Title: Vacua and Singular Supports

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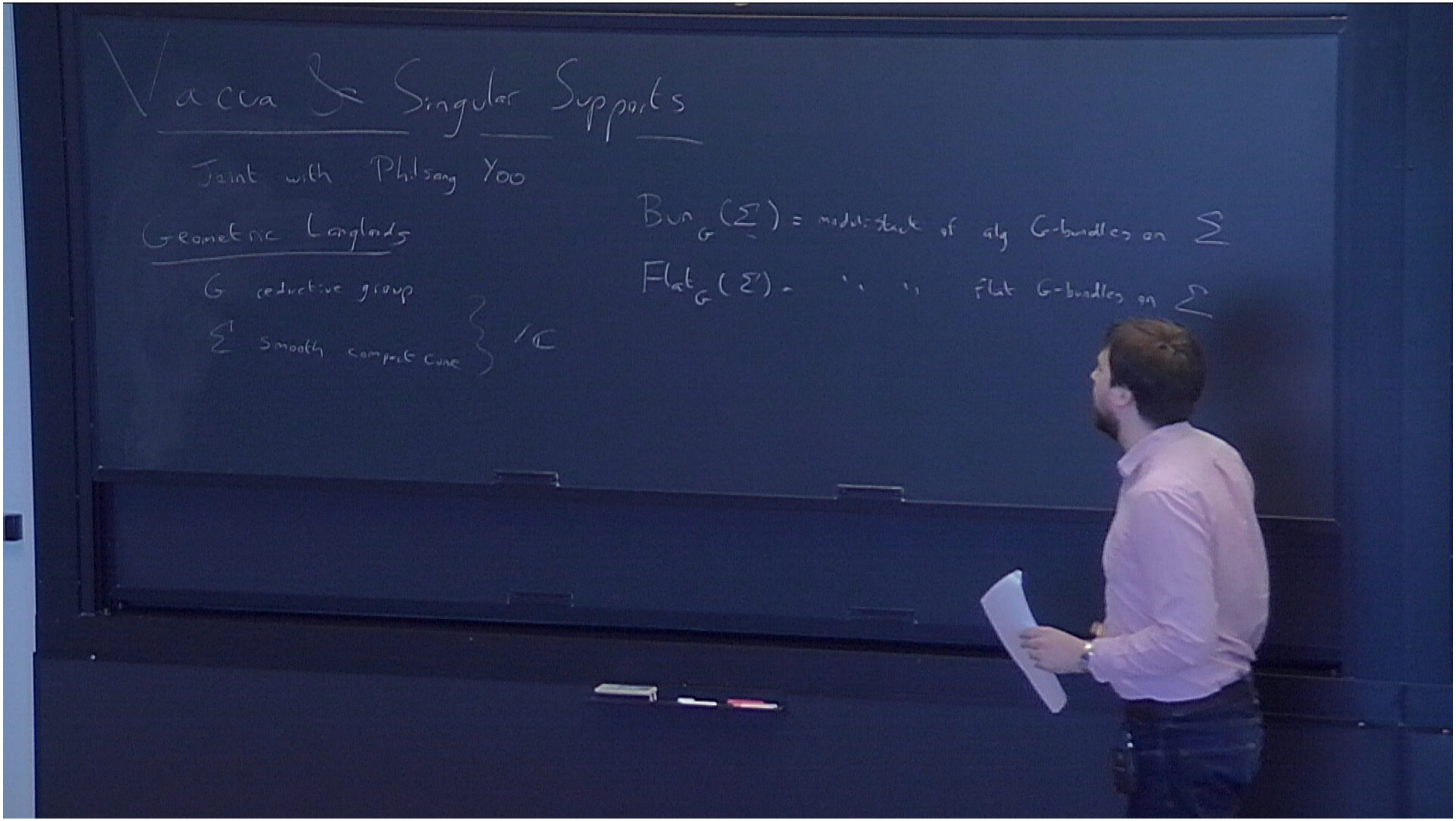
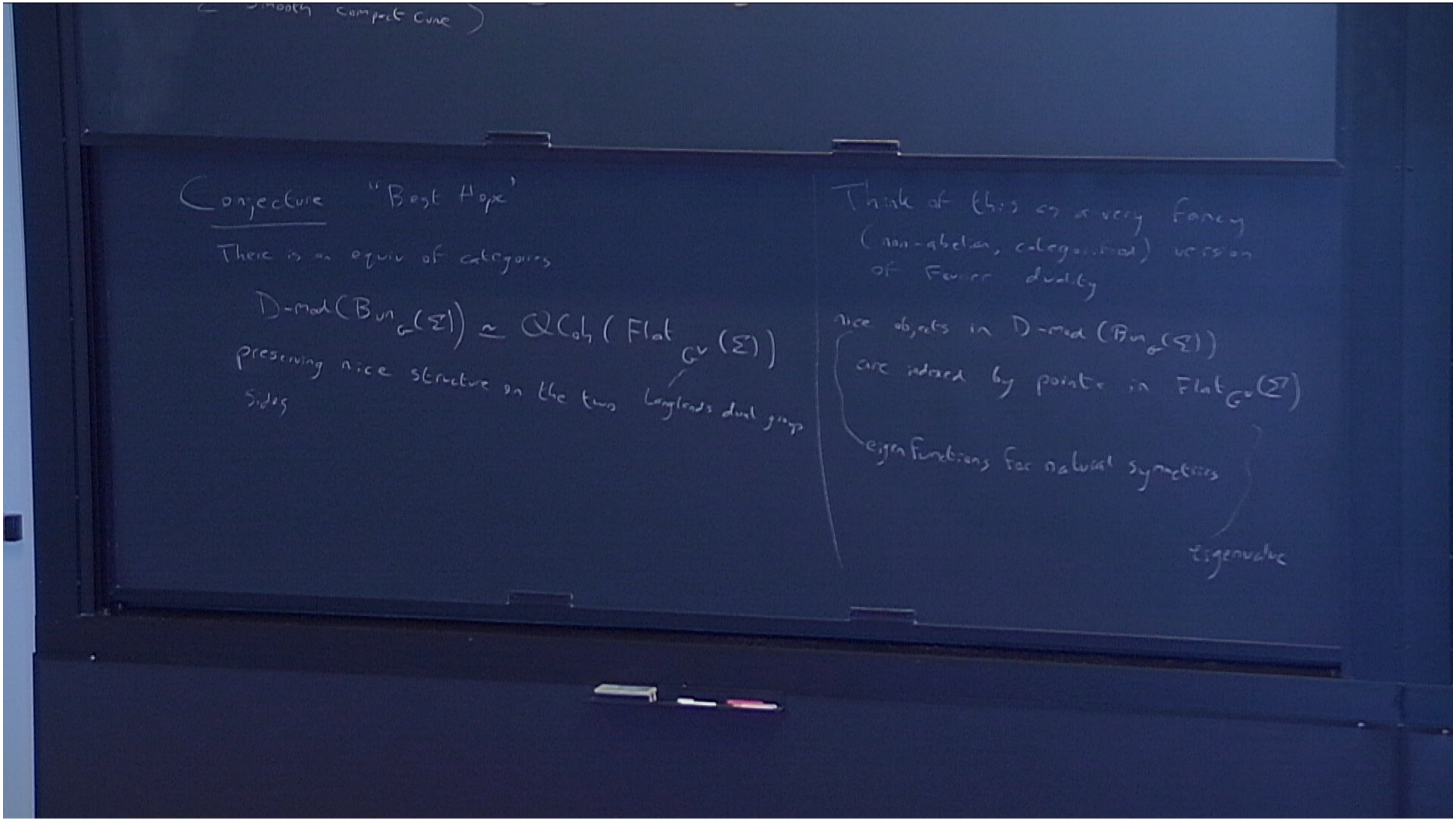
Abstract: <p>The notion of singular support for coherent sheaves was introduced by Arinkin and Gaitsgory in order to carefully state the geometric Langlands conjecture. This is a conjectural equivalence of categories of sheaves on certain moduli spaces: in order to make the conjecture reasonable one needs to restrict to sheaves which satisfy a certain "singular support condition". In this talk I'll explain how to think about this singular support condition from the point of view of boundary conditions in twisted N=4 gauge theory. Specifically, Arinkin and Gaitsgory's singular support condition arises by considering only those boundary conditions which are compatible with a natural choice of vacuum state. By allowing this vacuum state to move away from this natural choice we see aspects of a rich additional structure for the geometric Langlands correspondence. This work is joint with Philsang Yoo.</p>


