Title: Additivity for Poisson algebras

Date: May 11, 2017 04:00 PM

URL: http://pirsa.org/17050026

Abstract:

$\mathcal{O}(\mathrm{d}(\mathrm{r.t}(S))) = \left(\underset{0}{\mathcal{O}_{\mathcal{F}}} \xleftarrow{d_{DR}} \underset{-1}{\mathbb{T}_{\mathcal{F}}} \xleftarrow{d_{DR}} \underset{-2}{\wedge^2 \mathbb{T}_{\mathcal{F}}} \leftarrow \cdots \right)$

This is a cdga. Moreover, have Schouten bracket, becomes a P_0-algebra.

Def A <u>P_n-algebra</u> is a cdga w/ Poisson bracket of coh deg $1-n$.

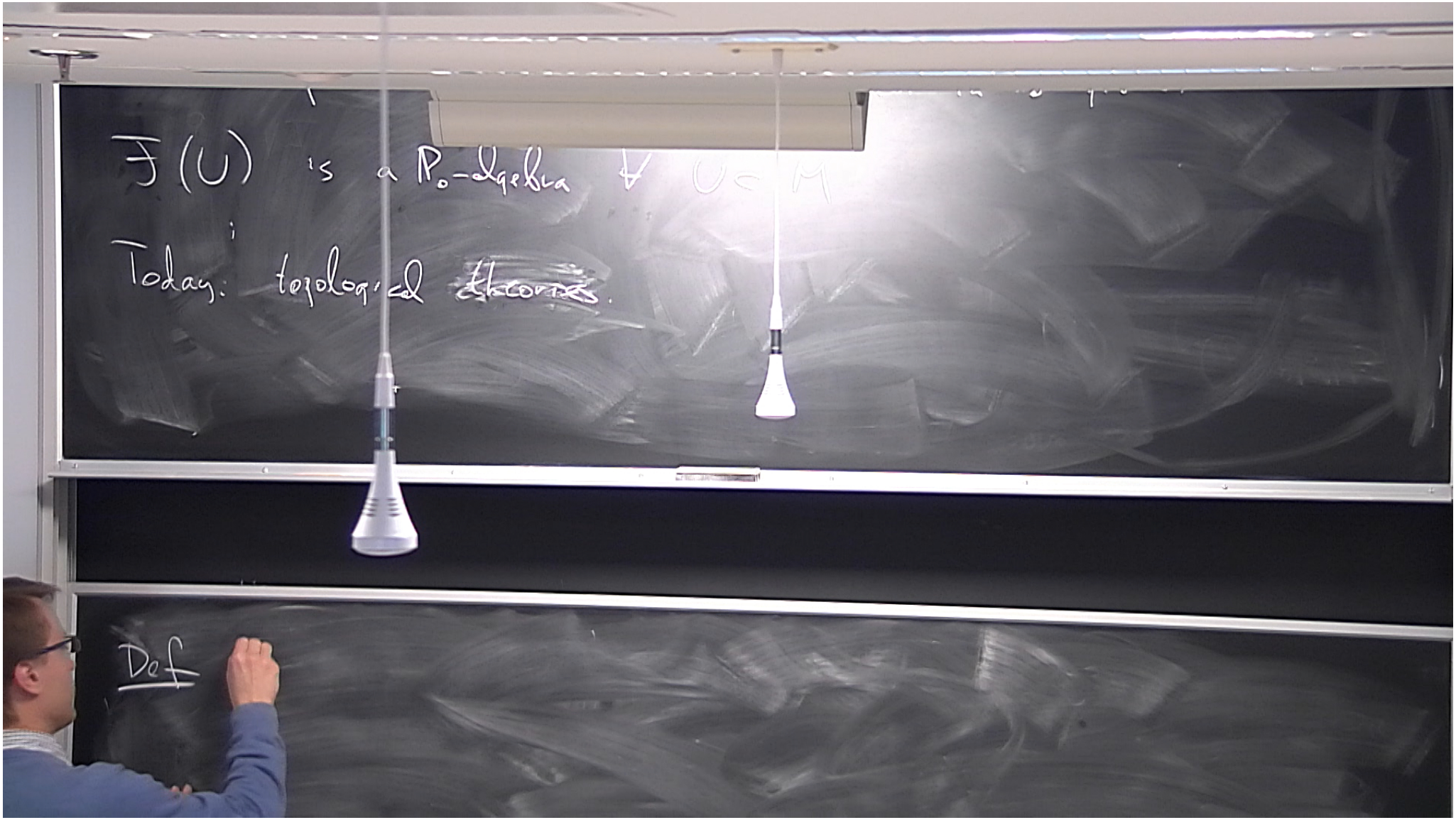
Also need locality.

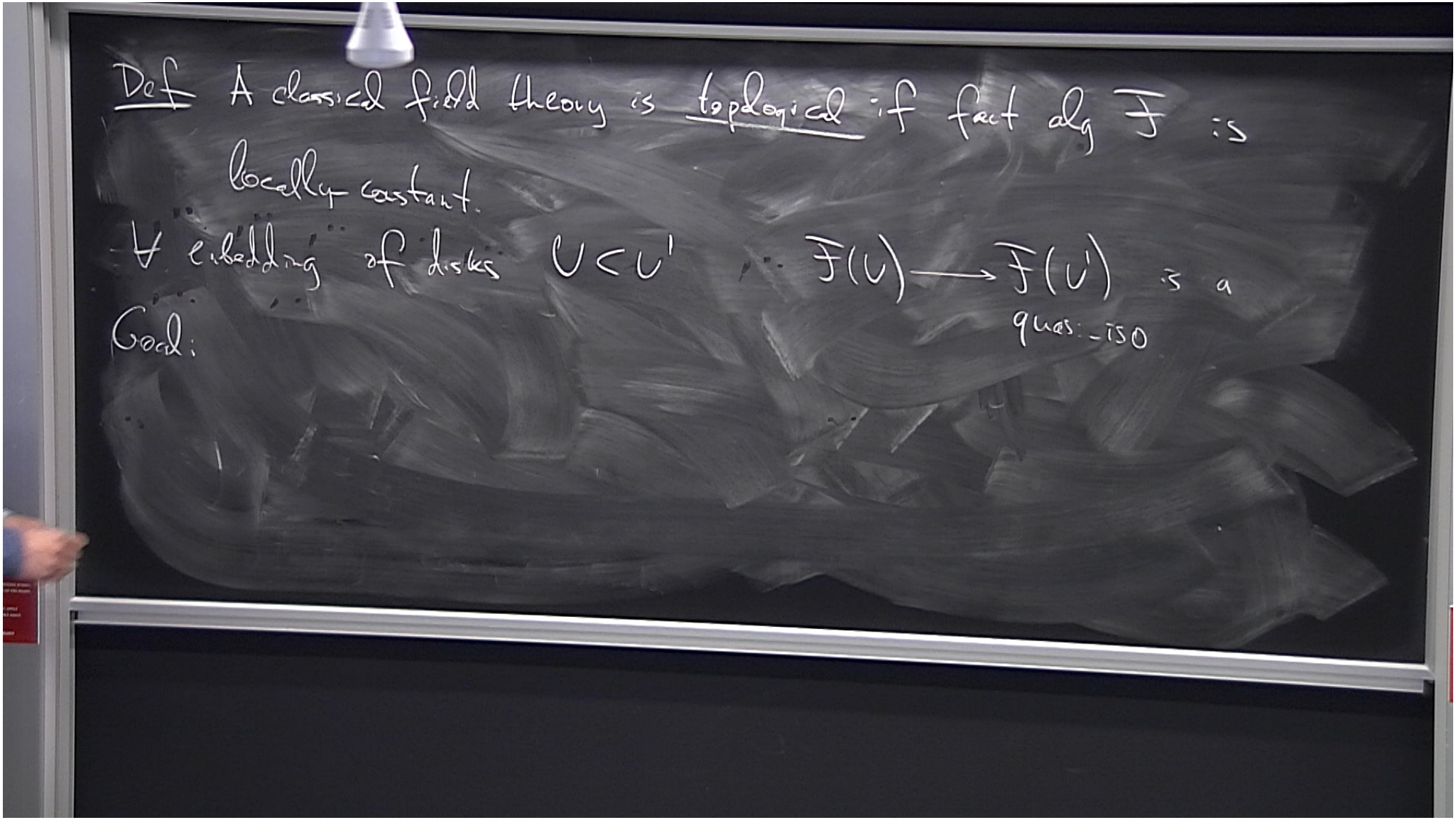
Def ((Costello–Gwilliam) A *classical field theory* on M is a factorization algebra on M valued in P_0-algebras.

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Def ((Costello-Gwilliam) A classical field theory on M is a factorization algebra on M valued in P_0-algebras.

$\mathcal{F}(U)$ is a P_0-algebra $\forall$ $U \subset M$.


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Today: topological theories.
Def


Def A classical field theory is topological if fact alg $\mathcal{F}$ is locally constant.
$\forall$ embedding of disks $U \subset U'$ $\mathcal{F}(U) \longrightarrow \mathcal{F}(U')$ is a quas.-iso
Goal:

Def A classical field theory is **topological** if fact alg $\mathcal{F}$ is locally constant.

$\forall$ embedding of disks $U \subset U'$ $\quad \mathcal{F}(U) \longrightarrow \mathcal{F}(U')$ is a quas.-iso

Goal: describe $\mathrm{Fact}^{lc}\left(\mathbb{R}^3, \mathrm{Alg}_{\mathbb{R}^3}\right)$

Th (Rozenblyum, S) 7 equivalences

$$\mathrm{Fact}^{lc}(\mathbb{R}^n;\ \mathrm{Alg}_{R_0}) \simeq \mathrm{Alg}_{R_n}$$

Th (Rozenblyum, S) ∃ equivalence

$$\mathrm{Fact}^{lc}(\mathbb{R}^n;\ \mathrm{Alg}_{\mathbb{R}_0}) \xrightarrow{\sim} \mathrm{Alg}_{\mathbb{P}_n}$$

$\mathbb{P}_1$-algebra

Example $n=1$. Classical mechanics (Poisson algebra + Hamiltonian H) gives examples of classical field theory.

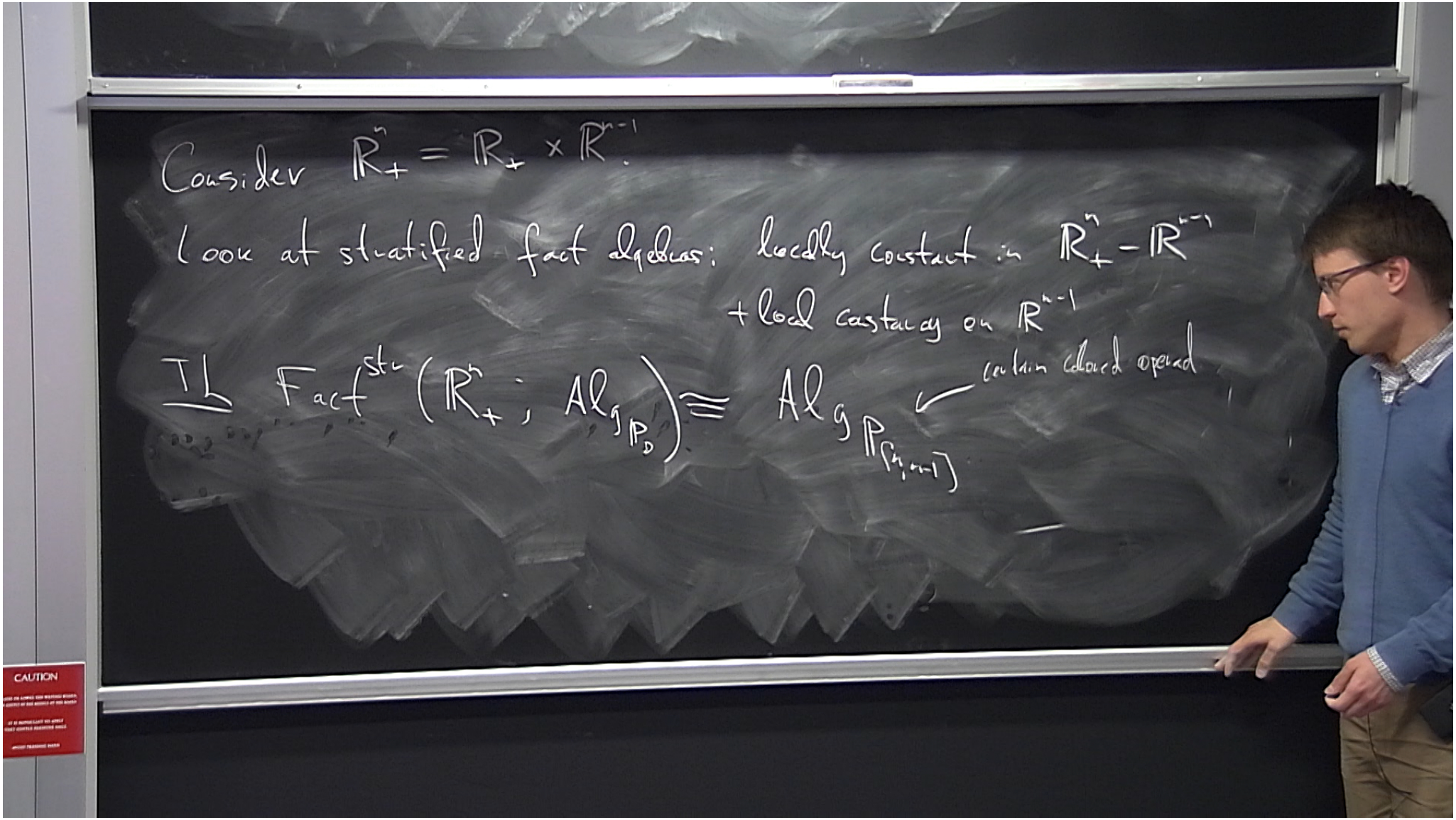
Topological. $H=0$.

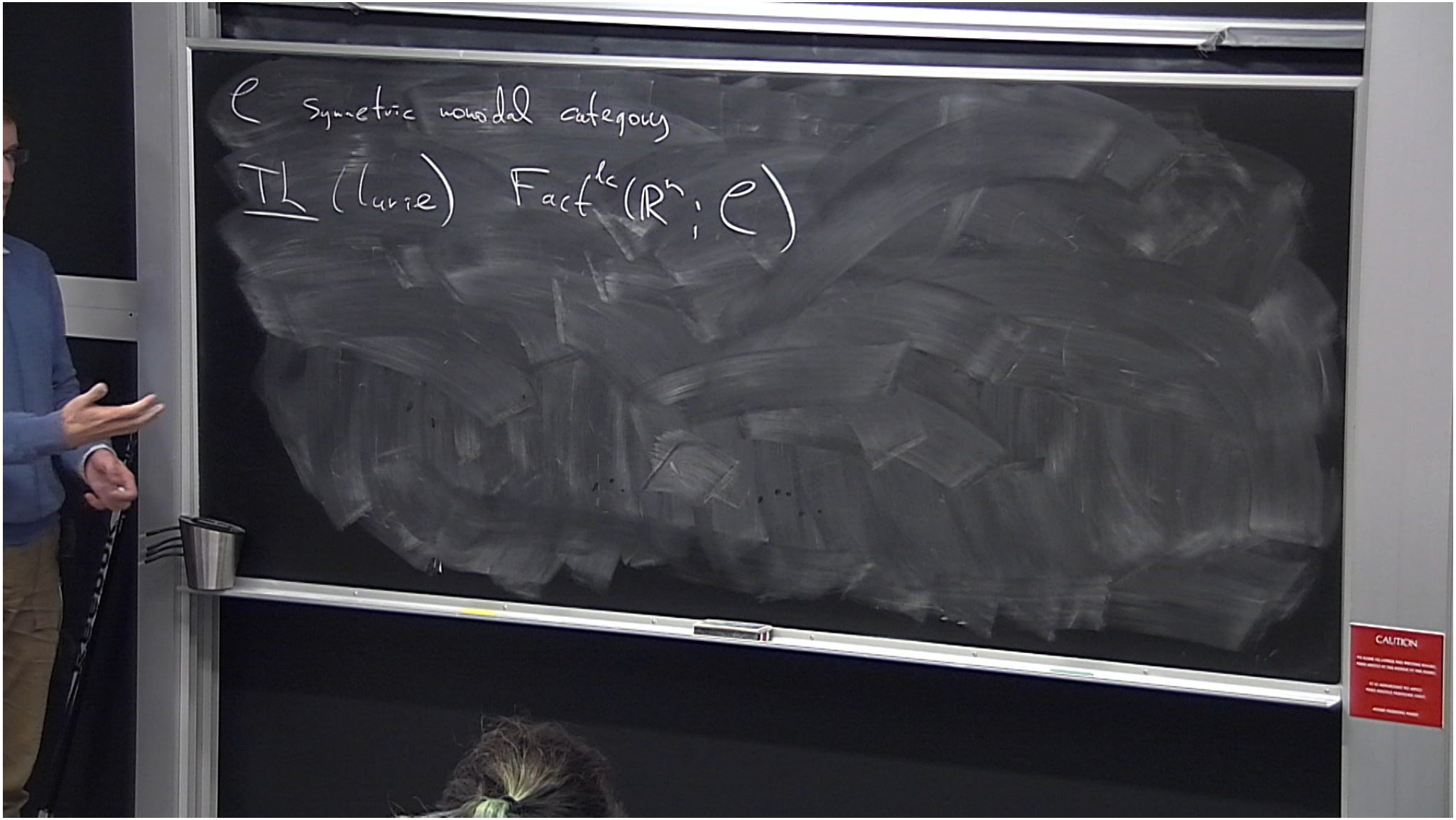
Consider $\mathbb{R}^{\cdot}_{+} = \mathbb{R}_{+} \times \mathbb{R}$.

