Title: Positive Representations of Split Real Quantum Groups

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Abstract: <p>The notion of Positive Representations is a new research program devoted to the representation theory of split real quantum groups, initiated in a joint work with Igor Frenkel. It is a generalization of the special class of representations considered by J. Teschner for Uq(sl(2,R)) in Liouville theory, where it exhibits a strong parallel to the finite-dimensional representation theory of compact quantum groups, but at the same time also serves some new properties that are not available in the compact case. In this talk, I will survey the recent developments and describe some of its relations to other areas of mathematics. </p>

Motivation

[Drinfeld, Jimbo (1985)]
Simple Lie algebra $\mathfrak{g} \rightsquigarrow$ quantum group $\mathcal{U}_q(\mathfrak{g})$

- Search for solutions to Yang-Baxter equation

Finite dimensional representation theory $\longrightarrow$ many applications!

- Knot and 3-manifolds invariant:
  Reshetikhin-Turaev's TQFT, braided tensor category
- Categorification: Khovanov homology, Nakajima's quiver variety...
- Kazhdan-Lusztig theory: $Rep(\widehat{\mathfrak{g}}) \longleftrightarrow Rep(\mathcal{U}_q(\mathfrak{g}))$
- Lusztig's Canonical basis / Kashiwara's Crystal basis
- Many more...

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Motivation

Classical Lie theory: two important real forms: $\mathfrak{g}_c$ and $\mathfrak{g}_{\mathbb{R}}$

$\mathfrak{g}_c \longleftrightarrow$ compact groups (e.g. $SU(n), SO(2n)$)

- Finite dimensional representation theory is well-behaved
  - Highest weight representations $V_\lambda : \lambda \in P^+$
  - Closure under tensor product:

$$V_\lambda \otimes V_\mu \simeq \bigoplus_\nu c_{\lambda\mu}^\nu V_\nu$$

  - Peter-Weyl's Theorem:

$$\mathbb{C}[G] \simeq \bigoplus_\lambda V_\lambda \otimes V_\lambda^*$$

- Generalized nicely to corresponding quantum groups $\mathcal{U}_q(\mathfrak{g}_c)$
  - Universal R matrix: Braiding: $V_\lambda \otimes V_\mu \simeq V_\mu \otimes V_\lambda$
    $\longrightarrow$ Braided Tensor Category

Motivation

$\mathfrak{g}_{\mathbb{R}} \longleftrightarrow$ split real groups (e.g. $SL(n,\mathbb{R}), SO(n,n)$)

- Much more complicated [Harish-Chandra]

  Ex: G=$SL(2,\mathbb{R})$:
  - Principal series P_λ^ϵ, discrete series, complementary series
  - Tensor product:

$$P_\lambda \otimes P_\mu \simeq \int_{\mathbb{R}+}^{\oplus} P_\nu d\mu(\nu) \bigoplus \text{(discrete part)}$$

  - Peter-Weyl's Theorem:

$$L^2(SL(2,\mathbb{R})) \simeq \int_{\mathbb{R}+}^{\oplus} P_\lambda \otimes P_\lambda^* d\mu(\lambda) \bigoplus \text{(discrete part)}$$

- Quantum group level - involving self-adjoint operators
- Largely open due to analytic difficulties and spectral properties (non-compactness, unbounded operators etc.)
- Foundation work from Physics by Faddeev, Kashaev, Volkov, ...

Representation theory of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$

Simplest case: $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$

Some special class of representations $\mathcal{P}_\lambda$ = GOOD!

Studied by Teschner *et al.* from quantum Liouville theory:

- Parameterized by $\lambda \in \mathbb{R}_{\geq 0}$
- Generators = positive self-adjoint operators on $L^2(\mathbb{R})$
- Integrable representations in the sense of [Schmüdgen (1999)].
- No classical limit $q \longrightarrow 1$

Representation theory of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$

Parallel to representation theory of compact case!!

- (Continuous) Braided tensor category structure [Ponsot-Teschner, Bytsko-Teschner]:

$$\mathcal{P}_\alpha \otimes \mathcal{P}_\beta \simeq \int_{\mathbb{R}_+}^{\oplus} \mathcal{P}_\gamma d\mu(\gamma)$$

$$\mathcal{P}_\alpha \otimes \mathcal{P}_\beta \simeq \mathcal{P}_\beta \otimes \mathcal{P}_\alpha$$

- Harmonic analysis gives Peter-Weyl type theorem [Ip (2013)]:

$$L^2(SL_q^+(2,\mathbb{R})) \simeq \int_{\mathbb{R}_+}^{\oplus} \mathcal{P}_\gamma \otimes \mathcal{P}_\gamma^* d\mu(\gamma)$$

Positive Representations of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$

New research program started in [Frenkel-Ip]

- Generalization of Teschner's representations to higher rank
- Generators = positive self-adjoint operators on $L^2(\mathbb{R}^N)$

Many new phenomena not present in compact case:

- Faddeev's modular double and quantum dilogarithm
- Langlands duality as simple analytic relations
- Theory of multiplier Hopf algebra and locally compact quantum group from C^*-algebra
- Discriminant variety for Positive Casimirs
- Connection to quantum Teichmüller theory and cluster algebra

We expect many applications of finite dimensional representation theory of $\mathcal{U}_q(\mathfrak{g}_c)$ can be generalized to $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$.

