Title: Dynamical systems approaches and methods in cosmology

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Abstract: I will with simple examples from spatially homogeneous and isotropic cosmology illustrate the importance of respecting the global features of a state space for a given model when reformulating field equations to useful dynamical systems. In particular I will use examples from f(R) gravity and GR with a minimally coupled scalar field. In this context I will also illustrate how various dynamical systems methods, such as, e.g., monotonic functions, center manifold techniques, averaging methods, can yield a global understanding of the solution spaces as well as approximations, complementing, e.g., the slow-roll approximation.

On dynamical systems approaches and methods in cosmology

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State space analysis: An example in $f(R)$ cosmology

A.Alho, S.Carloni, C.U: JCAP 1608 (2016) no.08, 064; e-Print: arXiv:1607.05715

- Action for vacuum $f(R)$ gravity:

$$\mathcal{S} = \int \frac{f(R)}{2\kappa^2}\sqrt{-\det g}\, d^4x$$

  where $\kappa^2 = 8\pi G$; $c = 1$; $\det g$ is the determinant of a Lorentzian 4-dimensional metric g, and R the associated curvature scalar

- General relativity with a cosmological constant Λ:

$$f(R) = R - 2\Lambda$$

- Typical motivation: Quantum corrections, geometric models for inflation and dark energy.
- Present motivation: Lessons for dynamical systems approaches.

- Illustrative example:
  - $ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$,
  - $f(R) = R + \alpha R^2\,, \qquad \alpha > 0$,
  - vacuum.

- Evolution equations

$$\dot{a} = Ha,$$
$$\dot{H} = -2H^2 + \frac{R}{6},$$
$$\ddot{R} = -3H\dot{R} - \frac{1}{F_{,R}}\left[F_{,RR}\dot{R}^2 + \frac{1}{3}(2f - FR)\right].$$

- Constraint

$$-6H\left(F_{,R}\dot{R} + FH\right) + FR - f = 0.$$

where

$$F = \frac{df}{dR}, \qquad F_{,R} = \frac{dF}{dR} = \frac{d^2 f}{dR^2}, \qquad F_{,RR} = \frac{d^2 F}{dR^2} = \frac{d^3 f}{dR^3}.$$

- Equation for a decouples $\Rightarrow$ Reduced closed system of first order equations for $(H, \dot{R}, R)$ + constraint $\Rightarrow$ 2-dimensional reduced (gauge projected) state space.
- Invariance under the transformation $(t, H) \to -(t, H)$.
- Fixed points: $-2H^2 + \frac{R}{6} = 0, \quad \dot{R} = 0, \quad 2f - FR = 0.$

Some general considerations

- Problems, e.g., when $F_{,R} = 0 \rightarrow$ classification:
  - (i) Models with $F_{,R} > 0$.
  - (ii) Models where $F_{,R} = 0$ for some value(s) of R.
- Einstein frame formulation:

$$\tilde{g}_{\mu\nu} = F g_{\mu\nu}, \qquad F = \frac{df}{dR} > 0,$$

$\rightarrow$ Einstein-Hilbert action + minimally coupled scalar field

$$\kappa\phi = \sqrt{\frac{3}{2}} \ln F, \qquad V(\phi) = \frac{RF - f}{2\kappa^2 F^2}$$

- $F = 0$ is not, in general, an invariant subset in the Jordan frame.

$$\dot{F}|_{F=0} = F_{,R}\dot{R}|_{F=0} = -\left.\frac{f}{6H}\right|_{F=0},$$

- Classification:
  - (i) Models with $F > 0$ everywhere.
  - (ii) Models for which F can change sign $\rightarrow$ solutions in the Einstein frame that can be conformally extended in the Jordan frame.

State space analysis of an illustrative example: $f(R) = R + \alpha R^2$, $\alpha > 0$

- Constraint equation

$$-12H\left(\dot{R} + HR + \frac{H}{2\alpha}\right) + R^2 = 0.$$

- Bring the constraint to a quadratic canonical form.
- Respect dimensional considerations:
  $([t], [H], [R], [\dot{R}], [\alpha]) = (L, L^{-1}, L^{-2}, L^{-3}, L^2)$:

$$H = \sqrt{\frac{\alpha}{12}}(\mathrm{t} - \mathrm{x})$$
$$\dot{R} + HR + \frac{H}{2\alpha} = \frac{1}{\sqrt{12\alpha}}(\mathrm{t} + \mathrm{x}),$$

  where t, x and R all have dimension L^{-2} $\rightarrow$ constraint:

$$-\mathrm{t}^2 + \mathrm{x}^2 + R^2 = 0,$$

- Jacobian determinant of $(H, \dot{R}, R) \rightarrow (\mathrm{t}, \mathrm{x}, R)$ is $1/6$.

