Title: Vertex algebras and BV master equation

Date: May 11, 2017 09:00 AM

URL: http://pirsa.org/17050025

Abstract:

Vertex algebras & BV master equation

<u>motivation</u>: $e^{u/t} = 1 + \frac{u}{t} + \frac{u^2}{2!\,t^2} + \frac{u^3}{3!\,t^3} + \cdots$

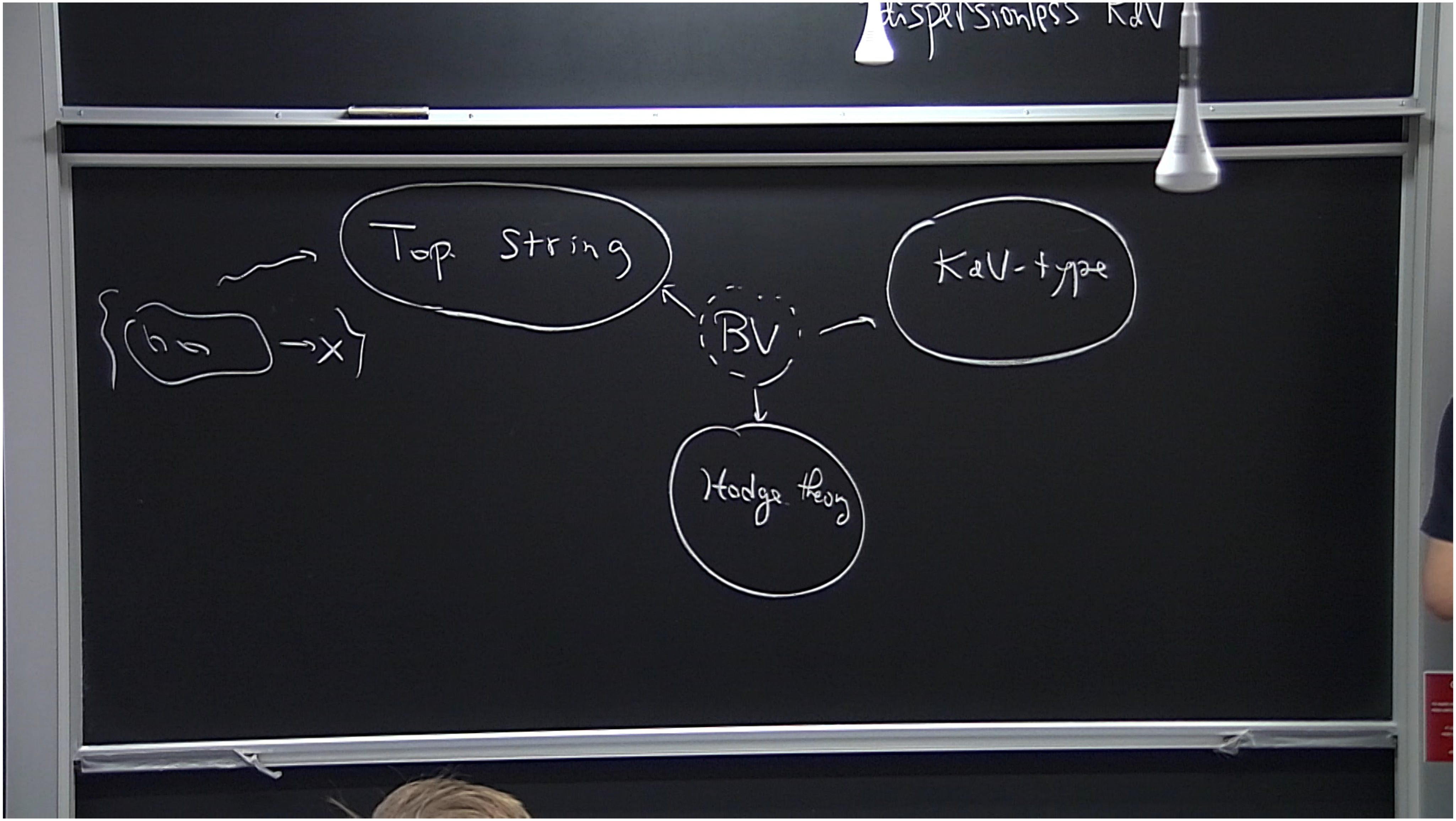
Vertex algebras & BV master equation

motivation: $e^{u/t} = 1 + \frac{u}{t} + \frac{u^2}{2!\,t^2} + \frac{u^3}{3!\,t^3} + \cdots$

promote $u \longrightarrow u(x)$, introduce Poisson bracket

$$\{u(x), u(y)\} = \partial_x \delta(x-y)$$

$$h_k = \frac{1}{(k+1)!}\oint dx\, u^{k+1}(x) \quad \Longrightarrow \quad \{h_k, h_m\} = 0 \ \forall k, m$$