Definition of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$

$q = e^{\pi i b^2}$ not root of unity, $0 < b^2 < 1$.

Definition [*Drinfeld-Jimbo*]

$\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ = *Hopf-* algebra* $\langle E, F, K^{\pm 1}\rangle$ *such that*

$$KE = q^2 EK, \qquad KF = q^{-2} FK, \qquad [E,F] = \frac{K - K^{-1}}{q - q^{-1}}$$

Coproduct:

$$\Delta(K) = K \otimes K, \qquad \Delta(E) = E \otimes K + 1 \otimes E, \qquad \Delta(F) = F \otimes 1 + K^{-1} \otimes F$$

Real form:

$$K^* = K, \qquad E^* = E, \qquad F^* = F$$

(Also counit ϵ, antipode S)

For higher rank, also Serre relations, $K_i E_j = q_i^{a_{ij}} E_j K_i$ etc.

Integrable Representations of Quantum Plane

For $q = e^{\pi i b^2}$, there is a canonical representation for the relation

$$UV = q^2 VU$$

where U, V are positive self adjoint operators:

$$U = e^{2\pi bx}, \qquad V = e^{2\pi bp} \qquad (i.e. (Vf)(x) = f(x - ib)$$

unbounded operators on $L^2(\mathbb{R})$, where $p = \frac{1}{2\pi i}\frac{d}{dx}$

- Acting on common dense core: $\mathcal{W} := \{e^{-\alpha x^2 + \beta x} P(x)\}_{Re(\alpha)>0, \beta\in\mathbb{C}}$
- Integrable representation [Schmüdgen]:

$$U^{is}V^{it} = q^{-2st} V^{it} U^{is}, \qquad \forall s, t \in \mathbb{R}$$

as unitary operators.

Faddeev's Modular Double

Recall $UV = q^2VU$,

$$q = e^{\pi i b^2}, \qquad U = e^{2\pi b x}, \qquad V = e^{2\pi b p}$$

Define

$$\widetilde{q} := e^{\pi i b^{-2}}, \qquad \widetilde{U} := U^{\frac{1}{b^2}}, \qquad \widetilde{V} := V^{\frac{1}{b^2}}$$

i.e. replacing b by b^{-1}. Then

- $\widetilde{U}\widetilde{V} = \widetilde{q}^2\widetilde{V}\widetilde{U}$
- $\{U, V\}$ commute (weakly) with $\{\widetilde{U}, \widetilde{V}\}$

Together $\langle U, V, \widetilde{U}, \widetilde{V}\rangle$ generates the Modular Double.

- Faddeev's Idea: should extend to quantum group level.
- Liouville theory: $q = e^{\pi i b^2} \longleftrightarrow c = 1 + 6(b + b^{-1})^2$

Ponsot-Teschner's representation

- $b \longleftrightarrow b^{-1}$ gives $\{\widetilde{E},\widetilde{F},\widetilde{K}\}$ a representation of $\mathcal{U}_{\widetilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$.
- $\{E,F,K\}$ commute (weakly) with $\{\widetilde{E},\widetilde{F},\widetilde{K}\}$
- Define

$$\mathbf{e} := \left(\frac{i}{q-q^{-1}}\right)^{-1} E, \qquad \mathbf{f} := \left(\frac{i}{q-q^{-1}}\right)^{-1} F,$$

we have

$$\mathbf{e}^{\frac{1}{b^2}} = \widetilde{\mathbf{e}}, \qquad \mathbf{f}^{\frac{1}{b^2}} = \widetilde{\mathbf{f}}, \qquad K^{\frac{1}{b^2}} = \widetilde{K},$$

called the "transcendental relations".

Hence $\mathcal{P}_\lambda$ is a representation of the Modular Double

$$\mathcal{U}_{q\widetilde{q}}(\mathfrak{sl}(2,\mathbb{R})) := \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R})) \otimes \mathcal{U}_{\widetilde{q}}(\mathfrak{sl}(2,\mathbb{R})).$$

Properties

Existence of universal R operator $(K =: q^H)$:

Theorem [*Bytsko-Teschner (2003)*]

$$R = q^{\frac{H\otimes H}{4}} g_b(\mathbf{e}\otimes\mathbf{f}) q^{\frac{H\otimes H}{4}}$$

$$R\Delta = \Delta^{op}R$$

- *R is a unitary operator on $L^2(\mathbb{R})\otimes L^2(\mathbb{R})$.*
- *R is invariant under $b \longleftrightarrow b^{-1}$.*

Here $g_b(x)$ is called the quantum dilogarithm.