- The reduced vacuum state space is a 2-dimensional double cone with a joint apex, the non-hyperbolic Minkowski fixed point.

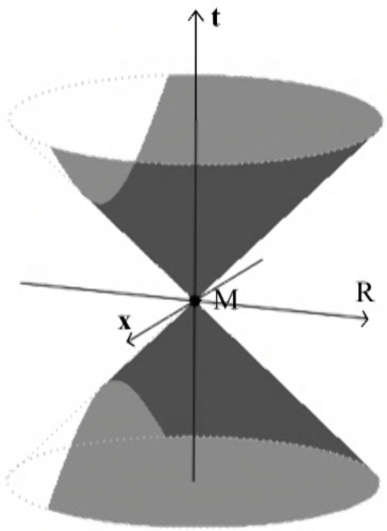


Figure: The state space light cone for $f(R) = R + \alpha R^2$, $\alpha > 0$. The shaded part denotes the state space domain of the Einstein frame, i.e., the state space of the Einstein frame is a (non-invariant) subset of that of the Jordan frame.

- $(t; H) \to -(t; H) \Rightarrow (t; \mathrm{t}, \mathrm{x}) \to -(t; \mathrm{t}, \mathrm{x})$.
- $\mathrm{t} > 0 \Rightarrow H \geq 0$.
- $H = 0 \Rightarrow \mathrm{t} = \mathrm{x}$, $R = 0$, where $\dot{R} = \mathrm{t}/\sqrt{3\alpha} > 0$.

- Consider the t > 0 part of the state space and introduce new dimensionless bounded variables:

$$(X, S) = \left(\frac{x}{t}, -\frac{R}{t}\right), \qquad T = \frac{1}{1+2\alpha t} \rightarrow$$

$$X^2 + S^2 = 1.$$

Solve the constraint:

$$X = \cos\theta, \qquad S = \sin\theta.$$

- Introduce a new dimensionless time variable $\bar{t}$:

$$\frac{dt}{d\bar{t}} = 2\sqrt{12\alpha}\, T.$$

- Dynamical system

$$T' = T(1-T)\left[T\sin\theta + (1-T)(1-\cos\theta)^2\right],$$
$$\theta' = -T(3+\cos\theta) - (1-T)(1-\cos\theta)\sin\theta.$$

- State space **S**, finite cylinder defined by

$$0 < T < 1.$$

- Due to regularity — extend the state space **S** to include the invariant boundaries $T = 0$ ($t \rightarrow \infty \Rightarrow H \rightarrow \infty$), and $T = 1$ ($t \rightarrow 0 \Rightarrow H \rightarrow 0$).

- Compactified state space $\bar{\mathbf{S}}$ of the future state space light cone + blow up the neighborhood of the non-hyperbolic Minkowski fixed point to a periodic orbit.

Dynamical systems analysis

- Monotone function: $J = \frac{(1-T)(3+\cos\theta)}{T} > 0$ monotonically decreasing $\rightarrow$ all solutions originate and end at the boundaries $T = 0$ and $T = 1$.

- On $T = 1$ boundary: Periodic orbit

$$\theta' = -(3 + \cos\theta) < 0.$$

- On the $T = 0$ boundary there are two fixed points:

$$\mathrm{R}: \quad \theta = \pi + 2n\pi, \quad \text{(hyperbolic source)},$$
$$\mathrm{d\bar{S}}: \quad \theta = 2n\pi, \quad \text{(non-hyperbolic with 2 zero eigenvalues)}.$$

- $\mathrm{d\bar{S}}$ is a nilpotent fixed point $\rightarrow$ known blow up methods $\rightarrow$ single "inflationary attractor solution" originating from $\mathrm{d\bar{S}}$ corresponding to an initial quasi-de Sitter state at $H \rightarrow \infty$. The center manifold structure explains the "attractor" nature of the solution.
- All other solutions originate from R at $H \rightarrow \infty$.
- An open set of solutions do not approach the inflationary attractor solution during the quasi-de Sitter inflationary stage.
- All solutions end at the limit cycle at $T = 1$, corresponding to the Minkowski state.