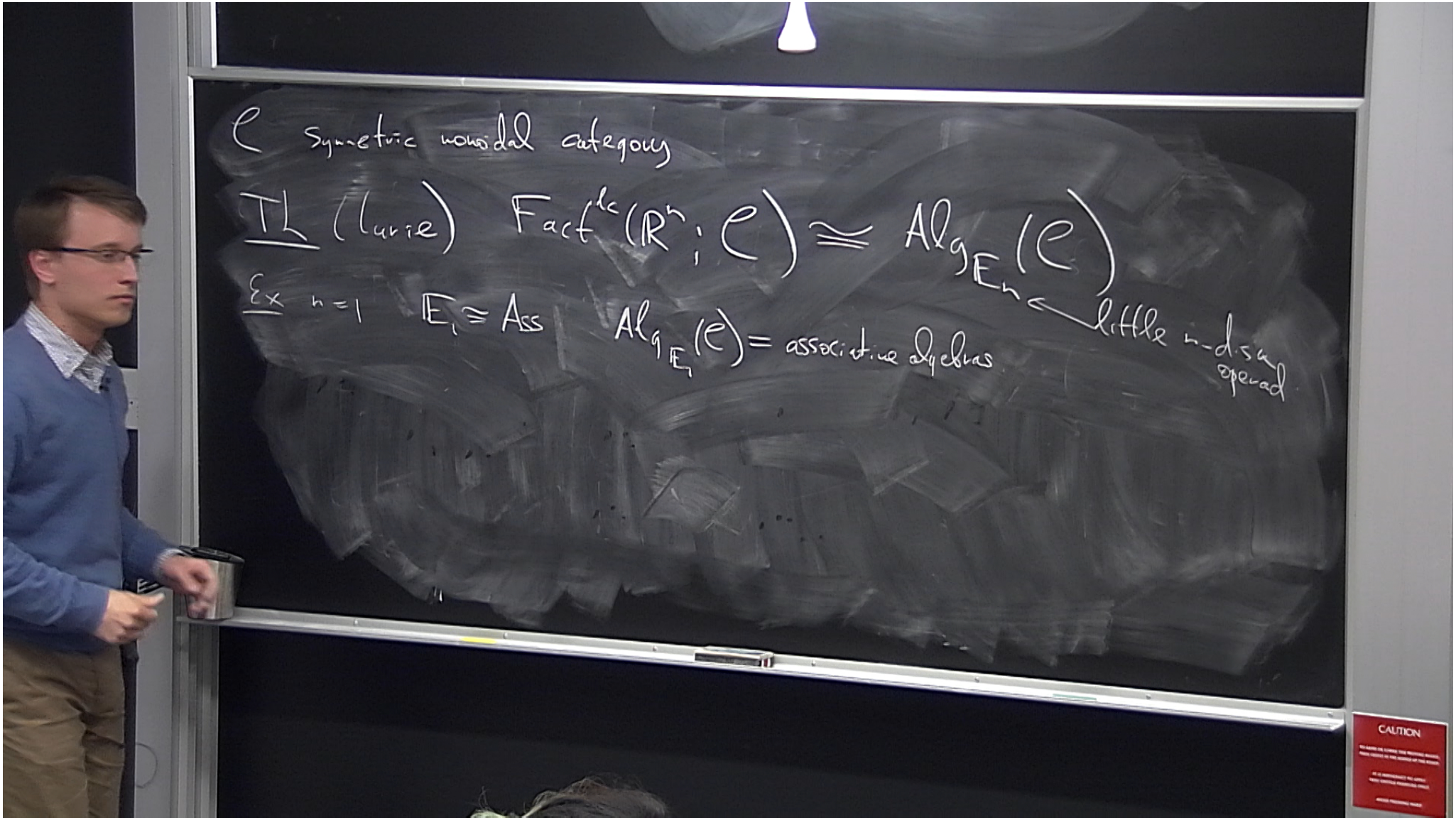
Look at stratified fact algebras: locally constant in $\mathbb{R}^{n}_{+} - \mathbb{R}^{n-1}$
+ local constancy on $\mathbb{R}^{n-1}$

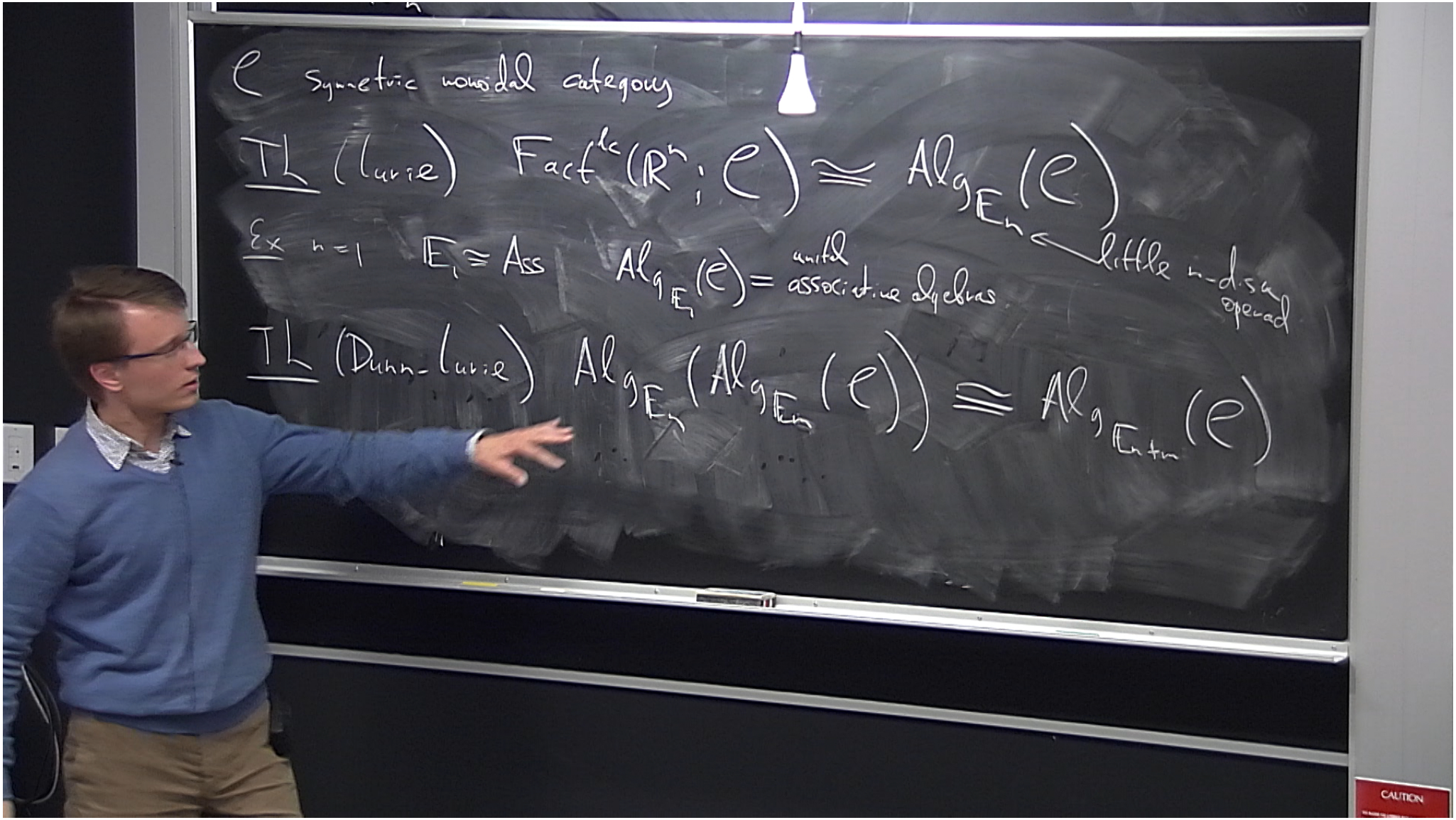
IL $\mathrm{Fact}^{\mathrm{str}}\left(\mathbb{R}^{n}_{+};\ \mathrm{Alg}_{/\mathbb{P}_0}\right) \simeq$

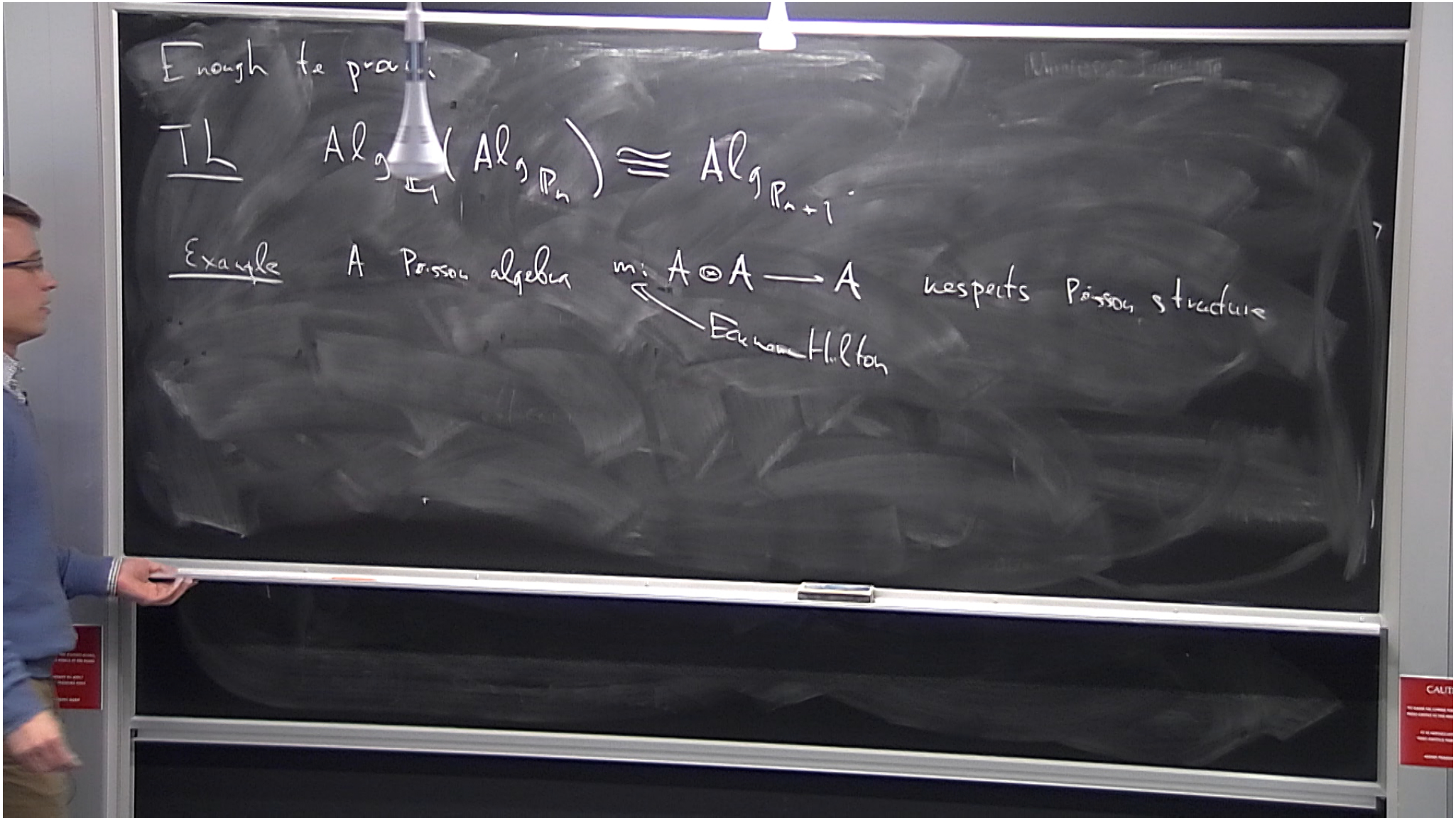
CAUTION

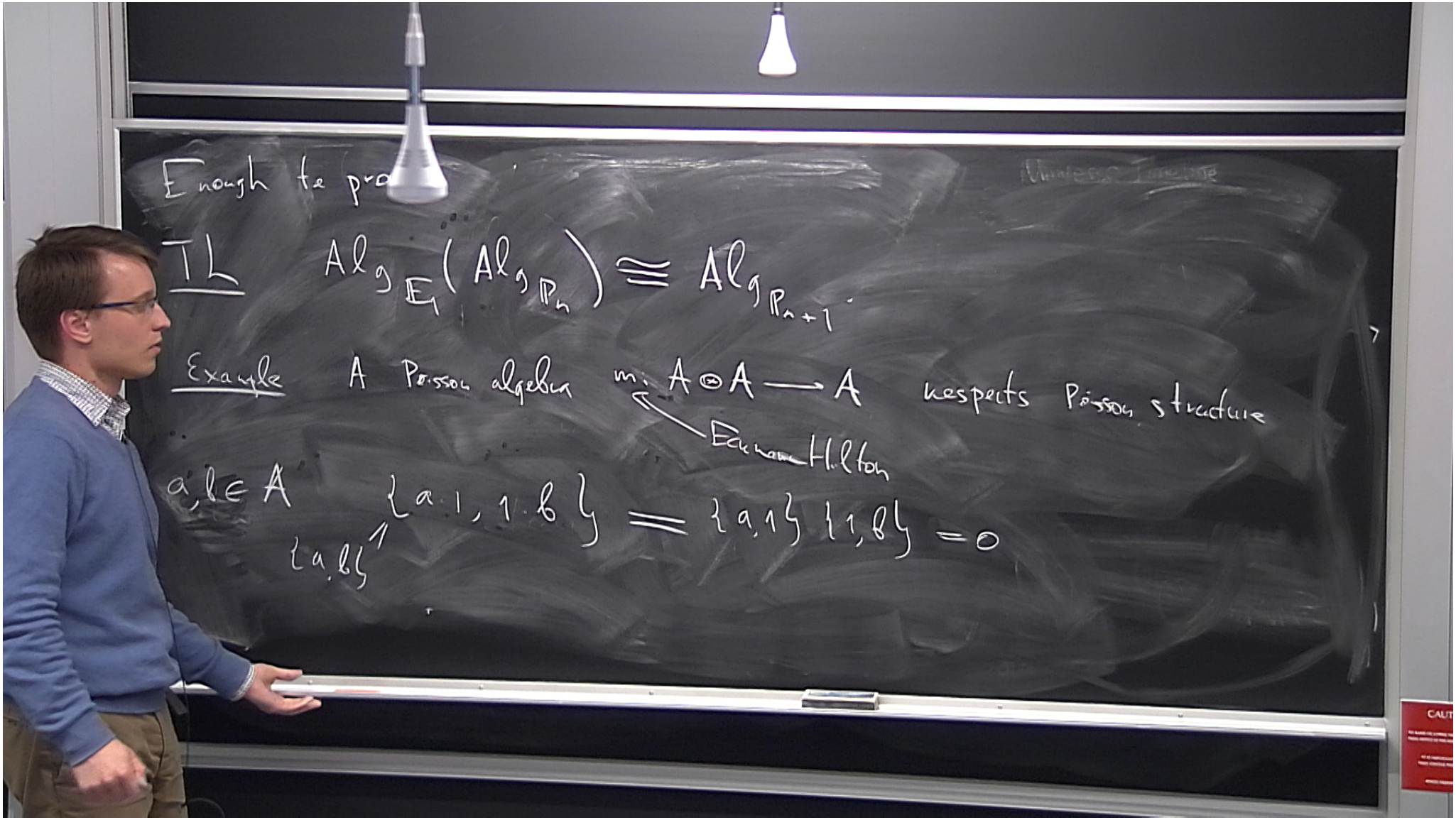

Consider $\mathbb{R}^n_+ = \mathbb{R}_+ \times \mathbb{R}^{n-1}$.
Look at stratified fact algebras: locally constant in $\mathbb{R}^n_+ - \mathbb{R}^{n-1}$
+ locally constant on $\mathbb{R}^{n-1}$
$\mathrm{Fact}^{\mathrm{str}}(\mathbb{R}^n_+; \mathrm{Alg}_{\mathbb{P}_D}) \simeq \mathrm{Alg}_{\mathbb{P}_{[n,n-1]}}$
contains coloured operad

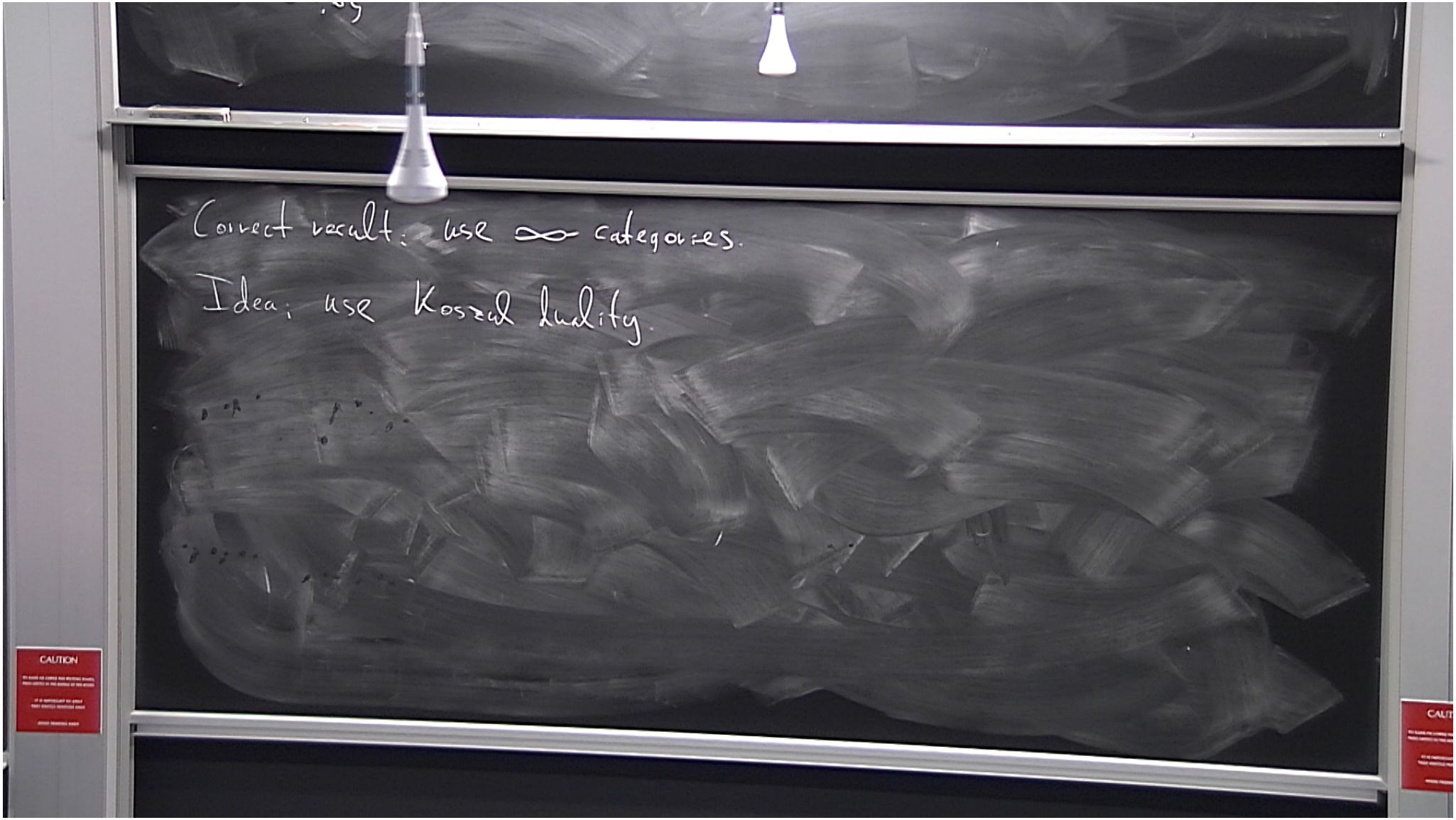

C Symmetric monoidal category
TL (Lurie) Fact^lc(R^n; C)
CAUTION


C symetric monoidal category
TL (Lurie) $Fact^{lc}(\mathbb{R}^n; C) \simeq Alg_{E_n}(C)$
Ex n=1 $E_1 \simeq Ass$ $Alg_{E_1}(C)$ = associative algebras
little n-d.s.m. operad
CAUTION