Vacua & Singular Supports
Joint with Philsang Yoo
Geometric Langlands
G reductive group
Σ smooth compact curve /ℂ
Bun_G(Σ) = moduli stack of alg G-bundles on Σ
Flat_G(Σ) = " " flat G-bundles on Σ


Conjecture "Best Hope"
There is an equiv of categories
D-mod(Bun_G(Σ)) ≃ QCoh(Flat_G∨(Σ))
preserving nice structure on the two sides
Langlands dual group
Think of this as a very fancy
(non-abelian, categorified) version
of Fourier duality
nice objects in D-mod(Bun_G(Σ))
are indexed by points in Flat_G∨(Σ)
eigenfunctions for natural symmetries
eigenvalue

- G abelian this is literally a Fourier-Mukai transform (Laumon, Rothstein)
- G non-abelian conjecture is false

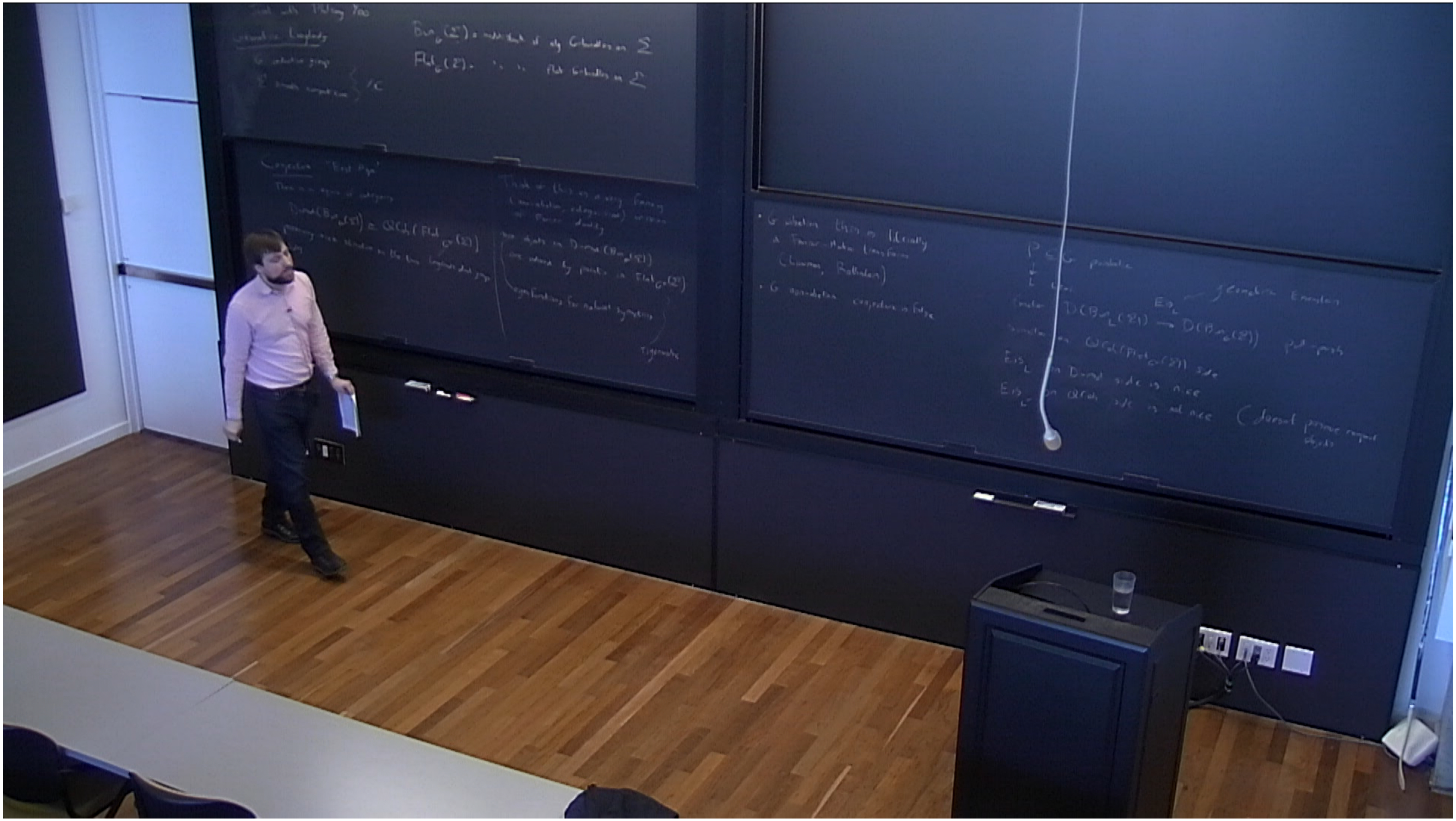
$P \subseteq G$ parabolic

$\downarrow$

L Levi

Eis_L ~ geometric Eisenstein functor $D(\mathrm{Bun}_L(\Sigma)) \to D(\mathrm{Bun}_G(\Sigma))$ pull-push

