Title: Weyl invariance and quantum gravity

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Abstract: I will discuss various ways of realizing the Weyl group in a theory of gravity, and the presence or absence of anomalies.

# Weyl invariance and quantum gravity

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$\phi = \langle \varphi \rangle$

## References

Based on:

- R. P., New J. Phys. **13** 125013 (2011) arXiv:1110.6758 [hep-th]
- A. Codello, G. D'Odorico, C. Pagani, R. P., Class. Quant. Grav. 30 (2013), arXiv:1210.3284 [hep-th]
- C. Pagani, R. P., Class.Quant.Grav. 31 (2014) 115005 arXiv:1312.7767 [hep-th]
- R.P. and G.P. Vacca, "The background scale Ward identity in quantum gravity" Eur.Phys.J. C77 (2017) no.1, 52 arXiv:1611.07005 [hep-th]

and work in progress with V. Skrinjar, C.M Nieto, R. De Leon Ardòn.

## Outline

1. Weyl invariance
2. Anomaly?
3. Gravity
4. Unimodular gravity
5. Conclusions

## Scale transformations

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$$

locally the same as

$$x^\mu \to \Omega x^\mu$$

## Dimensional analysis

Choose *dimensionless coordinates*, and $\hbar = c = 1$.

Invariance under

$$g_{\mu\nu} \to \Omega^2 g_{\mu\nu} \ , \quad \psi_a \to \Omega^{w_a} \psi_a \ , \quad g_i \to \Omega^{w_i} g_i$$

$$S(g_{\mu\nu}, \psi_a | g_i) = S(\Omega^2 g_{\mu\nu}, \Omega^{w_a} \psi_a | \Omega^{w_i} g_i)$$

fixes the weights $w_a = -d_a$, $w_i = -d_i$. Always true.

Invariance under *global scale transformations*

$$S(g_{\mu\nu}, \psi_a | g_i) = S(\Omega^2 g_{\mu\nu}, \Omega^{w_a} \psi_a | g_i)$$

not always true, must be $w_i = 0$.

## Weyl's idea

Interpret global Weyl transformations as changes of the unit of length. Allow choice of unit to depend on position.
Allow parallel transport to affect norm of vectors.

*"It is evident that two rods side by side, stationary with respect to each other, can be intercompared....this cannot be done for....rods with either a space- or time-like separation".*
*(Dicke 1962)*

Physics must be formulated in a way that is invariant under local changes of units, i.e. under *local* rescalings of the metric.

## Various routes to Weyl invariance

Global scale invariance is quite restrictive, local scale invariance even more so.

Construct Weyl invariant theories from non-invariant ones.

- Stückelberg gauging - can be applied to any theory
- Weyl gauging - can be applied to scale-invariant theories
- Ricci gauging - can be applied in special cases

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## Stückelberg gauging ("Fake" Weyl invariance)

Turn dimensionful couplings into fields.

Minimal version: one "dilaton" $\chi$ of weight $-1$:

$$\chi \to \Omega^{-1}\chi$$

Write $\hat{g}_{\mu\nu} = \chi^2 g_{\mu\nu}$, $\hat{\psi}_a = \chi^{w_a}\psi_a$ $\hat{g}_i = \chi^{w_i} g_i$

$$S(g_{\mu\nu}, \psi_a | g_i) \to S(\hat{g}_{\mu\nu}, \hat{\psi}_a | \hat{g}_i) := \hat{S}(g_{\mu\nu}, \psi_a, \chi | \hat{g}_i)$$

$\hat{S}$ is Weyl invariant by construction.

E.g.: $S = \int d^4x \sqrt{g}\, m^2\phi^2 \to \hat{S} = \int d^4x \sqrt{g}\, \hat{m}^2\chi^2\phi^2$

There is a Weyl gauge where $\chi = \mu$ (constant).

