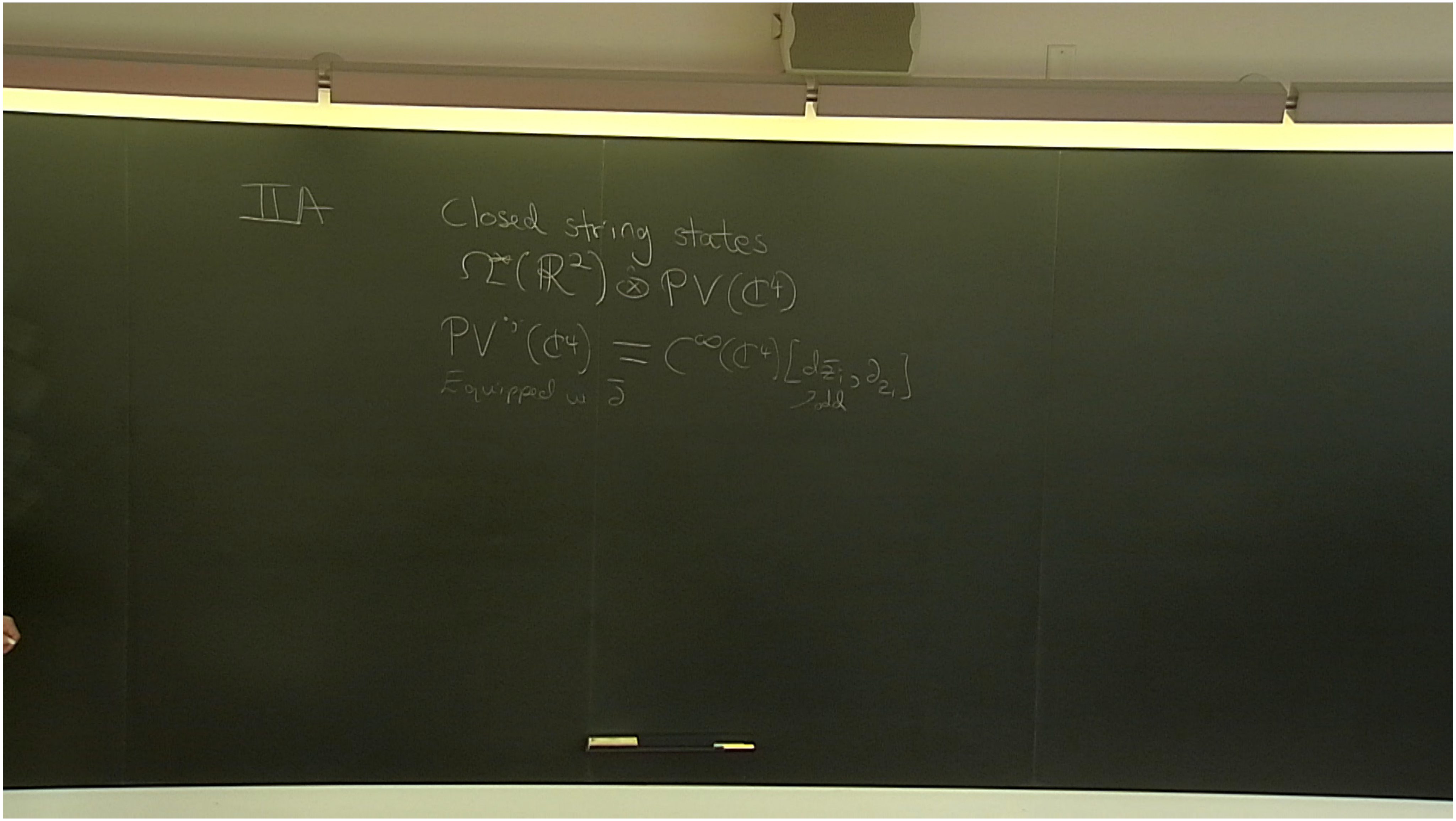
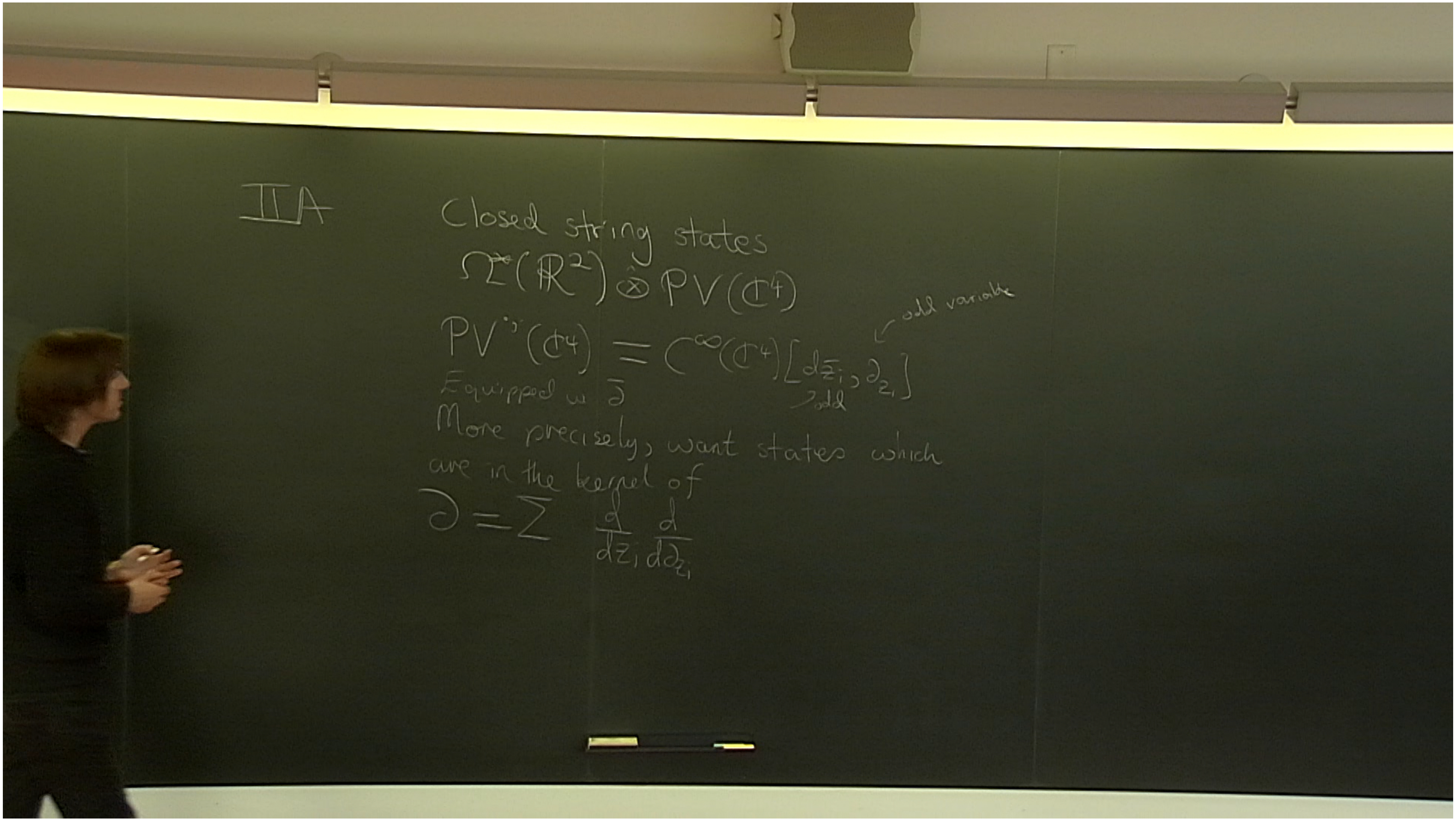
Title: String Theory for Mathematicians - Lecture 3

Date: May 02, 2017 02:30 PM

URL: http://pirsa.org/17050036

Abstract:


IIA
Closed string states
$\Omega^*(\mathbb{R}^2) \hat{\otimes} PV(\mathbb{C}^4)$
$PV^{*,*}(\mathbb{C}^4) = C^\infty(\mathbb{C}^4)[d\bar{z}_i, \partial_{z_i}]$
odd
Equipped w $\bar{\partial}$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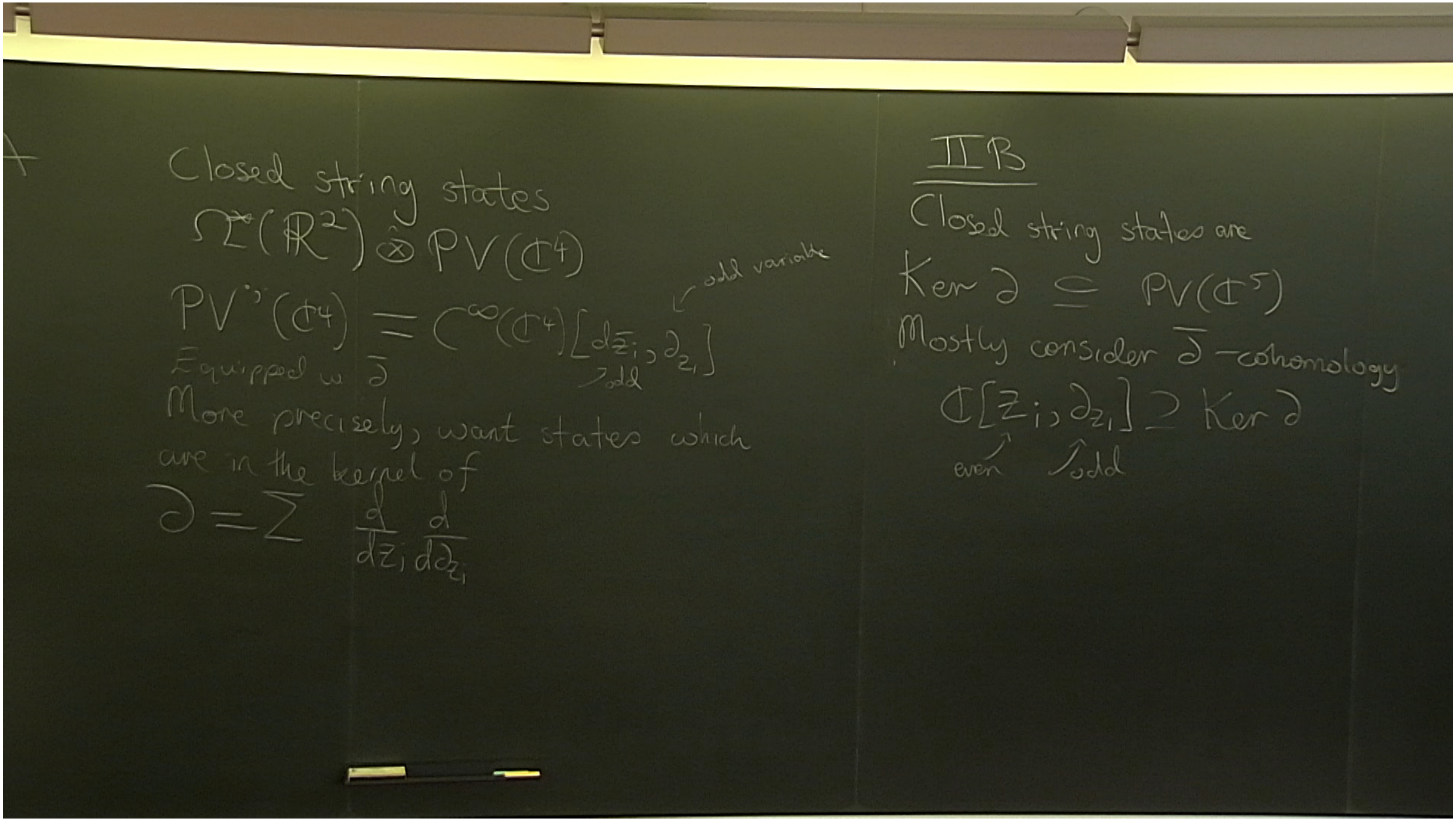
odd variable
Equipped w $\bar{\partial}$
odd
More precisely, want states which are in the kernel of
$\partial = \sum \frac{d}{dz_i} \frac{d}{d\partial_{z_i}}$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IIB
Closed string states are
$\mathrm{Ker}\, \partial \subseteq PV(\mathbb{C}^5)$
Mostly consider $\bar{\partial}$-cohomology
$\mathbb{C}[z_i, \partial_{z_i}] \supseteq \mathrm{Ker}\, \partial$
even
odd

Closed string states

$$\Omega^{*}(\mathbb{R}^2) \hat{\otimes} PV(\mathbb{C}^4)$$

$$PV^{*,*}(\mathbb{C}^4) = C^{\infty}(\mathbb{C}^4)[d\bar{z}_i, \partial_{z_i}]$$

(odd variable; odd)