- non-compact version of $\mathrm{Exp}_q(x)$ and $\Gamma_q(x)$.

Positive Representations of $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$

New research program started in [Frenkel-Ip]

- Generalization of Teschner's representations to higher rank

Positive representations

- ="Quantization of minimal principal series representations"

Construction:

- Induced rep. of $\mathcal{U}(\mathfrak{g})$ on $L^2(U^+)$ by differential operators
- Lusztig's total positive space $L^2(U^+_{>0}) \simeq L^2(\mathbb{R}_{>0}^{N=\dim U^+})$
- Mellin transformation: $L^2(\mathbb{R}^N_{>0}) \simeq L^2(\mathbb{R}^N)$
- $\mathcal{U}(\mathfrak{g})$ differential operator $\rightsquigarrow$ finite difference operator
- Quantization+"Wick rotation"
  $\Longrightarrow$positive operators $\mathbf{e}_i, \mathbf{f}_i, K_i \in \mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$

Positive Representations of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$

Theorem [*Ip (2012)*]

There exists a family of irreducible representations $\mathcal{P}_\lambda$ of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$:

- $\lambda \in \mathbb{R}_{\geq 0}P_+ \subset \mathfrak{h}_\mathbb{R}^*$ *of classical* $\mathfrak{g}_\mathbb{R}$ ($\Longleftrightarrow \lambda \in \mathbb{R}_{\geq 0}^{n=rank\mathfrak{g}}$).
- *Positivity:* $\{E_i, F_i, K_i\}$ *are represented by positive, essentially self-adjoint (unbounded) operators on* $L^2(\mathbb{R}^{N=\dim U^+})$
- *Transcendental relations: Define*

$$\widetilde{\mathbf{e}} := \mathbf{e}^{\frac{1}{b^2}}, \qquad \widetilde{\mathbf{f}} := \mathbf{f}^{\frac{1}{b^2}} \qquad \widetilde{K} := K^{\frac{1}{b^2}}$$

  - *simply-laced:* $\{\widetilde{E_i}, \widetilde{F_i}, \widetilde{K_i}\}$ *interchange* $b \longleftrightarrow b^{-1}$
  - *non-simply-laced:* $\{\widetilde{E_i}, \widetilde{F_i}, \widetilde{K_i}\}$ *generates the Langlands dual* $\mathcal{U}_{\widetilde{q}}({}^L\mathfrak{g}_\mathbb{R})$
- $\{E_i, F_i, K_i\}$ *commute weakly with* $\{\widetilde{E_i}, \widetilde{F_i}, \widetilde{K_i}\}$ *up to a sign.*
- *Does not depend on choice of reduced expression of* w_0

Positive Representations of $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$

Theorem [*Ip (2012) cont'd*]

Let

$$U_k = e^{2\pi b x_k}, V_k = e^{2\pi b p_k}$$

generate the quantum planes:

$$U_k V_k = q^2 V_k U_k.$$

Then $\{\mathbf{e}_i, \mathbf{f}_i, K_i\}_{i=1}^n$ = *Laurent polynomials in* $\{U_k, V_k\}_{k=1}^N$.

- *Positive coefficients* $\in \mathbb{Z}_{\geq 0}[q, q^{-1}]$
- K_i = *multiplication operators.*

(Ignoring $*$-structure:)
Extends the Feigin map of $\mathcal{U}_q(\mathfrak{b}) \hookrightarrow \mathbb{C}\langle\mathbb{T}^N\rangle$ to the whole $\mathcal{U}_q(\mathfrak{g})$.

Relations to Other Areas

Multiplier Hopf Algebra

G compact, $\Delta : C[G] \longrightarrow C[G] \otimes C[G]$:

$$\Delta f(g_1, g_2) = f(g_1 g_2)$$

G locally compact: $\Delta : C_0(G) \nrightarrow C_0(G) \otimes C_0(G)$!

Definition

Multiplier algebra $M(\mathcal{A})$ of a C^*-algbera $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is the C^*-algebra

$$M(\mathcal{A}) = \{b \in \mathcal{B}(\mathcal{H}) : b\mathcal{A} \subset \mathcal{A}, \mathcal{A}b \subset \mathcal{A}\}$$

Example

$\mathcal{A} = C_0(G), M(\mathcal{A}) = C_b(G)$, $\Delta : \mathcal{A} \longrightarrow M(\mathcal{A} \otimes \mathcal{A})$

Definition

Multiplier Hopf algbera $\mathcal{A}$: Δ, ϵ, S extends to homomorphisms of $M(\mathcal{A})$

Multiplier Hopf Algebra

- Idea: work with bounded operators

U, V	(positive unbounded operators)
$\rightsquigarrow U^{is}, V^{it}$	(unitary operators)
$\rightsquigarrow \iint_{\mathbb{R}\times\mathbb{R}} f(s,t)U^{is}V^{it}dsdt$	(bounded operators generated by U, V)

- Non-simple root:

$$T_i(\mathbf{e}_j) := \mathbf{e}_{ij} := \frac{q^{\frac{1}{2}}\mathbf{e}_j\mathbf{e}_i - q^{-\frac{1}{2}}\mathbf{e}_i\mathbf{e}_j}{q - q^{-1}}$$

Also positive self-adjoint in $\mathcal{P}_\lambda$!