ispersionless KdV
Top String
KdV-type
BV
Hodge theory

main Construction (w/ He, Too): Let X be CY

B-model (BCOV) $X \times \mathbb{C}$

$\downarrow$ = integrating out massive modes along X

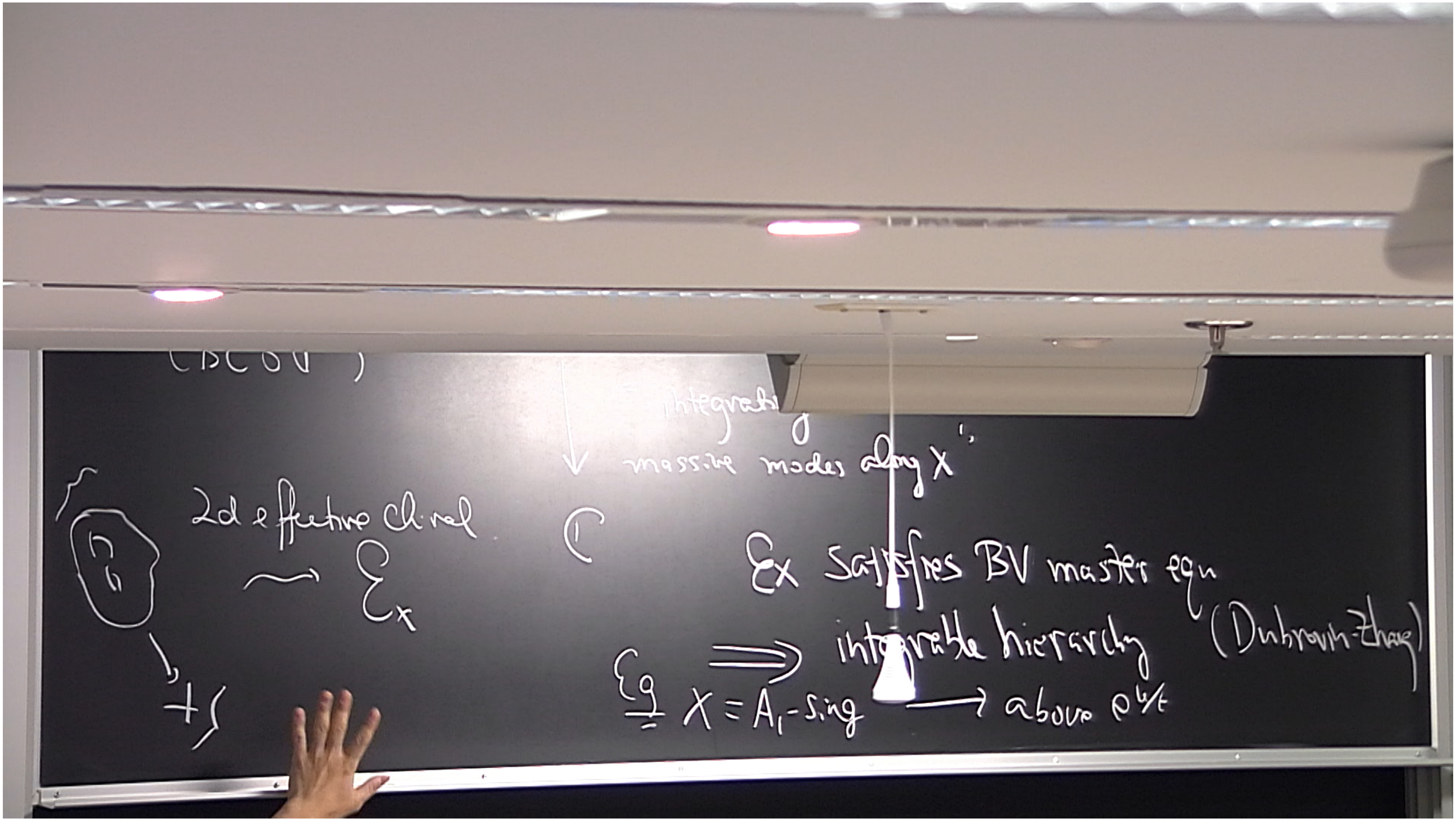
2d effective chiral $\mathbb{C}$

$\rightsquigarrow \mathcal{E}_X$

$\mathcal{E}_X$ satisfies BV master eqn

$\Longrightarrow$ integrable hierarchy (Dubrovin-Zhang)

Eg $X = A_1$-sing


integrating
massive modes along X
2d effective chiral
E_X
E_X satisfies BV master eqn
integrable hierarchy
(Dubrovin-Zhang)
Eg X = A_1-sing
above