Equipped w $\bar{\partial}_{\mathbb{C}^4} + d^{dR}_{\mathbb{R}^2}$

More precisely, want states which are in the kernel of

$$\partial = \sum \frac{d}{dz_i} \frac{d}{d\partial_{z_i}}$$

IIB

Closed string states are

$$\operatorname{Ker} \partial \subseteq PV(\mathbb{C}^5)$$

Mostly consider $\bar{\partial}$-cohomology

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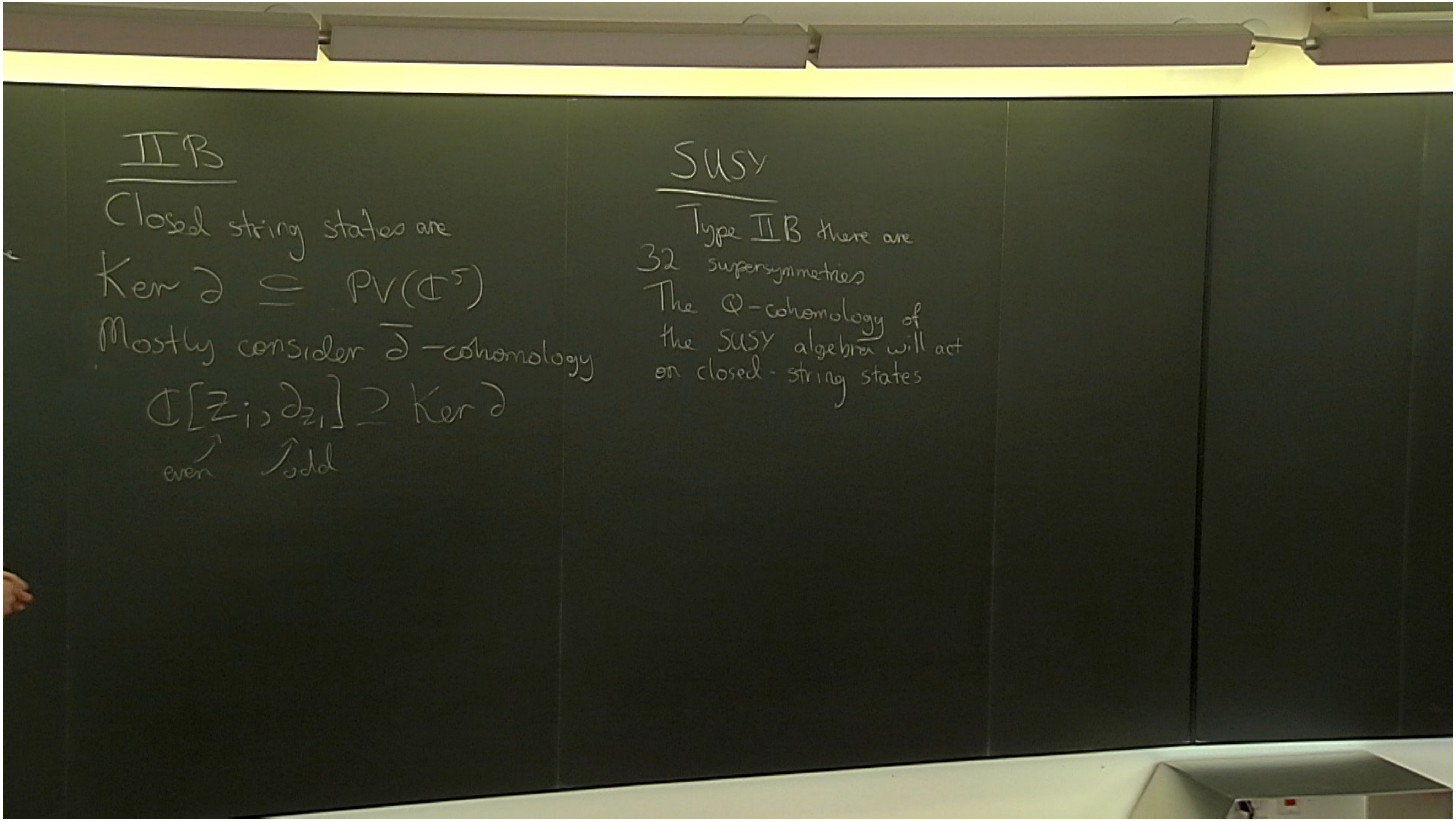
(even, odd)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SUSY
Type IIB there are
32 supersymmetries
The Q-cohomology of
the SUSY algebra will act
on closed-string states

IIB

Closed string states are

$\mathrm{Ker}\, \partial \subseteq PV(\mathbb{C}^5)$

Mostly consider $\bar{\partial}$-cohomology

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even ↗ ↗ odd

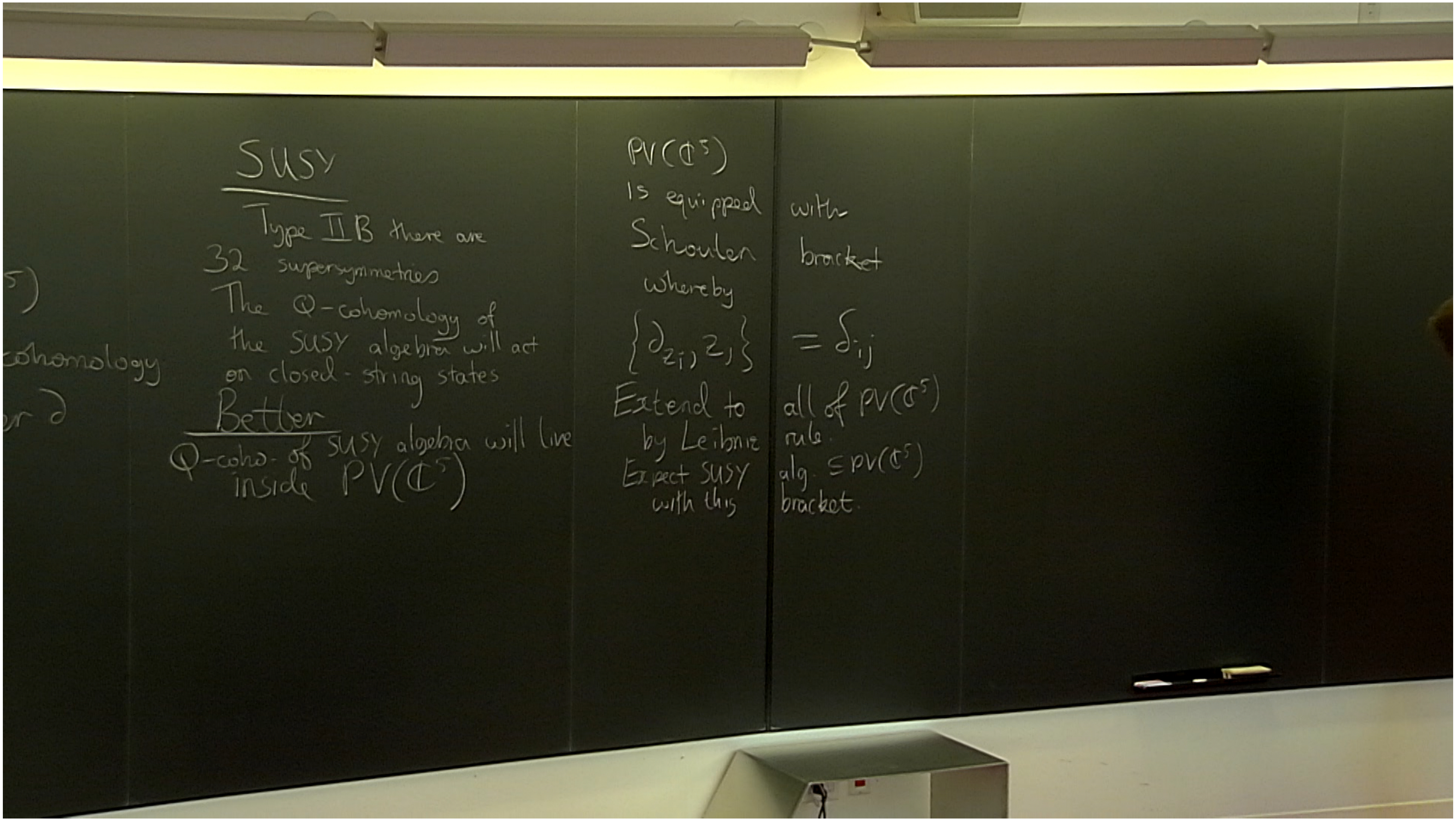
SUSY

Type IIB there are 32 supersymmetries

The Q-cohomology of the SUSY algebra will act on closed-string states

Better

Q-coho. of SUSY algebra will live inside $PV(\mathbb{C}^5)$


SUSY
Type IIB there are
32 supersymmetries
The Q-cohomology of
the SUSY algebra will act
on closed-string states
Better
Q-coho. of SUSY algebra will live
inside PV(ℂ^5)
PV(ℂ^5)
is equipped with
Schouten bracket
whereby
{∂_{z_i}, z_j} = δ_ij
Extend to all of PV(ℂ^5)
by Leibniz rule.
Expect SUSY alg. ⊆ PV(ℂ^5)
with this bracket.