- Continuous PBW basis:

$$\overrightarrow{\prod_{\alpha\in\Delta_+}} E_\alpha^{k_\alpha} \rightsquigarrow \overrightarrow{\prod_{\alpha\in\Delta_+}} \mathbf{e}_\alpha^{it_\alpha}$$

Drinfeld-Jimbo quantum groups in C^*-algebra!

Techniques from C^*-algebra and Non-Commutative Geometry:

- Multiplier Hopf algebra [van Daele]
- Locally compact quantum groups [Kustermans-Vaes]
- Multiplicative unitary W [Woronowicz]:

$$\Delta(x) = W(x \otimes 1)W^*, \qquad x \in \mathcal{A}$$

- Gelfand-Naimark-Segal (GNS) representations

Theorem [*Ip (2013)*]

Peter-Weyl theorem:

$$L^2(SL_{q\tilde{q}}^+(2,\mathbb{R})) \simeq \int_{\mathbb{R}_+}^{\oplus} \mathcal{P}_\gamma \otimes \mathcal{P}_\gamma^* d\mu(\gamma)$$

as regular $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ *representation.*
(GNS representation + Drinfeld Double + Hopf pairing)

Positive Casimirs

- Center of $\mathcal{U}(\mathfrak{sl}_2)$: $C = FE + \frac{1}{2}(H+1)$
- Center of $\mathcal{U}_q(\mathfrak{sl}_2)$: $C = FE + \left[\frac{1}{2}(H+1)\right]_q^2$

$$= FE + \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2} + const$$

- Rescale by $(\frac{i}{q-q^{-1}})^2$:

$$C = \mathbf{fe} - qK - q^{-1}K^{-1}$$

- $C \curvearrowright \mathcal{P}_\lambda$ as positive scalar:

$$C = e^{2\pi b\lambda} + e^{-2\pi b\lambda} > 0$$

Spectral decomposition of $\Delta(C)$ on $\mathcal{P}_\lambda \otimes \mathcal{P}_\mu$
$\Longrightarrow$ Tensor product decomposition!

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Positive Casimirs

Center of $\mathcal{U}_q(\mathfrak{g})$ generated by $n = \text{rank } \mathfrak{g}$ Casimir elements

- [Zhang-Gould-Bracken]

$$C_k := (Tr_q|_{V_k})(R_{21}R), \qquad k = 1, ..., n$$

- $V_k = k$th fundamental representation of $\mathcal{U}_q(\mathfrak{g})$

Theorem [*Ip (2016)*]

Each C_k acts on $\mathcal{P}_\lambda$ as positive scalar $C_k(\overrightarrow{\lambda})$

$$C_k(\overrightarrow{\lambda}) = \sum_{V_k^\mu \subset V_k} e^{-4\pi b \mu(\overrightarrow{\lambda} \cdot \overrightarrow{W})} \geq \dim V_k$$

where μ= weight of V_k^μ, $\overrightarrow{W}$=fundamental coweight.

Idea: virtual highest weights $\longleftrightarrow$ "analytic continuation" from $\mathcal{U}_q(\mathfrak{g})$!

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Example: Type A_2

Center of $\mathcal{U}_q(\mathfrak{sl}_3)$ generated by C_1 and C_2:

$$C_1 = K_1^{\frac{1}{3}} K_2^{-\frac{1}{3}} (q^{-2} K_1 K_2 + K_1^{-1} K_2 + q^2 K_1^{-1} K_2^{-1} - q^{-1} K_2 \mathbf{e}_1 \mathbf{f}_1 - q K_1^{-1} \mathbf{e}_2 \mathbf{f}_2 + \mathbf{e}_{21} \mathbf{f}_{12})$$

$$C_2 = K_1^{-\frac{1}{3}} K_2^{\frac{1}{3}} (q^{-2} K_1 K_2 + K_1 K_2^{-1} + q^2 K_1^{-1} K_2^{-1} - q K_2^{-1} \mathbf{e}_1 \mathbf{f}_1 - q^{-1} K_1 \mathbf{e}_2 \mathbf{f}_2 + \mathbf{e}_{12} \mathbf{f}_{21})$$

