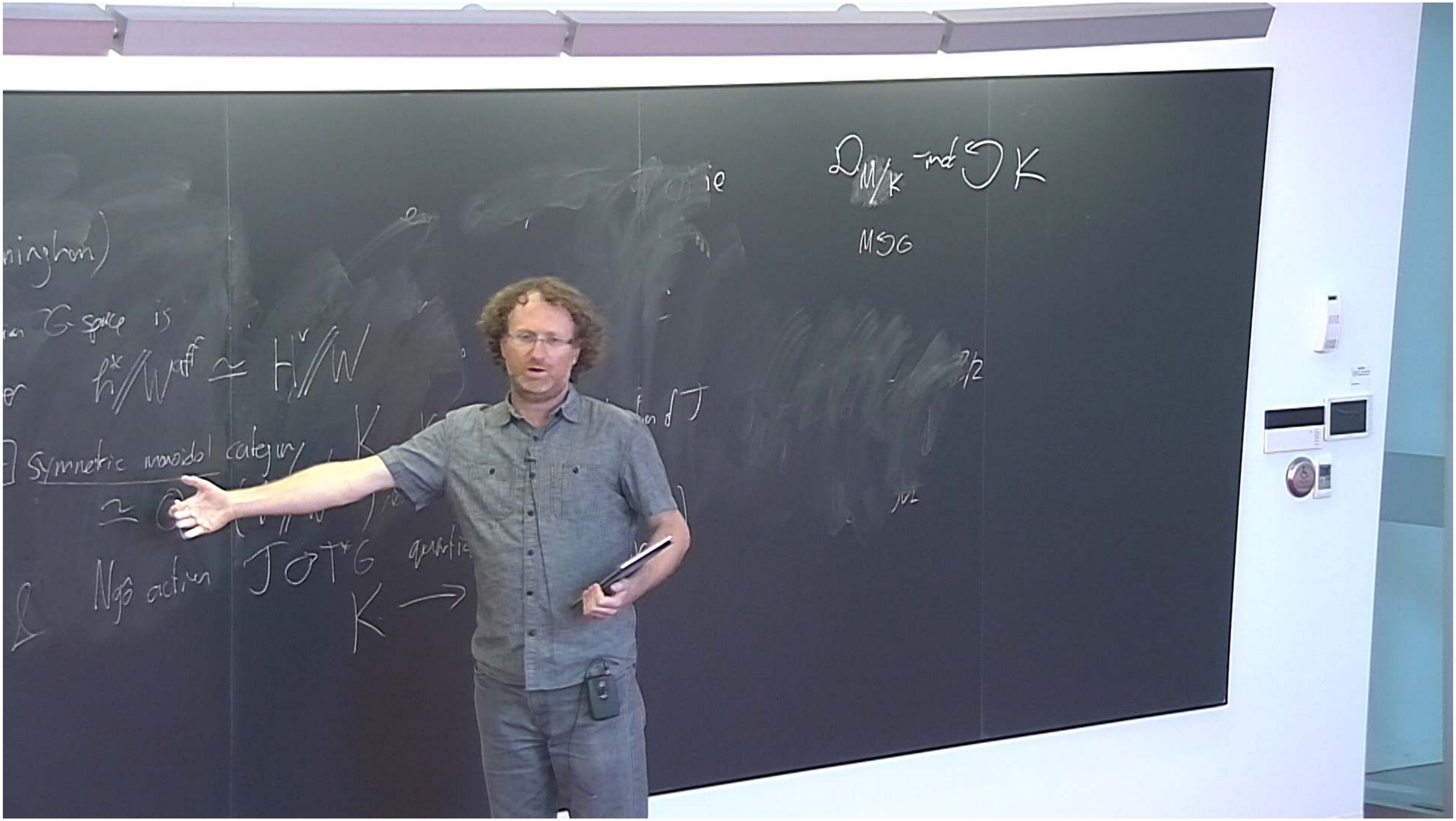
Any quantum Hamiltonian G-space is linear over $\mathfrak{h}^*/\!/W^{aff} \simeq H^\vee/\!/W$

More precisely: $\exists$ symmetric monoidal category K (Kostant category, quantization of J)

$\simeq QC(\mathfrak{h}^*/\!/W^{aff})_\otimes$

& Ngô action $J \circlearrowright T^*G$ quantizes to

$$K \longrightarrow Z(\mathcal{D}_G\text{-mod})$$


ie
MSG
Symmetric monoidal category
Ngô action

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$K \longrightarrow Z(\mathcal{D}_G\text{-mod})$

ie $\mathcal{D}_{M/K}\text{-mod} \circlearrowleft K$

MSG

$H^\vee/\!\!/W \sim G^\vee/\!\!/G^\vee$

$$\mathcal{K} = H_*^{G\times S^1}(Gr_{G^\vee})\text{-mod}$$

= line operators in 3d $\mathcal{N}=4$ in Ω-background

= quantized sheaves on J

= Whittaker Hecke category

$Z \longrightarrow A$ not central

"Spec A" doesn't live over Spec $Z = \mathfrak{h}^*/W$

e.g. "quantum fibers" $M_\lambda \simeq M_{\lambda'}$ $\quad \lambda \neq \lambda'$