$$\mathbb{R}^{10} \otimes_{\mathbb{R}} \mathbb{C} = V \oplus V^*$$

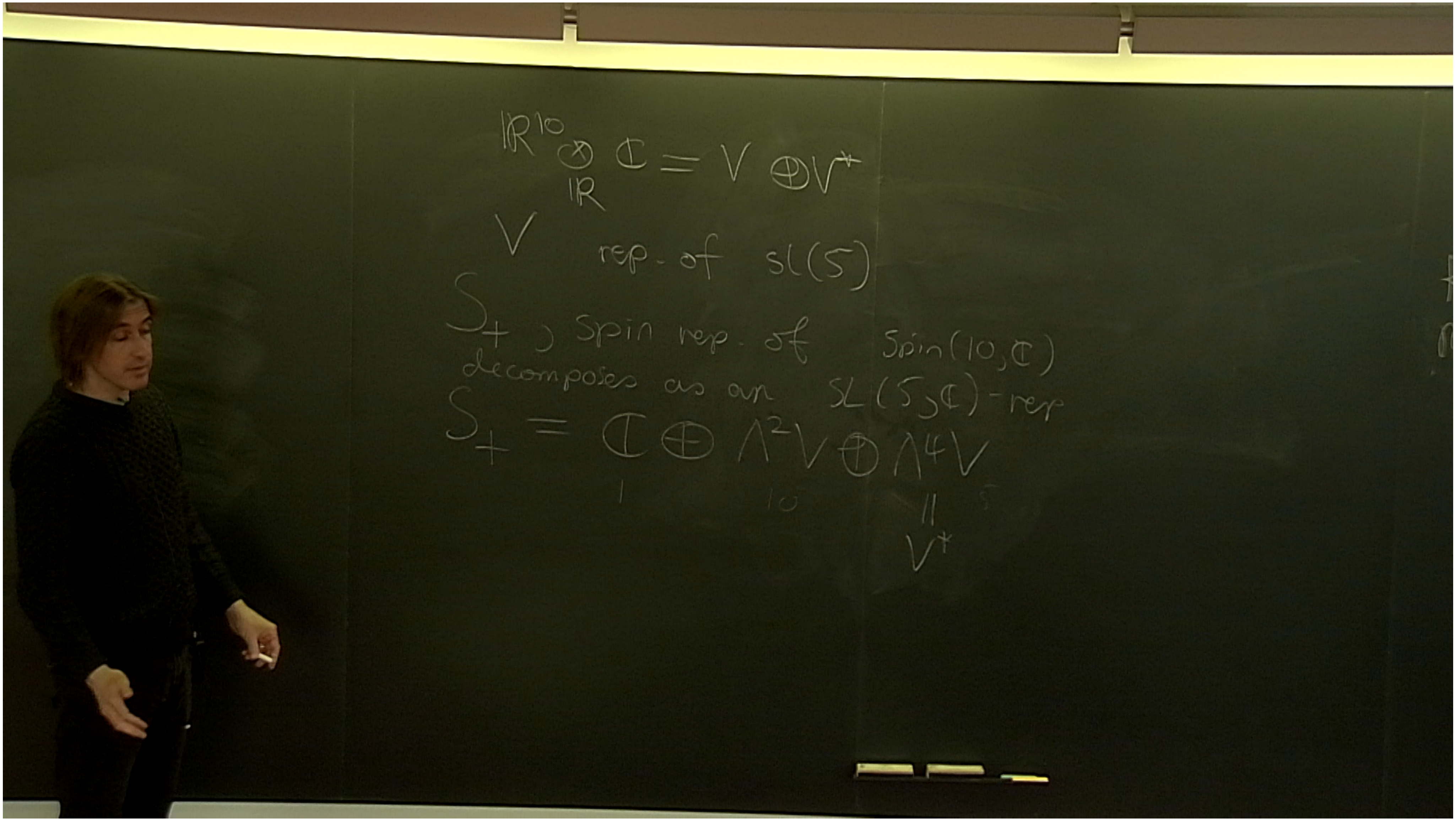
V rep. of $sl(5)$

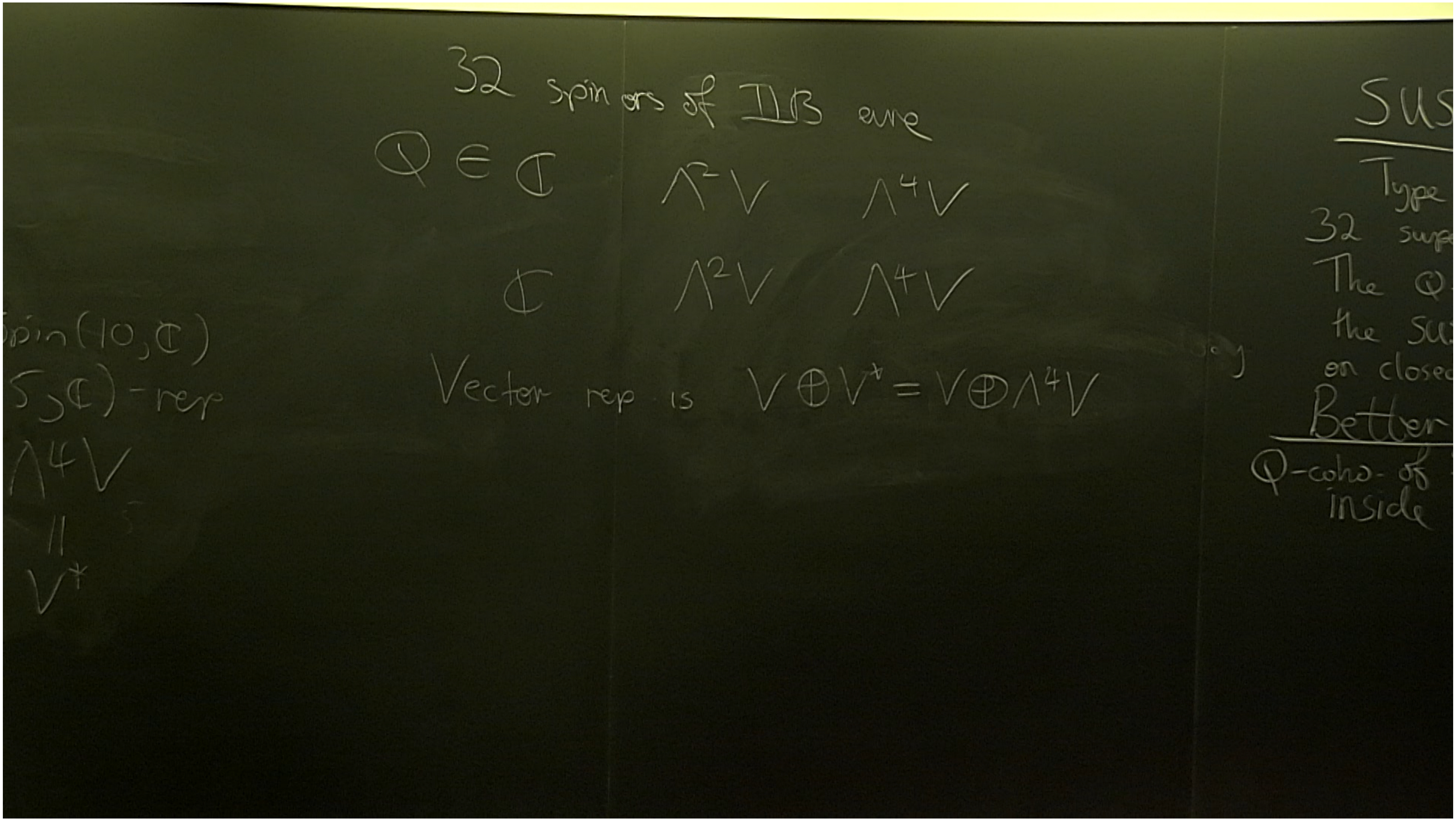
S_+, spin rep. of $Spin(10, \mathbb{C})$

decomposes as an $SL(5, \mathbb{C})$-rep

$$S_+ = \mathbb{C} \oplus \Lambda^2 V \oplus \Lambda^4 V$$

$$\Lambda^4 V = V^*$$


ℝ^10 ⊗_ℝ ℂ = V ⊕ V*
V rep. of sl(5)
S_+ , spin rep. of Spin(10, ℂ)
decomposes as an SL(5, ℂ)-rep
S_+ = ℂ ⊕ Λ²V ⊕ Λ⁴V
1 10 5
= V*


32 spinors of IIB are
Q ∈ ℂ   Λ²V   Λ⁴V
ℂ   Λ²V   Λ⁴V
Vector rep is V ⊕ V* = V ⊕ Λ⁴V
Spin(10,ℂ)
S,ℂ)-rep
Λ⁴V
||
V*
SUS
Type
32 sup
The Q
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on closed
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Q-coho. of
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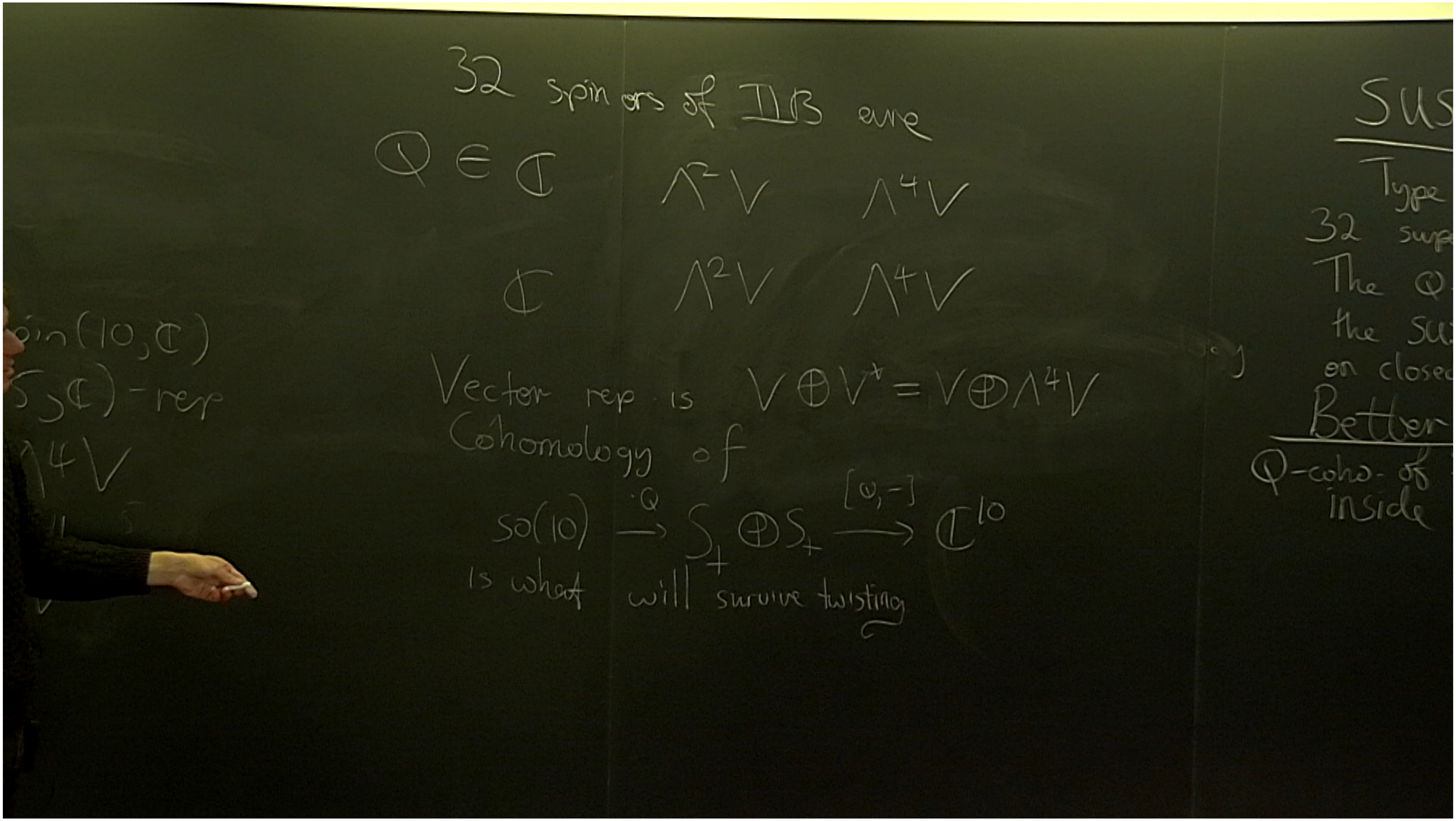
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Cohomology of
so(10) → S+ ⊕ S+ → ℂ¹⁰
is what will survive twisting
Spin(10, ℂ)
S, ℂ) - rep
Λ⁴V
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the su
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32 spinors of IIB are

$Q \in \mathbb{C} \quad \Lambda^2 V \quad \Lambda^4 V$

$\mathbb{C} \quad \Lambda^2 V \quad \Lambda^4 V$

Vector rep is $V \oplus V^* = V \oplus \Lambda^4 V$

Cohomology of

$$so(10) \xrightarrow{\cdot Q} S_+ \oplus S_+ \xrightarrow{[Q,-]} \mathbb{C}^{10}$$

is what will survive twisting

Ker $[Q,-]$.

$[Q,-] : \Lambda^4 V \cong V^*$

Ker $[Q,-]$ is

$\mathbb{C} \quad \Lambda^2 V$

$\mathbb{C} \quad \Lambda^2 V \quad \Lambda^4 V$

32 spinors of IIB are

$Q \in \mathbb{C} \quad \Lambda^2 V \quad \Lambda^4 V$

$\mathbb{C} \quad \Lambda^2 V \quad \Lambda^4 V$

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Cohomology of

$so(10) \xrightarrow{\cdot Q} S_+ \oplus S_+ \xrightarrow{[Q,-]} \mathbb{C}^{10}$

is what will survive twisting

Ker $[Q,-]$.

$[Q,-] : \Lambda^4 V \cong V^*$

Ker $[Q,-]$ is

$\mathbb{C} \longrightarrow \Lambda^2 V$ ← Image of Q under rotation

$\mathbb{C} \quad \Lambda^2 V \quad \Lambda^4 V$

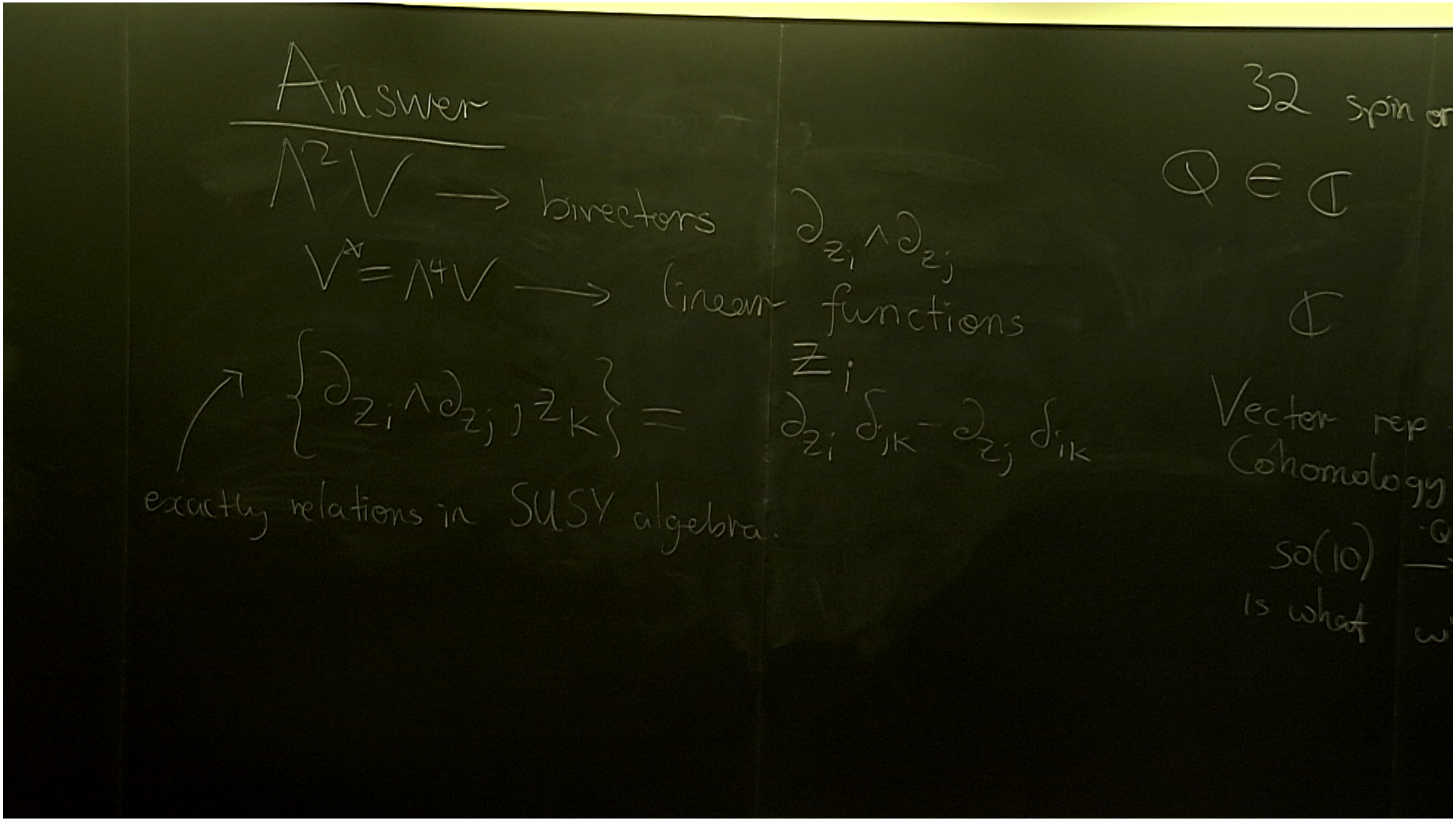
↗ These survive

Need $\Lambda^2 V \subseteq PV(\mathbb{C}^5)$

$V^* \subseteq PV(\mathbb{C}^5)$

Schouten bracket between them is

$\partial/\partial z_i$


Answer
$\Lambda^2 V \longrightarrow$ bivectors $\partial_{z_i} \wedge \partial_{z_j}$
$V^* = \Lambda^4 V \longrightarrow$ linear functions z_i
$\{\partial_{z_i} \wedge \partial_{z_j}, z_k\} = \partial_{z_i} \delta_{jk} - \partial_{z_j} \delta_{ik}$
exactly relations in SUSY algebra.
32 spin
$Q \in \mathbb{C}$
$\mathbb{C}$
Vecter rep
Cohomology
so(10)
is what

$\partial_{z_i} \wedge \partial_{z_j}$

...ear functions

z_i

$\partial_{z_i} \delta_{jk} - \partial_{z_j} \delta_{ik}$

...bra.