C symetric monoidal category
TL (Lurie) $Fact^{lc}(\mathbb{R}^n; C) \cong Alg_{E_n}(C)$
little n-d.sum operad
Ex n=1 $E_1 \cong Ass$ $Alg_{E_1}(C)$ = unital associative algebras
TL (Dunn-Lurie) $Alg_{E_n}(Alg_{E_m}(C)) \cong Alg_{E_{n+m}}(C)$
CAUTION


Enough te pra
TL
$Alg_{\mathbb{P}_n}(Alg_{\mathbb{P}_n}) \cong Alg_{\mathbb{P}_{n+1}}$.
Example
A Poisson algebra
$m: A \otimes A \longrightarrow A$
respects Poisson structure
Eckmann-Hilton


Enough te pro
TL
$Alg_{\mathbb{E}_1}(Alg_{\mathbb{P}_n}) \cong Alg_{\mathbb{P}_{n+1}}$
Example
A Poisson algebra
$m: A \otimes A \longrightarrow A$
respects Poisson structure
Eckmann-Hilton
$a, b \in A$
$\{a \cdot 1, 1 \cdot b\} = \{a, 1\}\{1, b\} = 0$
$\{a \cdot b\}$


Correct result: use ∞ categories.
Idea: use Koszul duality.
CAUTION

Correct result: use ∞-categories.

Idea: use Koszul duality.

Have dualities:

- commutative alg $\longleftrightarrow$ Lie coalgebras
- P_n-algebra $\longleftrightarrow$ P_n coalgebra
- P_n-algebra $\longleftrightarrow$ $(n-1)$-shifted Lie bialgebras
  eg Lie algebra, has cobracket of deg $n-1$

Correct result: use ∞-categories.

Idea: use Koszul duality.

Have dualities:

- commutative dg $\longleftrightarrow$ Lie coalgebras
- P_n-algebra $\longleftrightarrow$ P_n coalgebra
- P_n-algebra $\longleftrightarrow$ $(n-2)$-shifted Lie bialgebras

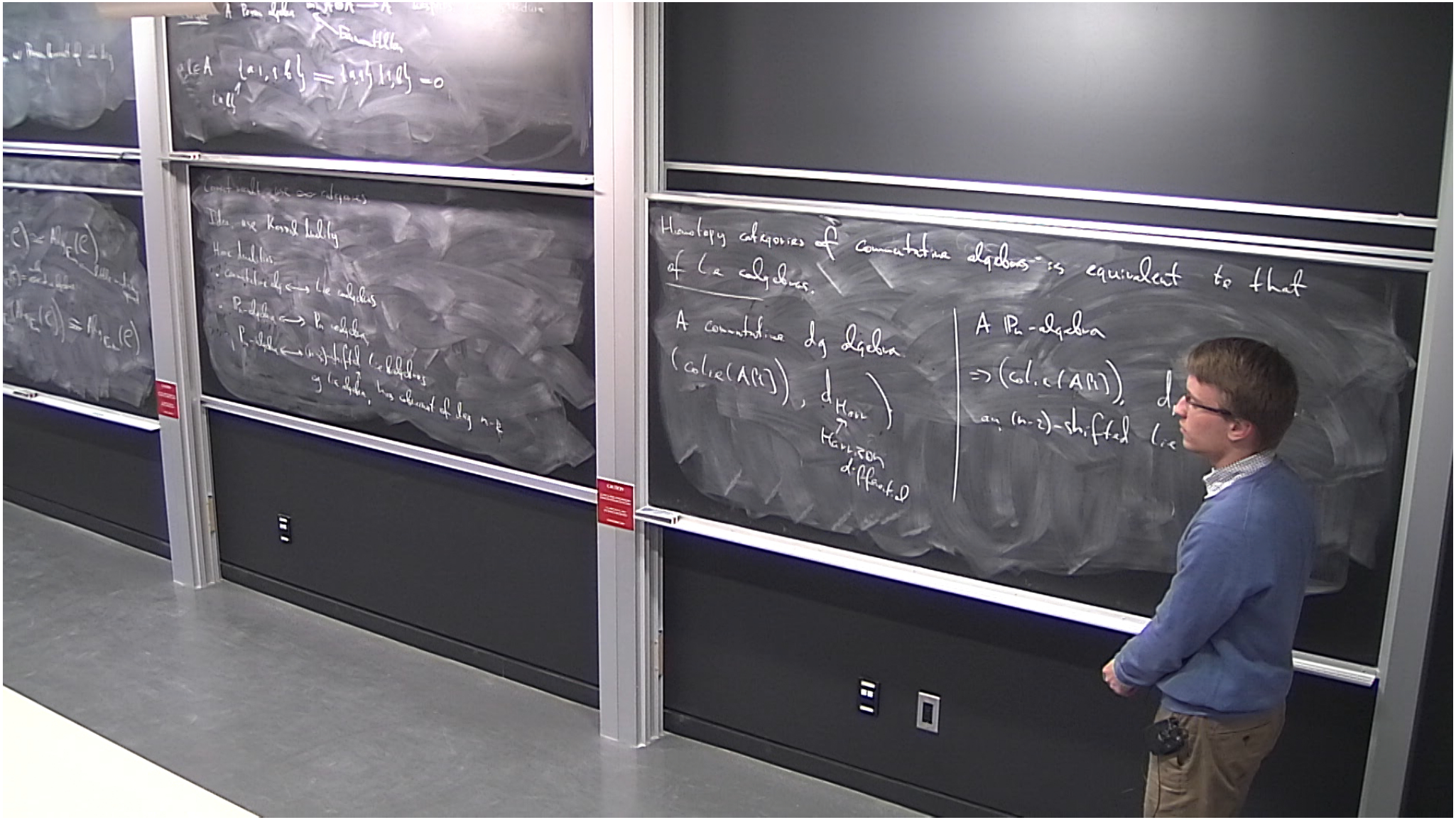
  dg Lie algebra, has cobracket of deg $n-2$

CAUTION

Cooler $\mathbb{R}_+^n = \mathbb{R}_+ \times \mathbb{R}^{n-1}$.

Homotopy categories of commutative algebras is equivalent to that of Lie algebras.

CAUTION


Homotopy categories of commutative dg algebras is equivalent to that of Lie algebras.
A commutative dg algebra
A $\mathbb{P}_n$-algebra

differential

A $\mathbb{P}_{n+1}$-algebra $\rightsquigarrow$ $\mathfrak{g}$ $(n-1)$-shifted Lie bialgebra

$\rightsquigarrow$ $U\mathfrak{g}$ universal enveloping algebra

it's an associative alg + cocomm coalgebra

differential

A $\mathbb{R}_{n+1}$-algebra $\rightsquigarrow$ $\mathfrak{g}$ (n-1)-shifted Lie bialgebra

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CAUTION

differential

A $\mathbb{P}_{n+1}$-algebra $\rightsquigarrow$ a (n-1)-shifted Lie bialgebra

$\rightsquigarrow$ U a universal enveloping algebra

it's an associative alg + cocomm comultiplication + Lie cobracket

= it's an assoc alg in $\mathbb{P}_n$-algebras $\rightsquigarrow$ assoc alg in $\mathbb{P}_n$-algebras

CAUTION

differential

A $\mathbb{P}_{n+1}$-algebra $\rightsquigarrow$ $\mathfrak{g}$ $(n-1)$-shifted Lie bialgebra

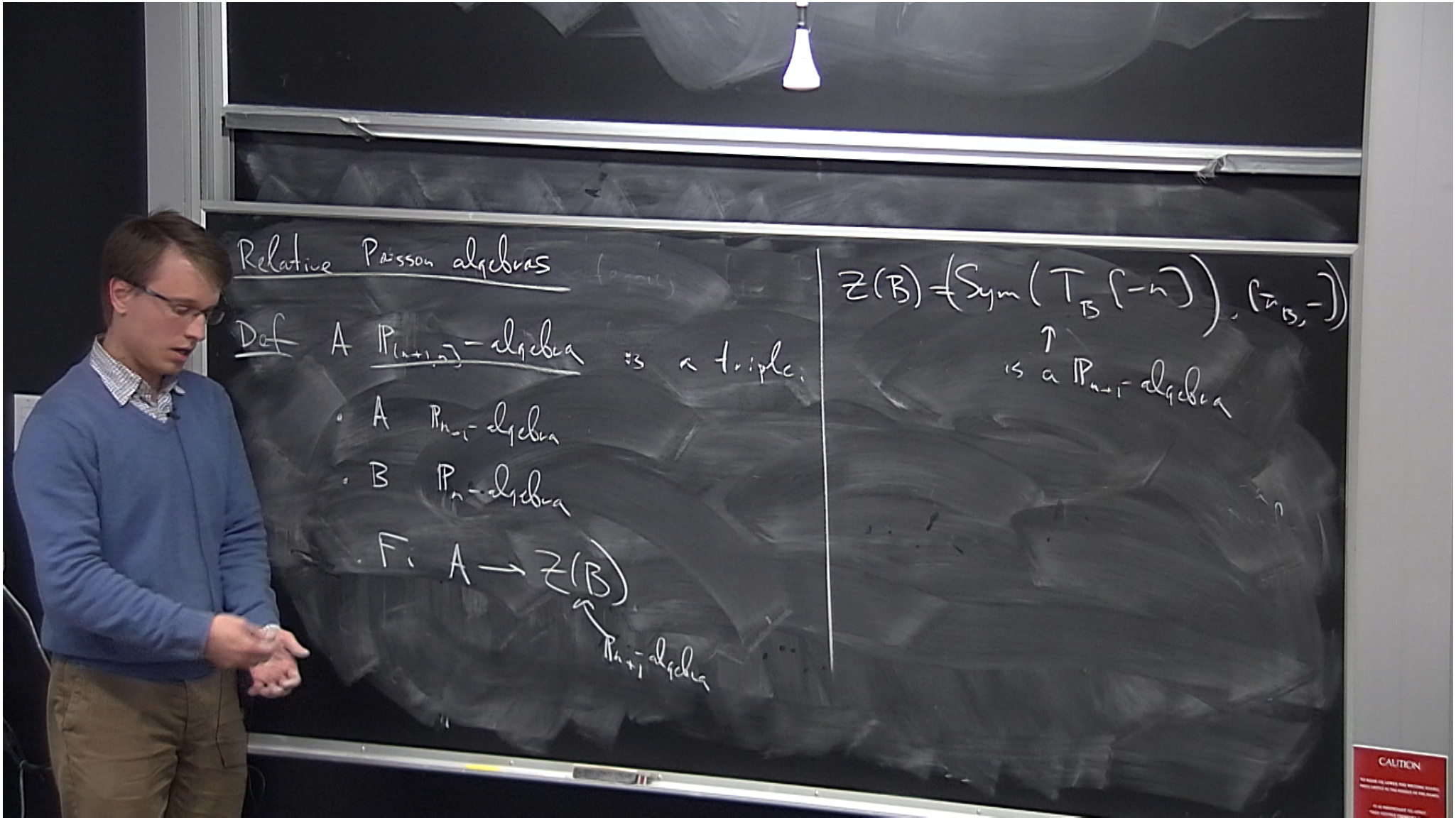
$\rightsquigarrow$ $U\mathfrak{g}$ universal enveloping algebra

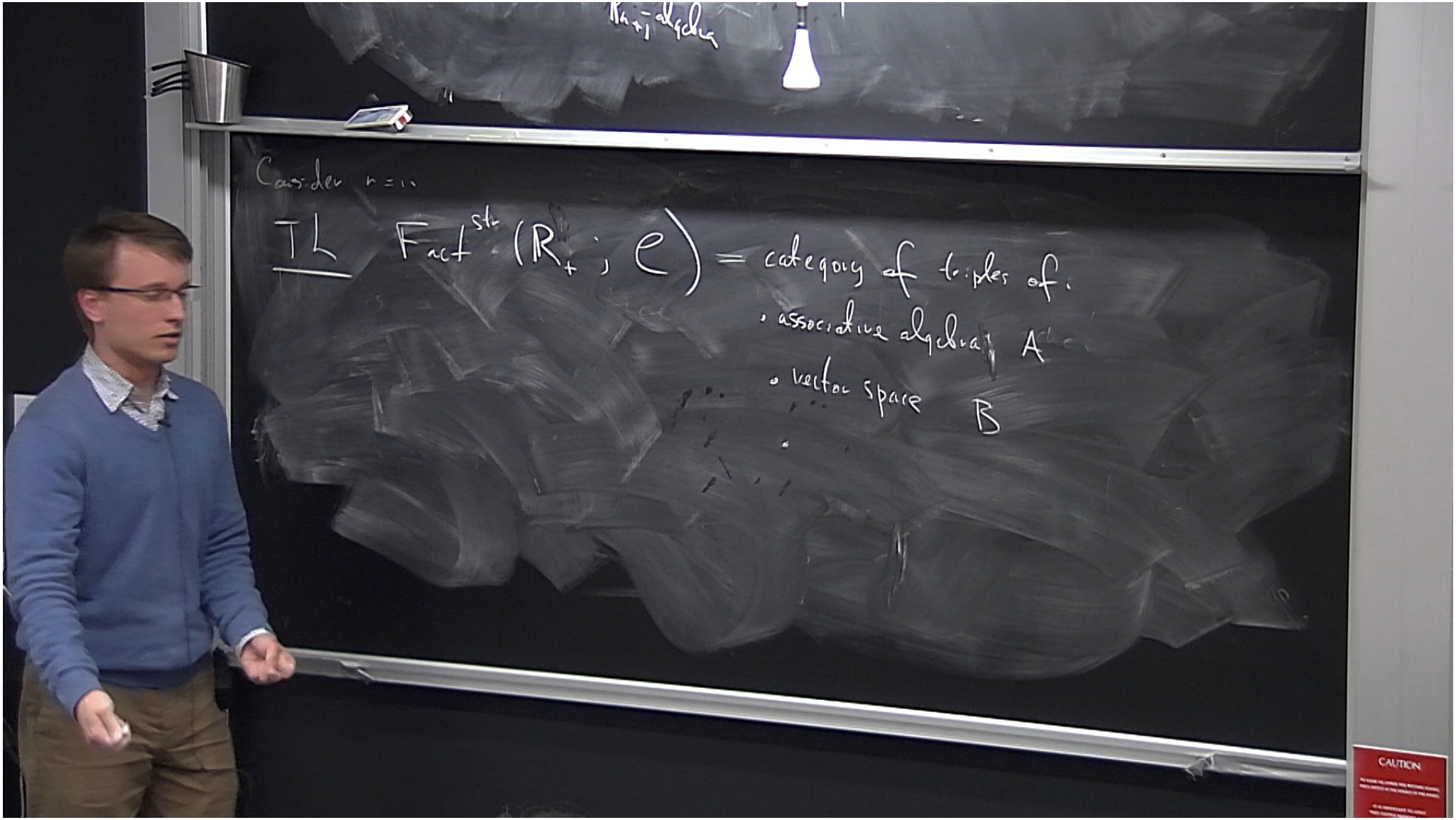
it's an associative alg + cocomm comultiplication + Lie cobracket