On $\mathcal{P}_\lambda$, each C_k reduced from 35 terms to positive scalar:

$$C_1 \curvearrowright \mathcal{P}_\lambda = e^{\frac{8\pi b\lambda_1}{3} + \frac{4\pi b\lambda_2}{3}} + e^{-\frac{4\pi b\lambda_1}{3} + \frac{4\pi b\lambda_2}{3}} + e^{-\frac{4\pi b\lambda_1}{3} - \frac{8\pi b\lambda_2}{3}} \geq 3$$

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Figure: Type A_2 Region

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Figure: Type A_2 Region

Discriminant Variety

Region in $\mathbb{R}^n$ defined by $\Phi : (\lambda_1, ..., \lambda_n) \mapsto (C_1(\overrightarrow{\lambda}), ..., C_n(\overrightarrow{\lambda}))$

- Singularity at $\Phi(0, ..., 0)$
- Boundary described by discriminant of some polynomial

Example of type A_2:

- Boundary:

$$(XY + 9)^2 = 4(X^3 + Y^3 + 27)$$
$$\iff \quad \mathrm{Disc}_z(z^3 + Xz^2 + Yz + 1) = 0$$

- Type A_2 singularity

Relation to simple (du Val) singularity theory?

Quantum Teichmüller Theory

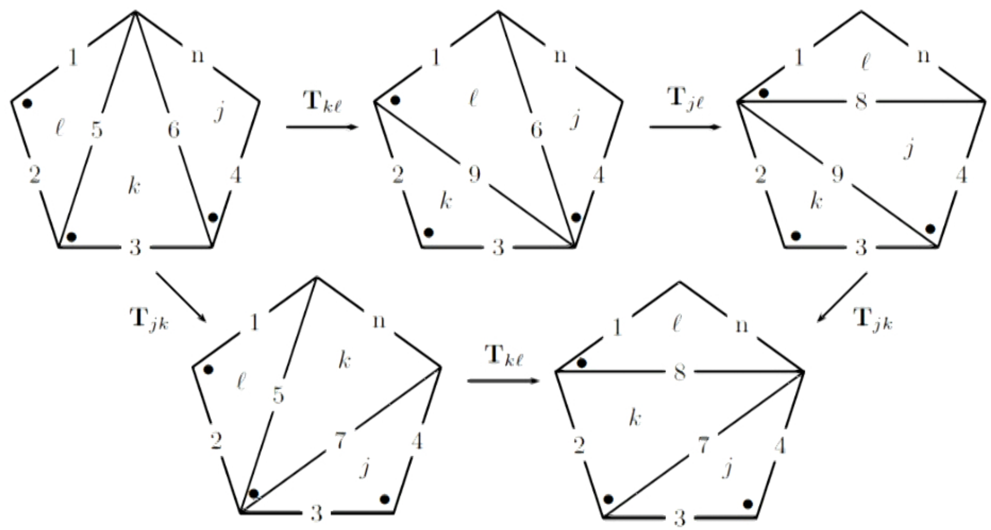
Given S = surface with punctures

- $\mathcal{T}_S$ = Teichmüller space with Weil-Petersson Poisson structure
- Mapping class group $MCG(S) \curvearrowright \mathcal{T}_S$
- Quantization $\mathcal{T}_S^q$ represented on space of states $\mathbf{H} = \otimes_{\tau\in\Delta}\mathcal{H}$
- $g \in MCG(S) \rightsquigarrow$ unitary operator $\rho(g) \curvearrowright \mathbf{H}$
- Goal: Projective unitary representations of $MCG(S)$

Remark

$$\boldsymbol{H} \simeq \textit{Space of conformal blocks in Liouville CFT}$$

Quantum Teichmüller Theory



The pentagon equation for $\mathbf{T}$

Quantum Teichmüller Theory

Two main approaches based on triangulations of S:

- Kashaev's coordinate: (x, p) on each Δ
  $\Longrightarrow$ Kashaev's groupid $\{\mathbf{T}_{jk}, \mathbf{A}_j\}_{j,k\in\Delta}$ associated to change of dotted ideal triangulations Δ
- Thurston's shear coordinate (Penner's lambda length) $\lambda_a, \lambda_b, \lambda_c$ on each edge
  $\Longrightarrow$ Fock-Goncharov cluster varieties $\mathcal{X}_{PSL(2,\mathbb{R}),S}$

Higher Rank Construction

Quantum plane = Borel part of $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$!

- Replace $\mathcal{H}$ by $\mathcal{P}_\lambda^{\mathfrak{b}}$, the restriction of $\mathcal{P}_\lambda$ from $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R})$ to $\mathcal{U}_{q\tilde{q}}(\mathfrak{b}_\mathbb{R})$
- $\mathcal{P}^{\mathfrak{b}} := \mathcal{P}_\lambda^{\mathfrak{b}}$ does not depend on λ

Theorem

$\mathcal{P}^{\mathfrak{b}}$ *is closed under tensor product:*

$$\mathcal{P}^{\mathfrak{b}} \otimes \mathcal{P}^{\mathfrak{b}} \simeq \mathcal{P}^{\mathfrak{b}} \otimes M$$

$\Longrightarrow$ *gives quantum mutation operators* $\boldsymbol{T}$.