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## Weyl calculus

Abelian gauge field transforming as $b_\mu \mapsto b_\mu + \Omega^{-1}\partial_\mu\Omega$.
For scalar field $\phi$ of weight $w$

$$D_\mu\phi = \partial_\mu\phi - w b_\mu\phi$$

More generally

$$\hat{\Gamma}_\mu{}^\lambda{}_\nu = \Gamma_\mu{}^\lambda{}_\nu - \delta^\lambda_\mu b_\nu - \delta^\lambda_\nu b_\mu + g_{\mu\nu} b^\lambda$$

is invariant under local Weyl transformations, hence for a tensor of weight $w$

$$D_\mu t = \hat{\nabla}_\mu t - w b_\mu t$$

is diffeomorphism and Weyl covariant.
Note $\hat{\nabla}_\lambda g_{\mu\nu} = 2 b_\lambda g_{\mu\nu}$ but $D_\lambda g_{\mu\nu} = 0$

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## Weyl curvature

$$[D_\mu, D_\nu]v^\rho = \mathcal{R}_{\mu\nu}{}^\rho{}_\sigma v^\sigma$$

$$\begin{aligned}\mathcal{R}_{\mu\nu\rho\sigma} \quad = \quad & R_{\mu\nu\rho\sigma} + (w-1)F_{\mu\nu}g_{\rho\sigma} \\ & + g_{\mu\rho}\left(\nabla_\nu b_\sigma + b_\nu b_\sigma\right) - g_{\mu\sigma}\left(\nabla_\nu b_\rho + b_\nu b_\rho\right) \\ & - g_{\nu\rho}\left(\nabla_\mu b_\sigma + b_\mu b_\sigma\right) + g_{\nu\sigma}\left(\nabla_\mu b_\rho + b_\mu b_\rho\right) \\ & - \left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) b^2\end{aligned}$$

$$\mathcal{R} = R + 2(d-1)\nabla^\mu b_\mu - (d-1)(d-2)b^2$$

$$F_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu$$

Weyl gauge field is generally non-integrable.

## Weyl gauging

Starting from any globally scale invariant action $S(g_{\mu\nu}, \psi_a | g_i)$ (all $g_i$ dimensionless) replace $\nabla \to D$ and $R \to \mathcal{R}$.

Resulting action $\hat{S}(g_{\mu\nu}, \psi_a, b_\mu | g_i)$ is Weyl invariant.

## Integrable Weyl gauging=Stückelberg gauging

If $F_{\mu\nu} = 0$ we can write

$$b_\mu = -\chi^{-1}\partial_\mu \chi$$

where the field $\chi$ transforming as $\chi \to \Omega^{-1}\chi$ can be identified with the dilaton.

Starting from any action $S(g_{\mu\nu}, \psi_a | g_i)$,
replace $g_i \to \chi^{d_i}\hat{g}_i$, $\nabla \to D$ and $R \to \mathcal{R}$.
Resulting action $\hat{S}(g_{\mu\nu}, \psi_a, \chi | \hat{g}_i)$ agrees with Stückelberg gauging.

## Ricci gauging

The Weyl variation of

$$R_{\mu\nu} - \frac{1}{2(n-1)} g_{\mu\nu} R$$

is the same as the Weyl variation of

$$-\nabla_\mu b_\nu - b_\mu b_\nu + \frac{1}{2} g_{\mu\nu} b^2$$

so if $b_\mu$ occurs in a Weyl-gauged action only in this combination, it can be replaced by a combination of Ricci tensors.

Works when the starting theory is conformal
(A. Iorio, L. O'Raifeartaigh, I. Sachs, C. Wiesendanger, Nucl. Phys. B495 (1997) 433-450, arxiv:hep-th/9607110
K. Farnsworth, M.A. Luty, V. Prilepina, arXiv:1702.07079 [hep-th] )

## Main question

Can Weyl invariance be preserved in quantum theory?

Can Weyl invariance be preserved by renormalization group flow?