BV field theory

$\mathcal{E} = \Gamma(X, E^{\bullet})$, Q, ω

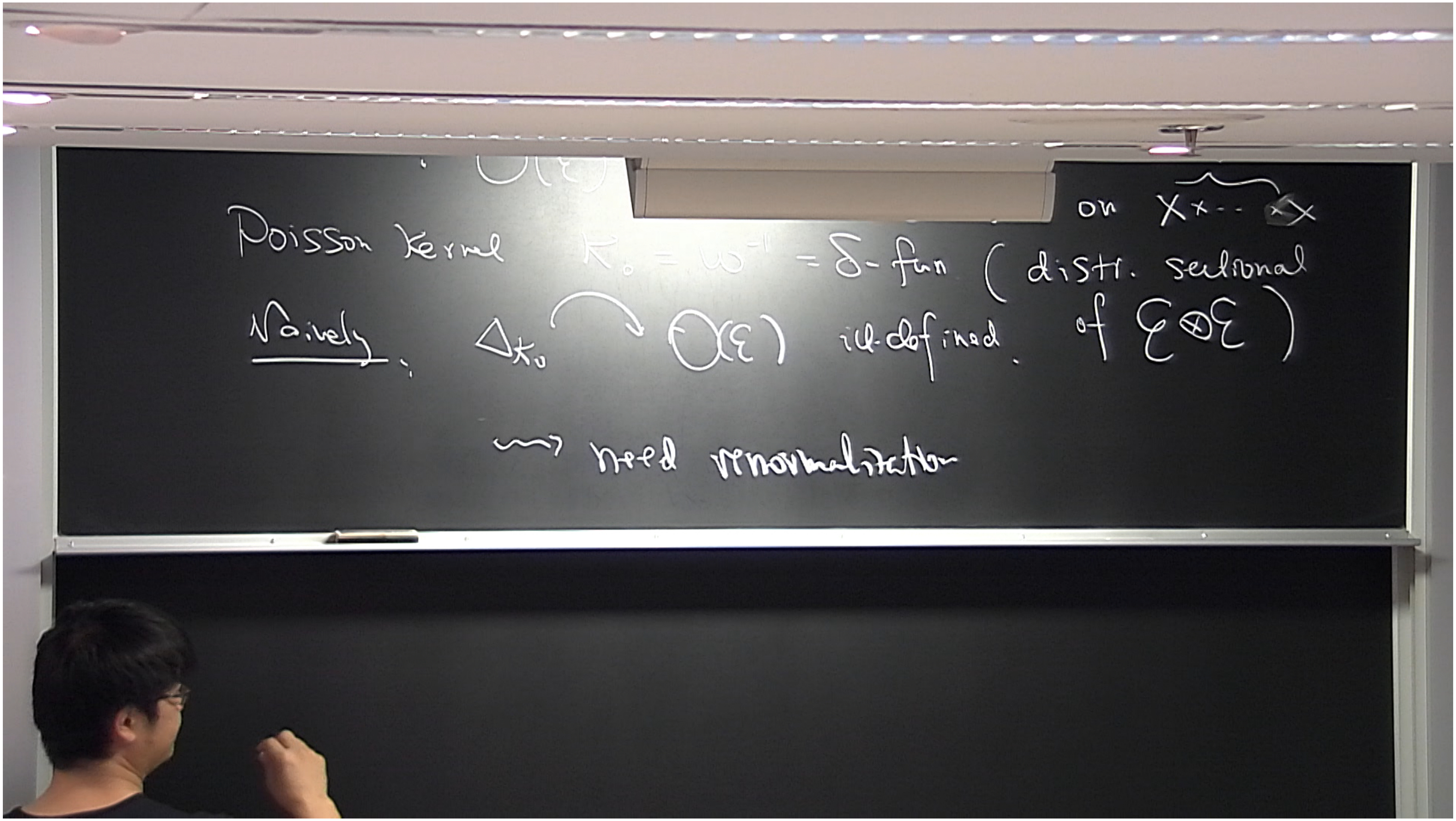
- $(\mathcal{E}, Q)$ elliptic complex
- ω local (-1)-symplectic $\quad \omega(\alpha, \beta) = \int_X \langle \alpha, \beta \rangle$
- $S_0 = \underbrace{\omega(Q-,-)}_{\text{free}} + I_0, \qquad I_0 = \int_X \mathcal{L}_0$

Satisfies CME: $\{S_0, S_0\} = 0 \iff Q I_0 + \frac{1}{2}\{I_0, I_0\} = 0$

Quantization. Naively: " $QI + \hbar \Delta I + \frac{1}{2}\{I, I\} = 0$ "

- $\mathcal{E}^* = \mathrm{Hom}(\mathcal{E}, \mathbb{R})$ = distributions
- $\mathcal{O}(\mathcal{E}) = \prod_n \mathrm{Sym}^n(\mathcal{E}^*)$ distr. on $\overbrace{X \times \cdots \times X}^{n}$

Poisson kernel $K_0 = \omega^{-1} = \delta$-fun. (distr. sectional of $\mathcal{E} \otimes \mathcal{E}$)


Poisson kernel K₀ = ω⁻¹ = δ-fun (distr. sectional
on X×···×X
of E⊗E)
Naively: Δ_{K₀} ↷ O(E) ill-defined.
⇝ need renormalization

Costello's homotopic renormalization

idea: $H^*(\text{dist}, Q) = H^*(\text{Smooth}, Q)$

$$\Longrightarrow \underbrace{K_0}_{\text{Sing}} = \underbrace{K_r}_{\text{Smooth}} + Q P_r .$$

$\Delta_{K_r} \curvearrowright \mathcal{O}(\mathcal{E})$ well-defined.

effective QME: $QI_r + \hbar\Delta_r I_r + \frac{1}{2}\{I_r, I_r\}_r = 0$

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<u>Naively</u>: $\Delta_{K_0} \curvearrowright \mathcal{O}(\mathcal{E})$ ill-defined.

$\rightsquigarrow$ need renormalization

$\Longrightarrow K$ ...

Sing. Smooth.