$\mathcal{D}(N_\psi \backslash G / N_\psi)$

$Z =$ Whittaker reduction of $U\mathfrak{g}$

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$\psi \quad \psi$

= modules for affine nil-Hecke algebra

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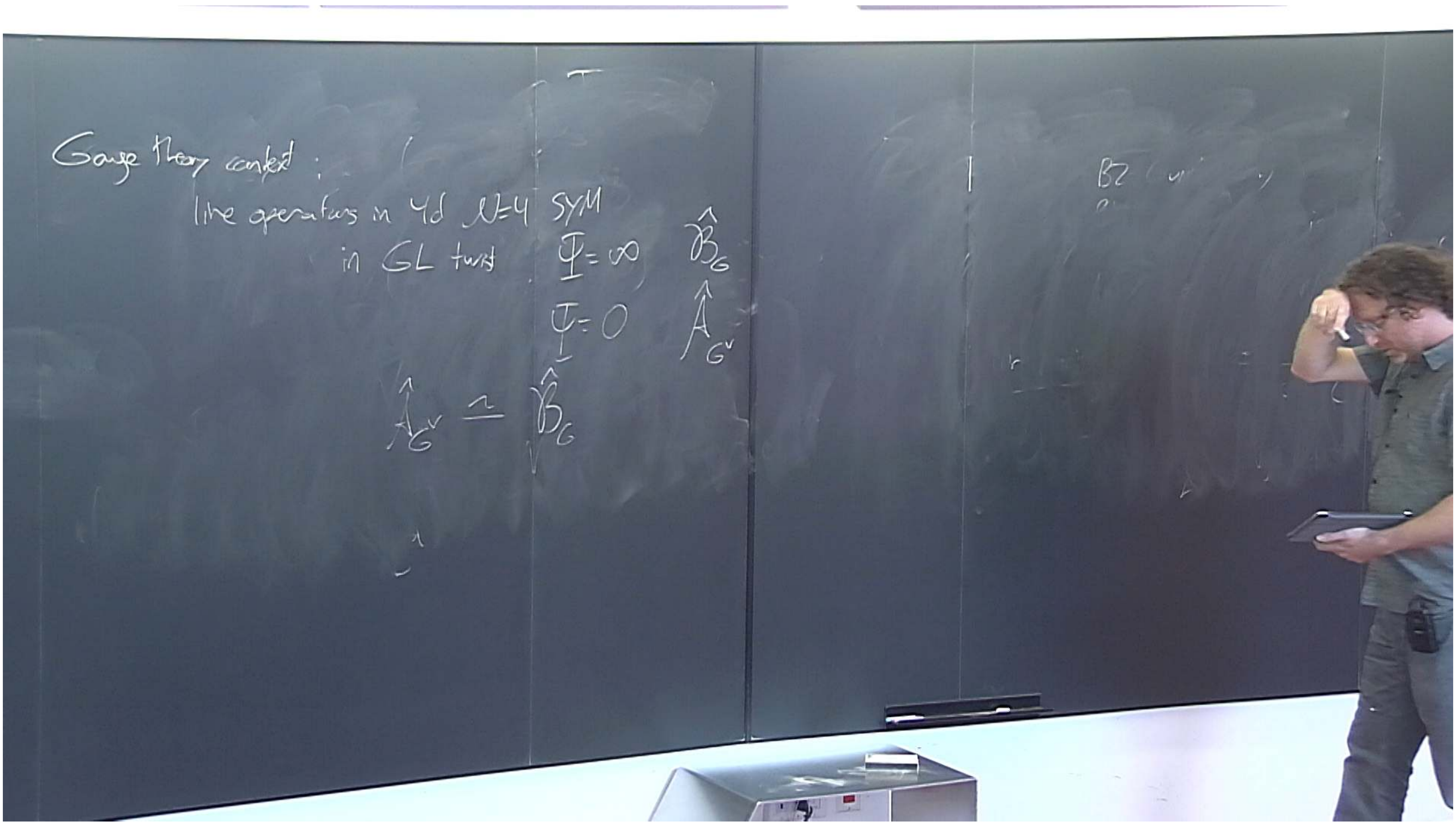
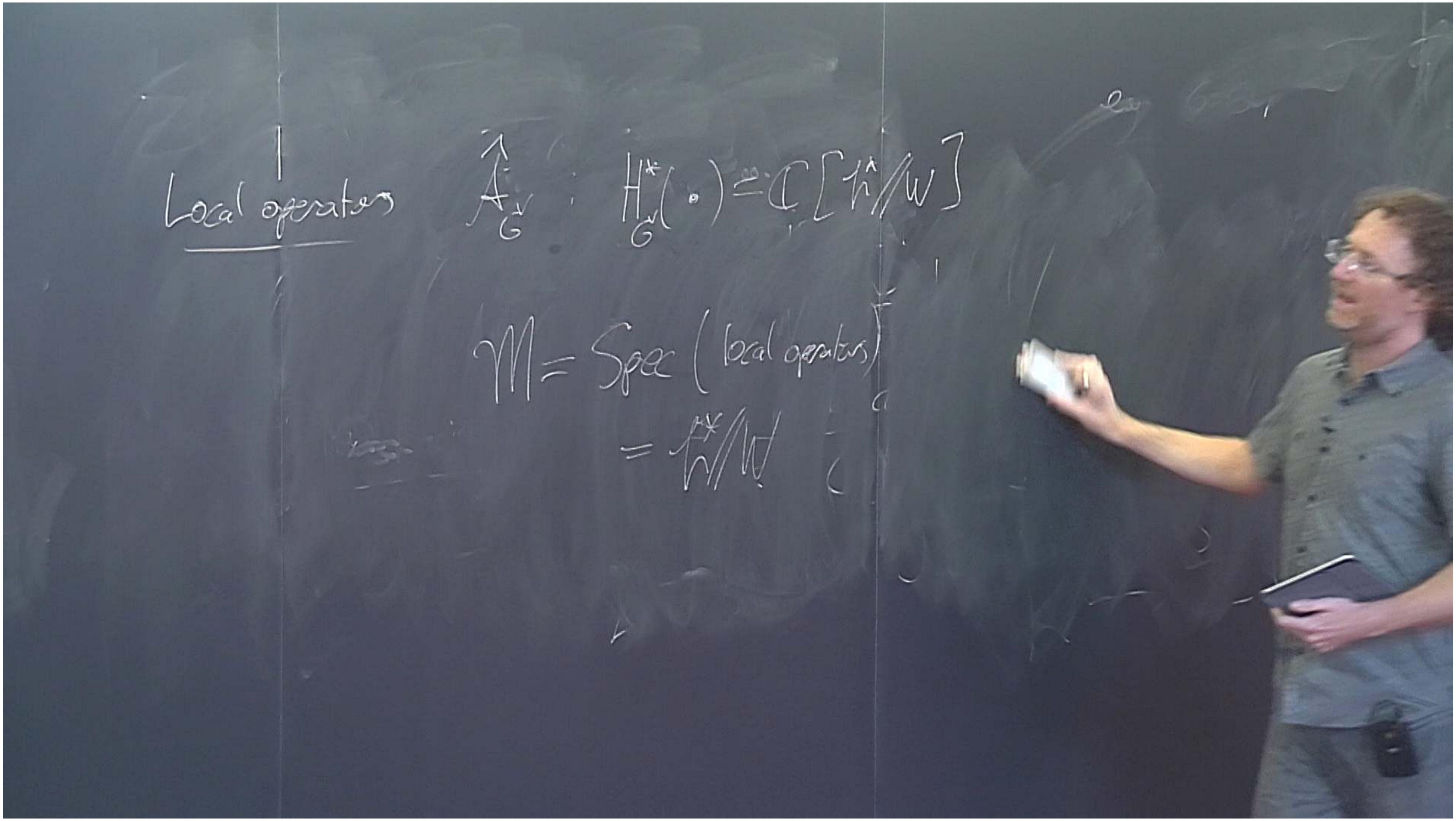