similar on $\mathrm{QCoh}(\mathrm{Flat}_{G^\vee}(\Sigma))$ side



Idea: ~~QCoh~~ IndCoh has nice 6-functors formalism

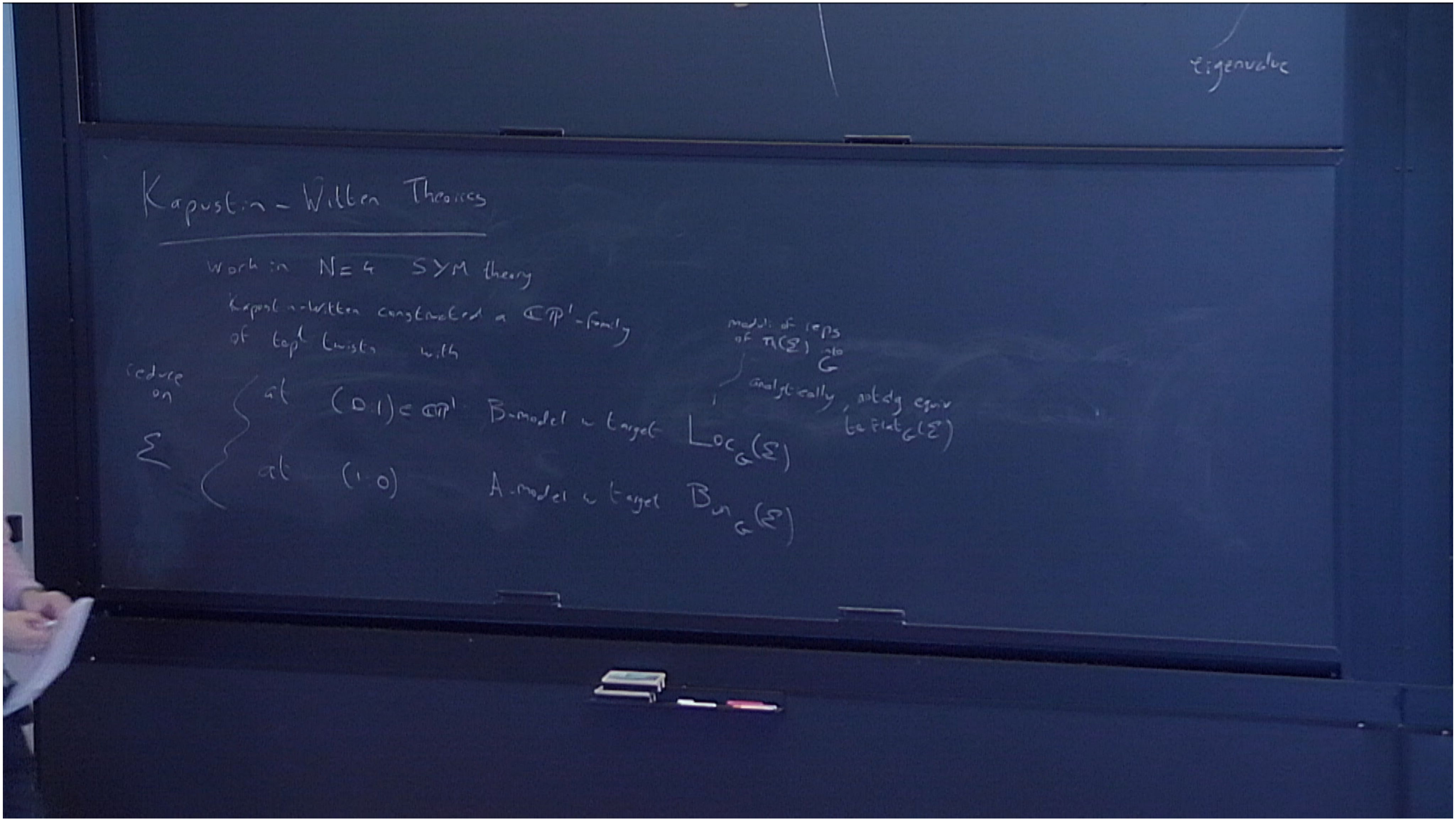
Example of $\Sigma = \mathbb{P}^1$ (V. Lafforgue) tells us $\mathrm{IndCoh}(\mathrm{Flat}_{G^\vee}(\Sigma))$ too big

Fix this by enlarging QCoh in the minimal way making all Eis_P preserve compact objects