(Ongoing) Construct Kashaev's $\mathbf{A}$ operators by dual representations.

$\Longrightarrow$ Candidate for quantum higher Teichmüller theory

$\Longrightarrow$ New projective unitary representations of $MCG(S)$.

(?) Relation to Fock-Goncharov's cluster varieties

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(?) Relation to Fock-Goncharov's cluster varieties

Cluster realization of $\mathcal{U}_q(\mathfrak{g})$

- Teichmüller theory $\longleftrightarrow$ "Moduli space of $PSL(2,\mathbb{R})$-local system"
- "Higher Teichmüller theory" $\longleftrightarrow$ "Moduli space of G-local system"

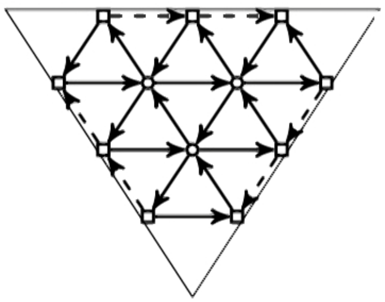
Fock-Goncharov's cluster varieties $\mathcal{X}_{G,S}$ on surface S

- Poisson structure on $\mathcal{X}_{G,S}$: described by quivers on Δ

$\rightsquigarrow$ Quantum torus algebra $\mathcal{X}_{G,S}^q$

Example

$\mathcal{X}^q_{\mathfrak{sl}_4,\Delta}$-*quiver:*



Cluster realization of $\mathcal{U}_q(\mathfrak{g})$

Positive Representations $\Longrightarrow$

Theorem [*Schrader-Shapiro, Ip (2016)*]

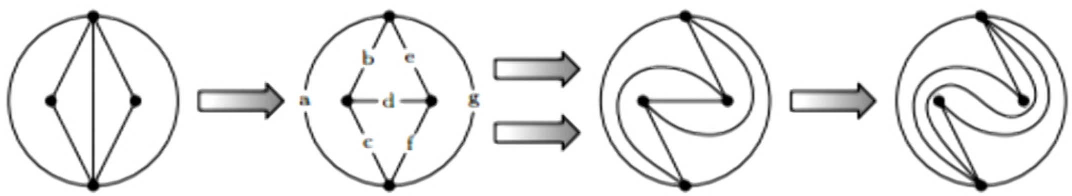
There is embedding for $\mathfrak{g}$ of any simple type:

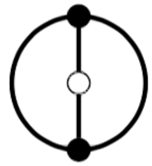
$$\mathcal{U}_q(\mathfrak{g}) \hookrightarrow \mathcal{X}^q_{G,S}/\sim$$

$S =$ *disk with 1 puncture and 2 marked points*

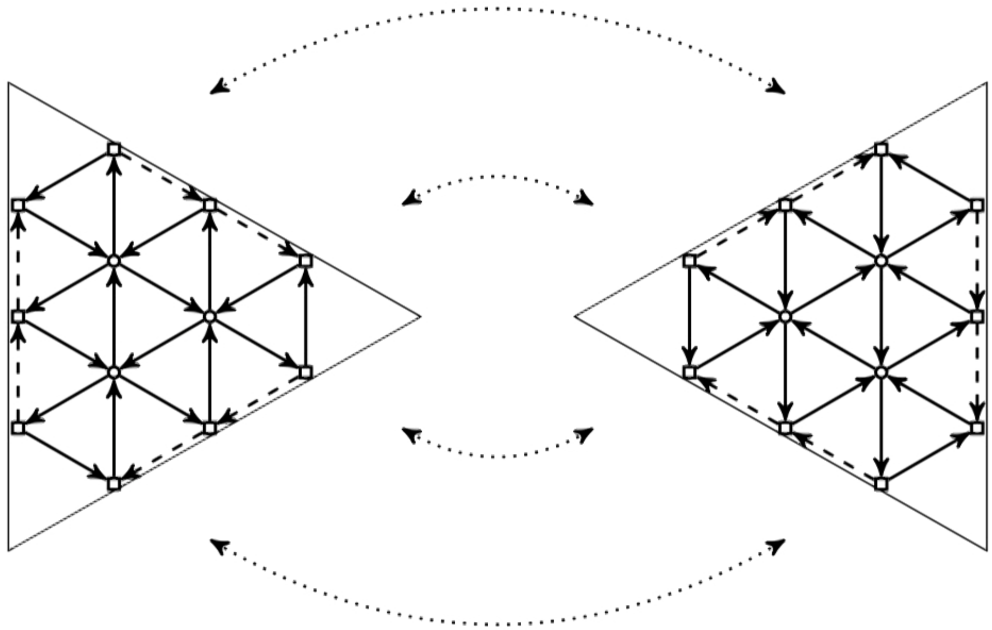
Theorem [*Schrader-Shapiro, Ip (2016)*]