Gauge theory context:
line operators in 4d $\mathcal{N}=4$ SYM
in GL twist
$\Psi=\infty$ $\hat{\mathcal{B}}_G$
$\Psi=0$ $\hat{A}_{G^\vee}$
$\hat{A}_{G^\vee} \simeq \hat{\mathcal{B}}_G$


Local operators
$\hat{A}_{G^\vee}$: $H^*_{G^\vee}(\cdot) \cong \mathbb{C}[\hat{\mathfrak{h}}/W]$
$\mathcal{M}$ = Spec (local operators)
$= \mathfrak{h}^*/W$

$\Omega_{M/K}$ -mod $\circlearrowright K$

MSG

$\cdot) \simeq \mathbb{C}[\hat{\mathfrak{h}}/\!/W]$

(local operators)

$\mathfrak{h}^*/\!/W$

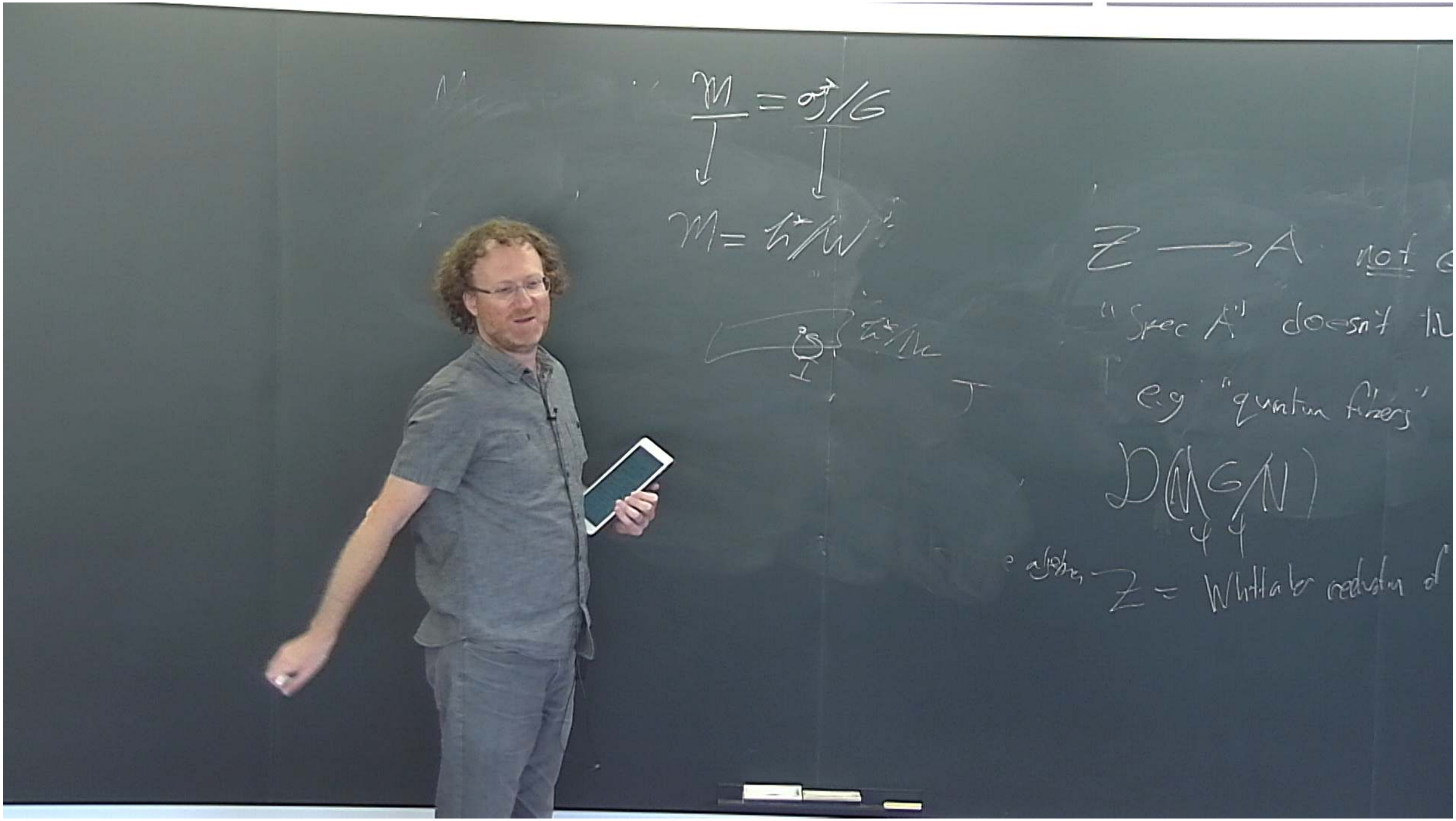
Line operators

$\hat{\mathfrak{g}}/G$

$\hat{B}(S^2) = QC(\mathrm{Loc}_G S^2)$, $*$