## Conformal anomaly

Regularization breaks Weyl invariance, but its effect can be offset if dilaton is present

Englert, C. Truffin and R. Gastmans, Nucl. Phys. B117, 407 (1976)
R.Floreanini and R. P., Nucl. Phys. B436, 141 (1995)
M. Shaposhnikov and I. Tkachev, Phys. Lett. B675, 403 (2009)

## Example: matter in background metric

$$S(\phi; g_{\mu\nu}) = \frac{1}{2}\int d^4x\sqrt{g}\,\phi\Delta^{(1/6)}\phi\ , \quad \Delta^{(1/6)} = -\Box + \frac{R}{6}$$

$$S(\psi; g_{\mu\nu}) = \int d^4x\sqrt{g}\,\bar{\psi}D\psi$$

$$S(A; g_{\mu\nu}) = \frac{1}{4}\int d^4x\sqrt{g}\,F_{\mu\nu}F^{\mu\nu}$$

## Standard scalar measure and path integral

$$(d\phi)^{\mathrm{I}} = \prod_x \frac{d\phi(x)}{\mu}$$

$$\Gamma^{\mathrm{I}}[g_{\mu\nu}] = -\log \int (d\phi) e^{-\int d^4x\sqrt{g}\phi\Delta^{(1/6)}\phi} = \frac{1}{2}\log\det\left(\frac{\Delta^{(1/6)}}{\mu^2}\right)$$

## Trace anomaly

$$\delta_\epsilon \Gamma^{\mathrm{I}}[g_{\mu\nu}] = \int dx \sqrt{g}\, \epsilon(x) \langle T^\mu{}_\mu \rangle := \mathcal{A}(\epsilon)$$

$$\langle T^\mu{}_\mu \rangle = \frac{2}{\sqrt{g}} g^{\mu\nu} \frac{\delta \Gamma^{\mathrm{I}}}{\delta g^{\mu\nu}} = c\, C^2 + aE$$

$$E = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2$$
$$C^2 = C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$$

$$c = \frac{1}{120(4\pi)^2} (n_S + 6 n_D + 12 n_M)$$

$$a = -\frac{1}{360(4\pi)^2} (n_S + 11 n_D + 62 n_M)$$

## The Weyl-invariant scalar measure

In presence of dilaton can define

$$(d\phi)^{\mathrm{II}} = \prod_x \frac{d\phi(x)}{\chi(x)}$$

$$\Gamma^{\mathrm{II}}[g_{\mu\nu}, \chi] = -\log \int (d\phi) e^{-\int d^4x\sqrt{g}\phi\Delta^{(1/6)}\phi} = \frac{1}{2}\log\det\left(\frac{1}{\chi^2}\Delta^{(1/6)}\right)$$

$$\frac{1}{\Omega^{-2}\chi^2}\Delta^{(1/6)}_{\Omega^2 g}(\Omega^{-1}\phi) = \Omega^{-1}\left(\frac{1}{\chi^2}\Delta^{(1/6)}\phi\right)$$

eigenvalues are Weyl invariant $\Rightarrow$ $\det\left(\frac{1}{\chi^2}\Delta^{(1/6)}\right)$ is Weyl invariant

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## Weyl invariant quantization

$\mu$ has been promoted to a field

dilaton acts as Stückelberg field in the *quantum* effective action

Note: trace of energy-momentum tensor still nonzero:

$$\begin{aligned} 0 = \delta_\epsilon \Gamma^{\mathrm{II}}[g_{\mu\nu}, \chi] &= \int dx \sqrt{g} \left( \frac{\delta \Gamma^{\mathrm{II}}}{\delta g_{\mu\nu}} \delta_\epsilon g_{\mu\nu} + \frac{\delta \Gamma^{\mathrm{II}}}{\delta \chi} \delta_\epsilon \chi \right) \\ &= \int dx \sqrt{g}\, \epsilon(x) \left( \langle T^\mu{}_\mu \rangle - \frac{\delta \Gamma^{\mathrm{II}}}{\delta \chi} \chi \right) \\ &= \mathcal{A}(\epsilon) - \int dx \sqrt{g}\, \epsilon(x) \frac{\delta \Gamma^{\mathrm{II}}}{\delta \chi} \chi \end{aligned}$$