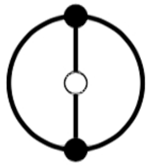
Universal R matrix $\longleftrightarrow$ *Quiver mutations giving half-Dehn twist*



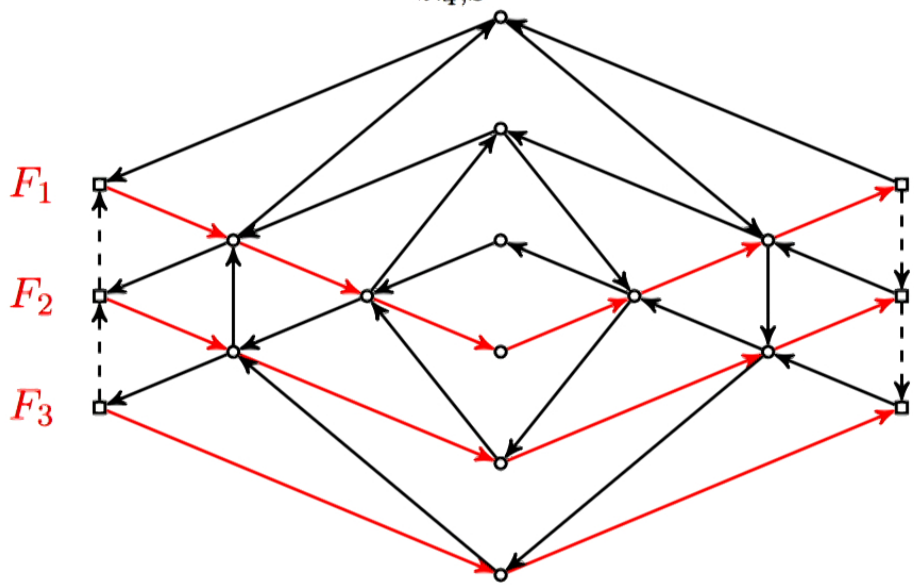
$S =$ 

The quantum cluster algebra $\mathcal{X}^q_{\mathfrak{sl}_4,S}$:



$S =$ 

The quantum cluster algebra $\mathcal{X}^q_{\mathfrak{sl}_4,S}$:



Embedding of $F_i \in \mathcal{U}_q(\mathfrak{sl}_4) \longrightarrow \mathcal{X}^q_{\mathfrak{sl}_4,S}$

[Schrader-Shapiro (2016)]

Cluster realization of $\mathcal{U}_q(\mathfrak{g})$

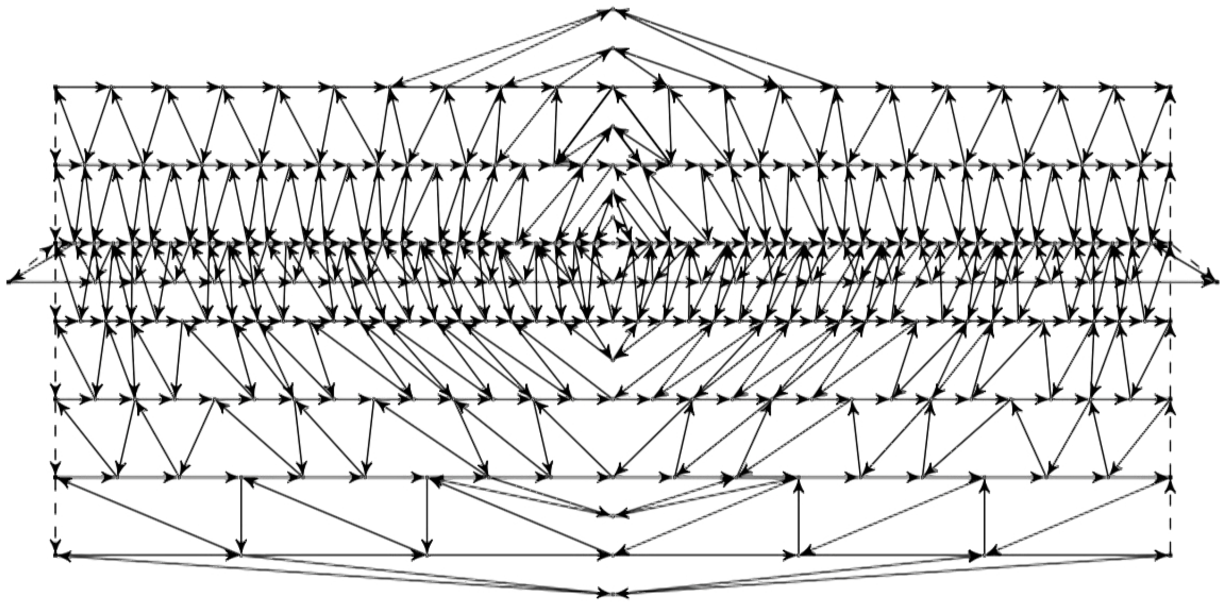


Figure: Cluster realization of Quantum Groups of type E_8

Modular Double of $\mathcal{U}_q(\mathfrak{osp}(1|2))$

Cl_2 = Clifford algebra $\langle\xi,\eta\rangle$ such that

$$\xi^2=\eta^2=1, \qquad \eta\xi+\xi\eta=1$$

Spinor trick: $Rep(\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R})) \longrightarrow Rep(\mathcal{U}_q(\mathfrak{osp}(1|2)))$