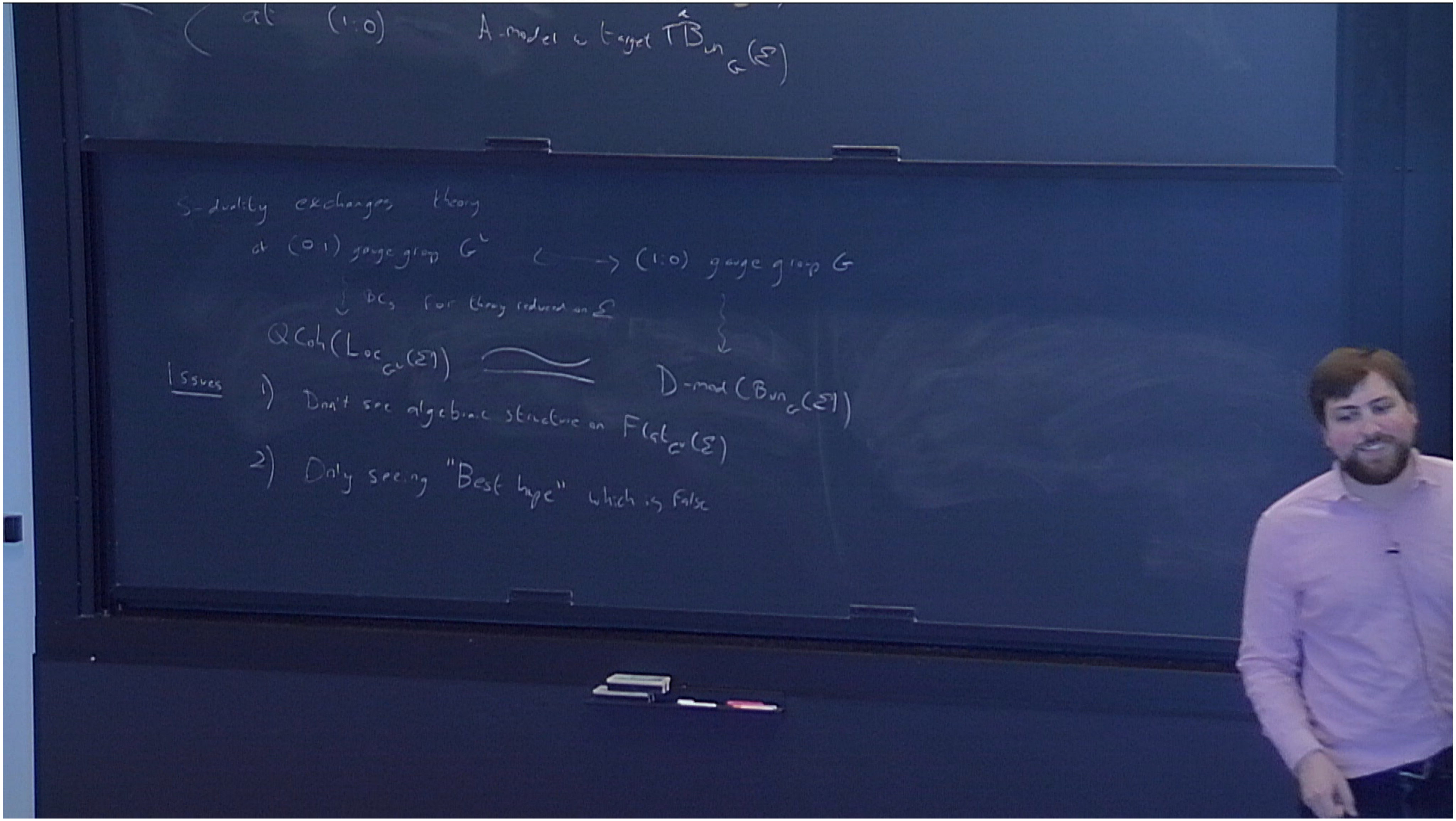
Result

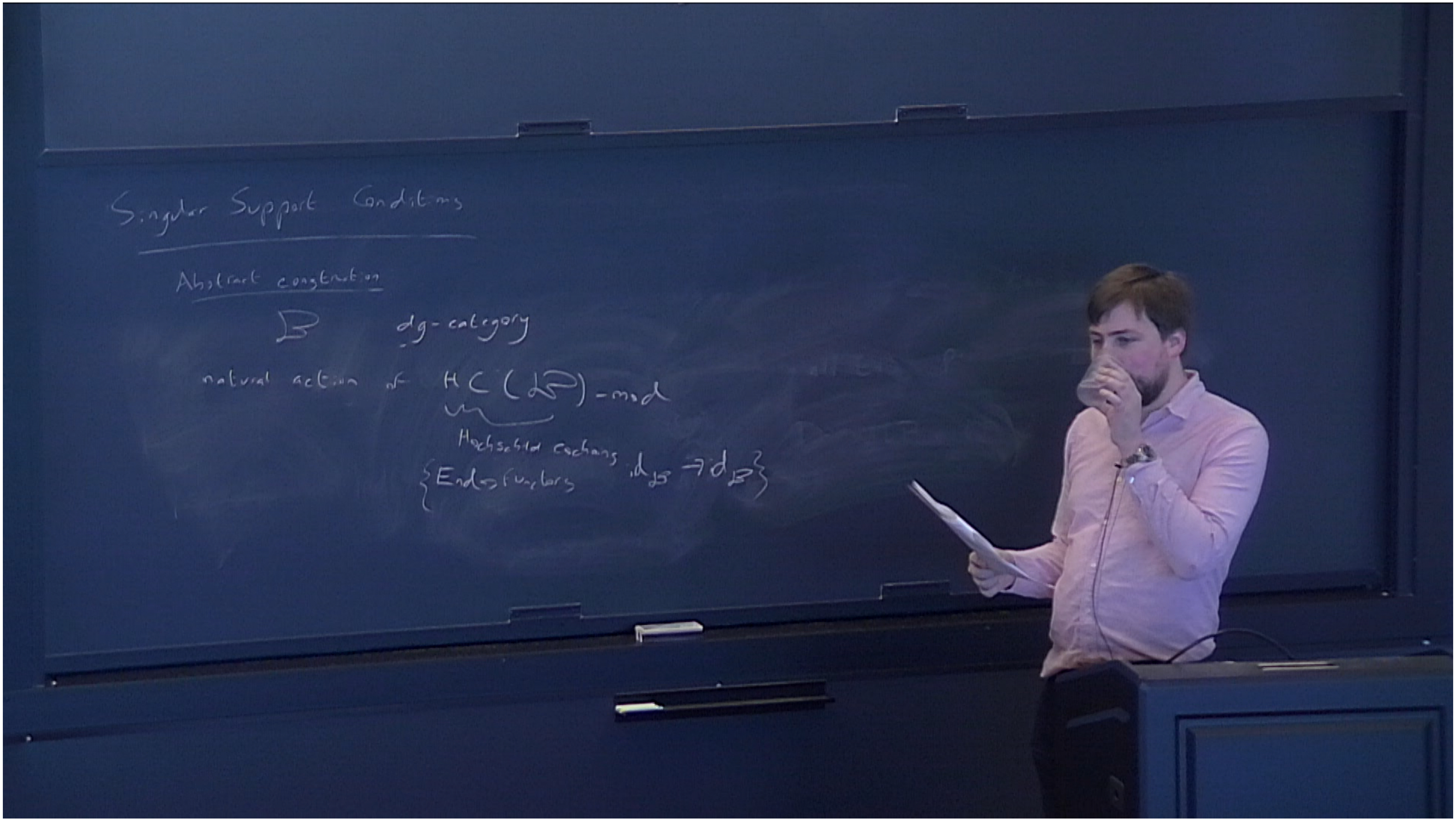
~~IndCoh~~ $\mathrm{IndCoh}_{\mathcal{N}_{G^\vee}}(\mathrm{Flat}_{G^\vee}(\Sigma)) \overset{\text{Conj}}{\underset{\text{Arinkin-Gaitsgory}}{\simeq}} \text{D-mod}(\mathrm{Bun}_G(\Sigma))$