## Another point of view

$$\Gamma^{\mathrm{I}}(g^{\chi}_{\mu\nu}) - \Gamma^{\mathrm{I}}(g_{\mu\nu}) = \Gamma_{WZ}(g_{\mu\nu}, \chi)$$

Wess-Zumino consistency condition:

$$\Gamma_{WZ}(g^{\Omega}_{\mu\nu}, \chi^{\Omega}) - \Gamma_{WZ}(g_{\mu\nu}, \chi) = -\Gamma_{WZ}(g_{\mu\nu}, \Omega)$$

where $g^{\Omega} = \Omega^2 g$, $\chi^{\Omega} = \Omega^{-1}\chi$
If we identify $\Gamma^{\mathrm{I}}(g_{\mu\nu})$ with $\Gamma^{\mathrm{II}}(g_{\mu\nu}, \chi = \mu)$,

$$\Gamma^{\mathrm{II}}(g_{\mu\nu}, \chi) = \Gamma^{\mathrm{I}}(g_{\mu\nu}) + \Gamma_{WZ}(g_{\mu\nu}, \chi)$$

## Summary so far

"Stückelberg gauging commutes with RG quantization"

$$\begin{array}{ccc} S(g_{\mu\nu}, \psi_a | g_i) & \longrightarrow & \Gamma(g_{\mu\nu}, \psi_a | g_i) \\ \downarrow & & \downarrow \\ \hat{S}(g_{\mu\nu}, \chi, \psi_a | \hat{g}_i) & \longrightarrow & \hat{\Gamma}(g_{\mu\nu}, \chi, \psi_a | \hat{g}_i) \end{array}$$

If classical theory is Weyl invariant without a dilaton (e.g. Weyl gauging with non-integrable Weyl field, Ricci gauging) it is necessary to introduce a dilaton to have Weyl invariance in the quantum theory.

## Generalization

Generalization to arbitrary interacting theories

Use Functional Renormalization Group Equation

## Exact Renormalization Group Equation

$$k\frac{d\Gamma_k}{dk} = \frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2\Gamma_k}{\delta\phi\delta\phi} + R_k\right)^{-1} k\frac{dR_k}{dk}$$

C. Wetterich, Phys. Lett. B **301** (1993) 90.

## ERGE used to define the QFT

EAA $\Gamma_k$ defined *formally* from functional integral,
but $\partial_t \Gamma_k$ is UV and IR finite.

Choose UV cutoff scale $\Lambda$ and bare action $\Gamma_\Lambda$.
Integrate RG flow equation from $k = \Lambda$ to $k = 0$ to obtain $\Gamma$.

UV limit can be studied integrating the FRGE for $\Lambda \to \infty$

## Anomalous global scale invariance

General matter field in external metric with scale-invariant classical action.
Scale invariance of $\Gamma_k$ broken for two reasons: the measure and the cutoff.
One finds anomalous Ward identity

$$\delta_\epsilon \Gamma_k = \mathcal{A}(\epsilon) + \epsilon k \frac{d\Gamma_k}{dk}$$

or

$$\delta^E_\epsilon \Gamma_k = \mathcal{A}(\epsilon)$$

where

$$\delta^E_\epsilon = \delta_\epsilon - \epsilon k \frac{d}{dk}$$

## The EAA in curved spacetime

$$\Delta S_k(\phi; g_{\mu\nu}) = \int d^d x \sqrt{g}\, \phi\, R_k(\Delta)\, \phi \ .$$

$$R_k(\Delta) = k^2 r(y) \ ; \qquad y = \Delta / k^2$$

Under global scale transformations

$$\delta_\epsilon \Delta = -2\epsilon\Delta \ .$$

The cutoff term transforms as

$$\delta_\epsilon \Delta S_k(\phi; g_{\mu\nu}) = \frac{1}{2}\epsilon \int d^d x \sqrt{g}\, \phi\, k \frac{dR_k}{dk}\, \phi \ .$$