– Other twists of IIB:

– make $\mathbb{C}^5$ non-commutative in some directions

– Or turn on a linear superpotential.

Ker

[φ, -

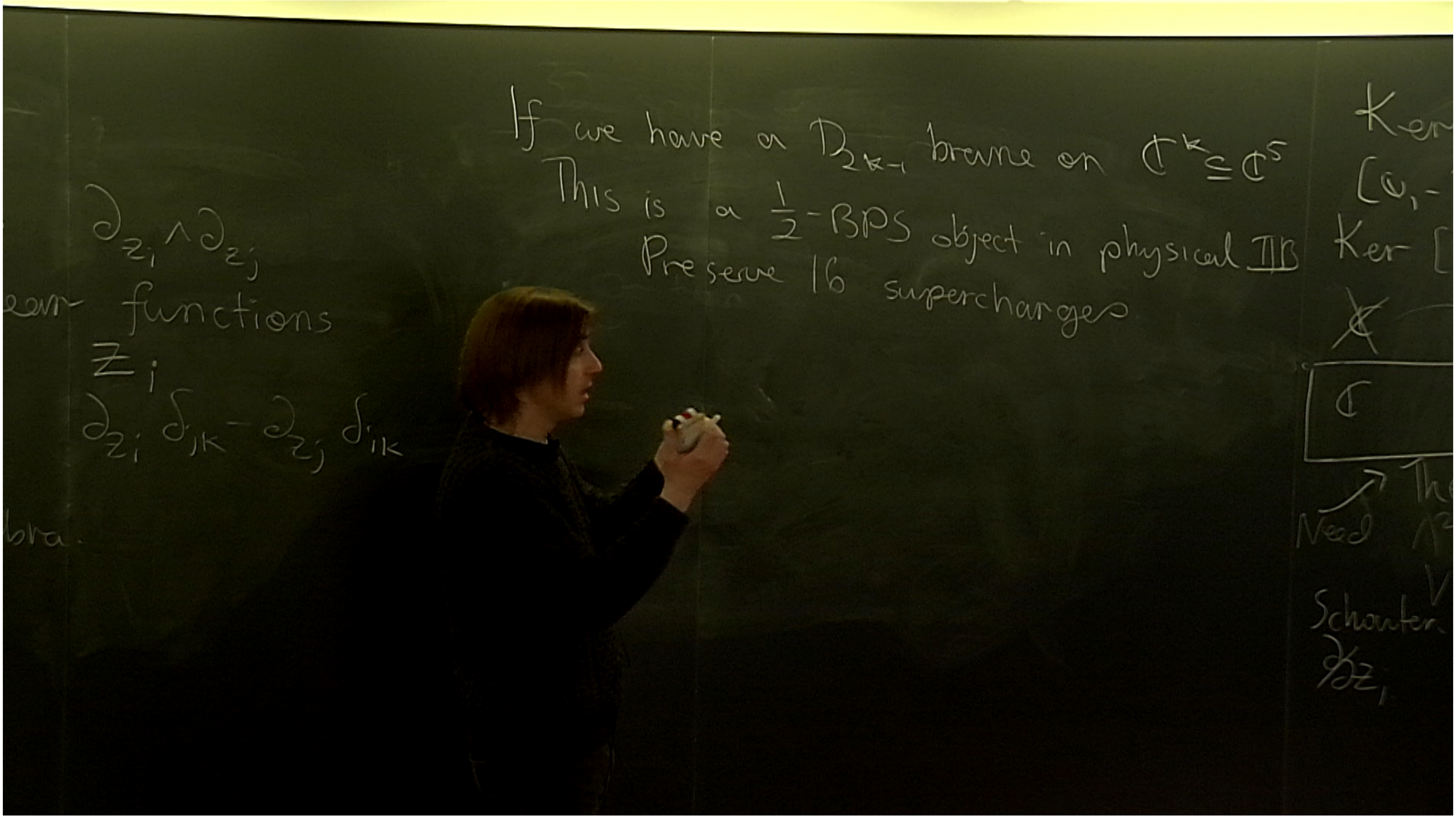
Ker [

$\mathbb{C}$

Need

Schouten

$\partial/\partial z_i$


If we have a D_{2k-1} brane on $\mathbb{C}^k \subseteq \mathbb{C}^5$
This is a $\frac{1}{2}$-BPS object in physical IIB
Preserve 16 supercharges
$\partial_{z_i} \wedge \partial_{z_j}$
functions
z_i
$\partial_{z_i} \delta_{jk} - \partial_{z_j} \delta_{ik}$
Ker
Ker
Need
Schouten

$\partial_{z_i} \wedge \partial_{z_j}$

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Preserve 16 supercharges

Can we see Q-coho. of these 16 supercharges in the twisted theory?

Ker

$[v_1, -$

Ker [

$\not{\mathbb{C}}$

$\boxed{\mathbb{C}}$

The

Need $\wedge^2$

V

Schouten

$\partial/\partial z_i$

$\partial_{z_i} \wedge \partial_{z_j}$

...ear functions

z_i

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$\mathbb{C}^k \subseteq \mathbb{C}^5$

w_i, z_j coordinates

RULE The brane is preserved by z_j, and $\partial_{w_i} \wedge \partial_{z_j}$

but not by the others.

Ker...

$[\omega_1, -$

Ker [

$\mathbb{C}$

$\mathbb{C}$

Need

Schouten

$\partial / \partial z_i$

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a $\frac{1}{2}$-BPS object in physical IIB
serve 16 supercharges
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5
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$$\{z_i, \partial_{w_j} \wedge \partial_{z_k}\} = -\partial_{w_j} \delta_{ik}$$

Translation on the brane

$PV(\mathbb{C}^5)$
is equipped
Schouten
whereby
$\{\partial_{z_i}, z_j\}$
Extend to
by Leibniz
Expect SUSY
with this

w
b
=
all
rul
alg
brac

we have a D_{2k-1} brane on $\mathbb{C}^k \subseteq \mathbb{C}^5$

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$\{z_i, \partial_{w_j} \wedge \partial_{z_k}\}$

$= -\partial_{w_j} \delta_{ik}$

Translation on the brane

There are

$(5-k) + (k)(5-k)$

supersymmetries preserving the brane.

Pv

Is

S

$\{z_i, \partial_{w_j} \wedge \partial_{z_k}\}$
$= -\partial_{w_j} \delta_{ik}$

↗ Translation on the brane

There are $(5-k) + (k)(5-k)$ supersymmetries preserving the brane.

Example: D3 brane

$N=4$ SUSY

16 supercharges in

$$S_+^{4d} \otimes (\mathbb{C} \oplus W) \oplus S_-^{4d} \otimes (\mathbb{C} \oplus W^*)$$

$\dim W = 3$

$Q = \psi \otimes 1 \in S_+^{4d} \otimes \mathbb{C}$

Q-coho:

$$\frac{\text{Q-coho}}{\text{Ker}[Q, -]} = S_+^{4d} \otimes (\mathbb{C} \oplus W) \oplus S_-^{4d} \otimes W$$

What do I get from $\mathbb{Q}$ by rotation

- Rotate by $SL_4(\mathbb{C})^R = Spin_6(\mathbb{C})$.

If we have a

This is a

Preserve

Can we see

Supercharges

$\mathbb{C}^k \subseteq \mathbb{C}^5$

w_i, z_j co

RULE The bra

z_j,

but not

What do I get from Q by rotation

- Rotate by $SL_4(\mathbb{C})^R = Spin_6(\mathbb{C})$
  will give $\psi \otimes W$
- Rotate by SO_4 get $S_+^{4d} \otimes \mathbb{C}$

Ker Q / Im rotation is

$$(S^+/\psi) \otimes W \oplus S^- \otimes W^*$$

3 ⟶ z_i

6 ⟶ $\partial_{w_i} \wedge \partial_{z_j}$

If we have a

This is a $\frac{1}{2}$

Preserve

Can we see

Supercharges

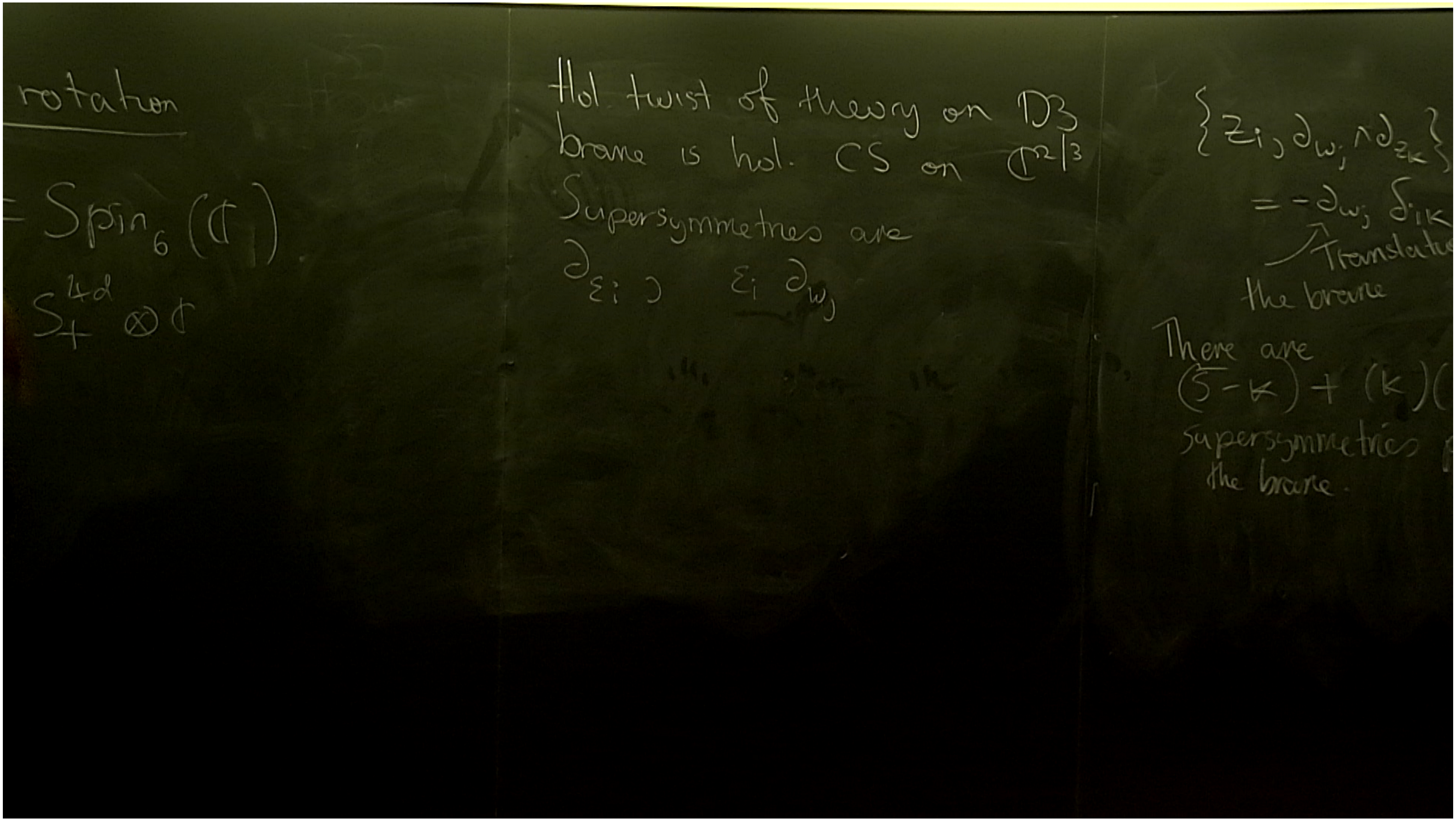
$\mathbb{C}^K \subseteq \mathbb{C}^5$

w_i, z_j coor

RULE The bran

z_j,

but not b


rotation
$= \mathrm{Spin}_6(\mathbb{C})$
$S_+^{4d} \otimes \mathbb{C}$
Hol. twist of theory on D3
brane is hol. CS on $\mathbb{C}^{2|3}$
Supersymmetries are
$\partial_{\varepsilon_i}, \quad \varepsilon_i \partial_{w_j}$
$\{z_i, \partial_{w_j} \wedge \partial_{z_k}\}$
$= -\partial_{w_j} \delta_{ik}$
Translatio
the brane
There are
$(5-k) + (k)($
supersymmetries
the brane.

rotation

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Hol. twist of theory on D3 brane is hol. CS on $\mathbb{C}^{2|3}$

Supersymmetries are

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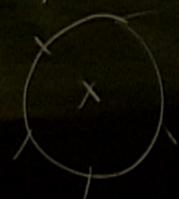
How do these match?