nilpotent singular support



eigenvalue
Kapustin – Witten Theories
Work in N=4 SYM theory
Kapustin-Witten constructed a CP'-family
of top' twists with
reduce on Σ
at (0:1) ∈ CP' B-model w target Loc_G(Σ)
at (1:0) A-model w target Bun_G(Σ)
moduli of reps of π₁(Σ) into G
analytically, not alg equiv to Flat_G(Σ)


at (1:0) A-model w target $T^*\mathrm{Bun}_G(\Sigma)$
S-duality exchanges theory
at (0:1) gauge group G^L $\longleftrightarrow$ (1:0) gauge group G
BCs for theory reduced on Σ
$QCoh(Loc_{G^L}(\Sigma)) \simeq D\text{-mod}(Bun_G(\Sigma))$
Issues
1) Don't see algebraic structure on $Flat_{G^L}(\Sigma)$
2) Only seeing "Best hope" which is False


Singular Support Conditions
Abstract construction
B dg-category
natural action of HC(B)-mod
Hochschild cochains
{Endofunctors id_B → id_B}

$\mathcal{B} = \mathrm{Ind}\,\mathrm{Coh}(X)$ nice enough derived stack

module for $HC^{\bullet}(\mathrm{Ind}\,\mathrm{Coh}(X)) =: HC^{\bullet}(X)$

Singular-support is support with respect to this action

There's a map $\mathcal{O}(\mathrm{Sing}\,X) \longrightarrow HC^{\bullet}(X)$ (X affine derived scheme)

where $\mathrm{Sing}\,X = \left(T^*[-1]X\right)^{cl}$ classical part at $(1:0)$ A-model w target $T^*\widehat{\mathrm{Bun}}_G(\Sigma)$

Definition $Y \subseteq \mathrm{Sing}\, X$ closed

$$\mathrm{IndCoh}_Y(X) = \mathrm{IndCoh}(X) \otimes_{\mathrm{QCoh}(\mathrm{Sing}\, X)} \mathrm{QCoh}(\mathrm{Sing}\, X)_Y$$

sing supp contained in Y

valid for X affine derived scheme
in general we use a smooth
cover by affines

$\mathcal{O}(\mathrm{Sing}\, X) \approx$ 2d local operators in B-model

What is $\mathcal{N}_G$ $\leftarrow$ global nilpotent cone

$$\mathcal{N}_G$$
$$\cap\!|$$
$$\mathrm{Sing}(\mathrm{Flat}_G(\Sigma))$$
$$\| \text{ closed points}$$
$$\{ (P, \nabla, \phi) : (P, \nabla) \text{ Flat bundle}, \ \phi \in H^0_\nabla(\Sigma; \underline{\mathfrak{g}}^*_P) \text{ Flat section} \}$$
$$\cup\!|$$
$$\{ (P, \nabla, \phi) : ev_x(\phi) \text{ is nilpotent in } \mathfrak{g}^*/G, \ x \in \Sigma \} = \mathcal{N}_G$$

Vacua in a TQFT are just points in $\mathrm{Spec}\ \mathrm{Obs}(D^\wedge) \ni v$

Local operators act on cat $\mathfrak{B}$ of BCs (say, along codim 2 manifold)

algebra $\mathrm{End}_{\mathfrak{B}}(\mathfrak{Y})$ models observables in the bulk-boundary system with BC $\mathfrak{Y}$