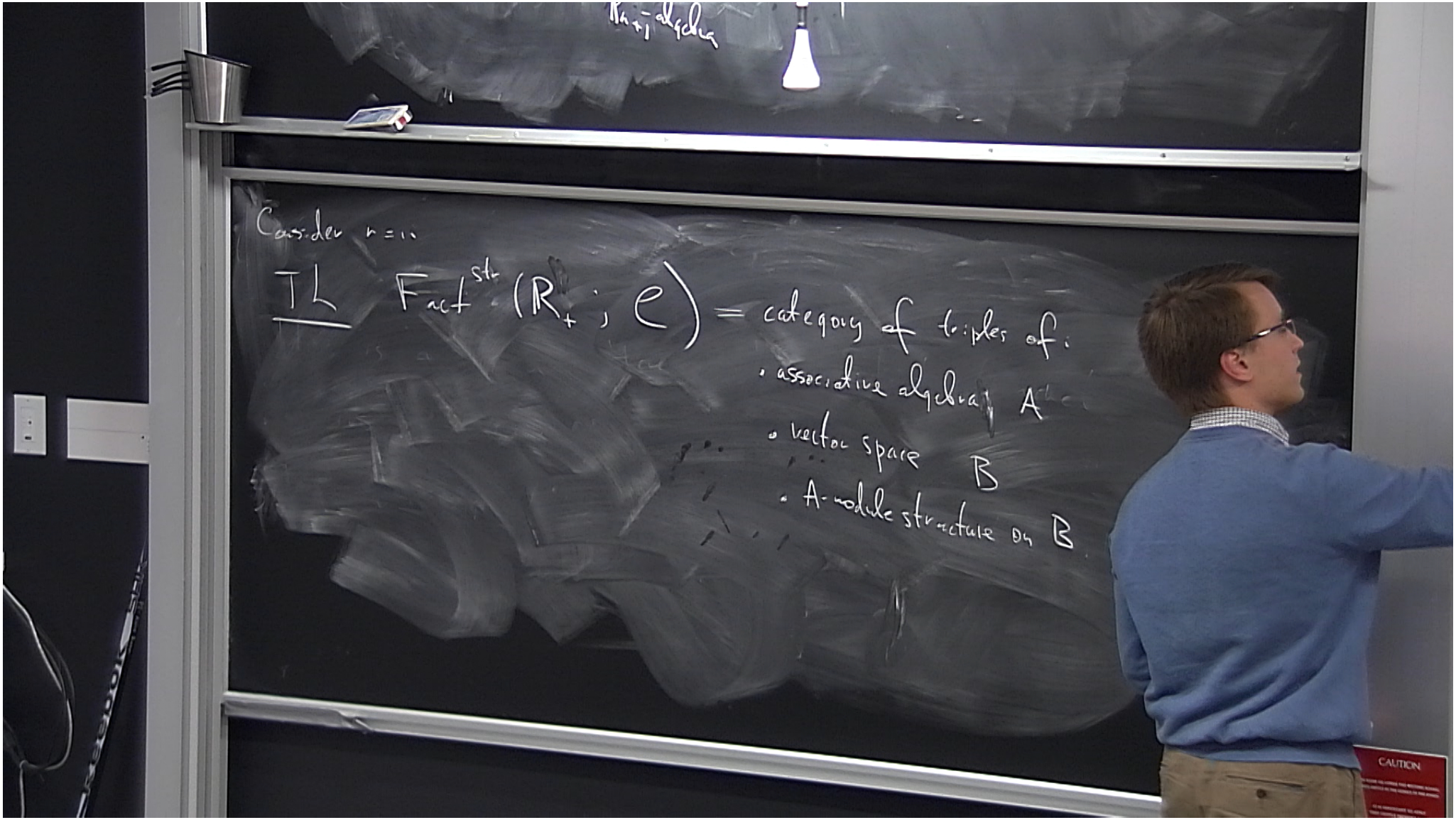
= it's an assoc alg in $\mathbb{P}_n$-coalgebras $\rightsquigarrow$ assoc alg in $\mathbb{P}_n$-algebras

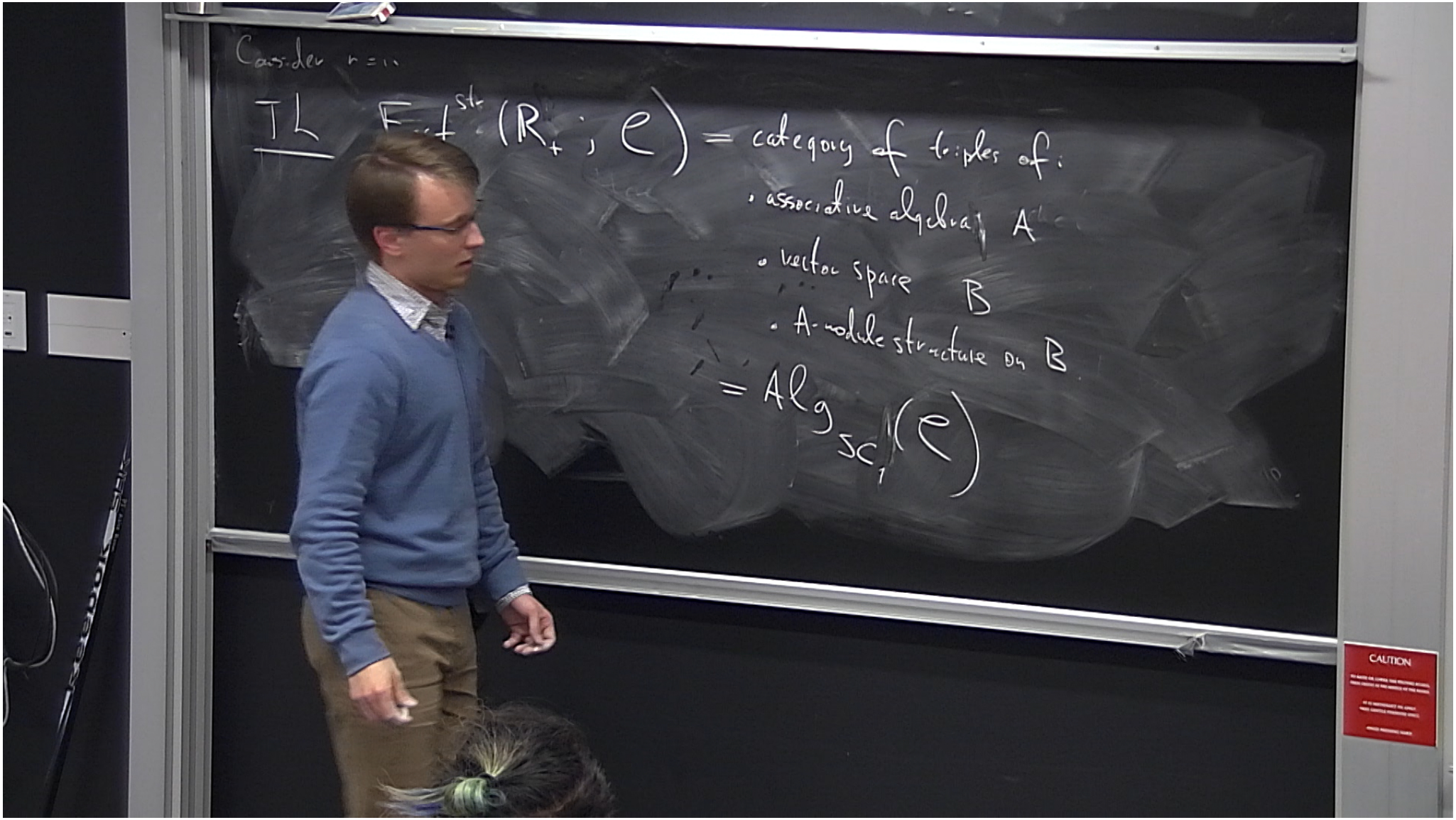
Claim The composite is an equivalence.

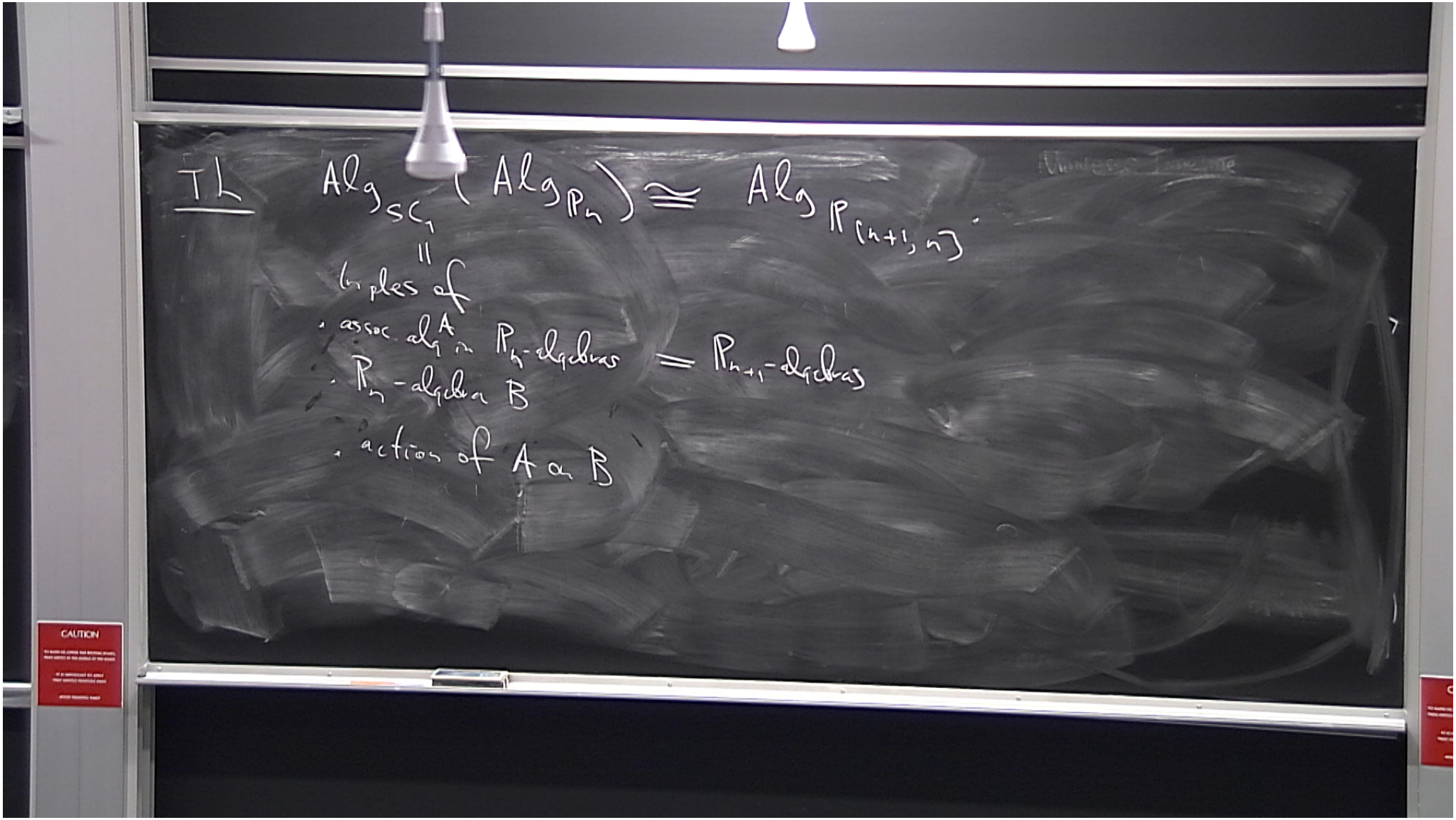
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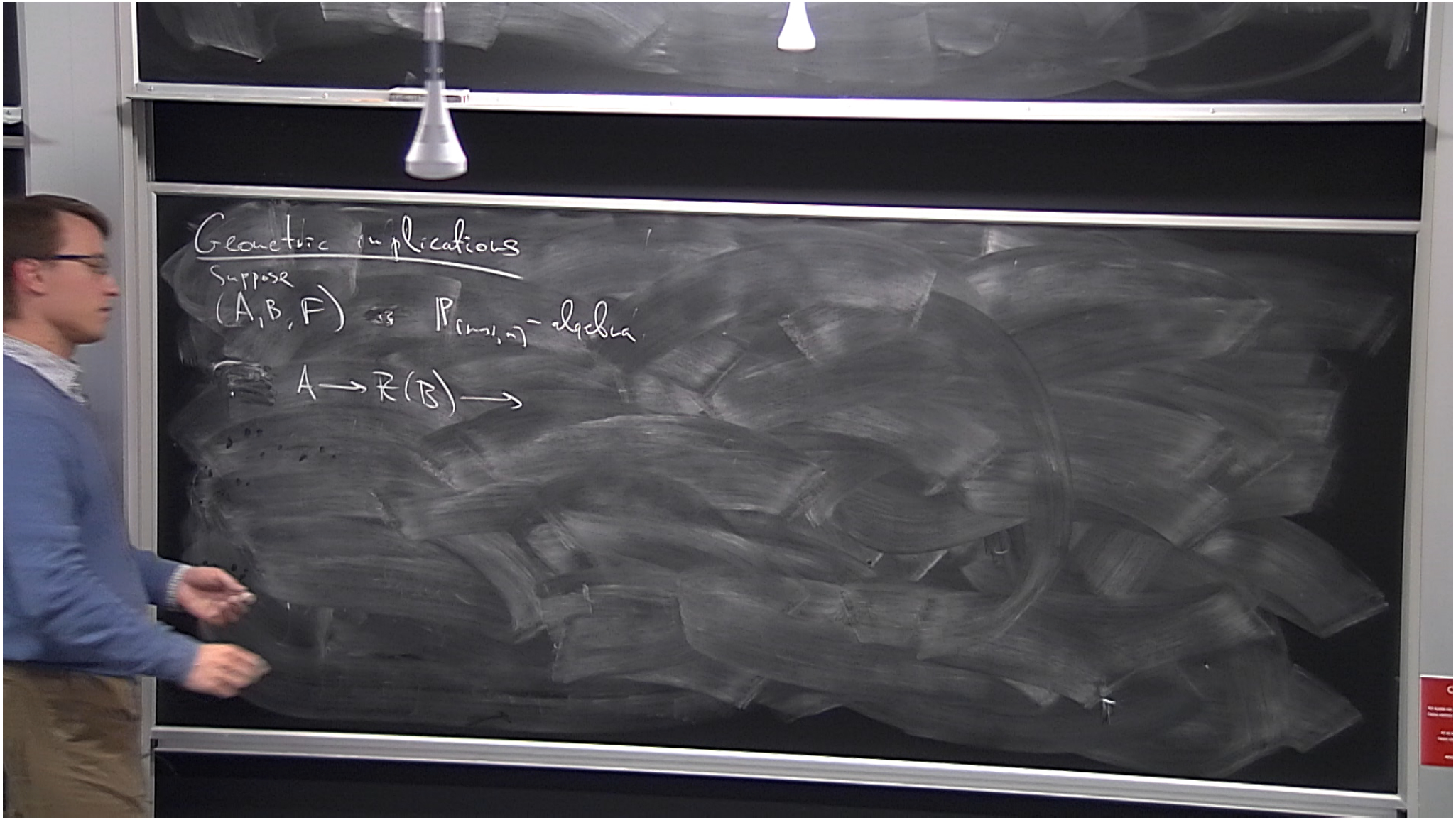

Relative Poisson algebras
Def A $\mathbb{P}_{[n+1,n]}$-algebra is a triple.
• A $\mathbb{P}_{n+1}$-algebra
• B $\mathbb{P}_n$-algebra
• $F: A \to Z(B)$
$\mathbb{P}_{n+1}$-algebra
$Z(B) = \{\mathrm{Sym}(T_B[-n]), [T_B, -]\}$
is a $\mathbb{P}_{n+1}$-algebra
CAUTION

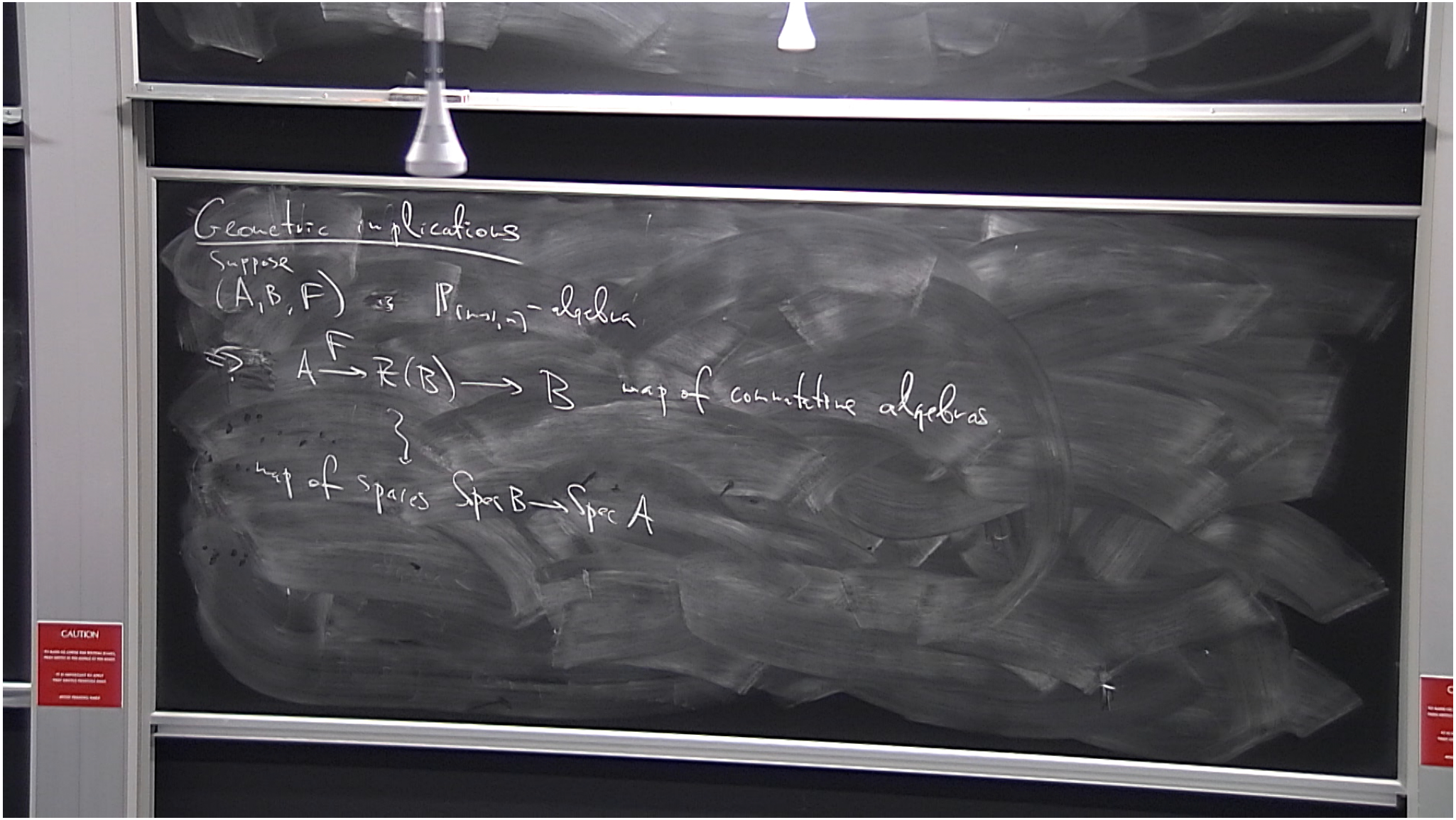

Consider n = 1.
TL Fact^str.(R+, e) = category of triples of:
• associative algebra A
• vector space B
CAUTION