$\Delta_{K_r} \curvearrowright \mathcal{O}(\mathcal{E})$ well-defined. $BV_r = (\mathcal{O}(\mathcal{E}), Q, \Delta_{K_r})$

effective QME: $Q I_r + \hbar \Delta_r I_r + \frac{1}{2}\{I_r, I_r\}_r = 0$

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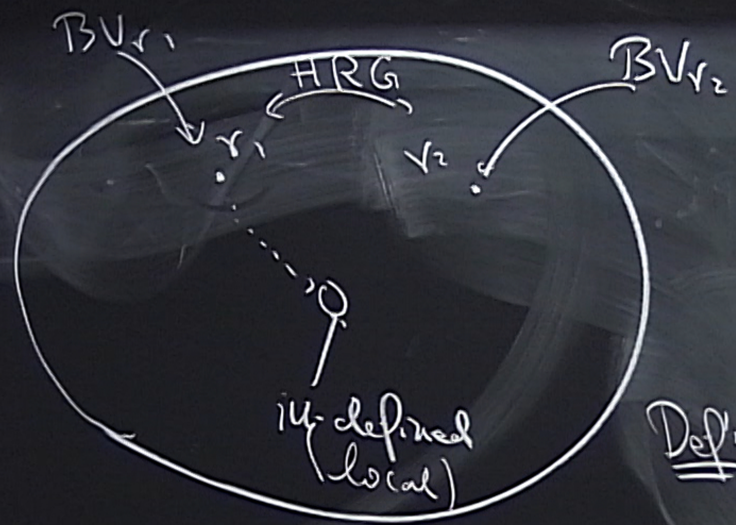
Poisson kernel $K_0 = \omega^{-1} = \delta$-fun (distr. sectional



effective sol'n of QME:

$\{I_r\}$, $I_r \in \mathcal{O}(\mathcal{E})[[\hbar]]$

solves QME at BV_r

compatible w/. HRG.

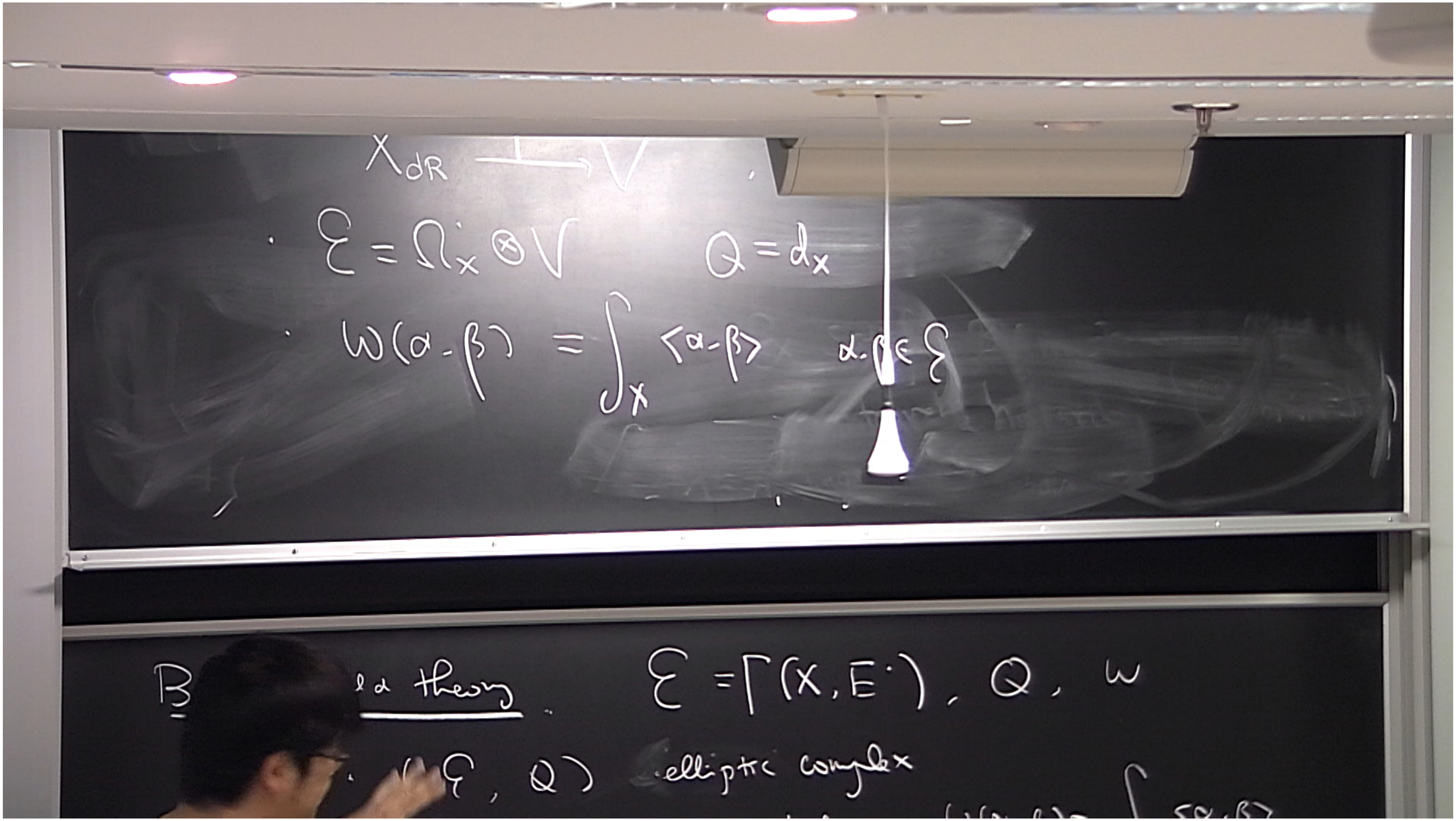
Def'n: We say $\{I_r\}$ is UV finite if $\lim_{r \to 0} I_r$ exists as a local func.