$\simeq QC(\mathfrak{g}^*[2]/G)$, $\otimes$ $H^\vee/\!/W \sim G^\vee/\!/G^\vee$,

$\longleftrightarrow$ sheaves on moduli space of Vacua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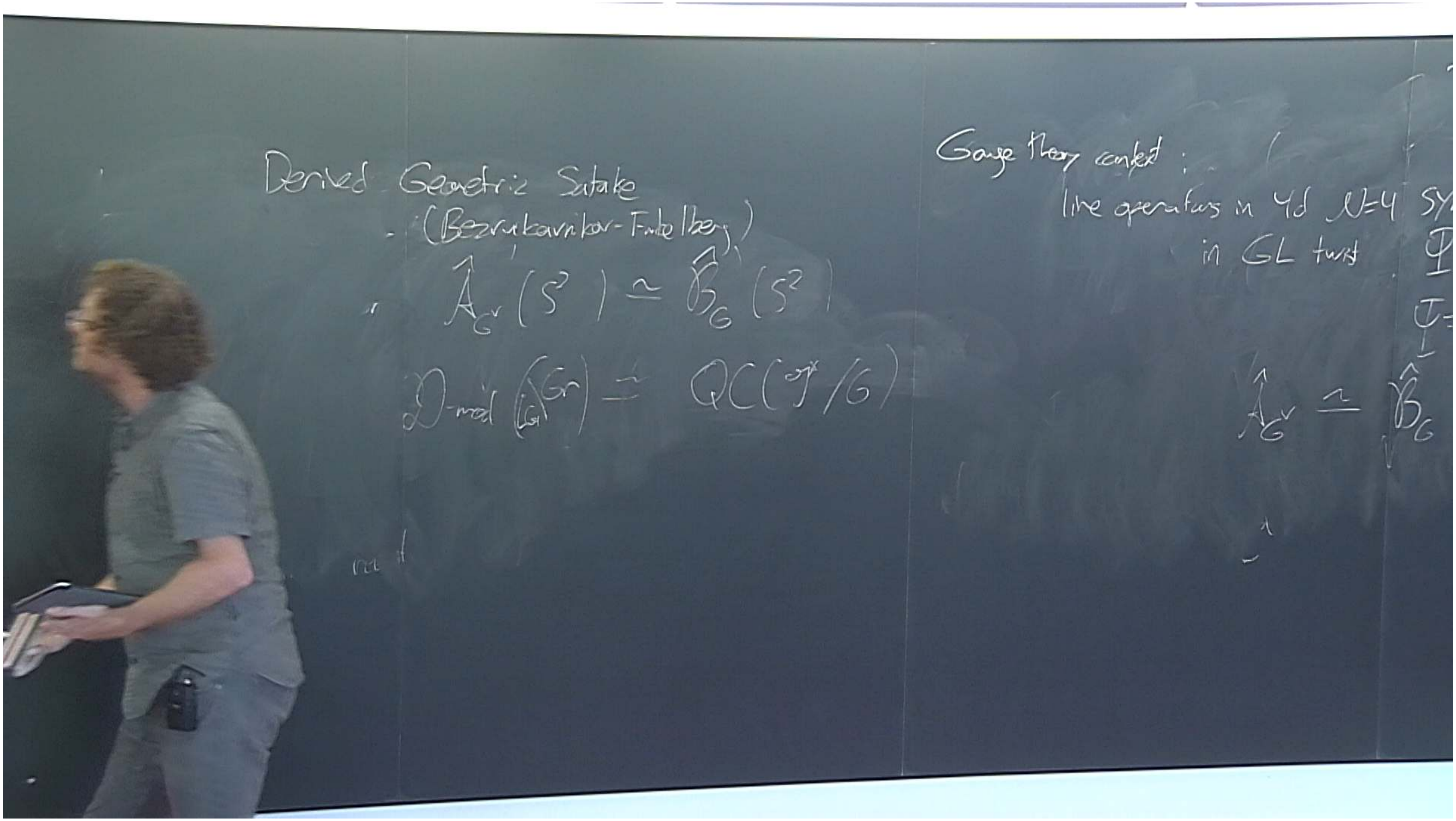

Derived Geometric Satake
(Bezrukavnikov-Finkelberg)
$\hat{A}_{G^\vee}(S^2) \simeq \hat{B}_G(S^2)$
$\mathcal{D}\text{-mod}(Gr) \simeq QC(\mathfrak{g}^*/G)$
Gauge theory context:
line operators in 4d $\mathcal{N}=4$ SY
in GL twist
$\hat{A}_{G^\vee} \simeq \hat{B}_G$

$$\begin{tikzcd} \mathcal{M} = \mathfrak{g}^*/G \arrow[d] \arrow[r, leftrightarrow] & T^*G \arrow[d] \\ M = \mathfrak{h}^*/W & \mathfrak{g}^* \end{tikzcd}$$