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where

$$\delta^E_\epsilon = \delta_\epsilon - \epsilon k \frac{d}{dk}$$

## Extension to local transformations

Assume there exists a dilaton field $\chi$.
$S(\psi, \chi, g_{\mu\nu}|g_i)$ invariant under Weyl transformations

$$\delta_\epsilon g_{\mu\nu} = 2\epsilon(x) g_{\mu\nu} \; ; \quad \delta_\epsilon \psi = -d_\psi \epsilon(x)\psi \; ; \quad \delta_\epsilon \chi = -\epsilon(x)\chi$$

Measure can be made invariant.
Make also the cutoff invariant:

$$\delta_\epsilon k = -\epsilon k$$

and transform $\Delta$ to a Weyl-covariant operator by $\nabla \to D$

## Construction of Weyl invariant flow

$$\Delta S_k = \int dx \sqrt{g}\, \phi\, k^2 r\left(\frac{1}{k^2}\Delta\right)\phi = \int dx \sqrt{g}\, \chi^2\, \phi\, u^2 r\left(\frac{1}{u^2}\mathcal{O}\right)\phi$$

with

$$\mathcal{O} = \frac{1}{\chi^2}\Delta$$

RG scale measured in units of dilaton:

$$u = k/\chi$$

$u$ dimensionless, Weyl-invariant.

Note this is just Stückelberg gauging of cutoff action.

## Local ERGE

$$\delta k \frac{\delta \Gamma_k}{\delta k} = \frac{1}{2}\mathrm{STr}\left[\left(\frac{\delta^2 \Gamma_k}{\delta\phi\delta\phi} + \mathcal{R}_k\right)^{-1} \delta k \frac{\delta \mathcal{R}_k}{\delta k}\right]$$

## Note: where is the RG?

Assume $u = k/\chi$ is constant. Think of the EAA as

$$\Gamma_k(\phi; g_{\mu\nu}, \chi) = \Gamma_u(\phi; g_{\mu\nu}, \chi)$$

It satisfies

$$u\frac{d\Gamma_u}{du} = \frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2\Gamma_u}{\delta\varphi\delta\varphi} + R_u\right)^{-1} u\frac{\delta R_u}{du}$$

## Beta functional is Weyl invariant

Structure of ERGE:

$$u\frac{d\Gamma_u}{du} = \frac{1}{2}\mathrm{Tr}\left(A^{-1}u\frac{dB}{du}\right)$$

with

$$A = \frac{\delta^2(\Gamma_u + \Delta S_u)}{\delta\phi\delta\phi}, \qquad B = \frac{\delta^2 \Delta S_u}{\delta\phi\delta\phi}$$

If $\phi$ has weight $w$ and $\Gamma_u$ and $\Delta S_u$ are Weyl invariant, $A$ and $B$ are Weyl covariant:

$$A \mapsto \Omega^{-w} A \Omega^{-w}\ , \quad B \mapsto \Omega^{-w} B \Omega^{-w}\ .$$

$\Rightarrow$ r.h.s. of ERGE is Weyl invariant.
RG flow preserves Weyl invariance.
If UV action is Weyl invariant, $\Gamma$ is Weyl invariant.

## Interpretation

In Weyl theory $\chi^{-1}$ interpreted as unit of length

In RG $k^{-1}$ used as unit of length

"Relational" form of RG depends only on invariant ratio $u = k/\chi$

## Quantum gravity

Definition of EAA requires a quantum-background split, e.g.