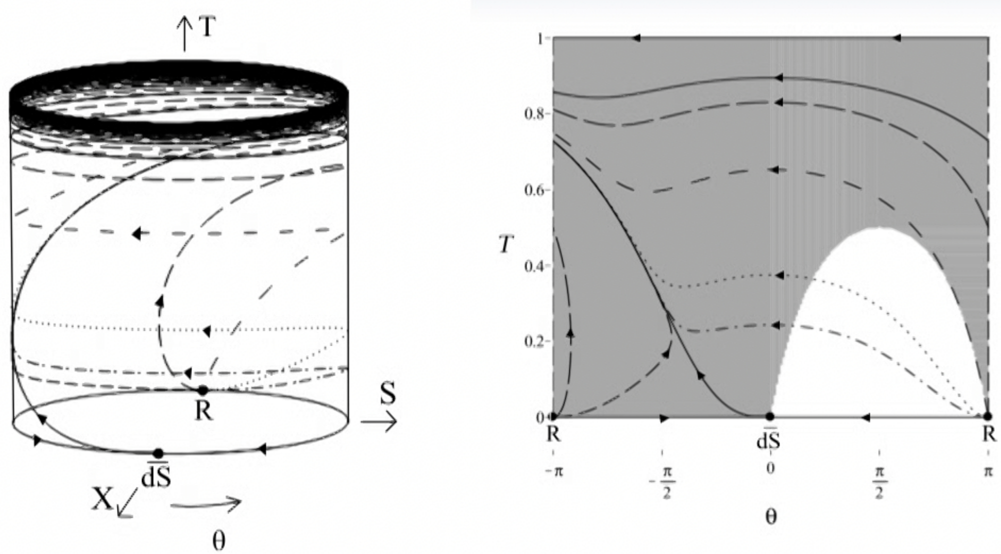


Figure: Representative solutions describing the solution space. Shaded area corresponds to the Einstein frame domain $F > 0$. All solutions in the Jordan frame state space end at the periodic orbit at $T = 1$, and they all originate from the fixed point R, except for 'the inflationary attractor solution' (solid line) that comes from $\bar{\mathrm{dS}}$.

- The Hubble and deceleration parameter H and q, respectively:

$$H = \frac{1}{2\sqrt{12\alpha}}\left(\frac{1-T}{T}\right)(1-\cos\theta), \qquad q = 1 + 4\left(\frac{T}{1-T}\right)\frac{\sin\theta}{(1-\cos\theta)^2} \quad \rightarrow$$

- oscillatory blow up of q.

Dynamics in the Einstein frame

- For $f(R) = R + \alpha R^2, \quad \alpha > 0$:

$$\kappa\phi = \sqrt{\frac{3}{2}}\ln(1+2\alpha R), \qquad V(\phi) = V_0\left(1 - e^{-\sqrt{\frac{2}{3}}\kappa\phi}\right)^2,$$

where $V_0 = \frac{1}{8\alpha\kappa^2} > 0$.

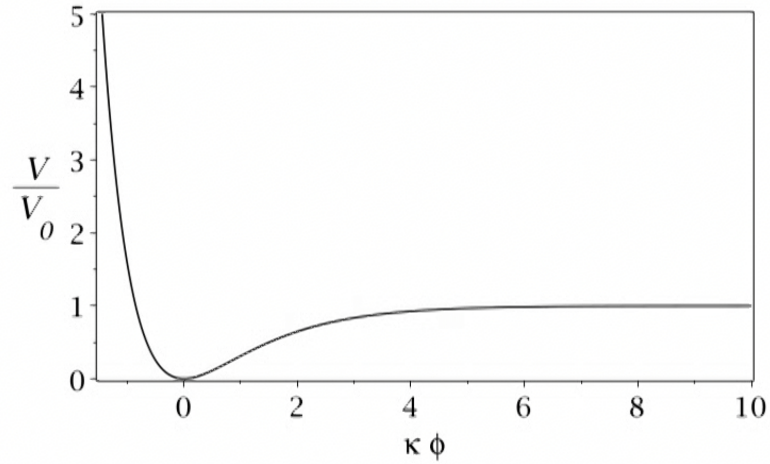


Figure: The potential $V(\phi)$ for the minimally coupled scalar field in the Einstein frame corresponding to the model $f(R) = R + \alpha R^2$, $\alpha > 0$.