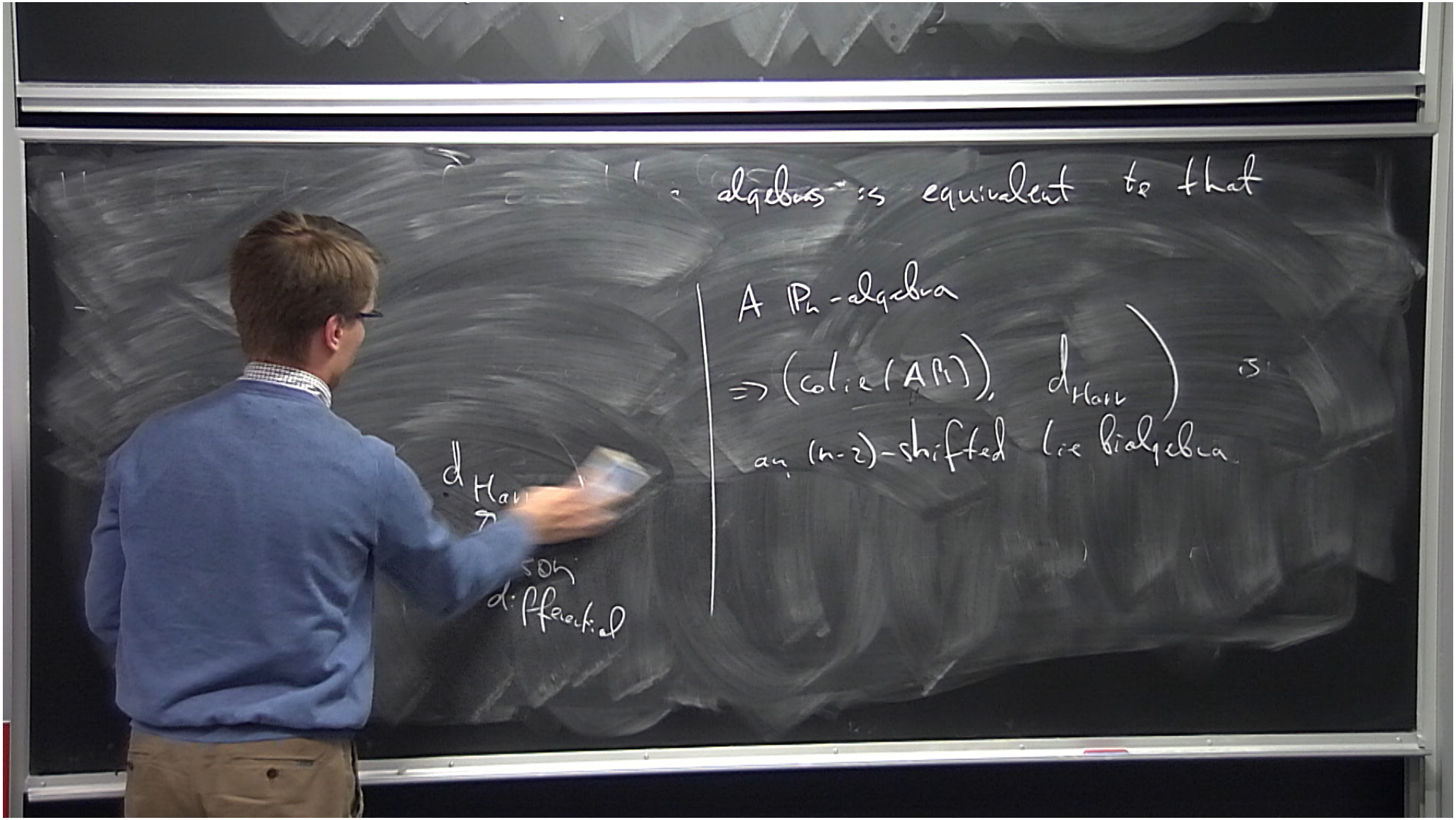

Consider n = 1.
TL Fact^str(R+; e) = category of triples of:
• associative algebra A
• vector space B
• A-module structure on B
CAUTION


Consider r = 1.
TL
Fct^str.(R+; C) = category of triples of:
• associative algebra A
• vector space B
• A-module structure on B
= Alg_sc,1(C)
CAUTION


Th $Alg_{SC_1}(Alg_{P_n}) \cong Alg_{P_{(n+1,n]}}$.
$= \{$ triples of
• assoc alg A in P_n-algebras $= P_{n+1}$-algebras
• P_n-algebra B
• action of A on B
CAUTION


Geometric implications
Suppose
(A, B, F) is $\mathbb{P}_{(m+1,n)}$-algebra
$A \longrightarrow R(B) \longrightarrow$


Geometric implications
Suppose
(A, B, F) is $\mathbb{P}_{(n-1,n)}$-algebra
$\Rightarrow$ $A \xrightarrow{F} R(B) \longrightarrow B$ map of commutative algebras
map of spaces $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$
CAUTION


algebras is equivalent to that
A P_n-algebra
$\Rightarrow (\mathrm{Cobar}(A^!), d_{Harr})$ is
an $(n-2)$-shifted Lie bialgebra
d_{Harr}
differential

Def $f: L \to X$ map of spaces. The space of n-shifted coisotropic structures

$Coiso(f,n) :=$ space of lifts of $\mathcal{O}(X) \to \mathcal{O}(L)$ to a $\mathbb{P}_{(n+1,n)}$-algebra

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By construction have

$$Coiso(f,n) \swarrow \quad \searrow Pois(X,n)$$

Homotopy categories of commutative algebras is equivalent to that of Lie algebras.

A commutative dg algebra

$(\mathrm{Coli}(A[1]), d_{\mathrm{Hall}})$

d_{Hall} ↗ Hall-Lon differential

A P_n-algebra

$\Rightarrow (\mathrm{Coli}(A[1]), d_{\mathrm{Hall}})$ is

an $(n-2)$-shifted Lie bialgebra.

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By construction have

$$\begin{array}{ccc} & Coiso(f,n) & \\ \swarrow & & \searrow \\ Pois(L,n-1) & & Pois(X,n) \end{array}$$

Example $f: L \hookrightarrow X$ smooth schemes

$Cois(f, 0)$ = set of Poisson structures on X for which L is coisotropic in usual sense.

$\mathrm{Pois}(f, 0)$ — set of Poisson structures on X for which

L is coisotropic in weak sense.

IL (Melani, S)

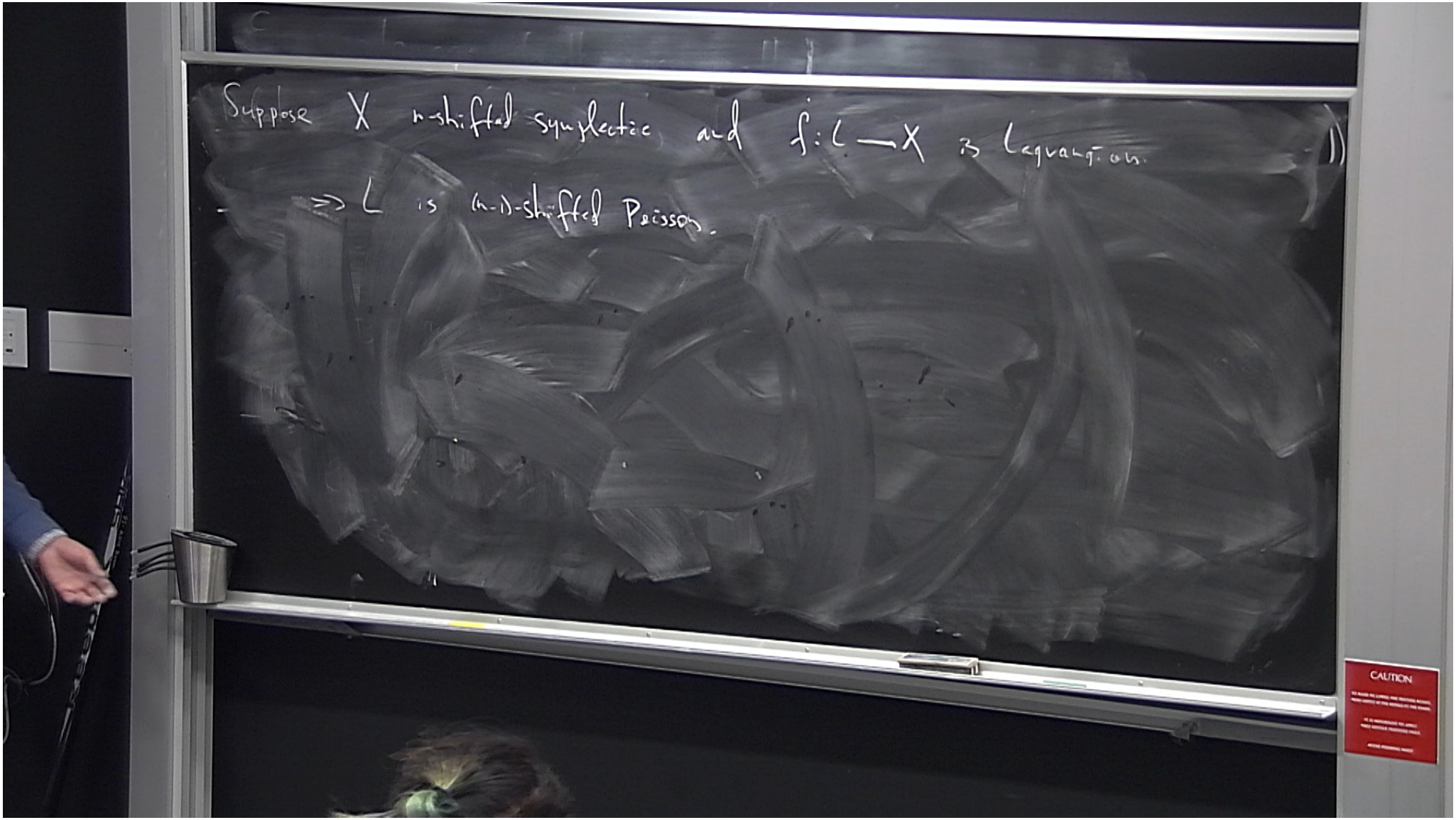
$\longrightarrow$ { n-shifted Lagrangians $f : L \longrightarrow X$ }

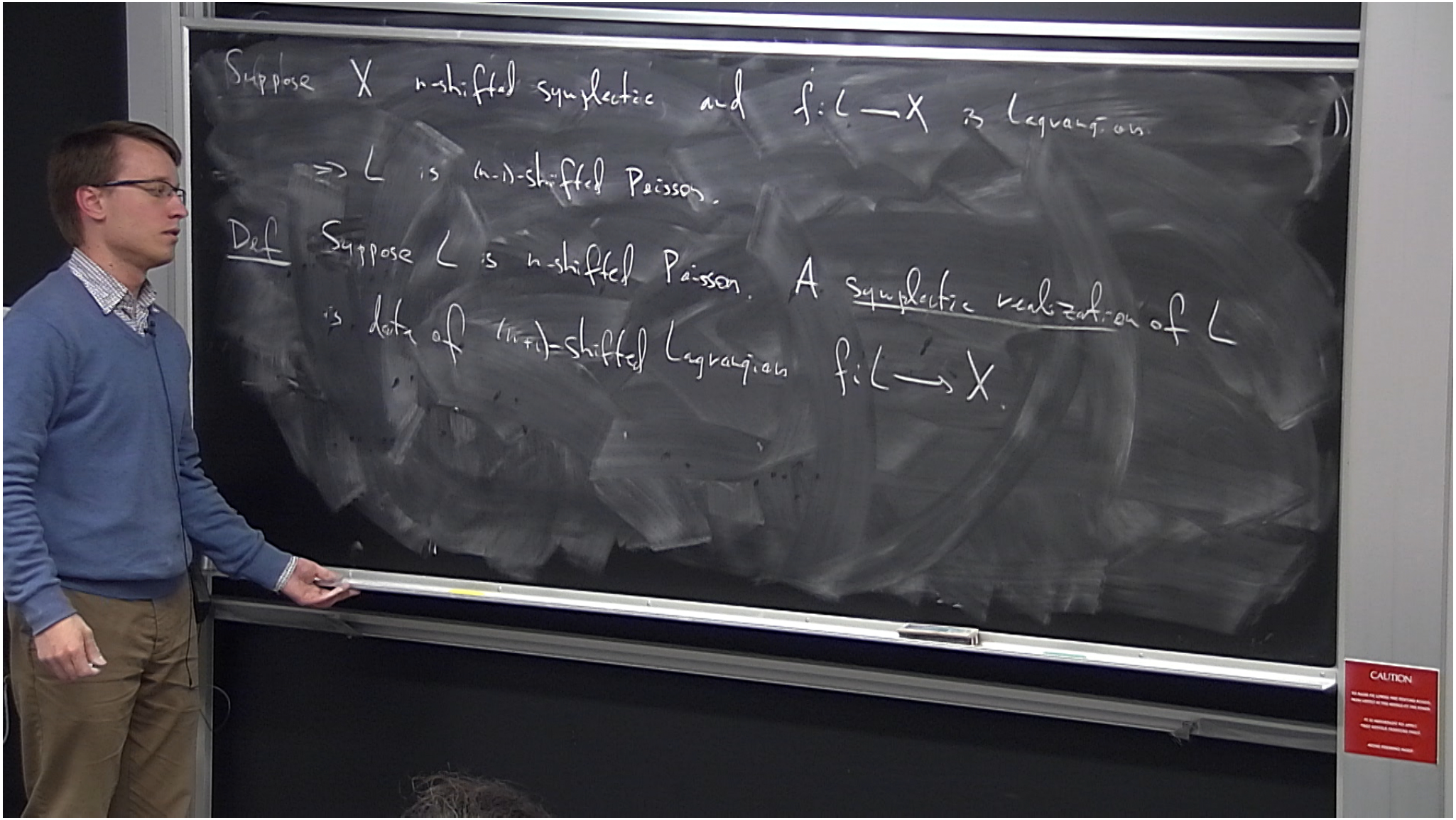
Example $f: L \hookrightarrow X$ smooth schemes

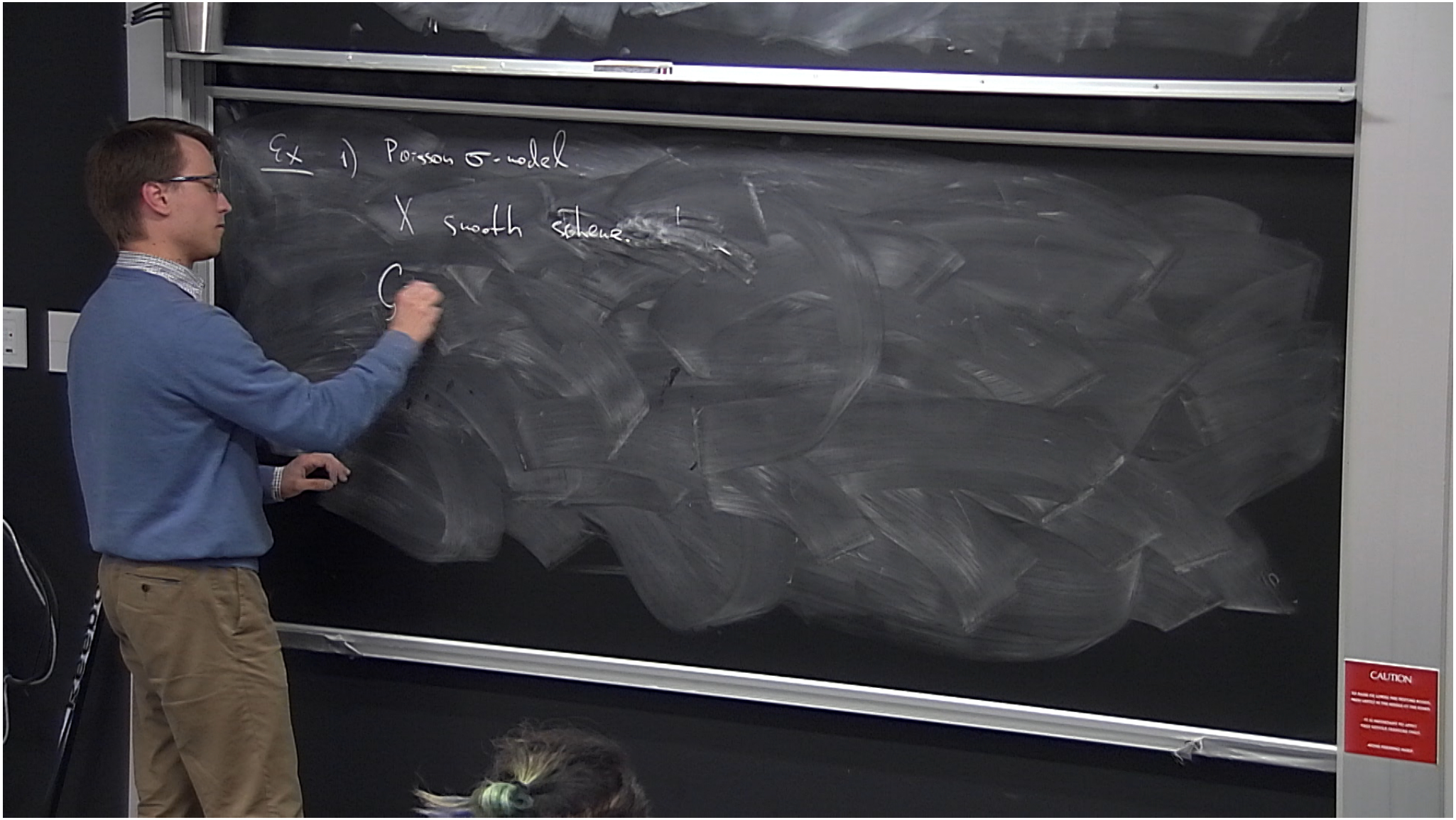
$Pois(f, 0)$ = set of Poisson structures on X for which L is coisotropic in usual sense.