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

Two generalizations

- "physical" scale transformation $\delta_\epsilon g_{\mu\nu} = 2\epsilon g_{\mu\nu}$
- "background" scale transformations $\delta_\epsilon g_{\mu\nu} = 0$

## Background transformations

$h_{\mu\nu}$ behaves like a matter field coupled to external metric $\bar{g}_{\mu\nu}$

Previous discussion carries over.

However, there is another important realization dictated by the requirement of background independence.

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## Split symmetry

Since $S = S(g)$,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

$S$ is invariant under

$$\delta\bar{g}_{\mu\nu} = \epsilon_{\mu\nu} \ , \qquad \delta h_{\mu\nu} = -\epsilon_{\mu\nu} \ .$$

but due to the presence of gauge and cutoff terms, the EAA $\Gamma_k(h; \bar{g})$ is not.

## Background-independence

- Write the anomalous Ward identity for the split symmetry (or a subgroup thereof)
- Solve it to eliminate from the EAA a number of fields equal to the number of parameters of the transformation
- Write the flow equation for the EAA depending on the remaining variables

This program still in early stages.
Global “background” scale transformations understood

$$
\begin{aligned}
\delta^B_\epsilon \bar{g}_{\mu\nu} &= 2\epsilon \bar{g}_{\mu\nu} \\
\delta^B_\epsilon h_{\mu\nu} &= -2\epsilon \bar{g}_{\mu\nu} \\
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## Unimodular gravity

$$\sqrt{|\det g|} = \omega \ .$$

$$S_{UG}(g) = \frac{1}{16\pi G} \int d^d x \, \omega \, R \ .$$

Invariant under $\mathcal{SD}iff$
Alternatively define

$$g_{\mu\nu} = \gamma_{\mu\nu} \left( \frac{|\det \gamma|}{\omega^2} \right)^{-1/d}$$

$$S(\gamma) = \frac{1}{16\pi G} \int d^d x \, |\gamma|^{1/d} \omega^{\frac{d-2}{d}} \Big[ R[\gamma] + \frac{(d-1)(d-2)}{4d^2} \left( |\gamma|^{-1} \nabla |\gamma| - 2\omega^{-1} \nabla \omega \right)^2 \Big]$$

Invariant under $\mathcal{SD}iff \times Weyl$

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Invariant under $\mathcal{SD}iff \times Weyl$

## Quantum UG

- Classical UG equivalent to GR, except for different rôle of cosmological constant.
- Do different formulations of classical UG lead to equivalent quantum theories?
- Are these quantum UG theories equivalent to quantum GR?

## Constrained formulation

Simple way to impose unimodularity constraint is to parametrize

$$g_{\mu\nu} = \bar{g}_{\mu\rho} \left(\exp h^T\right)^{\rho}{}_{\nu} \; ; \qquad \text{where} \qquad \sqrt{|\det \bar{g}|} = \omega$$

and $h^T$ is traceless.
Using York decomposition

$$h^T_{\mu\nu} = h^{TT}_{\mu\nu} + \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu + \bar{\nabla}_\mu \bar{\nabla}_\nu \sigma - \frac{1}{d} \bar{g}_{\mu\nu} \bar{\nabla}^2 \sigma$$

On Einstein background

$$S^{(2)} = \frac{Z_N}{2} \int d^d x \sqrt{\bar{g}} \left[ \frac{1}{2} h^{\mathrm{TT}}_{\mu\nu} \left( \Delta_{L2} - \frac{2\bar{R}}{d} \right) h^{\mathrm{TT}\mu\nu} - \frac{(d-1)(d-2)}{2d^2} \hat{\sigma} \Delta_{L0} \hat{\sigma} \right] .$$

