Title: Perturbative BV-BFV theories on manifolds with boundary Part 2

Date: May 09, 2017 09:00 AM

URL: http://pirsa.org/17050017

Abstract:

The polarization

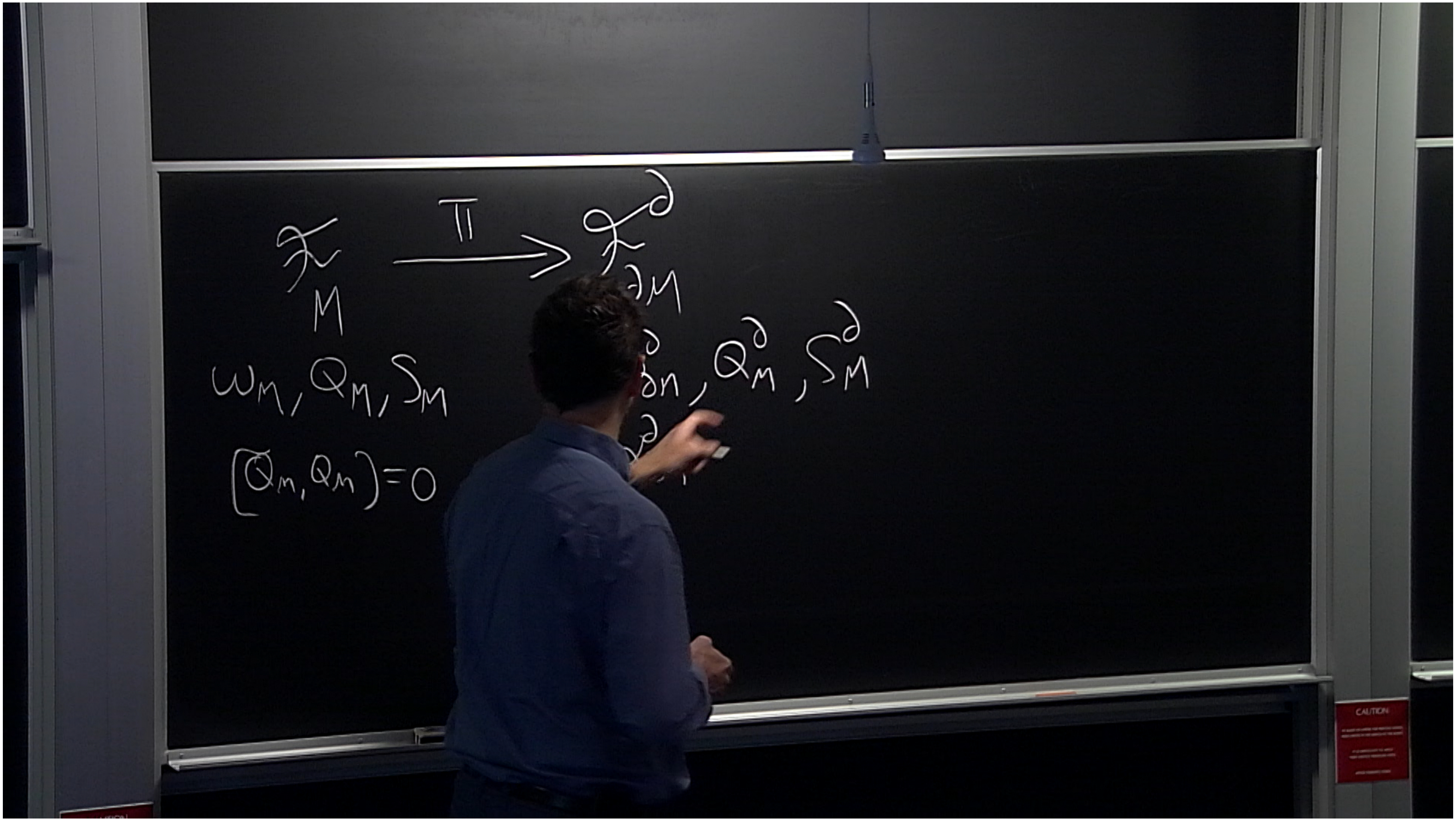
- Assume we have an involutive Lagrangian distribution $\mathcal{P}$ on $\mathcal{F}^\partial$, called a polarization, such that the restriction of α^∂ to its leaves is zero. We may use gauge transformations to adapt α^∂.
- For simplicity we assume $\mathcal{B} := \mathcal{F}^\partial/\mathcal{P}$ to be smooth.
- The crucial assumption now is that we have a splitting