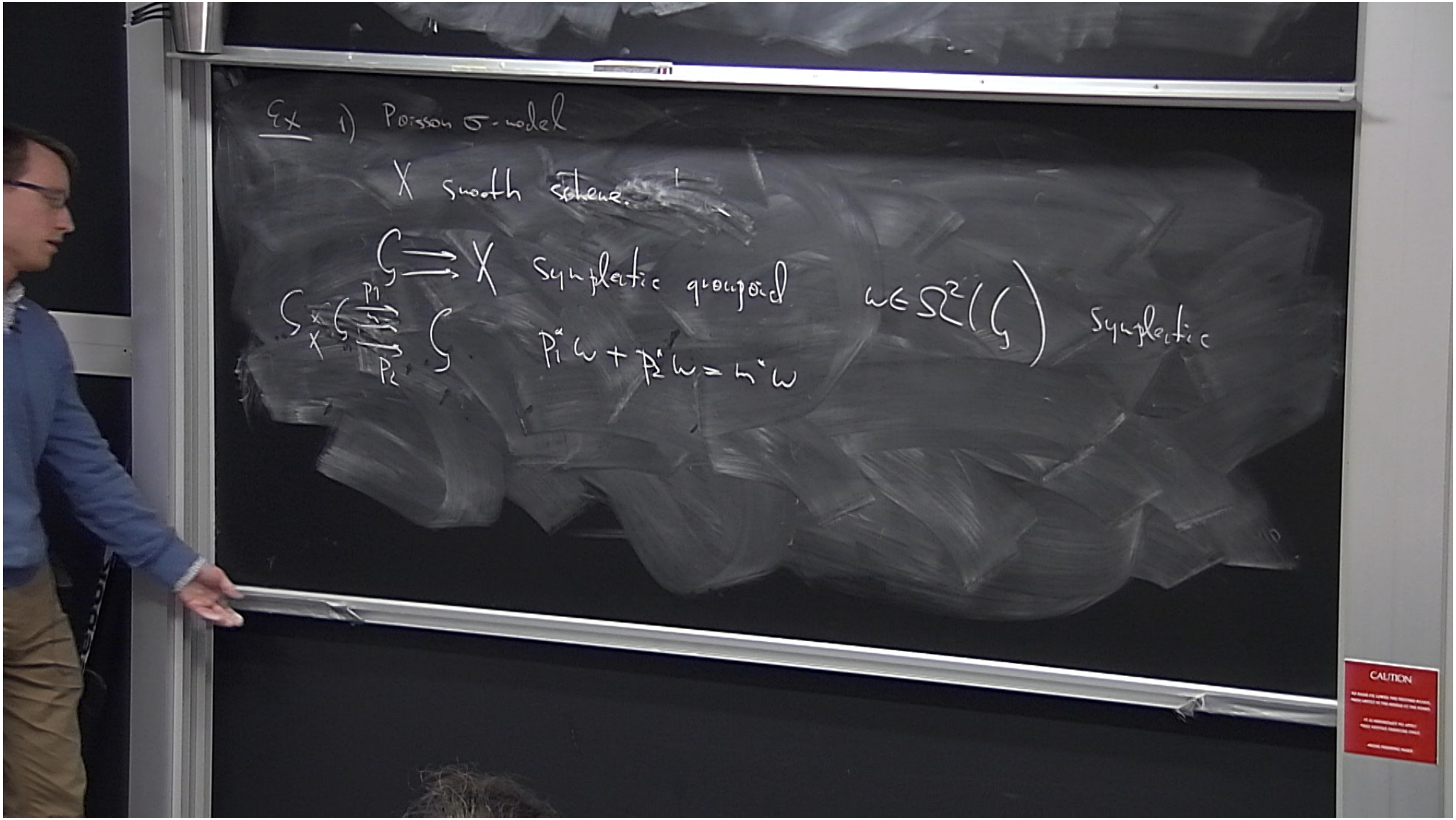
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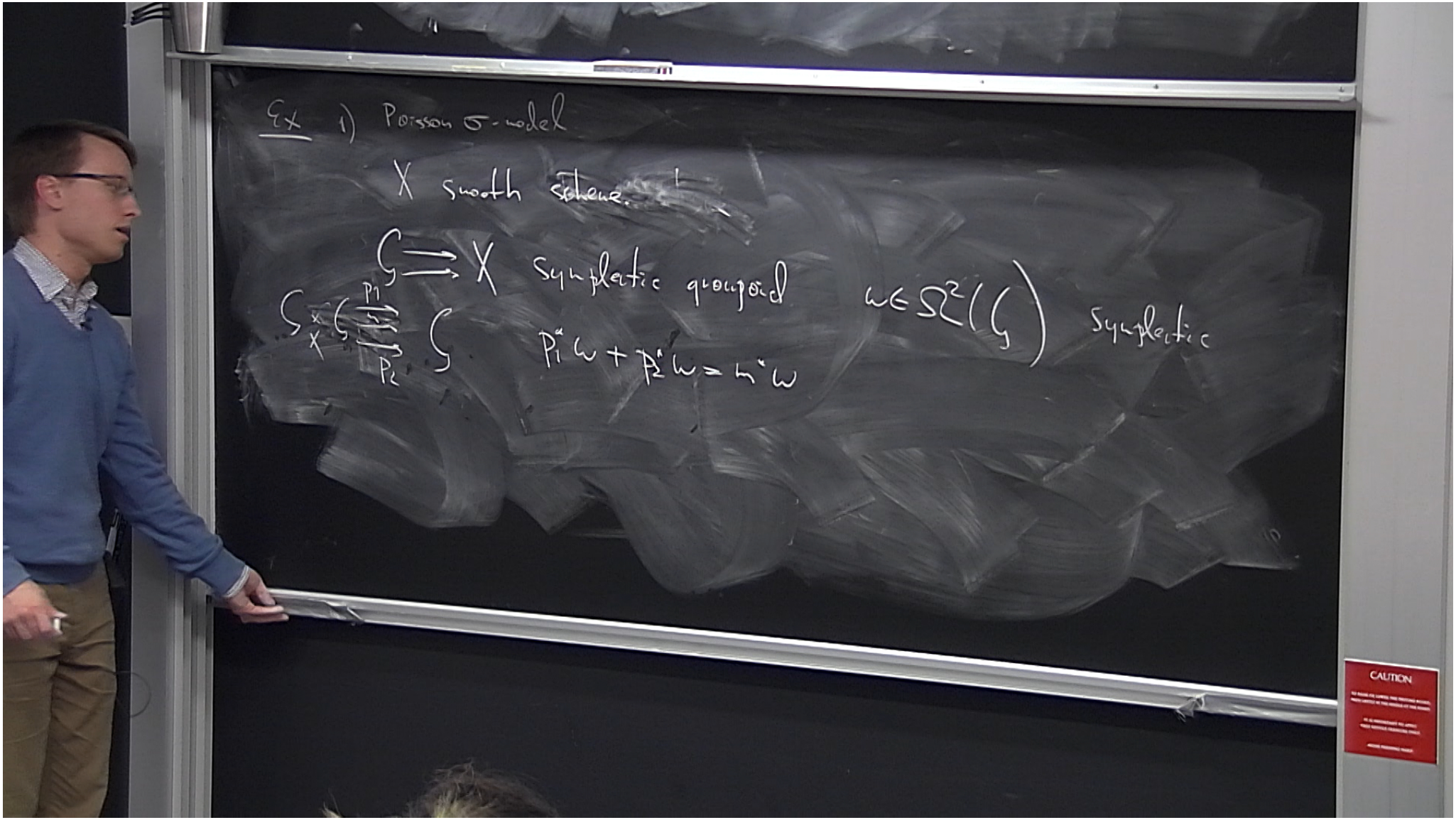
$\{$ n-shifted Lagrangians $f: L \to X \}$ $\subset$ $Pois(f, n)$

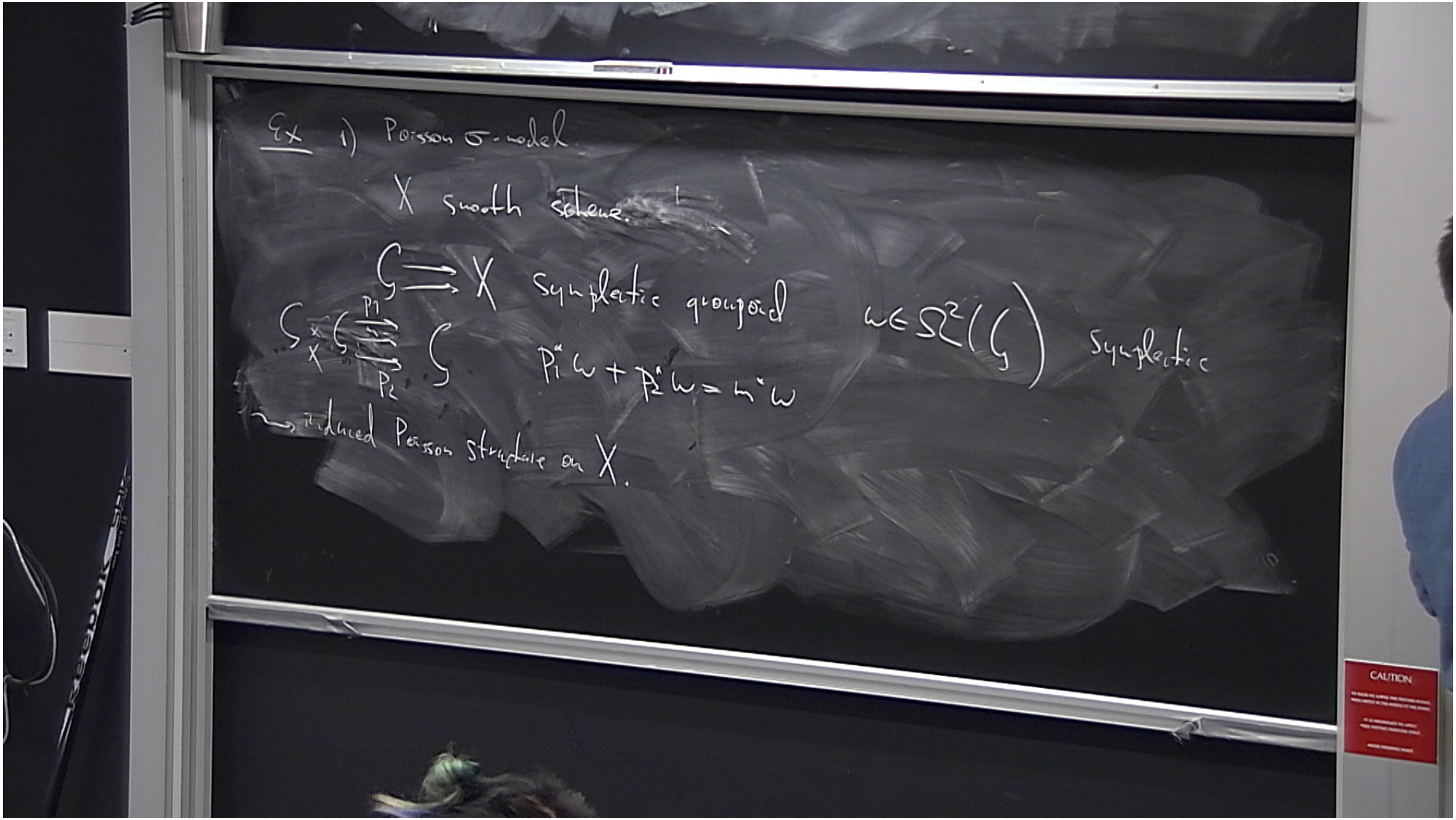

Suppose X n-shifted symplectic and f: L → X is Lagrangian.
⇒ L is (n-1)-shifted Poisson.
CAUTION

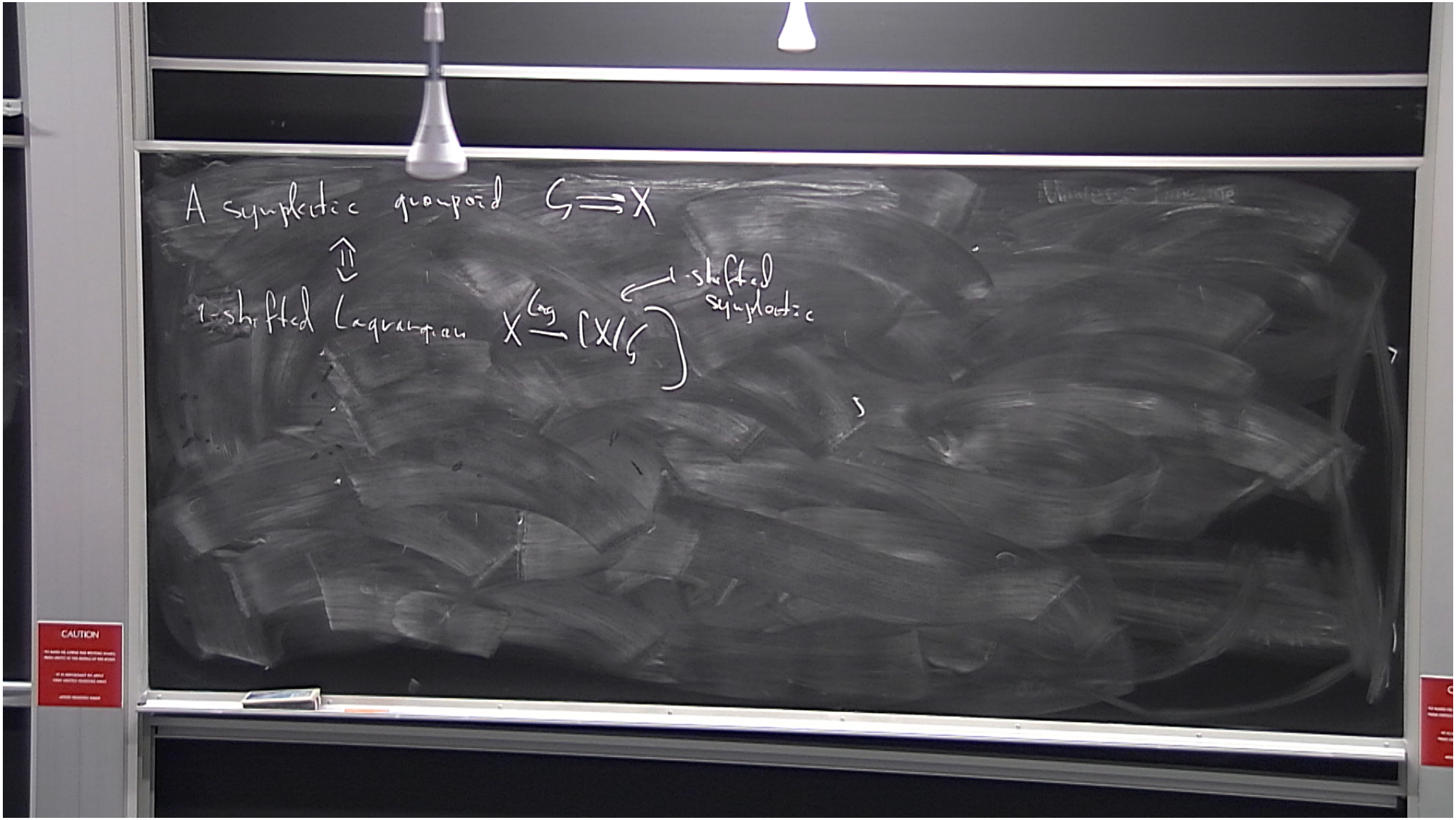

Suppose X n-shifted symplectic and f: L → X is Lagrangian
⇒ L is (n-1)-shifted Poisson.
Def Suppose L is n-shifted Poisson. A symplectic realization of L
is data of (n+1)-shifted Lagrangian f: L → X.
CAUTION