__Eg__: $X = 1\,d$ $(\mathbb{R}, S^1, (\ \ ), \ldots)$

$X_{dR} = (X, \Omega^{\bullet}_X)$ V: graded vector space w/ 0-symplectic pairing $\langle \cdot, \cdot \rangle$

$X_{dR} \xrightarrow{\varphi} V$, $\varphi \in \Omega^{\bullet}_X \otimes V$.

· $\mathcal{E} = \Omega^{\bullet}_X \otimes V$



$X_{dR} \longrightarrow V$,
· $\mathcal{E} = \Omega^{\cdot}_X \otimes V \quad Q = d_X$
· $\omega(\alpha, \beta) = \int_X \langle \alpha, \beta \rangle \quad \alpha, \beta \in \mathcal{E}$
theory $\quad \mathcal{E} = \Gamma(X, E^{\cdot}), \; Q, \; \omega$
$(\mathcal{E}, Q)$ elliptic complex

$$\begin{array}{ccc} \mathcal{O}(V) & \longrightarrow & \mathcal{O}_{loc}(\mathcal{E}) \\ \rotatebox{90}{$\in$} & & \\ I & \longrightarrow & \hat{I}(\varphi) = \int_X \varphi^*(I), \quad \varphi \in \mathcal{E} \end{array}$$

<u>Thm</u> (Grady-Li-L) $S = \text{free} + \hat{I}$ $\quad \boxed{\deg(\hat{I}) = \deg I - 1}$

($I \in \mathcal{O}(V)[[\hbar]]$)

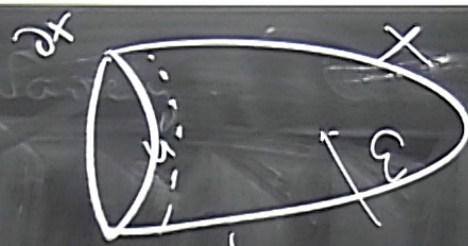
① theory is UV finite $\quad \deg I = 1$

② effective sol'n of QME $\Longleftrightarrow$ $[I, I]_\star = 0$

Commutator w.r.t. Moyal product on $\mathcal{O}(V)[[\hbar]]$

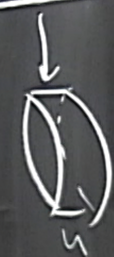
effective QME: $Q I_r + \hbar \Delta_r I_r + \frac{1}{2}\{ I_r, I_r \}_r = 0$

· Boundary



QME for $\mathcal{E}$

$\partial X \times I \sim$

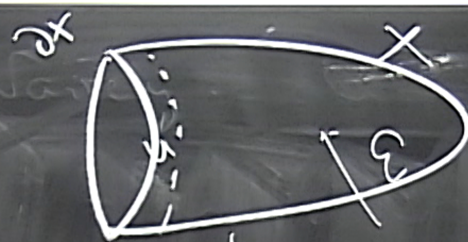


$\mathcal{E}|_u = \mathcal{E}^\partial \otimes \Omega^\cdot_I$

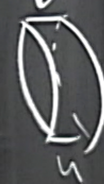
$\iff \Omega^\cdot_I \longrightarrow \mathcal{E}^\partial$

effective QME: $Q I_r + \hbar \Delta_r I_r + \frac{1}{2}\{I_r, I_r\}_r = 0$

· Boundary



QME for $\mathcal{E}^{\partial}$

QME $\rightsquigarrow$ $\partial X \times I \sim$ 

$\mathcal{E}|_U = \mathcal{E}^{\partial} \otimes \Omega^{\bullet}_I$

$\Longleftrightarrow \quad \Omega^{\bullet}_I \longrightarrow \mathcal{E}^{\partial}$

$\Downarrow$

$\deg I = 1 \qquad I \in \mathcal{O}(\mathcal{E}^{\partial})[[\hbar]] \qquad \underbrace{\mathcal{E}^{\partial} = T^*B}, \quad I \curvearrowright$ Fock rep

$I^2 = 0$

2d Chiral theory

free CFT's: $\int \partial\phi \wedge \bar\partial\phi$, $\int b \wedge \bar\partial c$, $\int \beta \wedge \bar\partial \gamma$;

boson — bc-system — $\beta\gamma$-system.

$$\boxed{S = \text{free} + \int \mathcal{L}^{hol}(\partial_z \phi, b, c, \beta, \gamma)}$$

only hol derivatives

$\underline{\text{Eg}}$: $\Sigma = 2d$. $\quad V$: 0-symplectic

- $\mathcal{E} = \Omega_{\Sigma}^{0,*} \otimes V$, $\quad Q = \bar{\partial}$

- $\omega(\alpha, \beta) = \int_{\Sigma} dz \cdot \langle \alpha, \beta \rangle$

$\Rightarrow (\mathcal{E}, \bar{\partial}, \omega)$.

$V \longrightarrow$ Vertex algebra $\mathcal{V}$

Choose $\{a^i\}$ basis of V^*

$\mathcal{V} = \mathbb{C}[[\partial_z^k a^i]]$ as $\mathbb{C}$-space

OPE : $a^i(z)\, a^j(w) \sim \hbar \dfrac{\langle a^i, a^j \rangle}{z-w}$

$\longrightarrow \mathcal{V}$ is a combination of bc, $\beta\gamma$-system.