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On Einstein background

$$S^{(2)} = \frac{Z_N}{2}\int d^dx\,\sqrt{\bar{g}}\left[\frac{1}{2}h^{\mathrm{TT}}_{\mu\nu}\left(\Delta_{L2} - \frac{2\bar{R}}{d}\right)h^{\mathrm{TT}\mu\nu} - \frac{(d-1)(d-2)}{2d^2}\hat{\sigma}\Delta_{L0}\hat{\sigma}\right].$$

## Gauge fixing

$$L^{\mu}{}_{\nu} = \bar{\nabla}^{\mu}\frac{1}{\bar{\nabla}^2}\bar{\nabla}_{\nu}\ ; \qquad T^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} - L^{\mu}{}_{\nu}$$

Gauge fixing term

$$S_{GF} = \frac{Z_N}{2a}\int d^dx\,\sqrt{\bar{g}}\,F_{\mu}T^{\mu\nu}F_{\nu}\,,$$

where

$$F_{\mu} = \bar{\nabla}_{\nu}h^{\nu}{}_{\nu}\ .$$

Using the York decomposition

$$S_{GF}=\frac{Z_N}{2a}\int d^dx\,\sqrt{\bar{g}}\,\hat{\xi}_{\mu}\left(\Delta_{L1} - \frac{2\bar{R}}{d}\right)\hat{\xi}^{\mu}\ .$$

Ghost term

$$S_{gh} = \int d^dx\,\sqrt{\bar{g}}\,\bar{C}_{\mu}\left(\Delta_{L1} - \frac{2\bar{R}}{d}\right)C^{\mu}\ .$$

## Constrained formulation

Simple way to impose unimodularity constraint is to parametrize

$$g_{\mu\nu} = \bar{g}_{\mu\rho} \left(\exp h^T\right)^{\rho}{}_{\nu} \; ; \qquad \text{where} \qquad \sqrt{|\det \bar{g}|} = \omega$$

and $h^T$ is traceless.
Using York decomposition

$$h^T_{\mu\nu} = h^{TT}_{\mu\nu} + \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu + \bar{\nabla}_\mu \bar{\nabla}_\nu \sigma - \frac{1}{d} \bar{g}_{\mu\nu} \bar{\nabla}^2 \sigma$$

On Einstein background

$$S^{(2)} = \frac{Z_N}{2} \int d^d x \sqrt{\bar{g}} \left[ \frac{1}{2} h^{\mathrm{TT}}_{\mu\nu} \left( \Delta_{L2} - \frac{2\bar{R}}{d} \right) h^{\mathrm{TT}\mu\nu} - \frac{(d-1)(d-2)}{2d^2} \hat{\sigma} \Delta_{L0} \hat{\sigma} \right] .$$

## Gauge fixing

$$L^{\mu}{}_{\nu} = \bar{\nabla}^{\mu}\frac{1}{\bar{\nabla}^2}\bar{\nabla}_{\nu} \; ; \qquad T^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} - L^{\mu}{}_{\nu}$$

Gauge fixing term

$$S_{GF} = \frac{Z_N}{2a}\int d^dx\,\sqrt{\bar{g}}\,F_{\mu}T^{\mu\nu}F_{\nu}\,,$$

where

$$F_{\mu} = \bar{\nabla}_{\nu}h^{\nu}{}_{\nu}\;.$$

Using the York decomposition

$$S_{GF}=\frac{Z_N}{2a}\int d^dx\,\sqrt{\bar{g}}\,\hat{\xi}_{\mu}\left(\Delta_{L1} - \frac{2\bar{R}}{d}\right)\hat{\xi}^{\mu}\;.$$

Ghost term

$$S_{gh} = \int d^dx\,\sqrt{\bar{g}}\,\bar{C}_{\mu}\left(\Delta_{L1} - \frac{2\bar{R}}{d}\right)C^{\mu}\;.$$