$\mathfrak{Y} \in \mathrm{ob}(\mathfrak{B})$

the localization of this algebra at v contains those observables which can be

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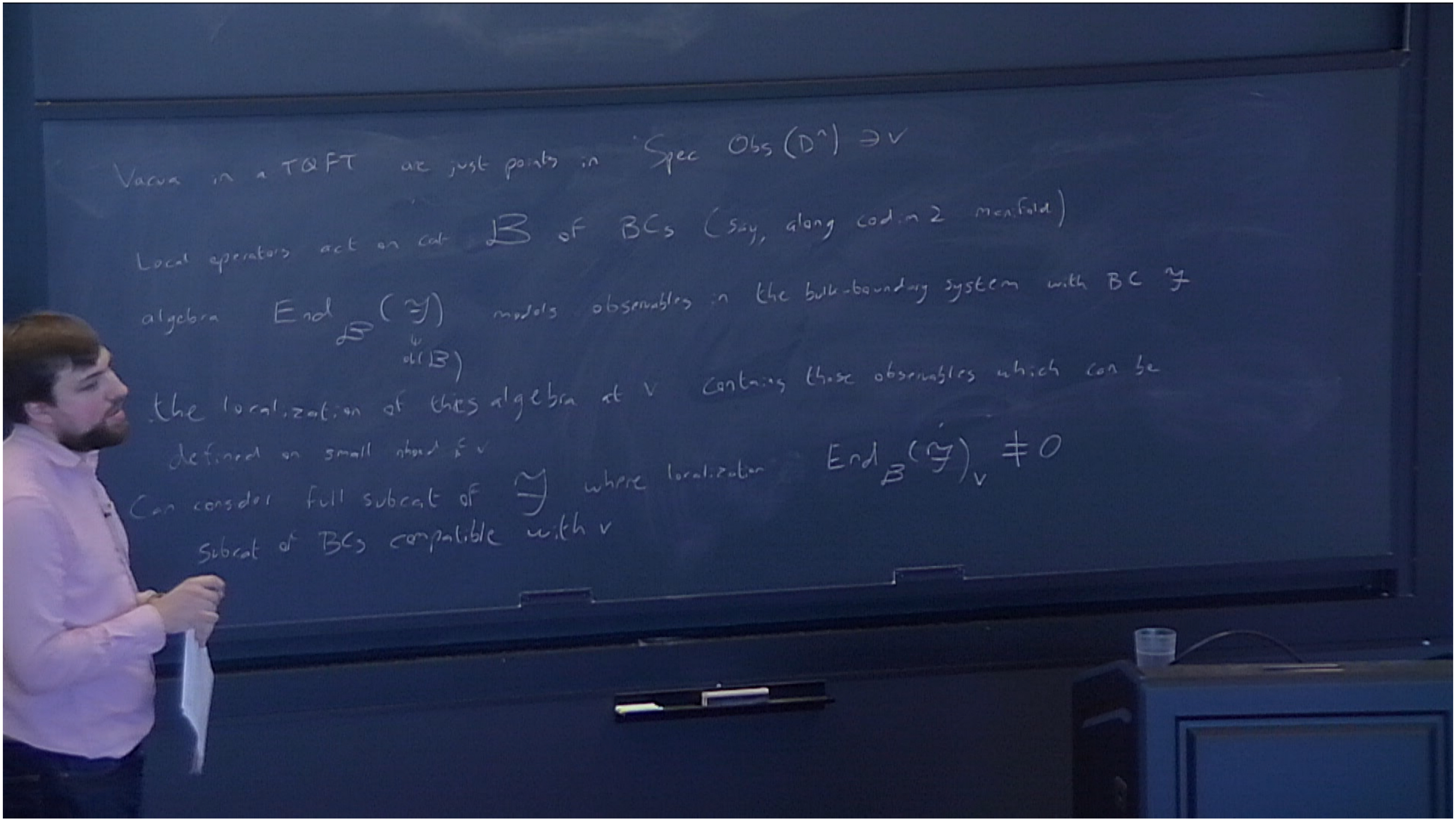

Vacua in a TQFT are just points in Spec Obs(D^n) ∋ v
Local operators act on cat B of BCs (say, along codim 2 manifold)
algebra End_B(Y) models observables in the bulk-boundary system with BC Y
∈ ob(B)
the localization of this algebra at v contains those observables which can be
defined on small nbhd of v
Can consider full subcat of Y where localization End_B(Y)_v ≠ 0
subcat of BCs compatible with v

Theorem

In B-twisted $N=4$ theory

- naive cat of BCs $\mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))$
- algebra of operators $\simeq \mathcal{O}(\mathfrak{h}^*[2]/W)$ acts on $\mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))$ in a natural way