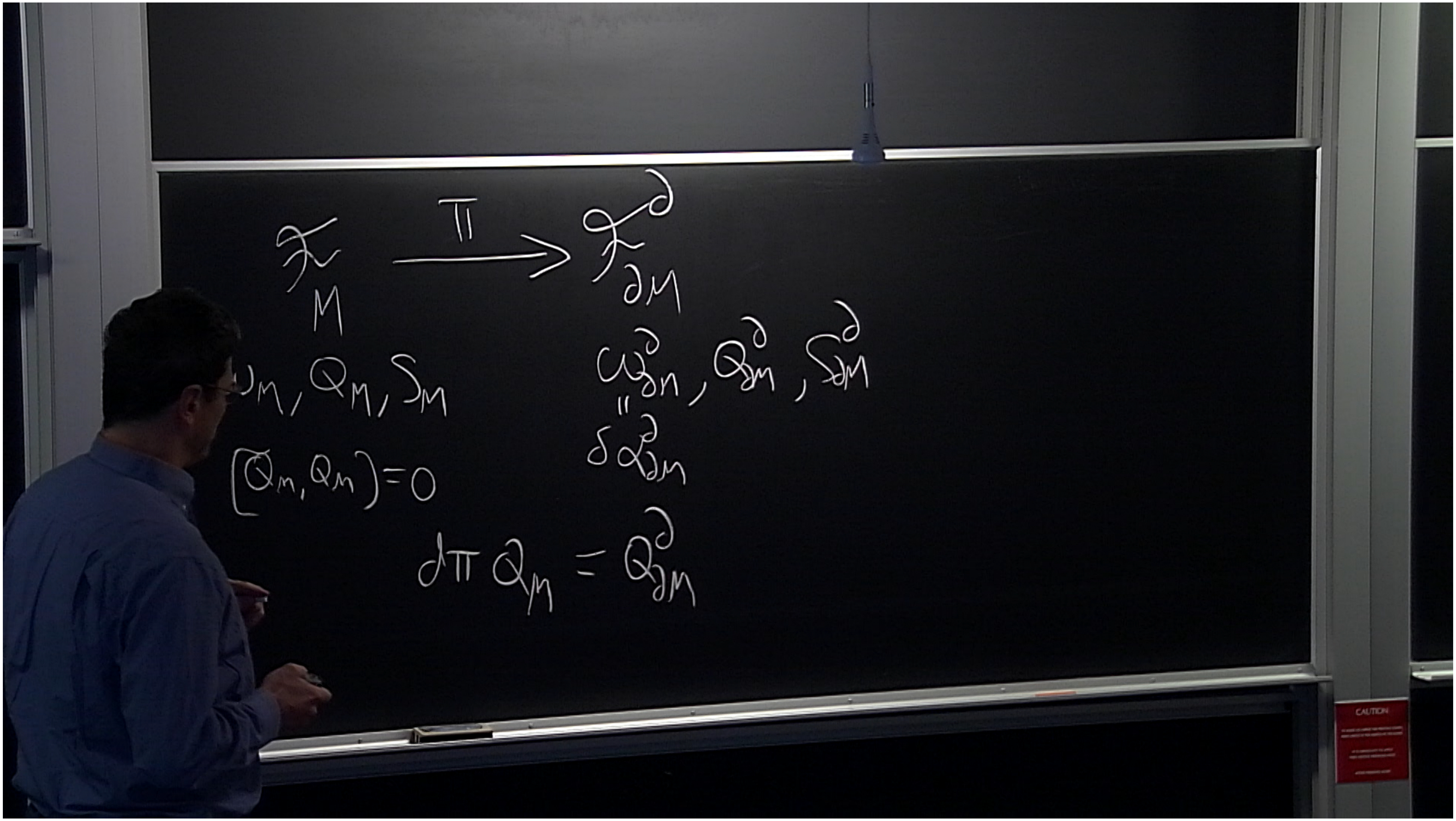
$$\mathcal{F} = \mathcal{Y} \times \mathcal{B}$$