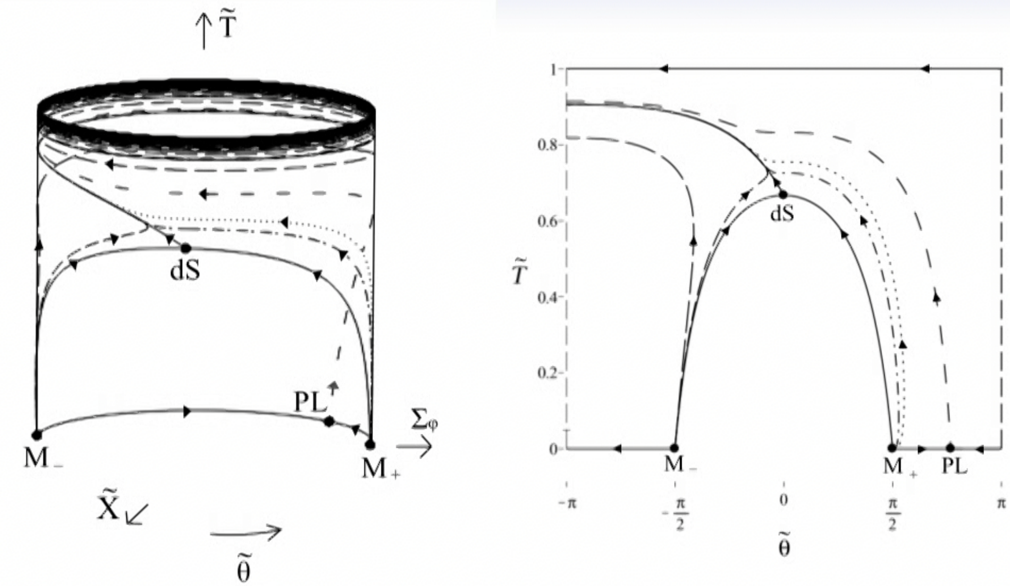


Figure: The extended state space $\bar{\tilde{\mathbf{S}}}$ of the Einstein frame, consisting of a finite cylinder with a removed non-physical $\tilde{T}F < 0$ region, and representative solutions.

- In the Einstein frame the inflationary attractor solution originates with $H_E = \sqrt{V_0/3}$.
- The Einstein frame exemplifies a situation where only part of the Jordan state space domain is covered $\Rightarrow$ state space coordinate singularities in the form of fixed points.

- Another example state space coordinates for $f(R) = R + \alpha R^2$:

$$z = \frac{1}{12\alpha H^2}, \qquad q = 1 - \frac{R}{6H^2},$$

$$\frac{dz}{dN} = 2(1+q)z, \qquad \frac{dq}{dN} = z - \frac{3}{2}(1-q^2), \qquad N = \ln a/a_0.$$

Jacobian determinant $1/(36\alpha H^5)$.

- Useful locally, but state space topology matters!

- Previous formulations in the literature: Worse.
- Need for physical interpretation!
- Sometimes non-local solution space structures matter!

Scalar fields

- Examples: $V = f^{2n}(\phi)$, e.g. α-attractor cosmology. A. Alho., C.U, J.Math.Phys. 56 (2015) no.1, 012502, A. Alho, J. Hell, Class.Quant.Grav. 32 (2015) no.14, 145005, A. Alho., C.U, Phys.Rev. D95 (2017) no.8, 083517,
- What can dynamical systems approaches do for you?

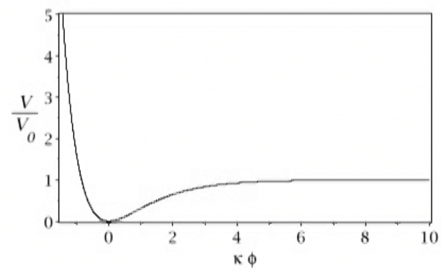


Figure: The potential $V(\phi) = V_0 \left(1 - e^{-\sqrt{\frac{2}{3}}\kappa\phi}\right)^2$ for a minimally coupled scalar field

$$3H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi) = \rho_\phi, \tag{5}$$

$$\dot{H} = -\frac{1}{2}\dot{\phi}^2, \tag{6}$$

$$0 = \ddot{\phi} + 3H\dot{\phi} + V_\phi. \tag{7}$$

- Heuristics: Particle with friction; particle in a potential with moving walls.

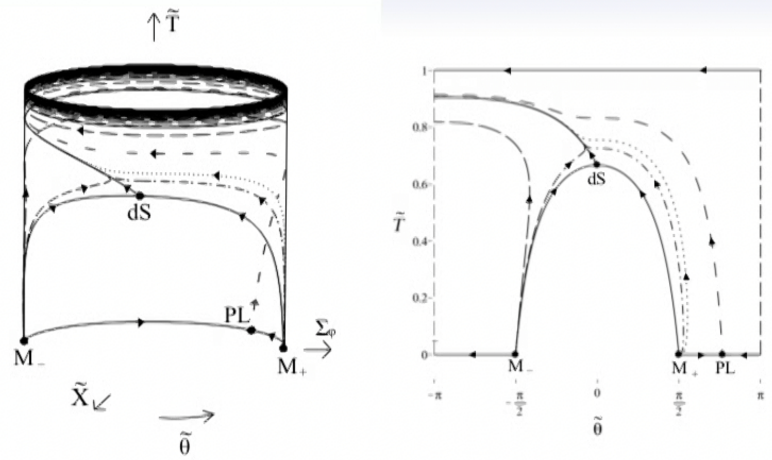


Figure: The extended state space $\tilde{\bar{S}}$ of the Einstein frame, consisting of a finite cylinder with a removed non-physical $\tilde{T}F < 0$ region, and representative solutions.