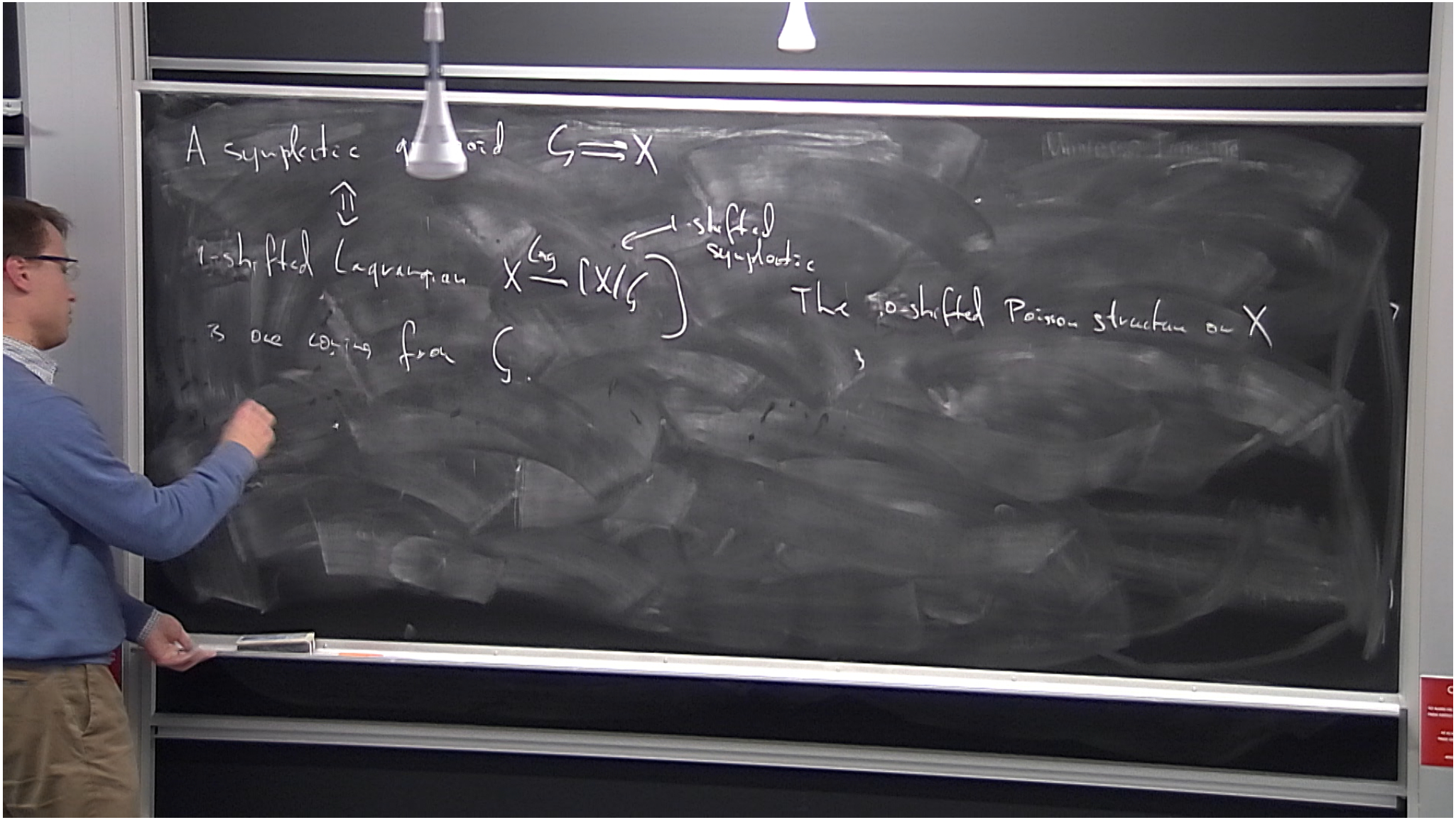

Ex 1) Poisson σ-model
X smooth scheme.
CAUTION

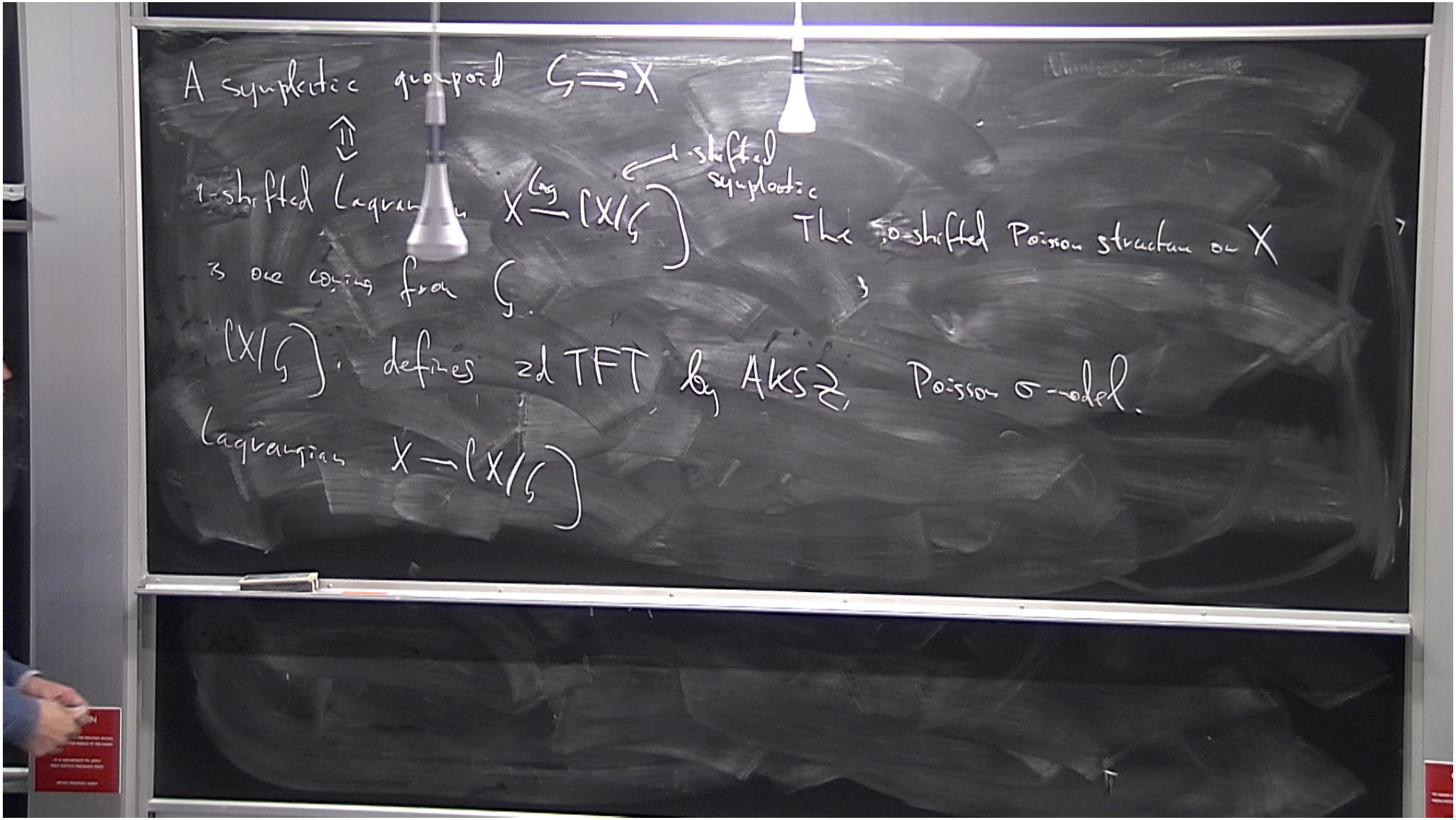

Ex 1) Poisson σ-model
X smooth scheme.
G ⇉ X symplectic groupoid
ω ∈ Ω²(G) symplectic
p₁*ω + p₂*ω = m*ω
CAUTION

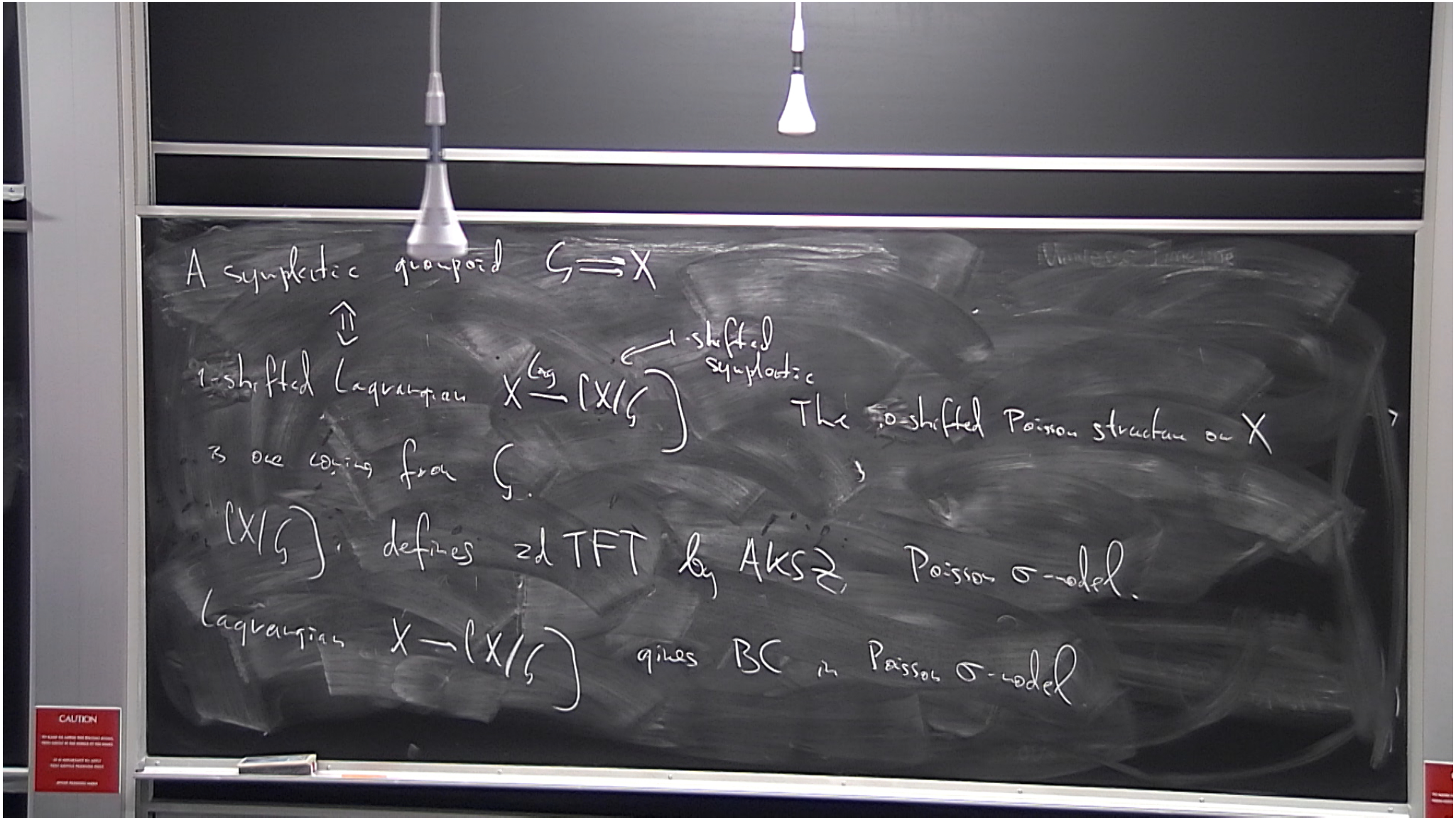

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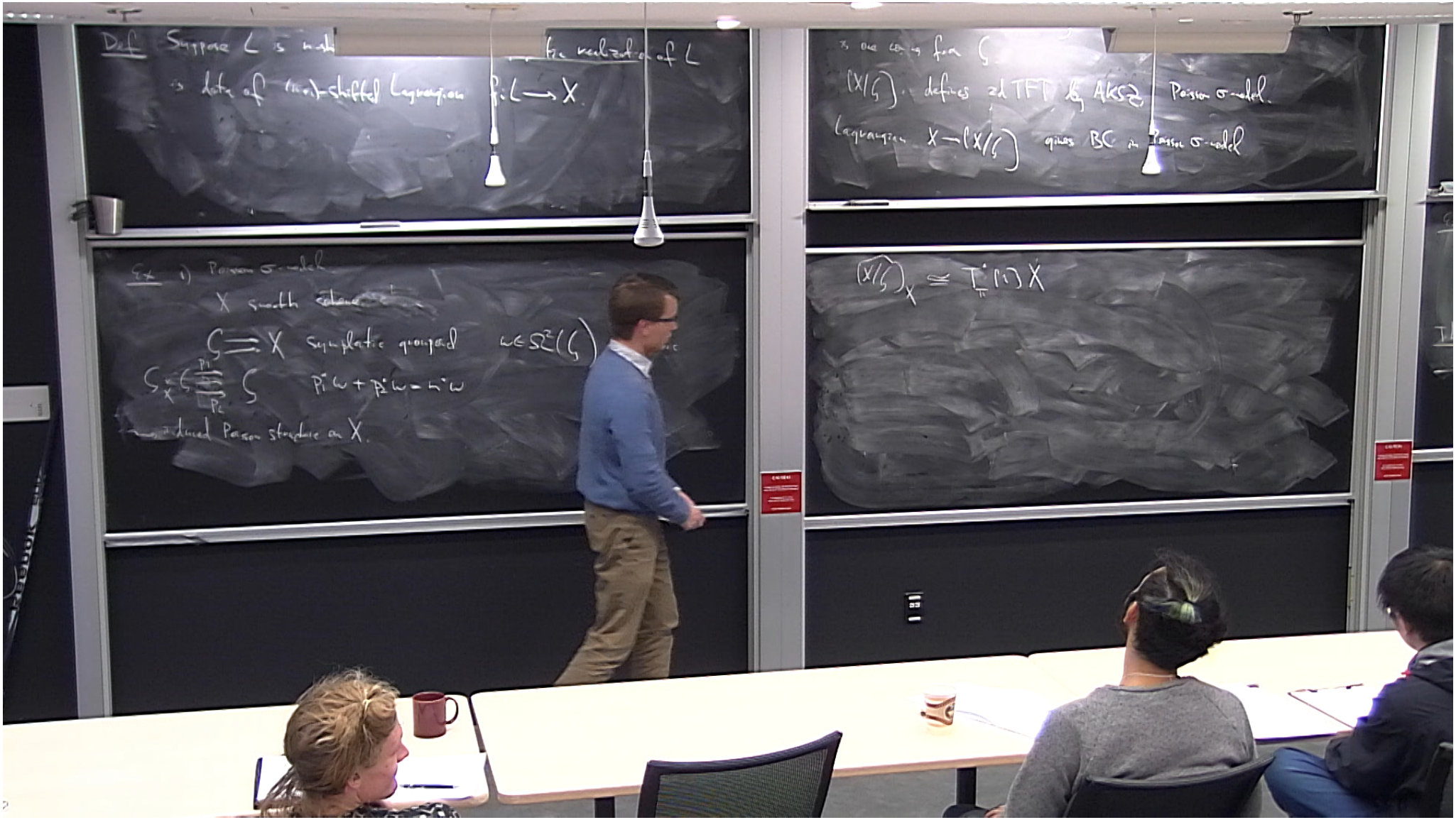

Ex 1) Poisson σ-model
X smooth scheme.
G ⇉ X symplectic groupoid
ω ∈ Ω²(G) symplectic
G ×_X G ⇉ G
p₁*ω + p₂*ω = m*ω
⇝ induced Poisson structure on X.
CAUTION

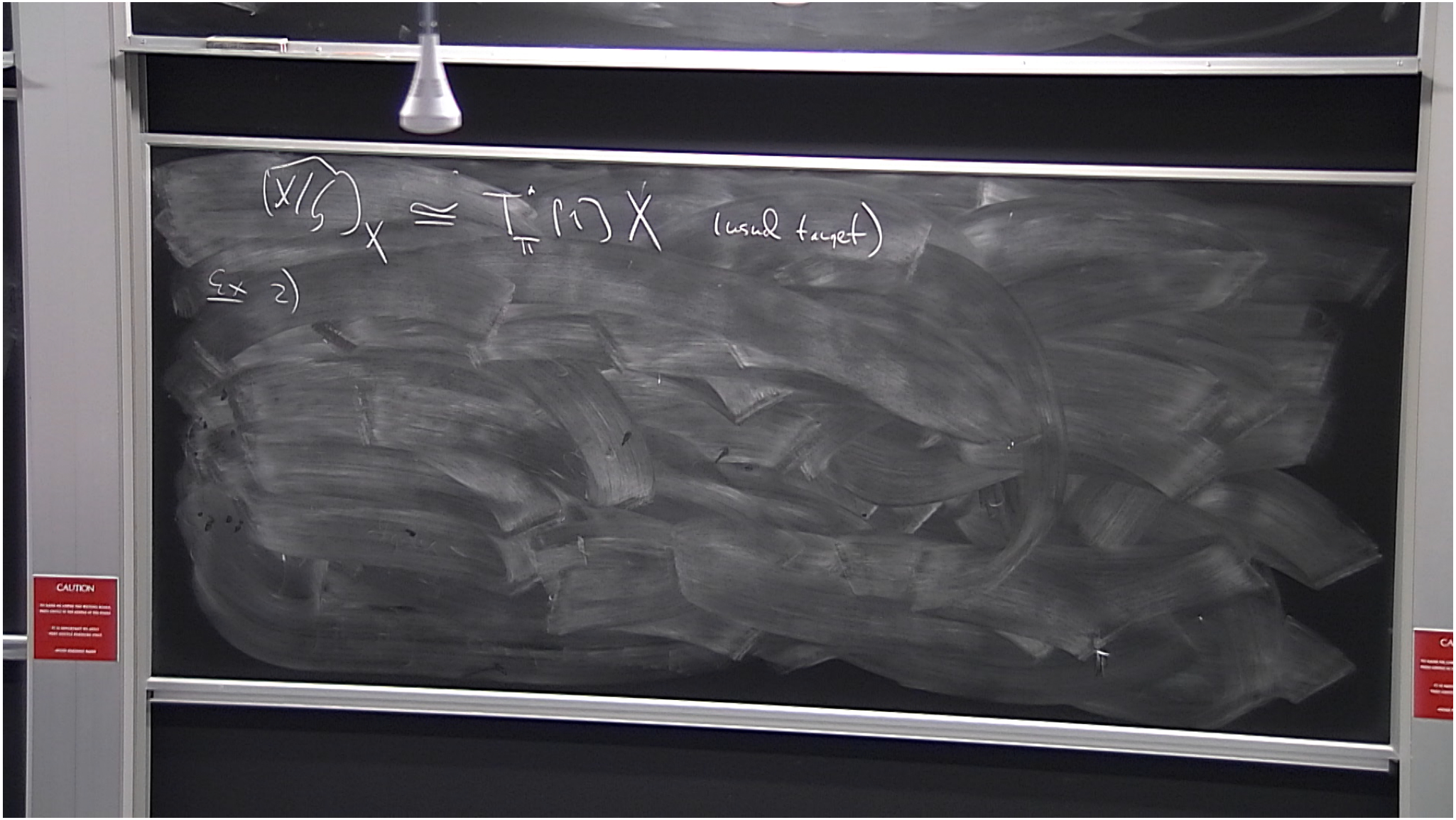

A symplectic groupoid $G \rightrightarrows X$
1-shifted Lagrangian $X \xrightarrow{Lag} [X/G]$
1-shifted symplectic