## One loop effective action

$$\Gamma = S + \frac{1}{2}\mathrm{Tr}\log\left(\Delta_{L2} - \frac{\bar{R}}{2}\right) + \frac{1}{2}\mathrm{Tr}\log\left(\Delta_{L0}\right) - \frac{1}{2}\mathrm{Tr}\log\left(\Delta_{L1} - \frac{2\bar{R}}{d}\right)$$

whereas for GR

$$\Gamma = S + \frac{1}{2}\mathrm{Tr}\log\left(\Delta_{L2} - \frac{\bar{R}}{2}\right) - \frac{1}{2}\mathrm{Tr}\log\left(\Delta_{L1} - \frac{2\bar{R}}{d}\right)$$

## Final result for divergences

On Einstein background

$$\Gamma_k = \int d^d x \sqrt{\bar{g}} \left[ \frac{A_1 k^d}{16\pi d} + \frac{B_1 k^{d-2}}{16\pi(d-2)} \bar{R} + \frac{k^{d-4}}{d-4} (C_1 \bar{R}^2 + D_1 \bar{R}_{\mu\nu\rho\sigma} \bar{R}^{\mu\nu\rho\sigma}) \right]$$

$$C_1 = -\frac{57}{80(4\pi)^2} , \quad D_1 = \frac{71}{60(4\pi)^2} .$$

whereas for pure gravity (Christensen, Duff, N.P.B170, 480 (1980))

$$C_1 = -\frac{58}{80(4\pi)^2} , \quad D_1 = \frac{53}{45(4\pi)^2} .$$

## Weyl gauged version, Exponential parametrization

$$S(\gamma) = S(\Omega^2\gamma)$$

Use exponential split

$$\gamma_{\mu\nu} = \bar{g}_{\mu\rho}\left(\exp(h)\right)^{\rho}{}_{\nu}$$

and decomposing fluctuation into $h_{\mu\nu} = h^T_{\mu\nu} + \frac{1}{d}\bar{g}_{\mu\nu}h$. Then

$$\gamma_{\mu\nu} = e^{\frac{1}{d}h}\tilde{g}_{\mu\nu}\ , \qquad \tilde{g}_{\mu\nu} = \bar{g}_{\mu\rho}\left(\exp(h^T)\right)^{\rho}{}_{\nu}$$

$$\det\gamma = \det\bar{g}\, e^{h} = \omega^2 e^{h}$$

Then

$$S(\gamma) = \frac{1}{16\pi G}\int dx\omega R[\tilde{g}]\ .$$

we are back to the constrained formulation.

## Weyl gauged version, linear parametrization

$$S = Z_N \int d^d x \sqrt{\bar{g}} \Big[ \frac{1}{4} h_{\mu\nu} \bar{\Delta}_2 h^{\mu\nu} - \frac{1}{2} \bar{\nabla}^\mu h_{\mu\nu} \bar{\nabla}_\rho h^{\rho\nu} - \frac{1}{d} h \bar{\nabla}_\mu \bar{\nabla}_\nu h^{\mu\nu} - \frac{d+2}{4d^2} h \bar{\Delta}_0 h - \frac{1}{2d} \bar{R} \left( h_{\mu\nu} h^{\mu\nu} - \frac{1}{d} h^2 \right) \Big]$$

using York decomposition, leads to same Hessian as the constrained formulation

Conclusion: all three formulations of quantum UG lead to the same result, and it differs from quantum GR

## Summary

- In the presence of dilaton, there is a way of quantizing the theory such that Weyl invariance is preserved.
- Presence of dilaton is natural in a relational theory.
- In the variation of the EA, the anomaly of the trace is cancelled by the contribution of the dilaton. The physical effects of the trace anomaly are still present.
- cutoff/renormalization scale will depend on position. A theory with nonconstant cutoff is equivalent to a theory with constant cutoff but a conformally related metric.
- Most natural to discuss QFT in frame where cutoff is constant. In other frames, it is equivalent but may look contrived.
- Unimodular gravity quantum inequivalent to GR