$QC(\mathfrak{g}^*/G)$ Gaitsgory

$\simeq$ Morita

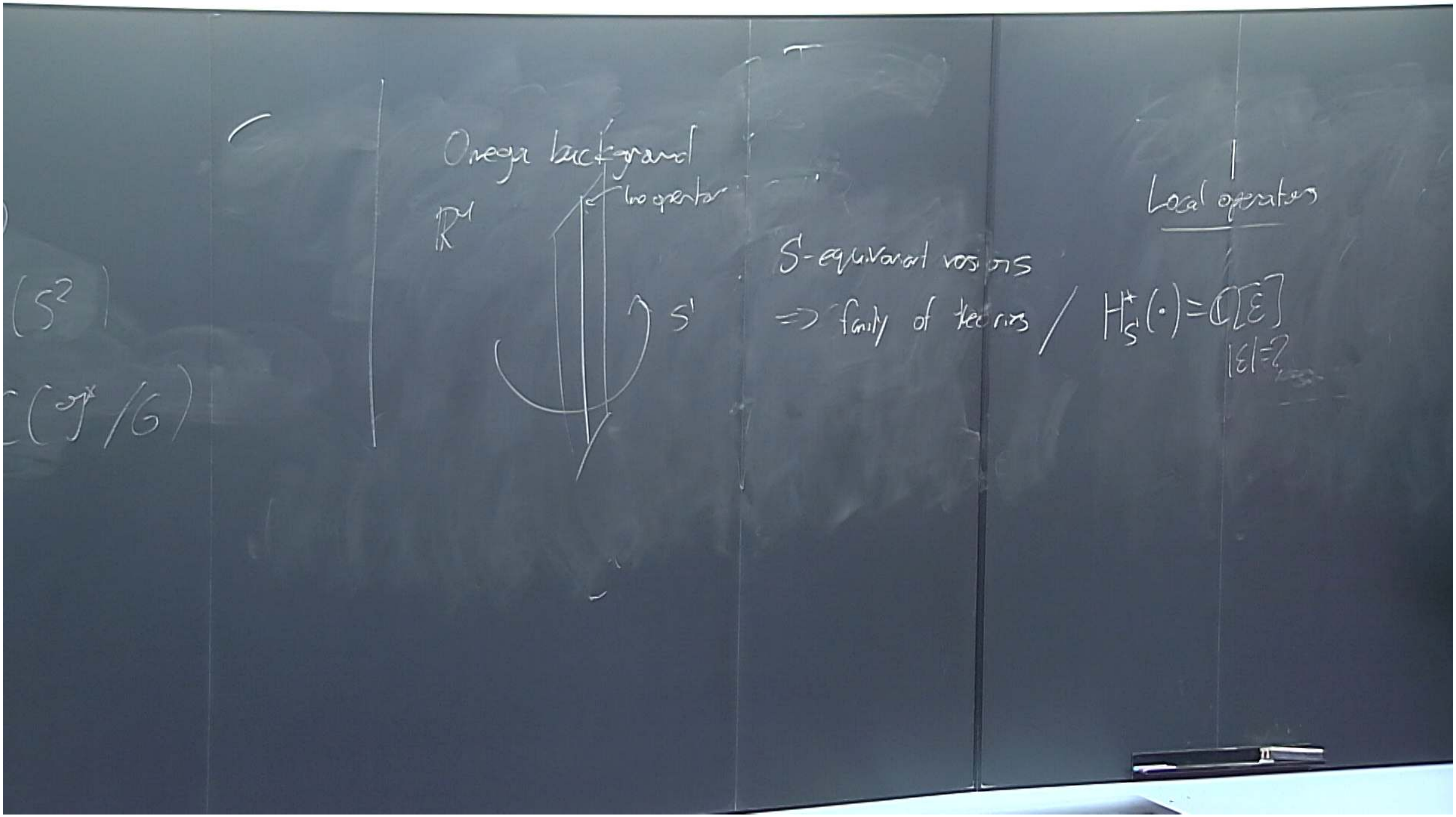
$QC(T^*G)$

Derived Geometric Satake

(Bezrukavnikov-Finkelberg)

$\hat{A}_{G^\vee}(S^2) \simeq \hat{B}_G(S$

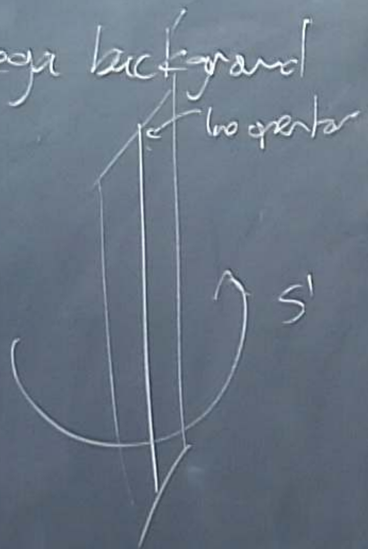
$\mathcal{D}\text{-mod}(\mathcal{L}G \backslash Gr) \simeq QC($



Omega background
line operator
$\mathbb{R}^4$
S^1
S-equivariant versions
$\Rightarrow$ family of theories / $H^*_{S^1}(\cdot) = \mathbb{C}[\varepsilon]$
$|\varepsilon| = 2$
Local operators
(S^2)
$C(\mathfrak{g}^*/G)$