A closed string state will give a single-trace def. of the theory on the brane



$\{z_i, \partial_{w_j} \wedge \partial_{z_k}\}$

$= -\partial_{w_j} \delta_{ik}$

Translations [illegible]

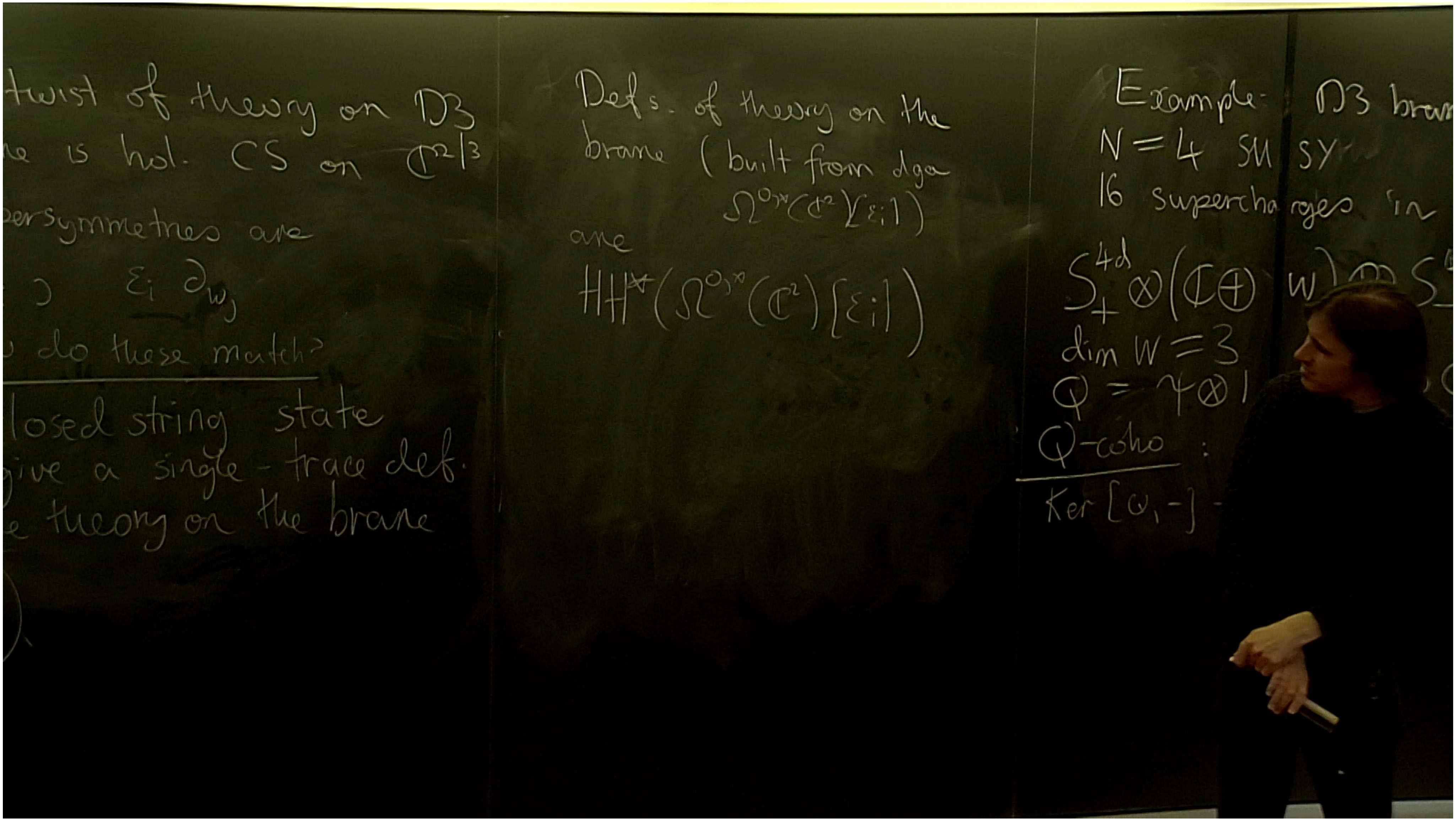
the brane

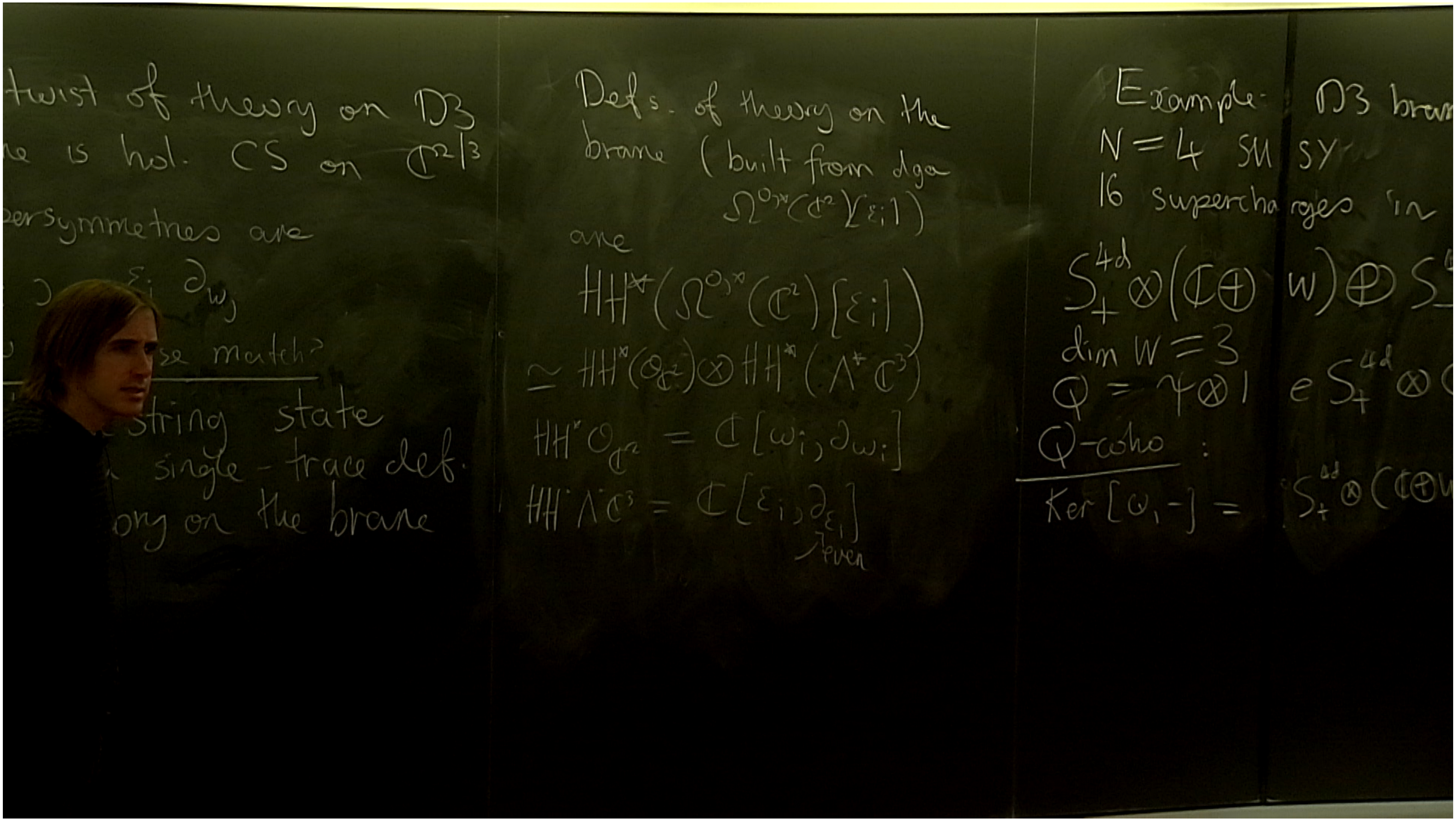
There are

$(5-k) + (k)($ [illegible]

supersymmetries [illegible]

the brane.


twist of theory on D3
is hol. CS on $\mathbb{C}^{2|3}$
persymmetries are
$\varepsilon_i \partial_{w_j}$
do these match?
losed string state
give a single-trace def.
e theory on the brane
Defs. of theory on the brane (built from dga $\Omega^{0,*}(\mathbb{C}^2)[\varepsilon_i]$)
are
$HH^*(\Omega^{0,*}(\mathbb{C}^2)[\varepsilon_i])$
Example: D3 bran
N = 4 SU SY
16 supercharges in
$S_+^{4d} \otimes (\mathbb{C} \oplus W)$
dim W = 3
$Q = \psi \otimes 1$
Q-coho :
Ker [Q, -]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$HH^*(\Omega^{0,*}(\mathbb{C}^2)[\varepsilon_i])$
$\simeq HH^*(\mathcal{O}_{\mathbb{C}^2}) \otimes HH^*(\wedge^* \mathbb{C}^3)$
$HH^* \mathcal{O}_{\mathbb{C}^2} = \mathbb{C}[\omega_i, \partial_{\omega_i}]$
$HH^* \wedge \mathbb{C}^3 = \mathbb{C}[\varepsilon_i, \partial_{\varepsilon_i}]$
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$Q = \psi \otimes 1 \in S_+^{4d} \otimes$
Q-coho :
Ker [Q, -] =
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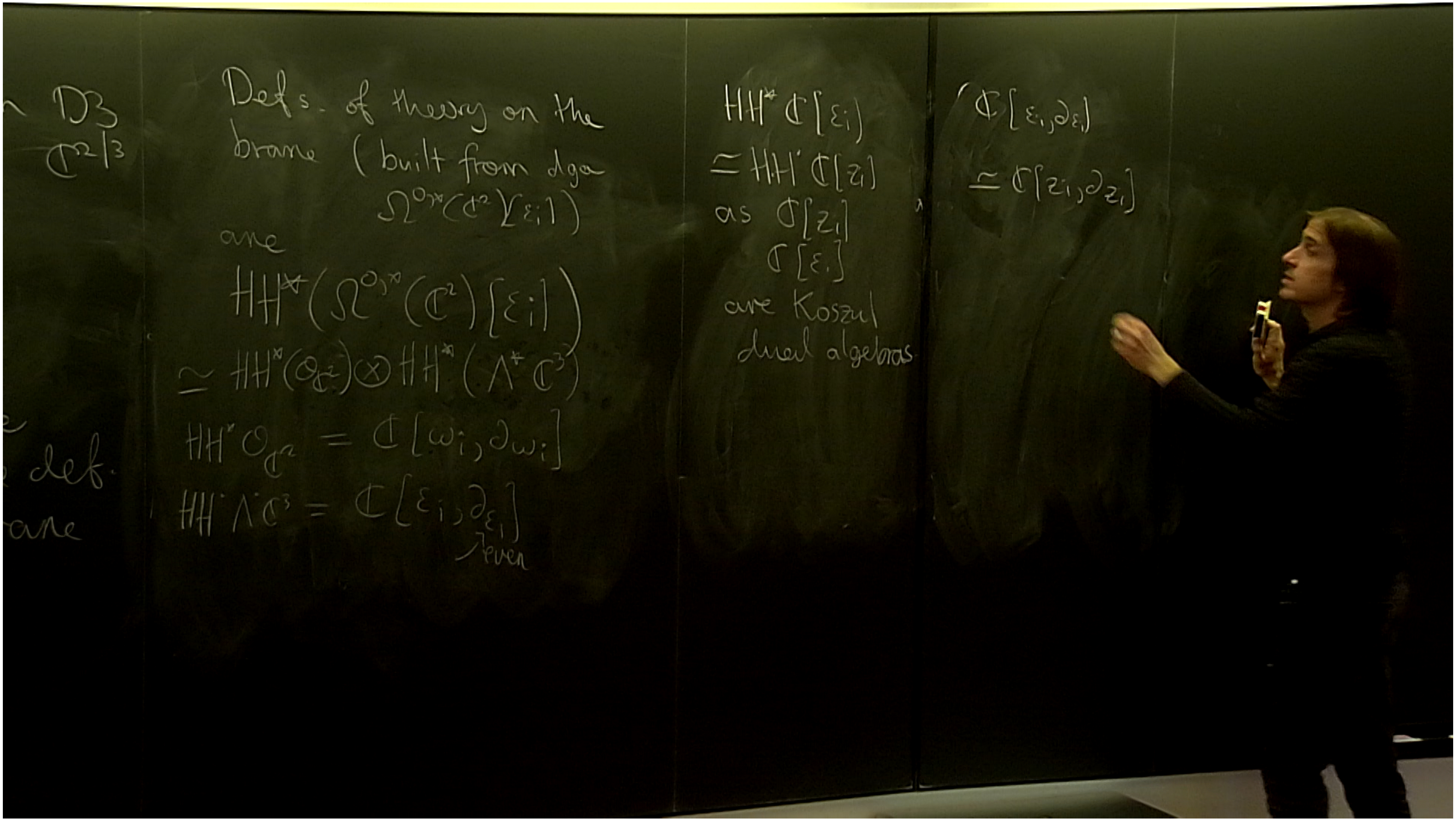
D3
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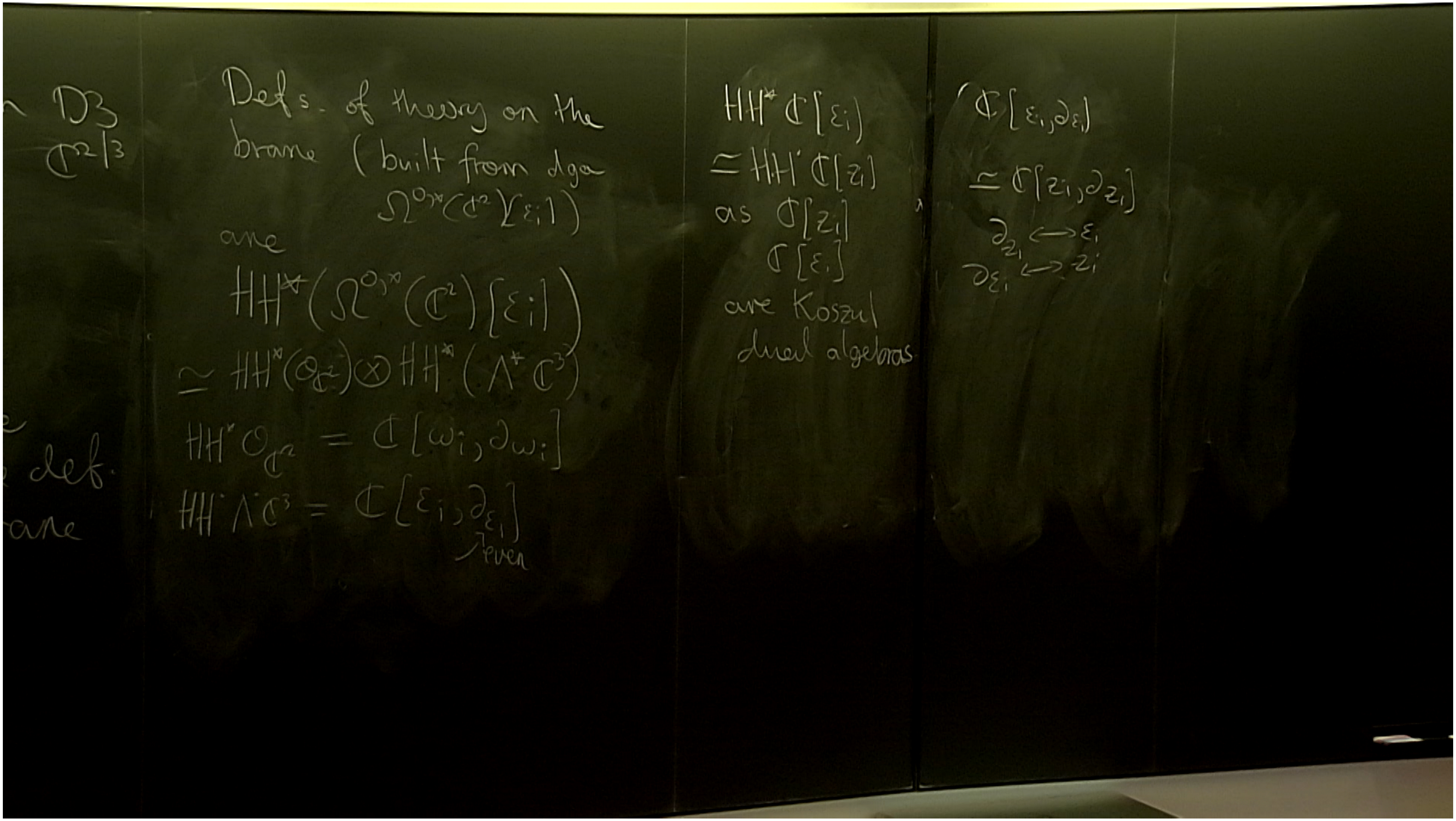
$\simeq HH^*(\mathcal{O}_{\mathbb{C}^2}) \otimes HH^*(\Lambda^* \mathbb{C}^3)$
$HH^* \mathcal{O}_{\mathbb{C}^2} = \mathbb{C}[\omega_i, \partial_{\omega_i}]$
$HH^\cdot \Lambda^\cdot \mathbb{C}^3 = \mathbb{C}[\varepsilon_i, \partial_{\varepsilon_i}]$
even
$HH^* \mathbb{C}[\varepsilon_i]$
$\simeq HH^\cdot \mathbb{C}[z_i]$
as $\mathbb{C}[z_i]$
$\mathbb{C}[\varepsilon_i]$
are Koszul dual algebras
$\mathbb{C}[\varepsilon_i, \partial_{\varepsilon_i}]$
$\simeq \mathbb{C}[z_i, \partial_{z_i}]$