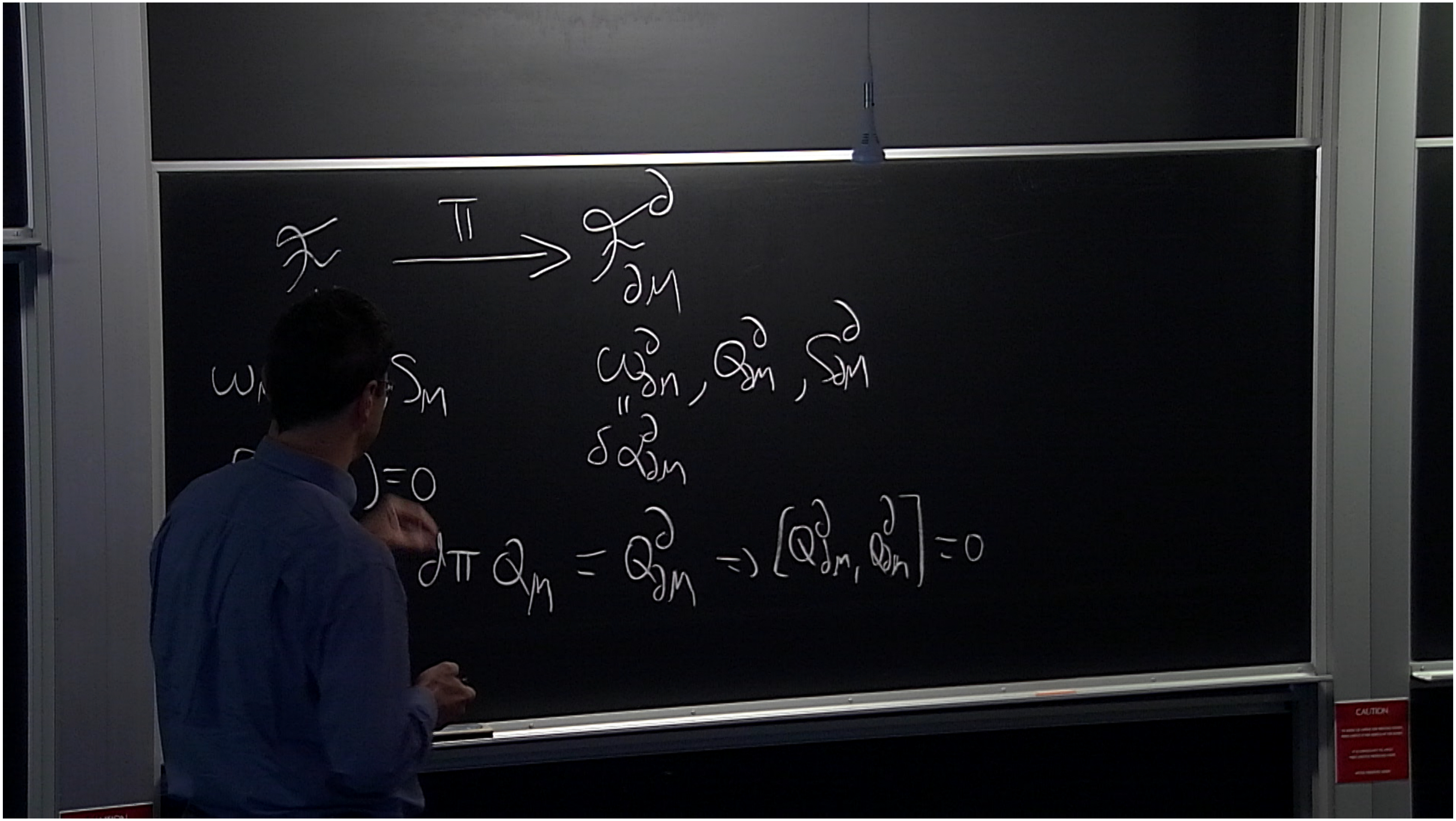
such that the BV form ω only has components along $\mathcal{Y}$ and is constant on $\mathcal{B}$. (A splitting is always possible locally; the crucial condition is on ω.)