$\tilde{T} = \text{const} \to \text{energy} = \text{const}$

- Dynamical system yielding a proven global solution space structure.
- Approximation methods, e.g. center manifold theory as a complement to the slow-roll approximation; averaging methods as a complement to the WKB approximation. Global solution approximations!
- *Generalizability.*

First principles and general structures

Example: Generic singularities in GR and beyond

- Work with J. Wainwright, H. van Elst, J.M. Heinzle, W.C. Lim, G.F.R. Ellis, N. Röhr, P. Sandin, (review C.U. Gen. Rel. Grav. 45 (2013) 1669-1710), J. Hell.
- New developments for Bianchi types VIII and IX: Thesis by B. Brehm, arXiv:1606.08058

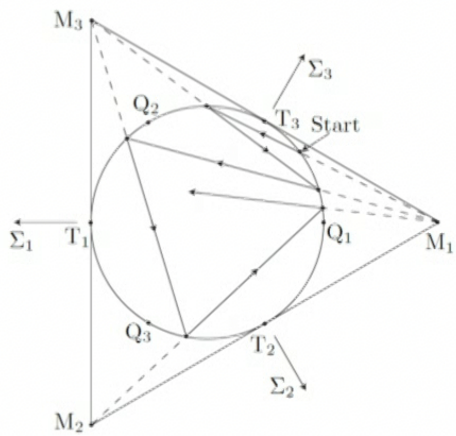
- Why study *generic* singularities?
- Why classical asymptotical behavior?
- Why generic Belinskiĭ, Khalatnikov and Lifshitz (BKL) and BKL related infinitely recurring transient spikes?

Connections with first principles!

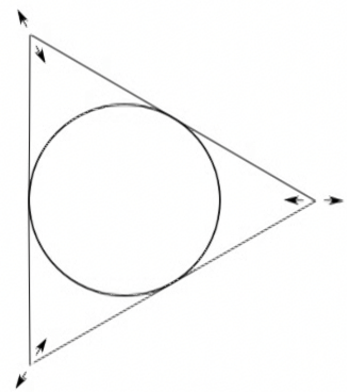
- Causal structure and its consequences for asymptotics in extreme gravity.
- General covariance and (conformal) scale invariance and their connection with symmetries and state space hierarchies.

- Asymptotic silence (an "anti-Newtonian" limit) and breaking thereof.
- Asymptotic silence $\rightarrow$ irrelevance of spatial topology.
- Asymptotic BKL locality = asymptotic causal decoupling, and partial locality.
- The relevance and irrelevance of spatial topology.
- Matter properties, causal propagation and spin.
- Spacetime isometries and homothetic symmetries are Einstein tensor collinations $\rightarrow$ symmetry induced state space hierarchies of invariant subsets.

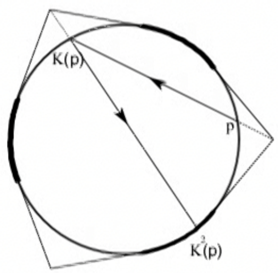
- BKL in the conformal Hubble-normalized dynamical systems approach to GR; asymptotic dynamics toward a generic singularity determined by first principles:
- Monotone functions pushing dynamics to boundaries of boundaries where symmetries (almost) determine the asymptotic solution structure.
- The vacuum solution structure of Bianchi types I and II is completely determined; relevance shown by B. Brehm.



- Ongoing work with Juliette Hell, deforming first principles. Example: Horava-Lifshitz gravity



- The subcritical case: Chaos on a Cantor set (similar to the tertiary Cantor set). GR = a bifurcation where chaos becomes generic in the extreme gravity regime.
- Yet another example that local dynamical systems analysis does not suffice.



Trends and developments:

- Finite and infinite dimensional dynamical systems approaches.
- Solution generating techniques and solution hierarchies.
- Hidden asymptotic symmetries(?) $\rightarrow$ Asymptotic quantization?