Defs. of theory on the brane (built from dga $\Omega^{0,*}(\mathbb{C}^2)[\varepsilon_i]$) are
$HH^*(\Omega^{0,*}(\mathbb{C}^2)[\varepsilon_i])$
$\simeq HH^*(\mathcal{O}_{\mathbb{C}^2}) \otimes HH^*(\Lambda^* \mathbb{C}^3)$
$HH^* \mathcal{O}_{\mathbb{C}^2} = \mathbb{C}[\omega_i, \partial_{\omega_i}]$
$HH^* \Lambda^* \mathbb{C}^3 = \mathbb{C}[\varepsilon_i, \partial_{\varepsilon_i}]$
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$\partial_{z_i} \longleftrightarrow \varepsilon_i$
$\partial_{\varepsilon_i} \longleftrightarrow z_i$
D3
$\mathbb{C}^{2|3}$
def.
rane

$\mathbb{C}^2_{w_i} \subseteq \mathbb{C}^5_{z_i}$

IIB Supersymmetries are

$\partial_{w_i} \wedge \partial_{z_j}$

z_j

$z_j \longrightarrow \partial_{\varepsilon_j}$

$\partial_{z_j} \longrightarrow \varepsilon_j$

Then, $\partial_{w_i} \wedge \partial_{z_j} \longrightarrow \varepsilon_j \partial_{w_i}$

$z_j \longrightarrow \partial_{\varepsilon_j}$

$\mathbb{C}^2_{w_i} \subseteq \mathbb{C}^5_{z_i}$

IIB Supersymmetries are

$\partial_{w_i} \wedge \partial_{z_j}$

z_j

$z_j \longrightarrow \partial_{\varepsilon_j}$

$\partial_{z_j} \longrightarrow \varepsilon_j$

Then, $\partial_{w_i} \wedge \partial_{z_j} \longrightarrow \varepsilon_j \partial_{w_i}$

$z_j \longrightarrow \partial_{\varepsilon_j}$

These are precisely the symmetries of $\mathbb{C}^{2|3}$ which come from SUSY of $N=4$ YM.

...es of $\mathbb{C}^{2|3}$

$N=4$ YM.

Example Kapustin–Witten twist

Comes from family of supercharges

$$\lambda \partial_{\varepsilon_3} + \mu(\varepsilon_1 \partial_{z_1} + \varepsilon_2 \partial_{z_2})$$

Def s...

brane...

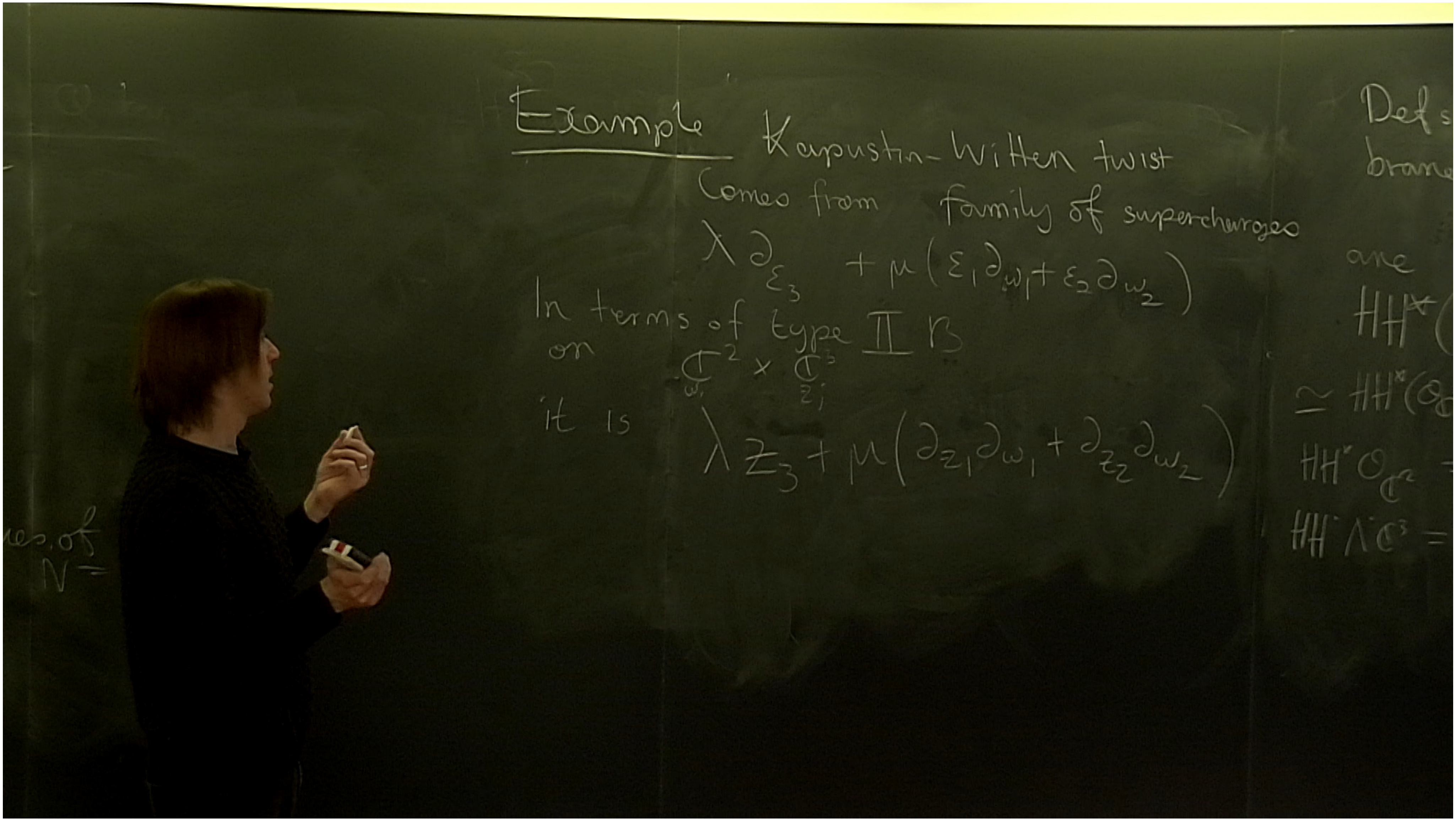
are

$HH^*($...

$\simeq HH^*(\mathcal{O}$...

$HH^* \mathcal{O}_{\mathbb{C}^2}$ ...

$HH^\cdot \Lambda^\cdot \mathbb{C}^3 =$ ...