Omega background

$\mathbb{R}^4$



S^1-equivariant versions

$\Rightarrow$ family of theories / $H^*_{S^1}(\cdot) = \mathbb{C}[\varepsilon]$

$|\varepsilon| = 2$

Local operators

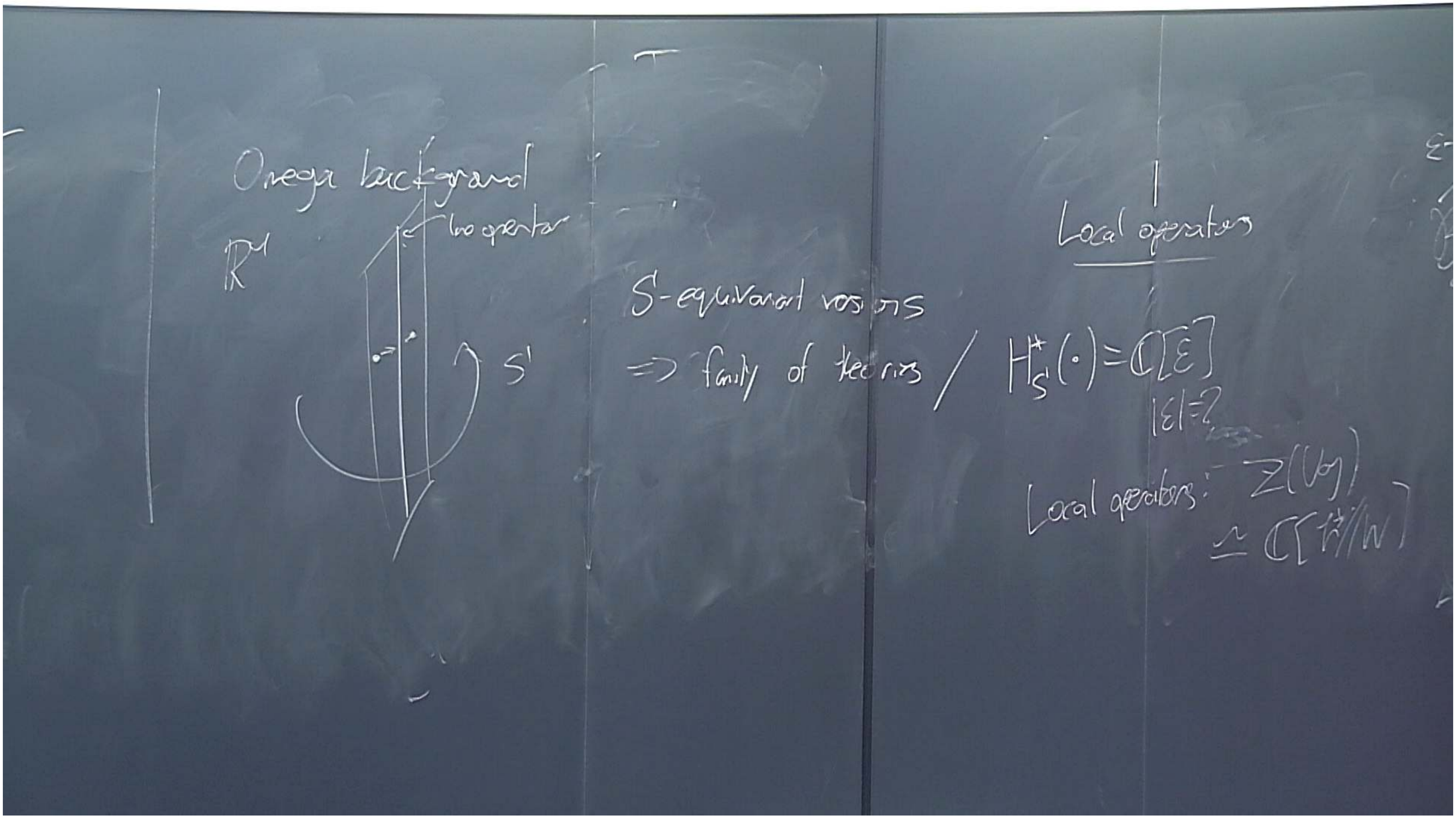
ε-line operators

$\hat{\mathcal{B}}_\varepsilon(S^2) = \text{HC bimodules}$

$= (U\mathfrak{g}\text{-mod})^G$