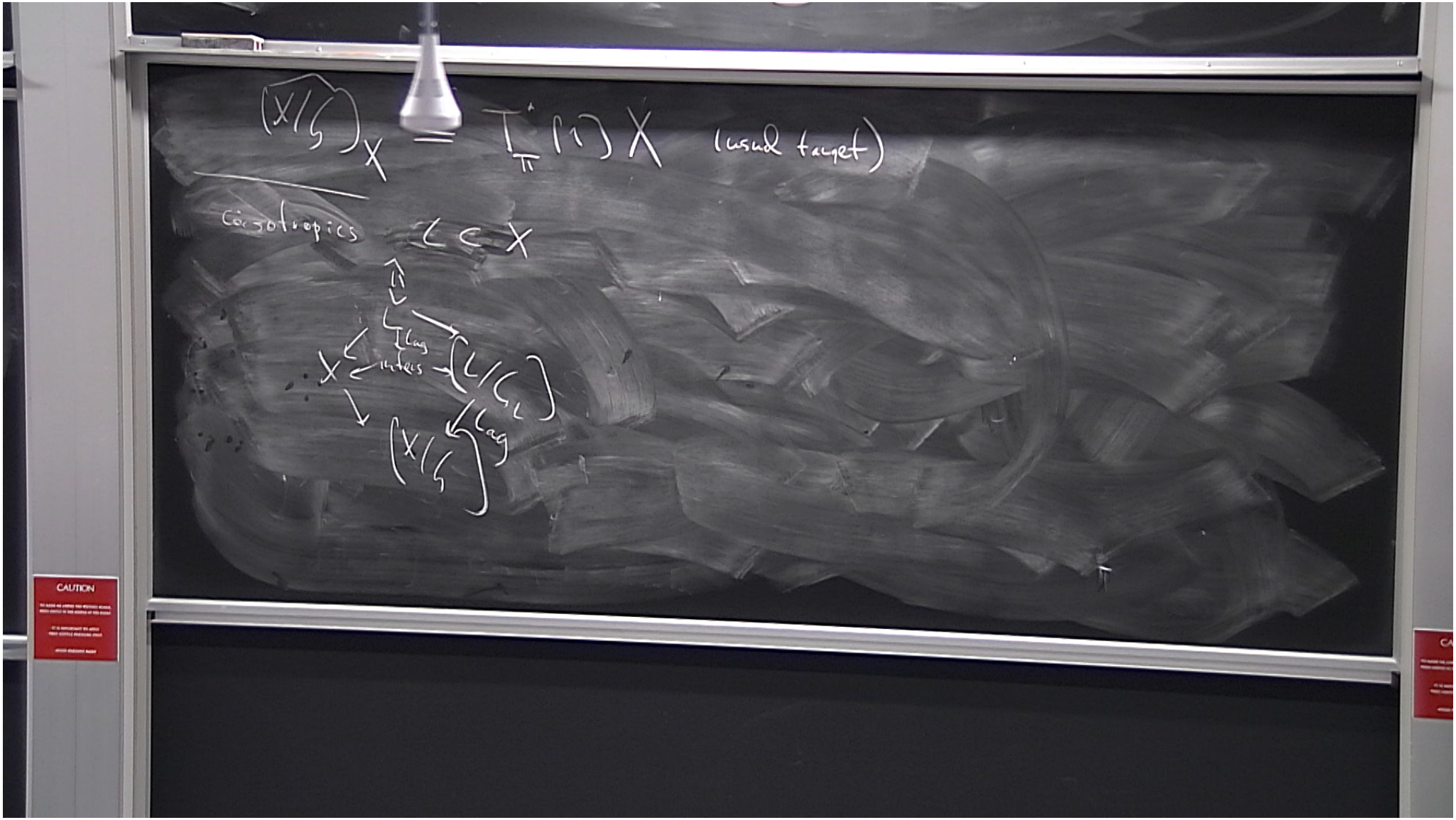

A symplectic
1-shifted Lagrangian
1-shifted symplectic
The 0-shifted Poisson structure on X
is the coming from G.


A symplectic groupoid $\mathcal{G} \rightrightarrows X$
1-shifted Lagrangian $X \xrightarrow{\text{Lag}} [X/\mathcal{G}]$
1-shifted symplectic
The 0-shifted Poisson structure on X
is the one coming from $\mathcal{G}$.
$[X/\mathcal{G}]$ defines 2d TFT by AKSZ, Poisson σ-model.
Lagrangian $X \to [X/\mathcal{G}]$


A symplectic groupoid $\mathcal{G} \rightrightarrows X$
1-shifted Lagrangian $X \xrightarrow{\text{Lag}} [X/\mathcal{G}]$
1-shifted symplectic
The 0-shifted Poisson structure on X
is one coming from $\mathcal{G}$.
$[X/\mathcal{G}]$ defines 2d TFT by AKSZ
Poisson σ-model.
Lagrangian $X \to [X/\mathcal{G}]$ gives BC in Poisson σ-model
CAUTION


Def
Ex 1) Poisson σ-model
X smooth
symplectic groupoid
defines 2d TFT
Poisson σ-model.
gives BC in Poisson σ-model


$(X/G)_x \cong T^*_{\pi(x)} X$ (usual tangent)
Ex 2)
CAUTION


$(X/G)_X = T^*_{\pi(x)} X$ (usual target)
Coisotropics $L \subset X$
Lag
Lag
L/G_L
(X/G)

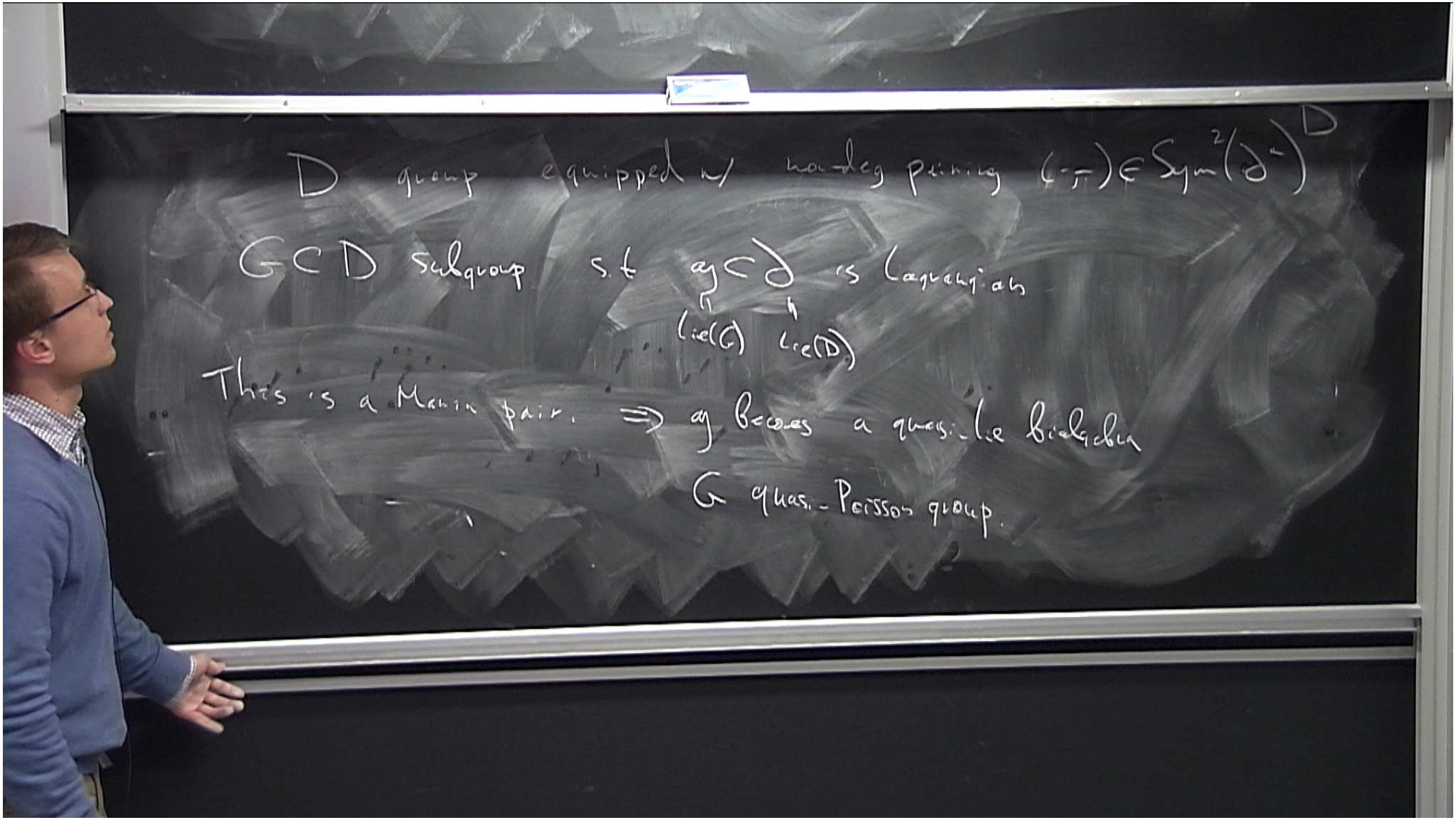
Ex 2) (Split) Chern-Simons

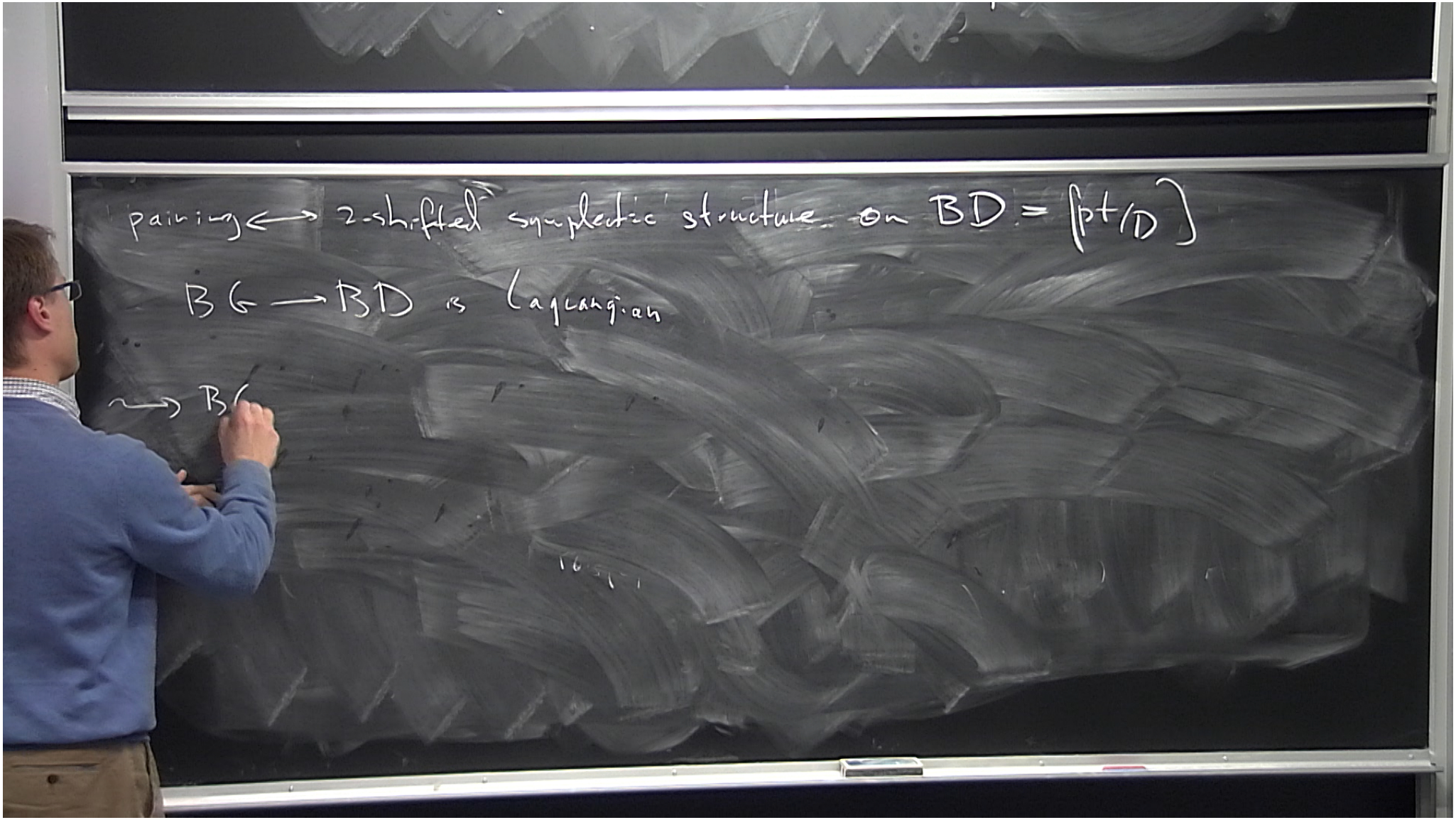
D group equipped w/ non-deg pairing $(-,-) \in \mathrm{Sym}^2(\mathfrak{d}^*)^D$

$G \subset D$ subgroup s.t. $\mathfrak{g} \subset \mathfrak{d}$ is Lagrangian

($\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{d} = \mathrm{Lie}(D)$)

This is a Manin pair.


D group equipped w/ mod-deg pairing $(\cdot,\cdot) \in \mathrm{Sym}^2(\mathfrak{d}^*)^D$
$G \subset D$ subgroup s.t. $\mathfrak{g} \subset \mathfrak{d}$ is Lagrangian
$\mathrm{Lie}(G)$ $\mathrm{Lie}(D)$
This is a Manin pair. $\Rightarrow$ $\mathfrak{g}$ becomes a quasi-Lie bialgebra
G quasi-Poisson group.


pairing ⟷ 2-shifted symplectic structure on BD = [pt/D]
BG ⟶ BD is Lagrangian
⇝ B(

pairing $\longleftrightarrow$ 2-shifted symplectic structure on $BD = [pt/D]$

$BG \longrightarrow BD$ is Lagrangian

$\rightsquigarrow$ BG is 1-shifted Poisson $\Longleftrightarrow$ quasi-Poisson structure on G.

This is the one obtained from Manin pair.

pairing $\longleftrightarrow$ 2-shifted symplectic structure on $BD = [pt/D]$

$BG \longrightarrow BD$ is Lagrangian

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BD defines Chern-Simons theory by AKSZ

$\uparrow$

BG

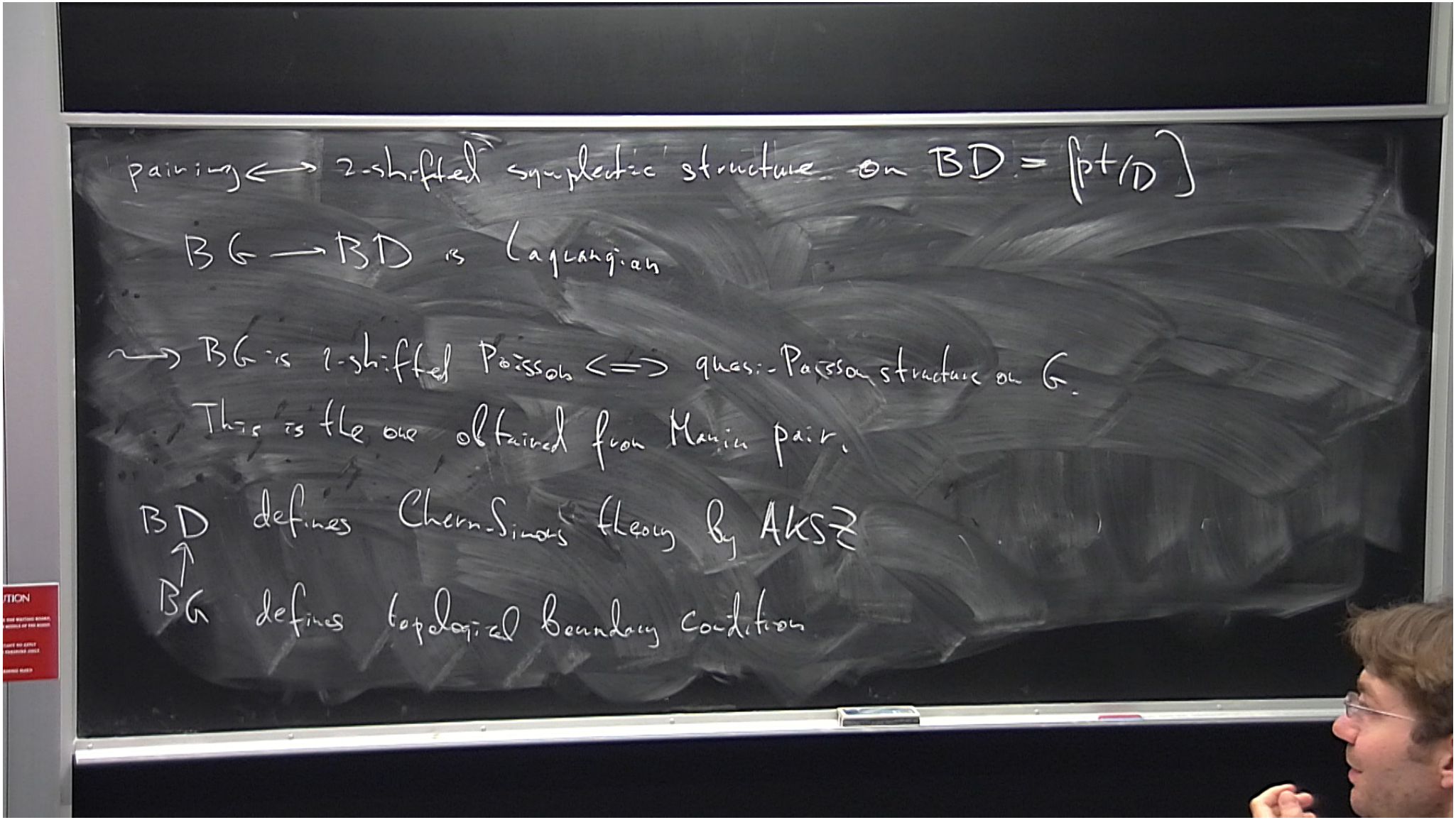
$BG \longrightarrow BD$ is Lagrangian

$\rightsquigarrow$ BG is 2-shifted Poisson $\Leftrightarrow$ quasi-Poisson structure on G.

This is the one obtained from Manin pair.

BD defines Chern-Simons theory by AKSZ

BG defines topological boundary condition


pairing ⟷ 2-shifted symplectic structure on BD = [pt/D]
BG → BD is Lagrangian
⇝ BG is 1-shifted Poisson ⟺ quasi-Poisson structure on G.
This is the one obtained from Manin pair.
BD defines Chern-Simons theory by AKSZ
BG defines topological boundary condition