Example Kapustin-Witten twist
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In terms of type II B
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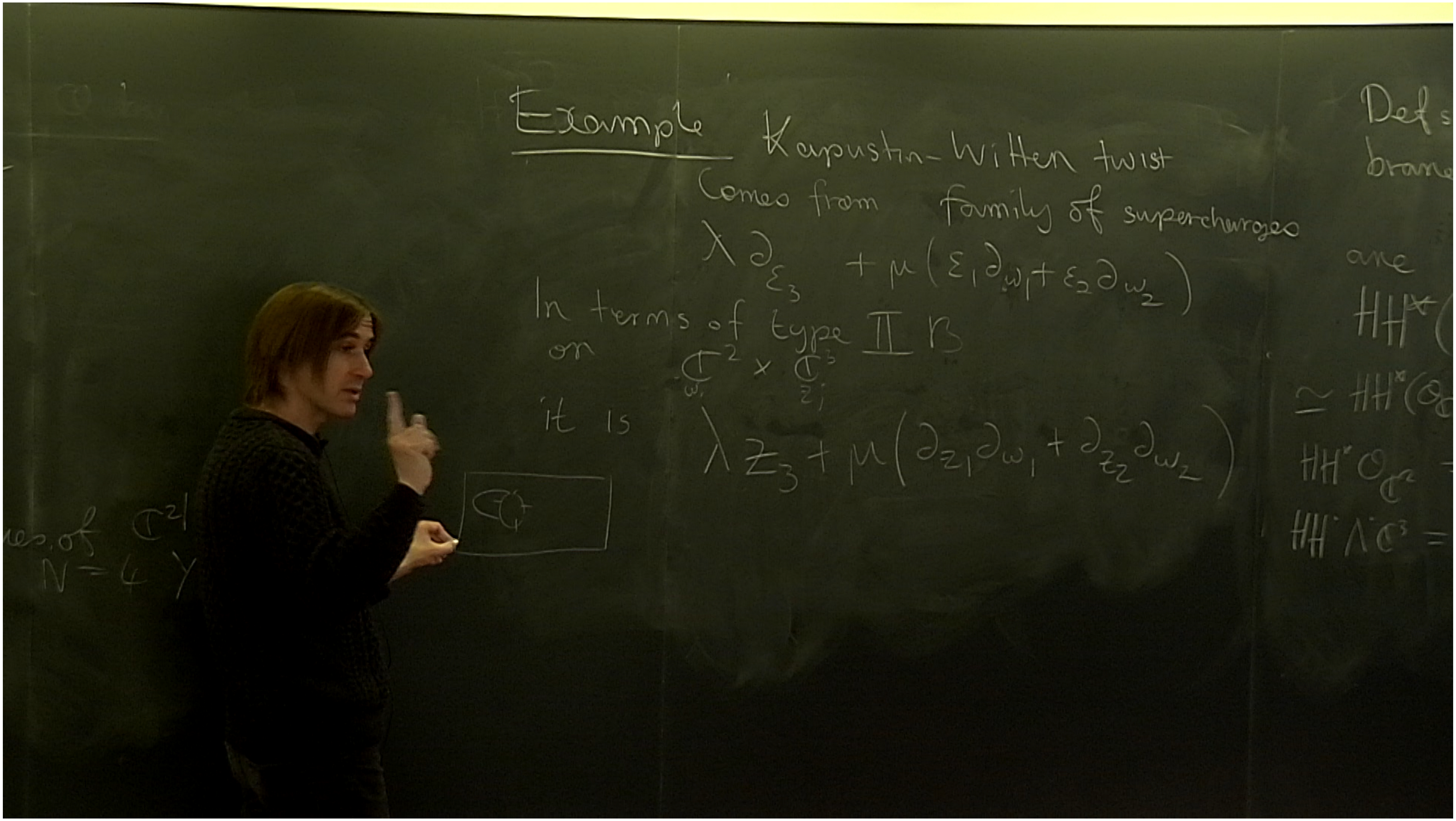
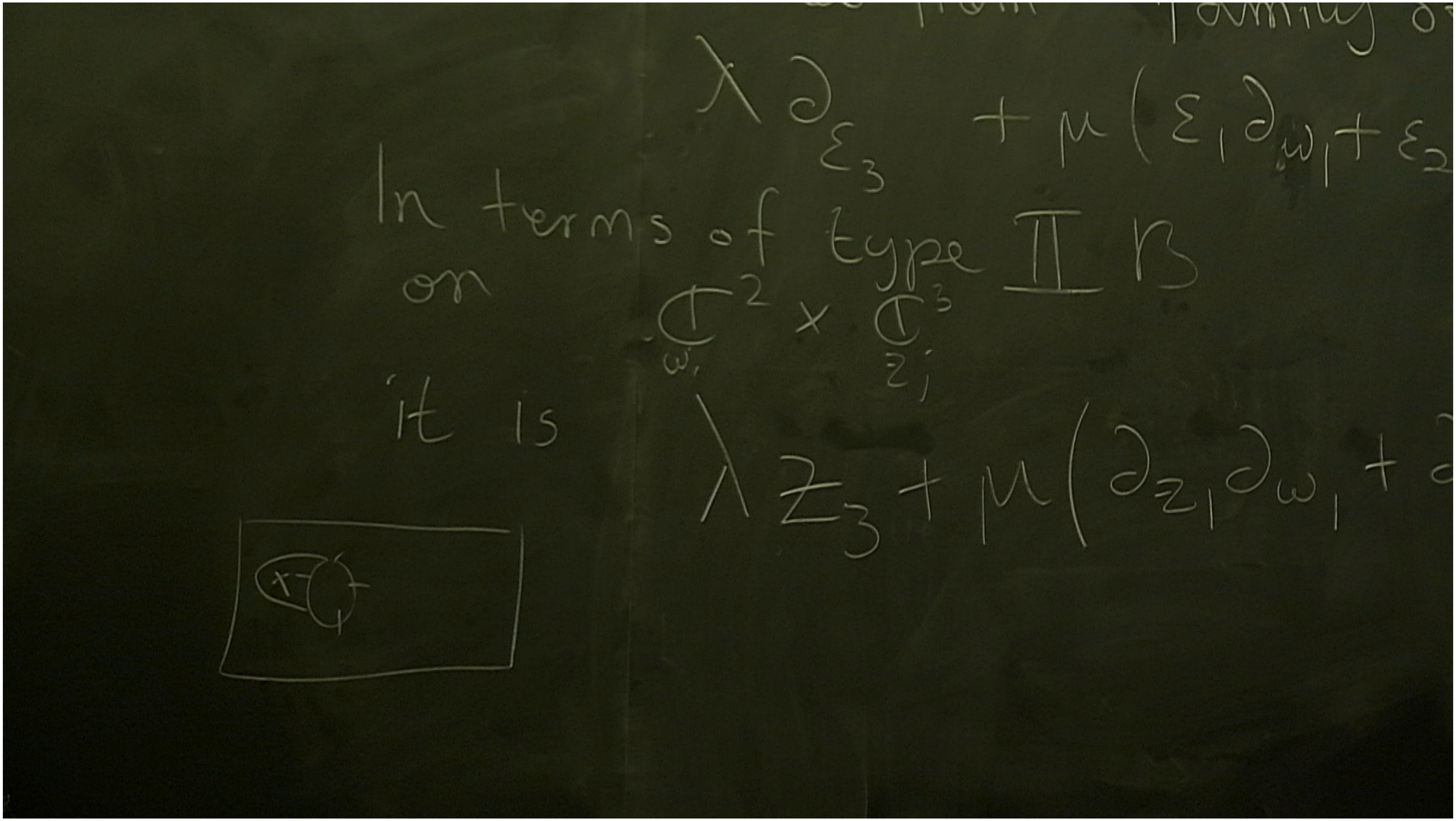
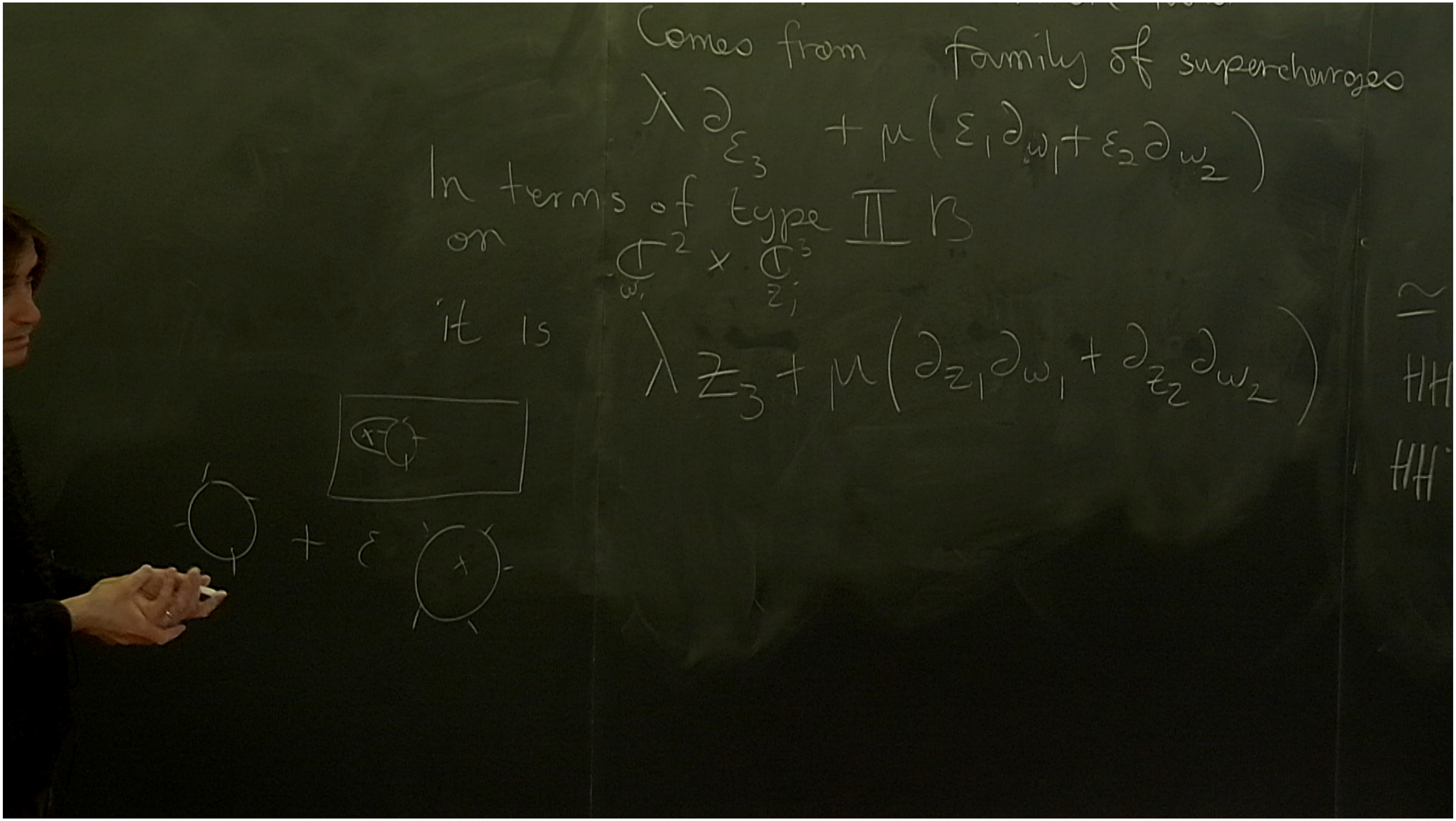
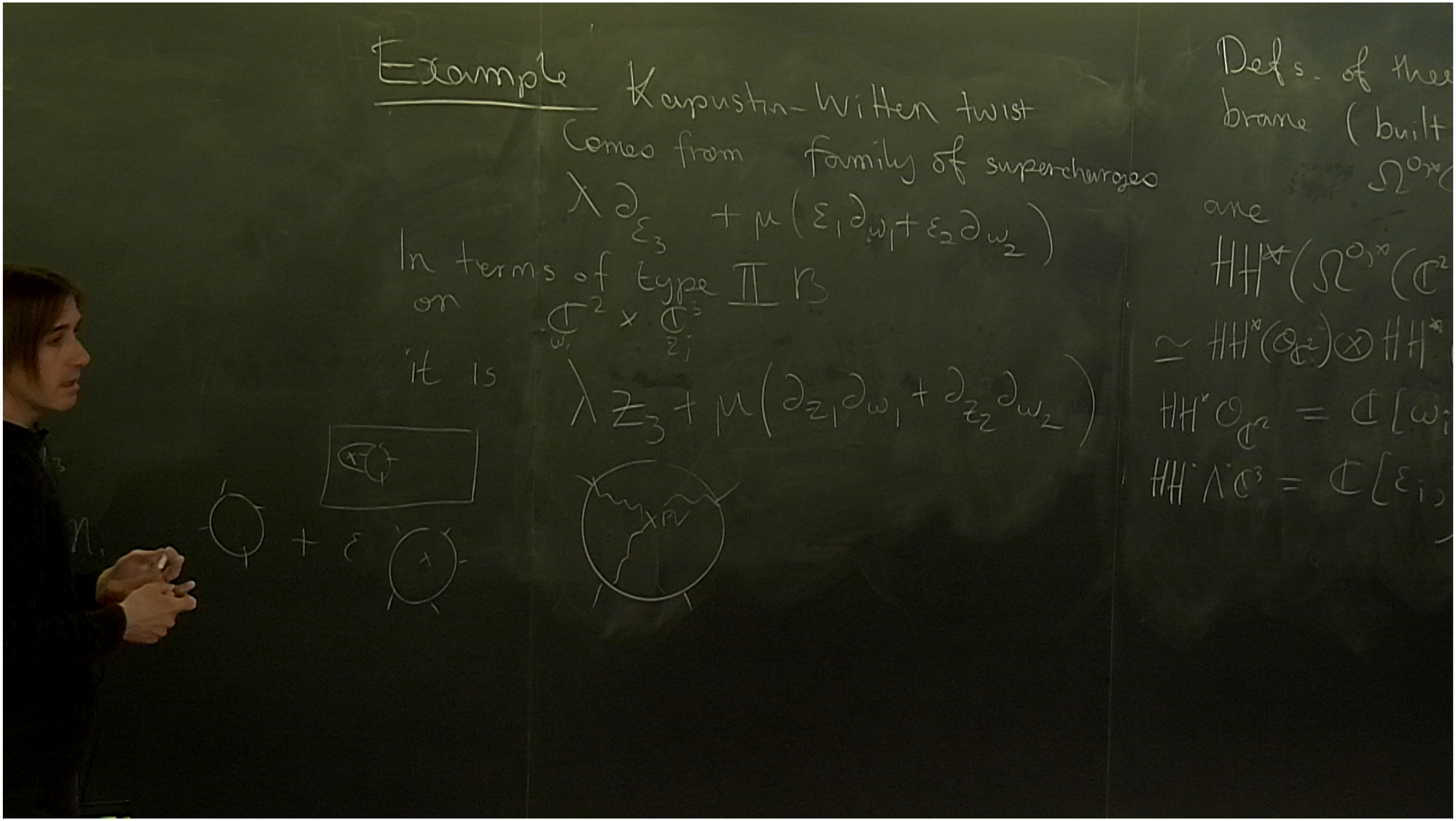