$(\mathbb{C}[\mathfrak{g}^*]\text{-mod})^G$

$= G\backslash \mathcal{D}_G\text{-mod}/G$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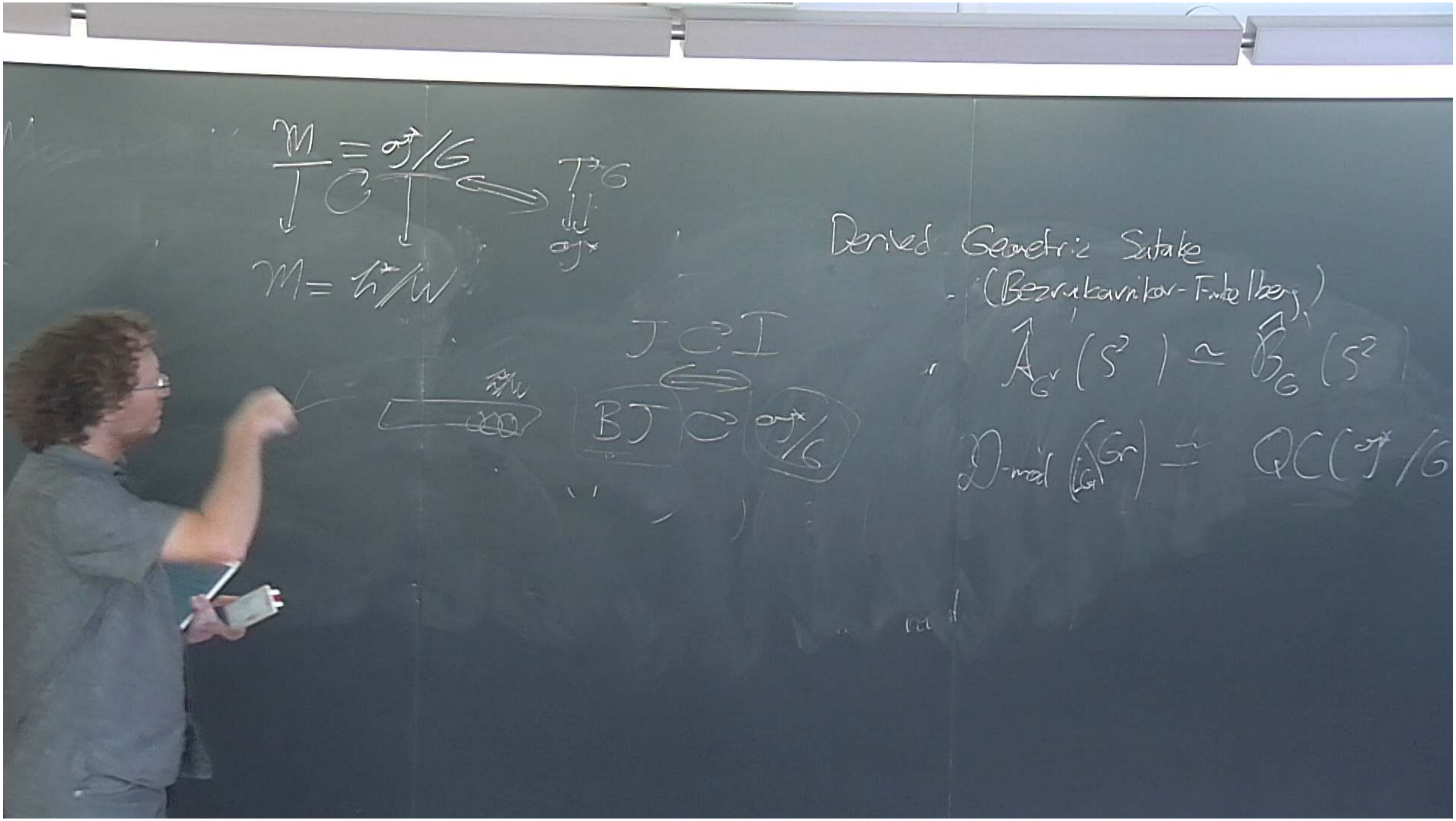
Derived Geometric Satake
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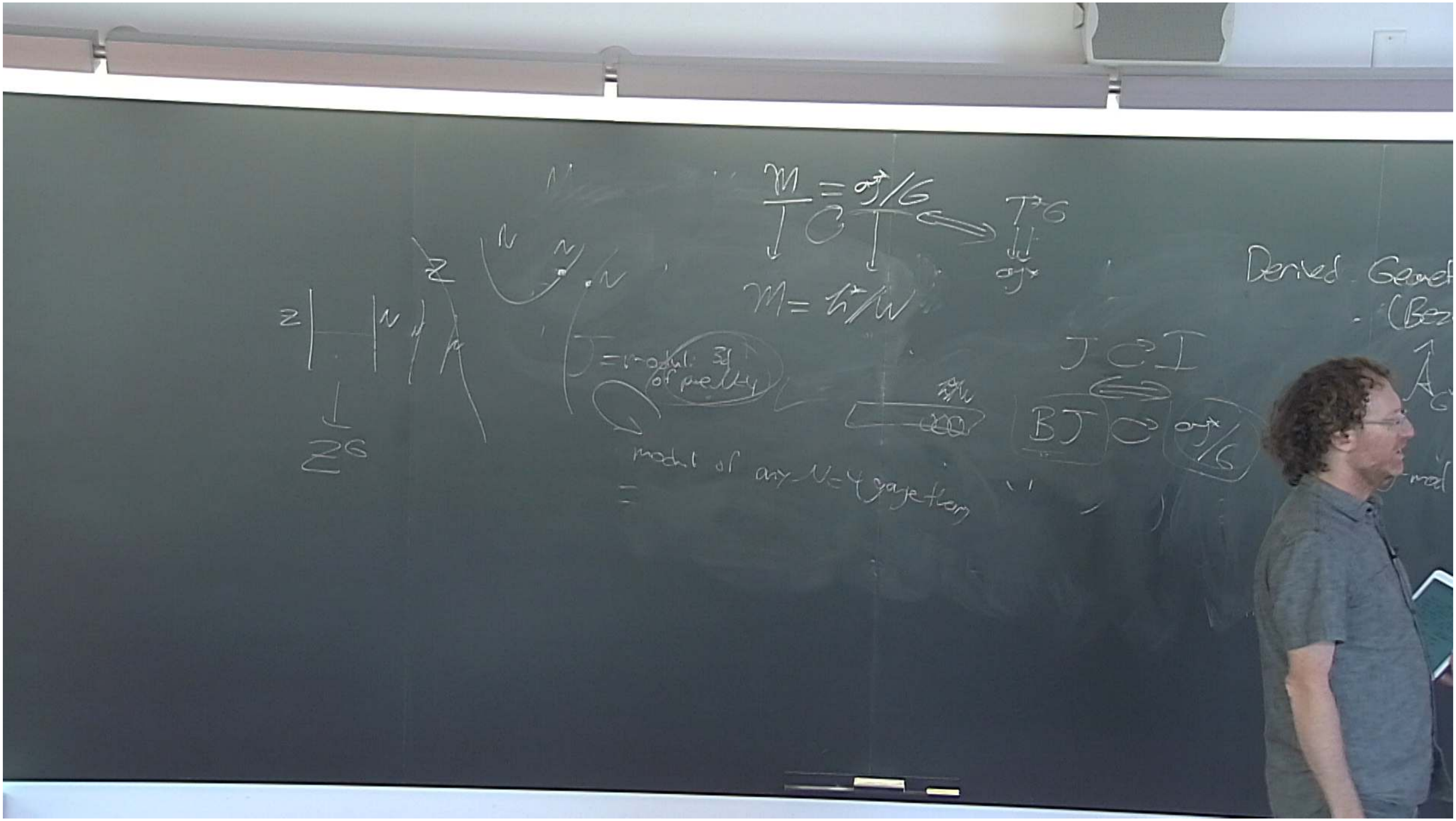