Lie algebra $\oint \mathcal{V}$

$= \mathrm{Span}_{\mathbb{C}} \left\{ \oint z^k A(z) \right\}_{A \in \mathcal{V},\ k \in \mathbb{Z}}$

$[\cdot,\cdot]$ is determined by OPE

$\underline{I}^2 = 0$

$$\oint V \longrightarrow \mathcal{O}_{loc}(\Sigma) \qquad \varphi \in \Omega^{0,*}_{\Sigma} \otimes V$$

$$I = \oint \partial_z^{k_1} a^{i_1} \cdots \partial_z^{k_n} a^{i_n} \longrightarrow \underline{\hat{I}}(\varphi) = \int_{\Sigma} dz \; \partial_z^{k_1} a^{i_1}(\varphi) \cdots \partial_z^{k_n} a^{i_n}(\varphi)$$

$$\underline{I}^2 = 0$$

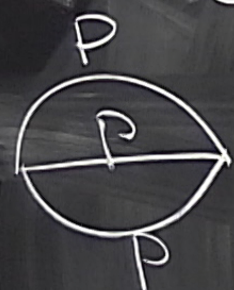
$$I = \oint \partial_z^{k_1} a^{i_1} \cdots \partial_z^{k_n} a^{i_n} \longrightarrow \hat{\underline{I}}(\varphi) = \int_\Sigma dz\, \partial_z^{k_1} a^{i_1}(\varphi) \cdots \partial_z^{k_n} a^{i_n}(\varphi)$$

$$\boxed{\deg \hat{\underline{I}} = \deg \underline{I} - 1}$$

· effective sol'n of OmE $\Longleftrightarrow$ $[I, I] = 0$ in $\oplus V [[\hbar]]$

Eg $\Sigma = E_\tau = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$.

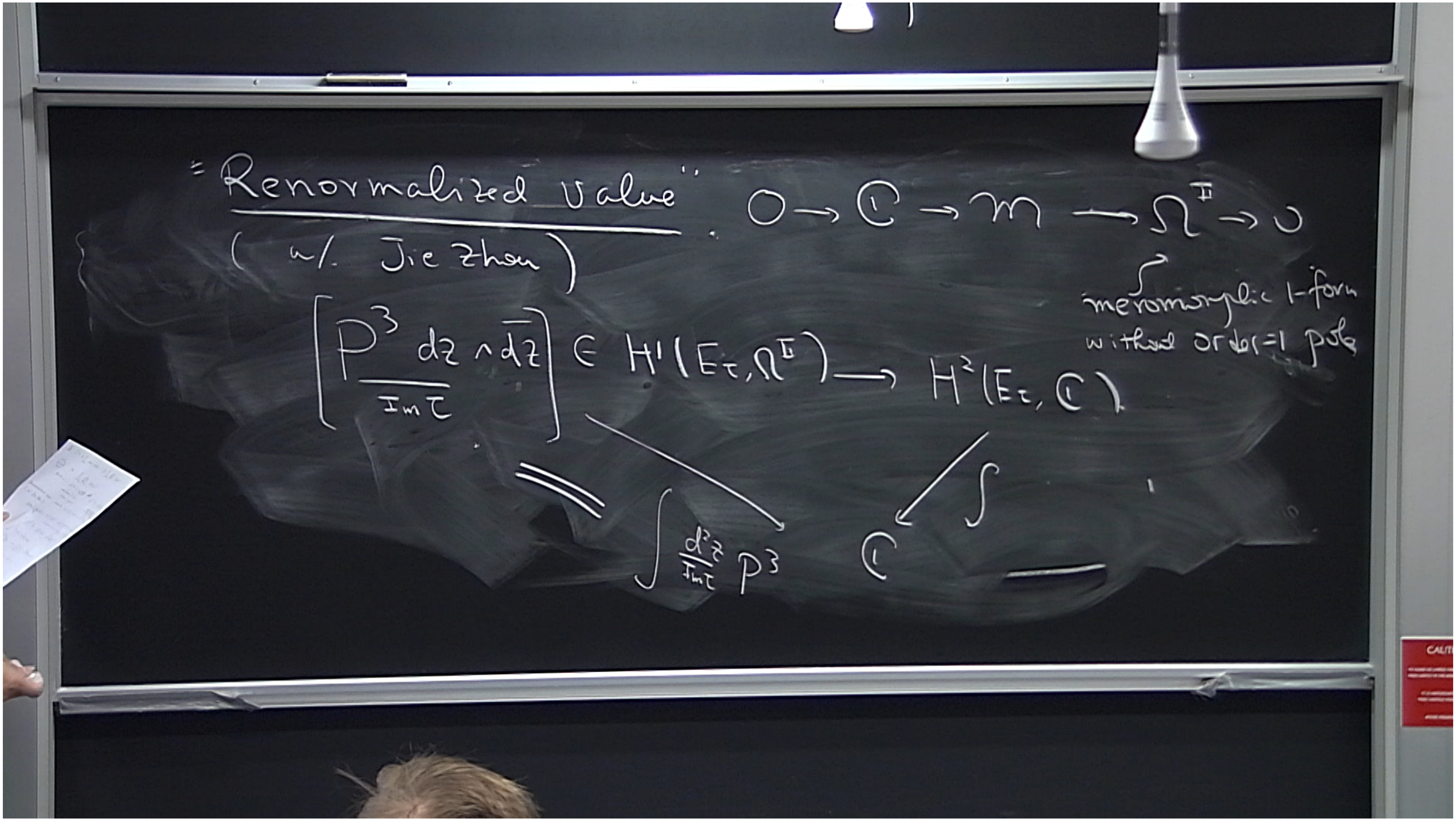
$$S = \int \partial\phi \wedge \partial\phi + \int \frac{d^2z}{\mathrm{Im}\,\tau} (\partial_z \phi)^3$$