Example Kapustin-Witten twist
Comes from family of supercharges
$\lambda \partial_{\varepsilon_3} + \mu(\varepsilon_1 \partial_{w_1} + \varepsilon_2 \partial_{w_2})$
In terms of type IIB
on $\mathbb{C}^2_{w_i} \times \mathbb{C}^3_{z_j}$
it is
$\lambda z_3 + \mu(\partial_{z_1}\partial_{w_1} + \partial_{z_2}\partial_{w_2})$


$\lambda \partial_{\varepsilon_3} + \mu(\varepsilon_1 \partial_{\omega_1} + \varepsilon_2$
In terms of type II B
on $\mathbb{C}^2_{\omega_i} \times \mathbb{C}^3_{z_i}$
it is $\lambda z_3 + \mu(\partial_{z_1}\partial_{\omega_1} + \partial$


Comes from family of supercharges
$\lambda \partial_{\varepsilon_3} + \mu(\varepsilon_1 \partial_{w_1} + \varepsilon_2 \partial_{w_2})$
In terms of type II B
on $\mathbb{C}^2_{w_i} \times \mathbb{C}^3_{z_j}$
it is $\lambda z_3 + \mu(\partial_{z_1}\partial_{w_1} + \partial_{z_2}\partial_{w_2})$


Example
Kapustin-Witten twist
Comes from family of supercharges
In terms of type II B
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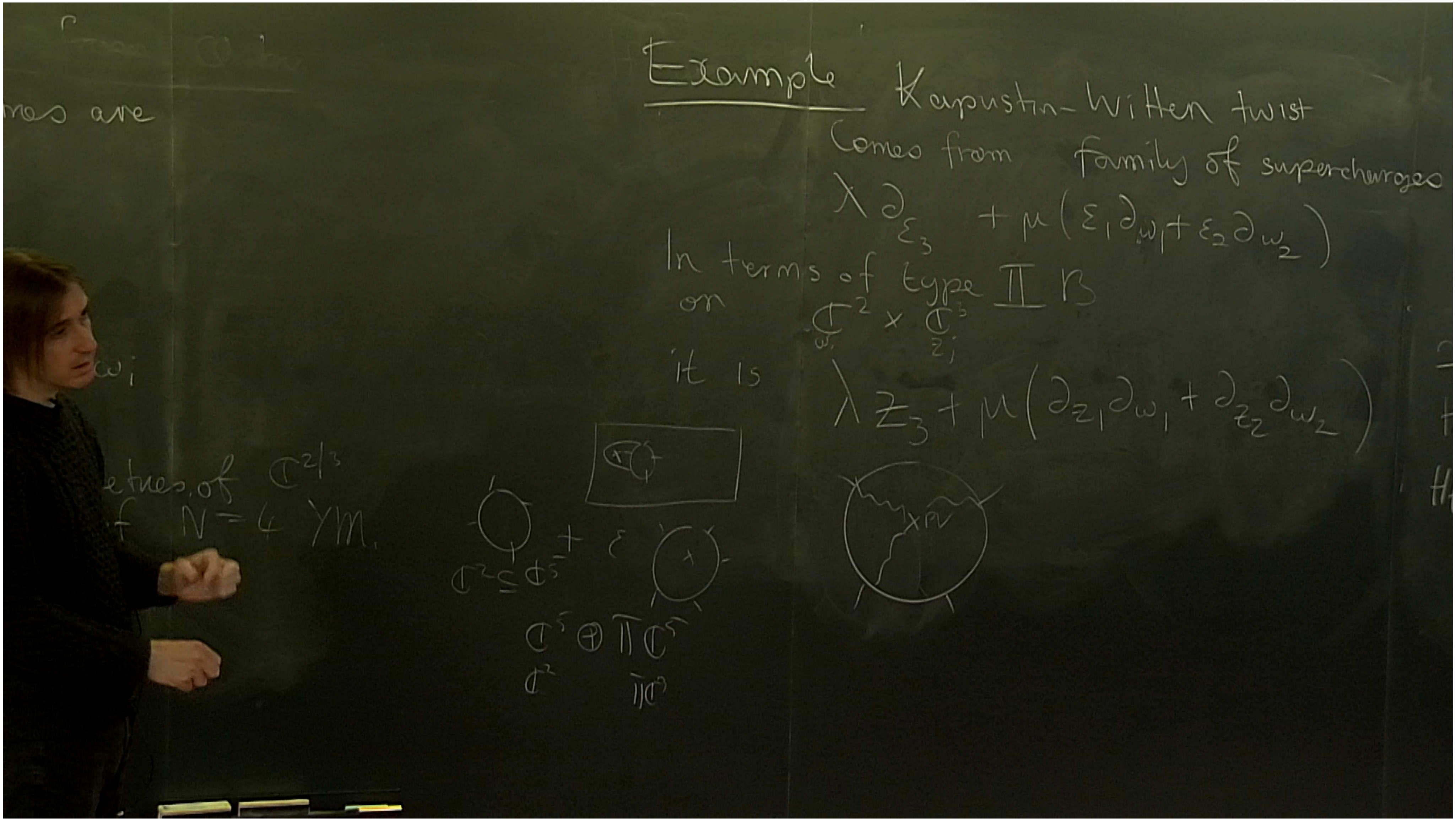
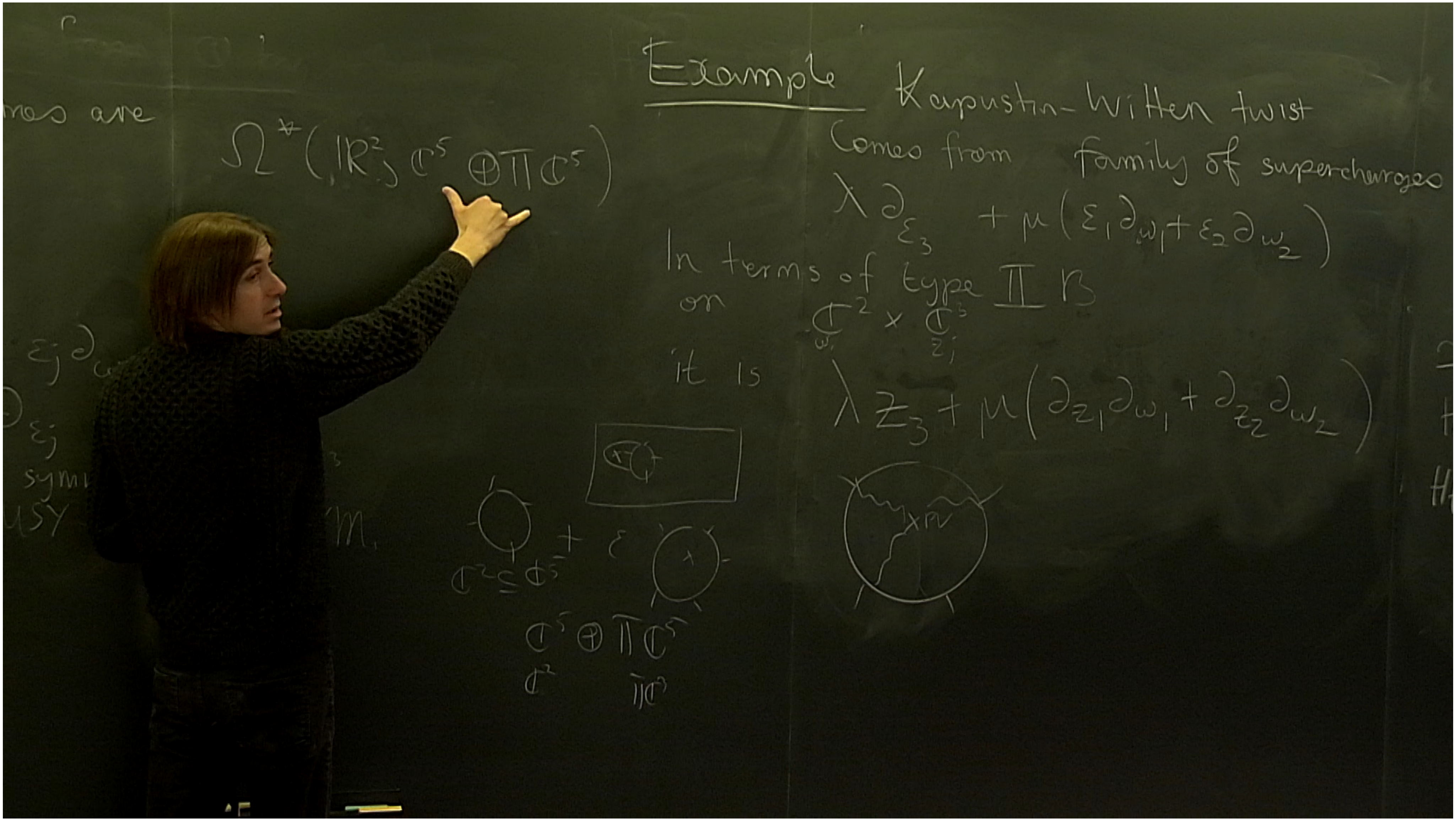
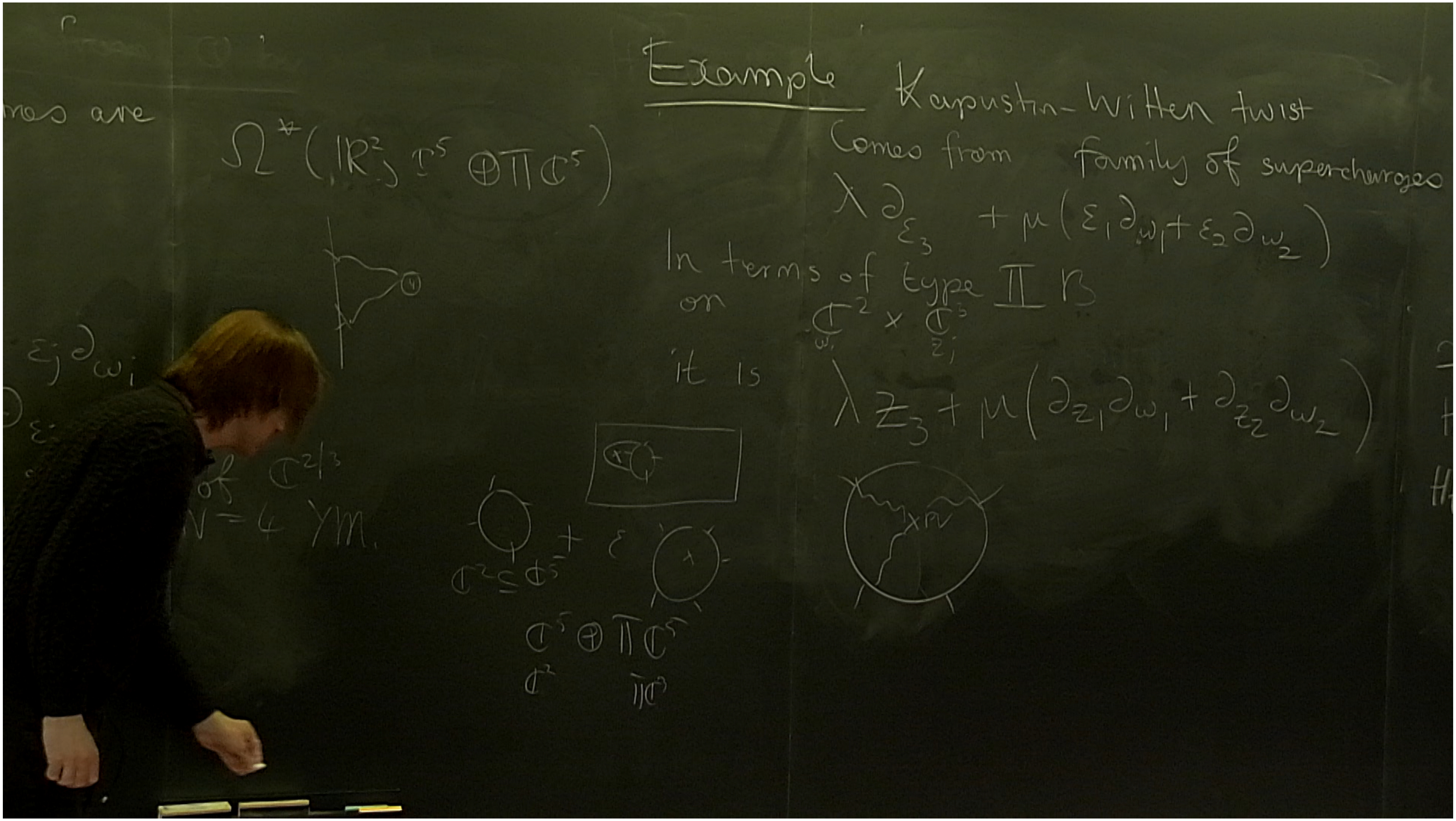


Example Kapustin-Witten twist
Comes from family of supercharges
λ ∂_{ε_3} + μ(ε_1 ∂_{ω_1} + ε_2 ∂_{ω_2})
In terms of type II B
on ℂ² × ℂ³
it is
λ z_3 + μ(∂_{z_1} ∂_{ω_1} + ∂_{z_2} ∂_{ω_2})
N = 4 YM.


Example Kapustin-Witten twist
Comes from family of supercharges
In terms of type IIB
it is


Example
Kapustin-Witten twist
Comes from family of supercharges
In terms of type IIB
it is
N = 4 YM.