Theorem [*Ip-Zeitlin (2013)*]

Let $q_*=iq$. *Given a representation of* $\mathcal{U}_{q_*}(\mathfrak{sl}(2,\mathbb{R}))$, *there exists a representation of* $\mathcal{U}_q(\mathfrak{osp}(1|2))=\langle\mathcal{E},\mathcal{F},\mathcal{K}^{\pm1}\rangle$ *by*

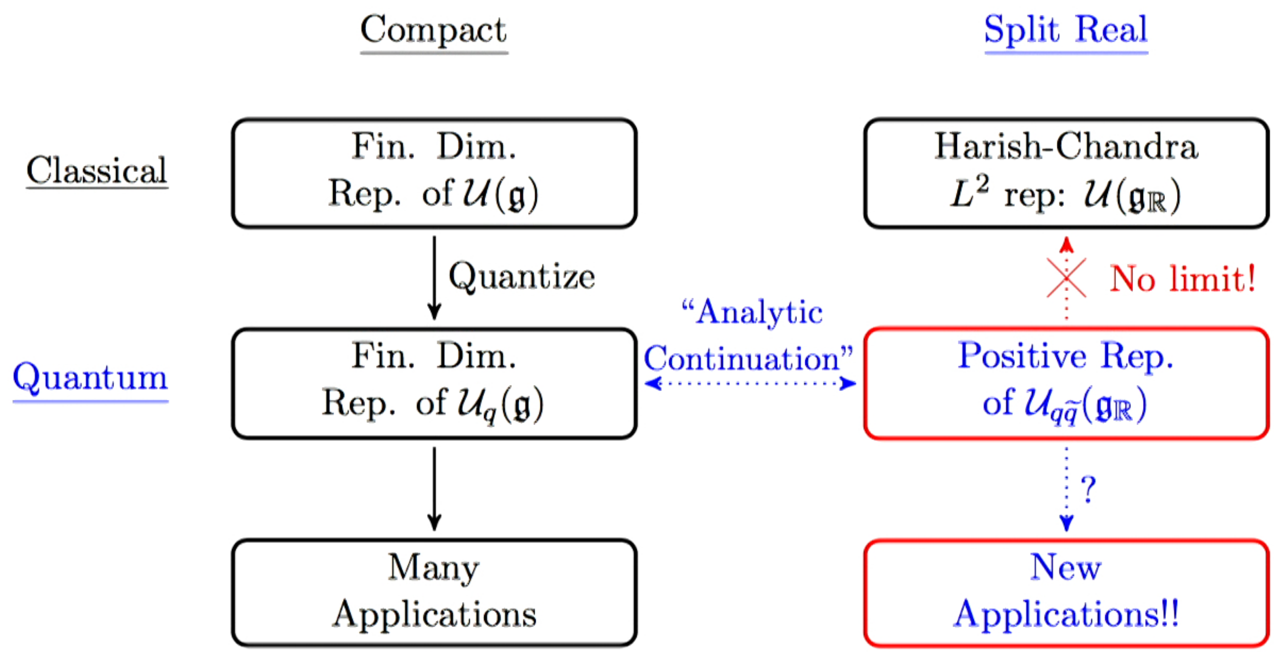
$$\mathcal{E}=\alpha E\xi, \qquad \mathcal{F}=F\eta, \qquad \mathcal{K}=Ki\eta\xi$$

where $\alpha=i\frac{q+q^{-1}}{q-q^{-1}}>0$.

In particular, induces Modular Double structure:

$$\mathcal{U}_{q_*,\widetilde{q_*}}(\mathfrak{sl}(2,\mathbb{R})) \Longrightarrow \mathcal{U}_{q,\tau(q)}(\mathfrak{osp}(1|2)), \qquad \tau(q)=-i\widetilde{q_*}$$

Summary


Compact
Split Real
Classical
Quantum
Fin. Dim. Rep. of $\mathcal{U}(\mathfrak{g})$
Harish-Chandra L^2 rep: $\mathcal{U}(\mathfrak{g}_\mathbb{R})$
Quantize
No limit!
"Analytic Continuation"
Fin. Dim. Rep. of $\mathcal{U}_q(\mathfrak{g})$
Positive Rep. of $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R})$
?
Many Applications
New Applications!!

Future Perspectives