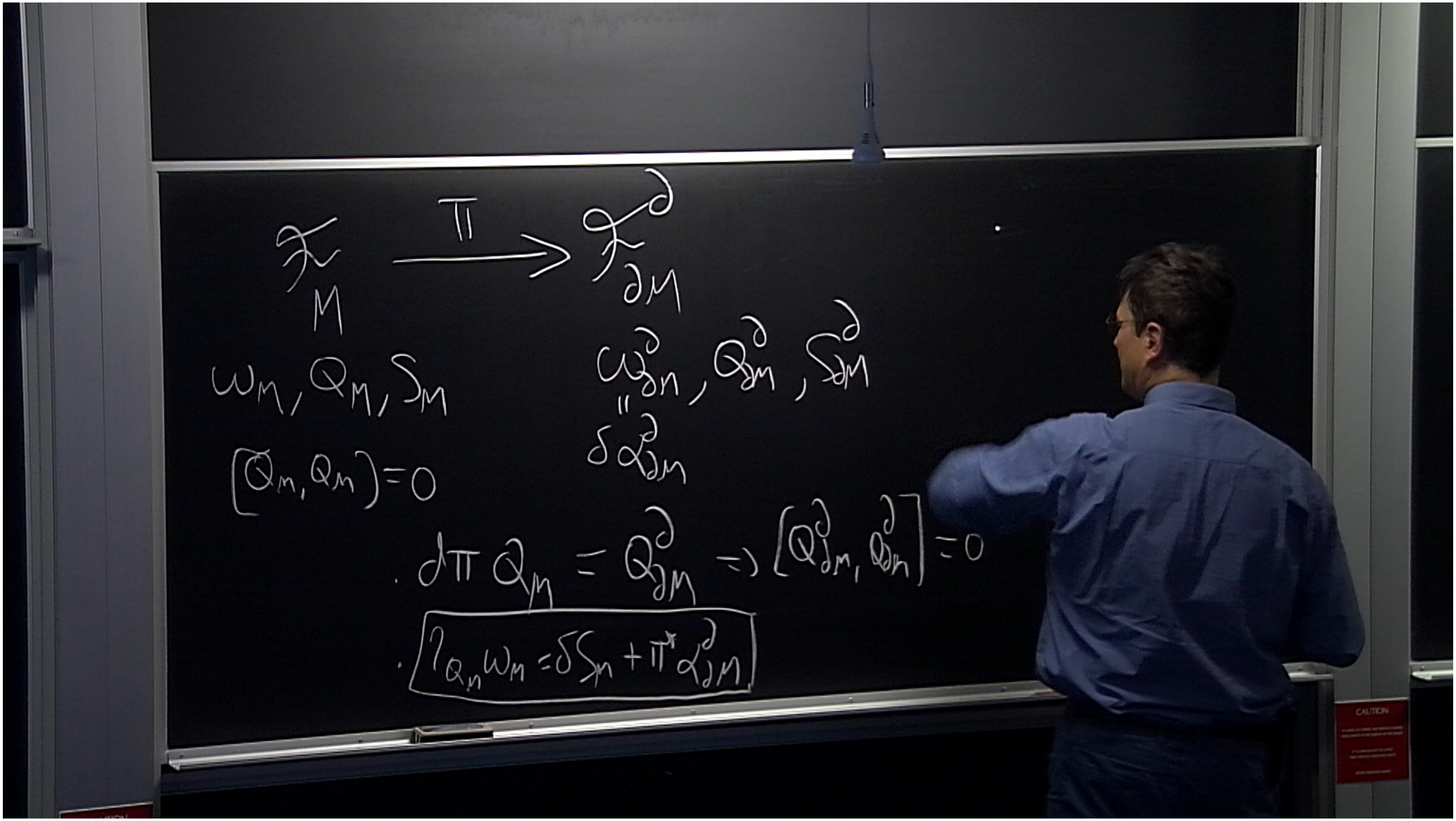
Remark

In the infinite dimensional case (e.g., in field theory), it is possible to have a nondegenerate ω with this property. In the finite-dimensional case (e.g., in a discretized field theory), ω is then necessarily degenerate, but we still require it to be nondegenerate on $\mathcal{Y}$, which is enough to define BV integration.


F_M π→ F^∂_∂M
ω_M, Q_M, S_M
Q^∂_M, S^∂_M
[Q_M, Q_M] = 0


$\mathcal{F}_M \xrightarrow{\pi} \mathcal{F}^\partial_{\partial M}$
ω_M, Q_M, S_M
$\omega^\partial_{\partial M}, Q^\partial_{\partial M}, S^\partial_{\partial M}$
$= \delta\alpha^\partial_{\partial M}$
$[Q_M, Q_M] = 0$
$d\pi\, Q_M = Q^\partial_{\partial M}$


π
ω_∂M^∂ , Q_∂M^∂ , S_∂M^∂
δα_∂M^∂
S_M
)=0
dπ Q_M = Q_∂M^∂ ⇒ [Q_∂M^∂ , Q_∂M^∂] = 0
CAUTION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ω_M, Q_M, S_M
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$[Q_M, Q_M] = 0$
$d\pi\, Q_M = Q^\partial_{\partial M} \Rightarrow [Q^\partial_{\partial M}, Q^\partial_{\partial M}] = 0$
$\iota_{Q_M}\omega_M = \delta S_M + \pi^* \alpha^\partial_{\partial M}$