$$= \int \frac{d^2z}{\mathrm{Im}\,\tau} P^3 \qquad P = \wp(z,\tau) + \frac{\pi^2}{3} E_2^*.$$

Weierstrass $\wp$ $\left(E_2^* = E_2 - \frac{3}{\pi} \frac{1}{\mathrm{Im}\,\tau} \right)$

I


"Renormalized value"
(w/. Jie Zhou)
$0 \to \mathbb{C} \to \mathcal{M} \to \Omega^{\mathrm{II}} \to 0$
meromorphic 1-form without order=1 pole
$\left[\frac{P^3}{\mathrm{Im}\,\tau} dz \wedge d\bar{z} \right] \in H^1(E_\tau, \Omega^{\mathrm{II}}) \longrightarrow H^2(E_\tau, \mathbb{C})$
$\int \frac{d^2 z}{\mathrm{Im}\,\tau} P^3$
$\mathbb{C}$
$\int$

$$\int p^3 \rightsquigarrow \frac{2^2}{3^3\cdot 5}E_6 + \frac{2}{3^2\cdot 5}E_4 E_2^* - \frac{2}{3^5}(E_2^*)^3$$

$$\Big\downarrow\; E_2^* \to E_2$$

1-descendant GW $\#\Big\{$  $\longrightarrow E \Big\}$ (Dijkgraaf)

$$\text{1-descendant GW} \;\#\; \{ \Sigma \to E \} \quad (\text{Dijkgraaf})$$

$$\Downarrow$$

$$\text{turning more terms} \rightsquigarrow \text{full GW theory} \quad (L)$$

$$V \longrightarrow \text{Vertex algebra } \mathcal{V}$$

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Lie algebra $\oint \mathcal{V}$