(Cat of BCs compatible with vacuum $0 \in \mathfrak{h}^*[2]/W$) $\simeq \mathrm{IndCoh}_{\mathcal{N}_G}(\mathrm{Flat}_G(\Sigma))$

$$\begin{array}{ccc} \mathcal{N}_G & \longrightarrow & \mathrm{Arth}_G(\Sigma) \\ \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \mathfrak{h}^*/W \end{array}$$

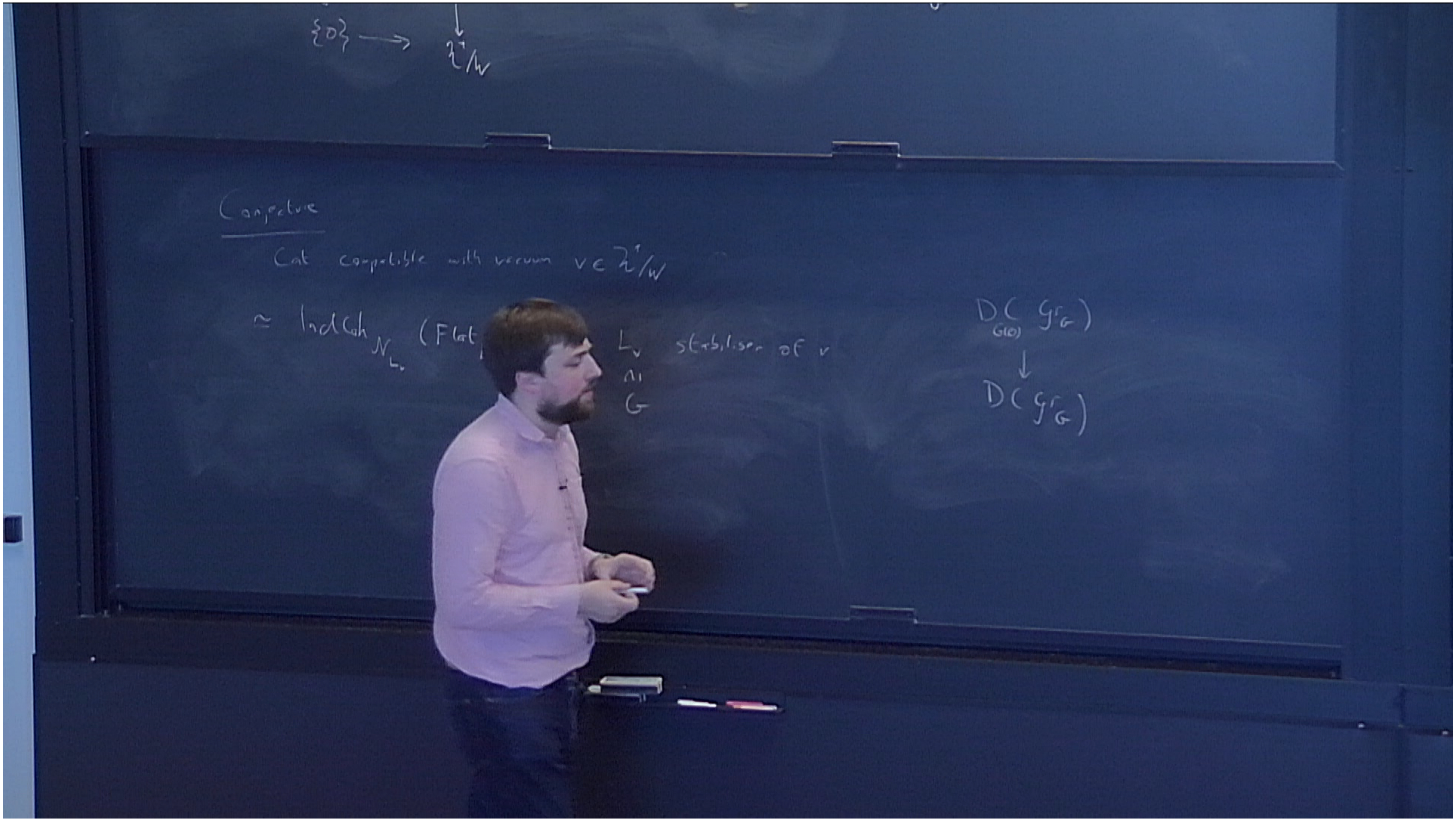
$$\{0\} \longrightarrow \mathfrak{t}^*/W$$

<u>Conjecture</u>

Cat compatible with vacuum $v \in \mathfrak{t}^*/W$

$$\simeq \mathrm{IndCoh}_{\mathcal{N}_{L_v}}\left(\mathrm{Flat}_{L_v}(\Sigma)\right) \qquad L_v \text{ stabiliser of } v$$

$$L_v \subset G$$


{0} → t*/W
Conjecture
Cat compatible with vacuum v ∈ t*/W
≃ IndCoh_{N_{L_v}} (Flat
L_v stabiliser of v
∩
G
D_{G(O)}(Gr_G)
D(Gr_G)