Derived Geomet
modul of any N=4 gauge theory

Derived: G

$QC(BT) = \mathrm{Rep}\, T$

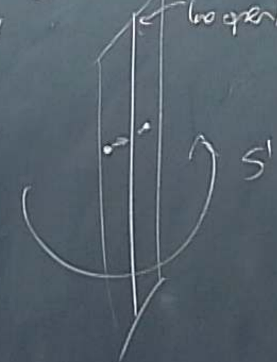
$= H_*^G(Gr)$-comod

$\sim Loc^G(Gr)$

$= Loc([S^2, BG])$

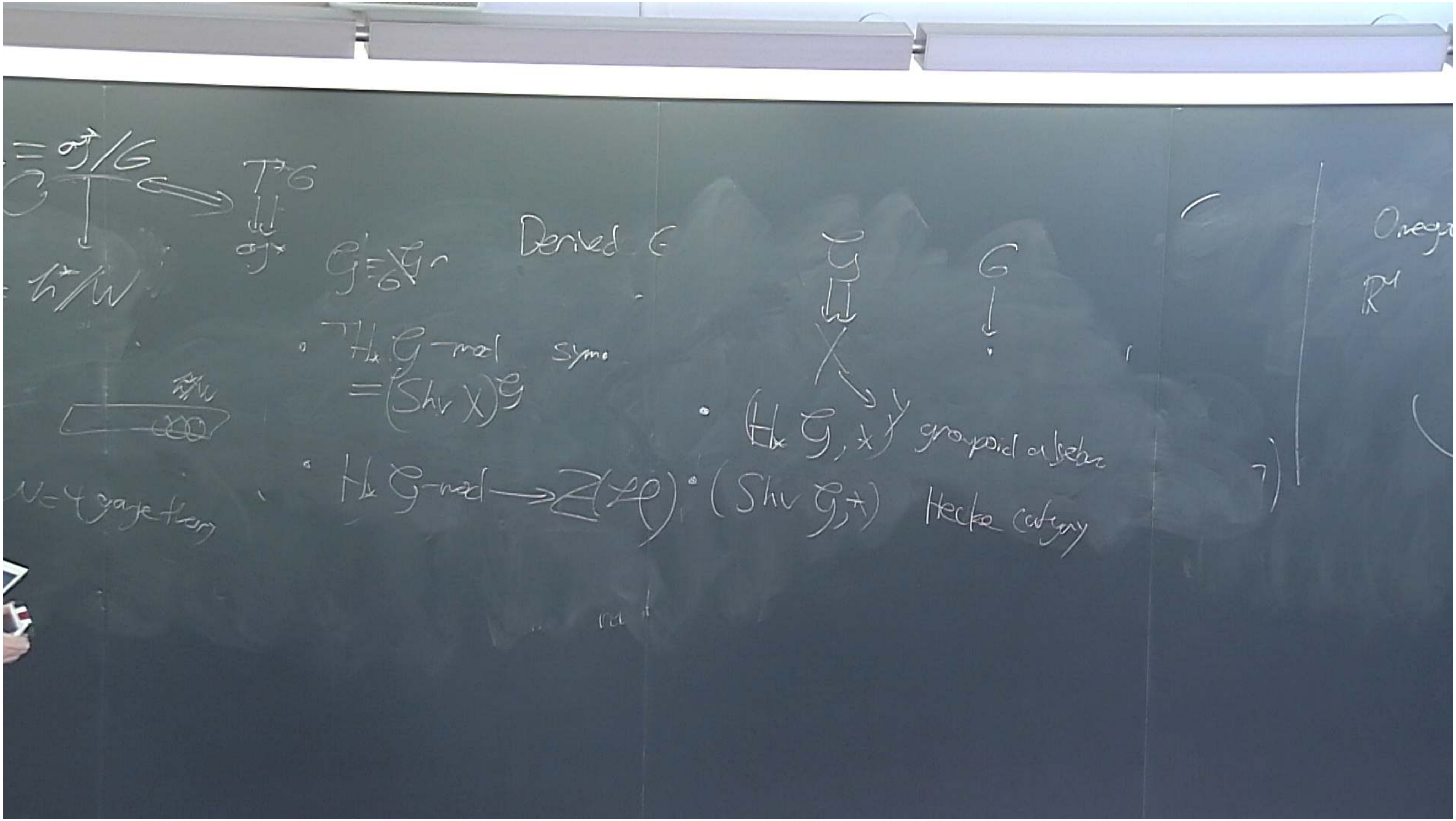
Omega background

$\mathbb{R}^4$



S-equivariant vos

=> family of the


Derived G
sym.
groupoid algebra
Hecke category
N=4 gauge theory