$= \mathrm{Span}\{ \oint z^k A(z) \}$

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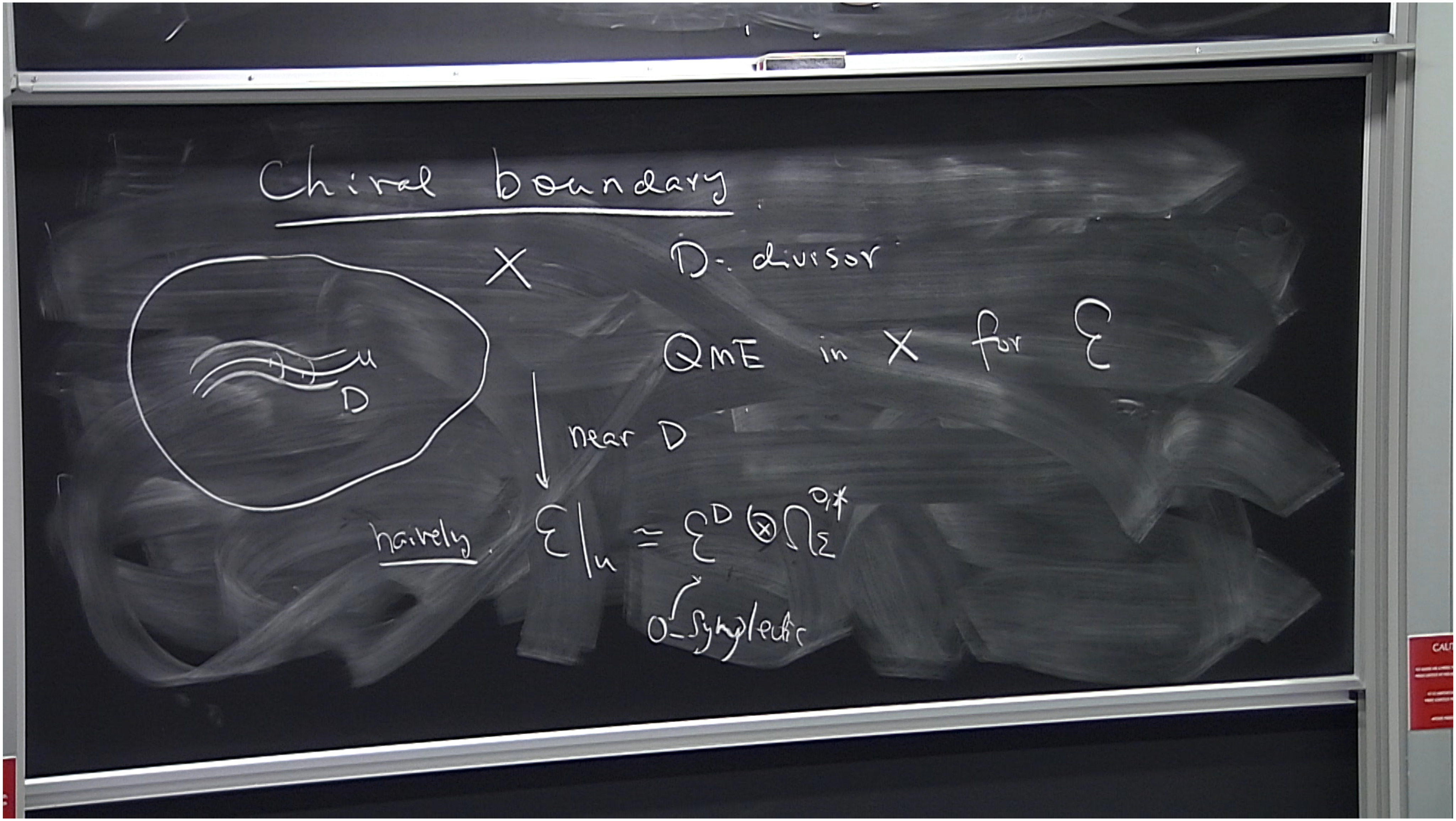
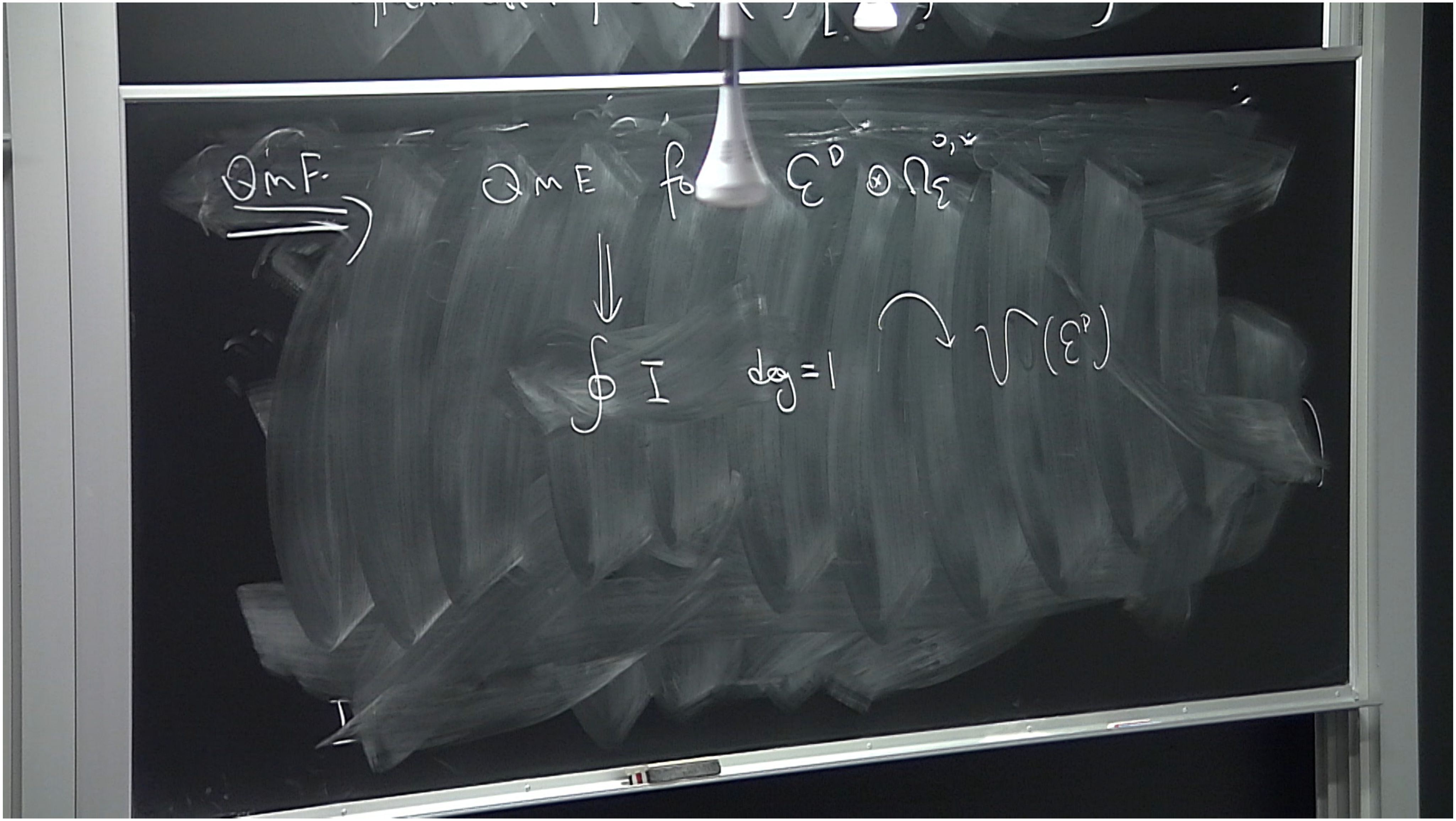

Chiral boundary
X
D - divisor
U
D
QME in X for ℰ
near D
naively
ℰ|U ≃ ℰ^D ⊗ Ω_Σ^{0,*}
0-symplectic


QMF
QME for
∮ I
deg=1