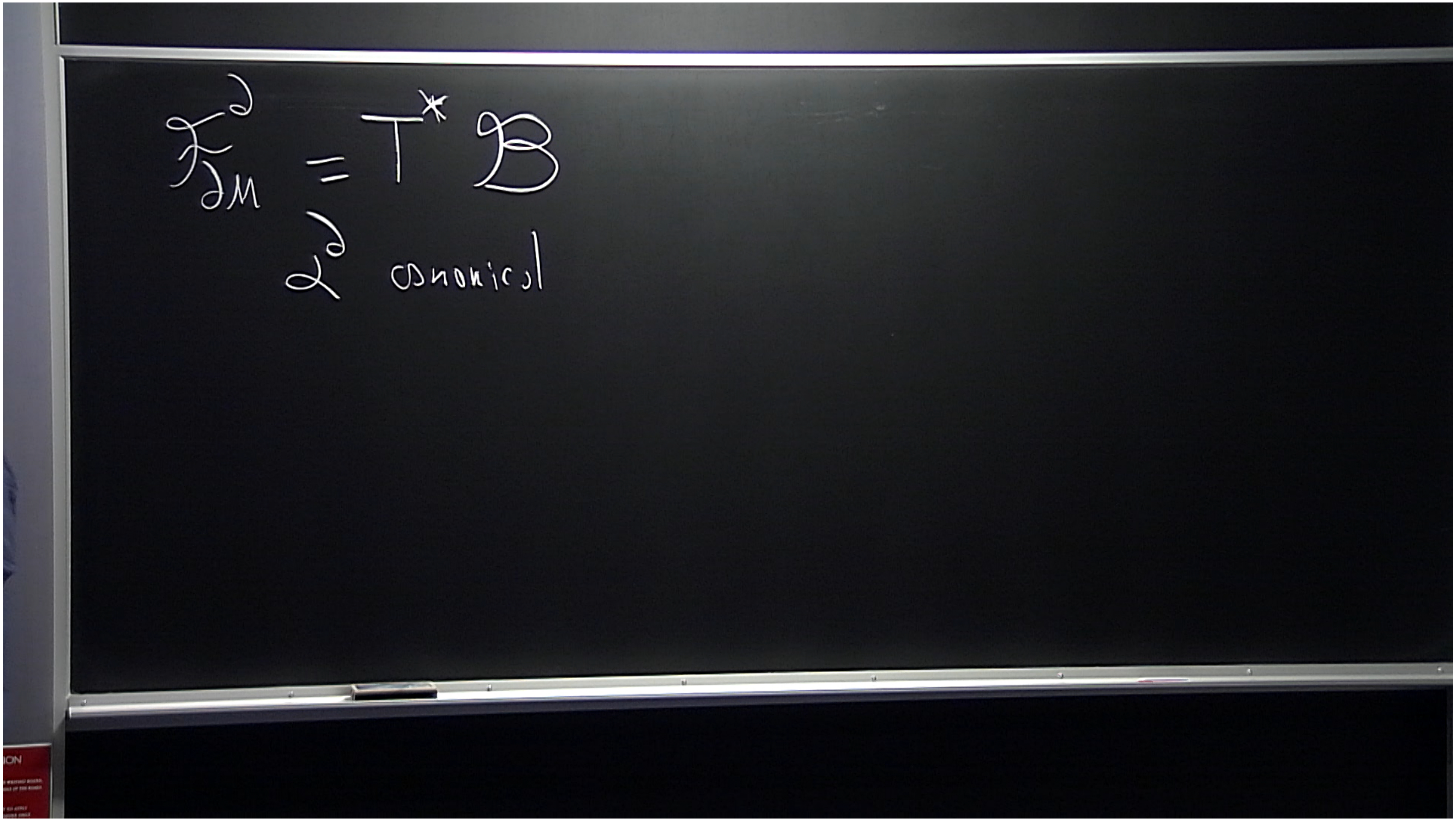
$$\mathcal{F}_M \xrightarrow{\ \pi\ } \mathcal{F}^\partial_{\partial M}$$

$$\omega_M,\ Q_M,\ S_M \qquad\qquad \omega^\partial_{\partial M} = \delta\alpha^\partial_{\partial M},\ Q^\partial_{\partial M},\ S^\partial_{\partial M}$$

$$[Q_M, Q_M] = 0$$

$$\Rightarrow \frac{1}{2}\iota_Q \iota_Q \omega_M = \pi^* S^\partial_{\partial M}$$

- $d\pi\, Q_M = Q^\partial_{\partial M} \Rightarrow [Q^\partial_{\partial M}, Q^\partial_{\partial M}] = 0$
- $\boxed{\iota_{Q_M}\omega_M = \delta S_M + \pi^*\alpha^\partial_{\partial M}}$ $\qquad \iota_{Q^\partial_{\partial M}}\omega^\partial_{\partial M} = \delta S^\partial_{\partial M}$


$\mathcal{F}^{\partial}_{\partial M} = T^{*}\mathcal{B}$
ω^{∂} canonical

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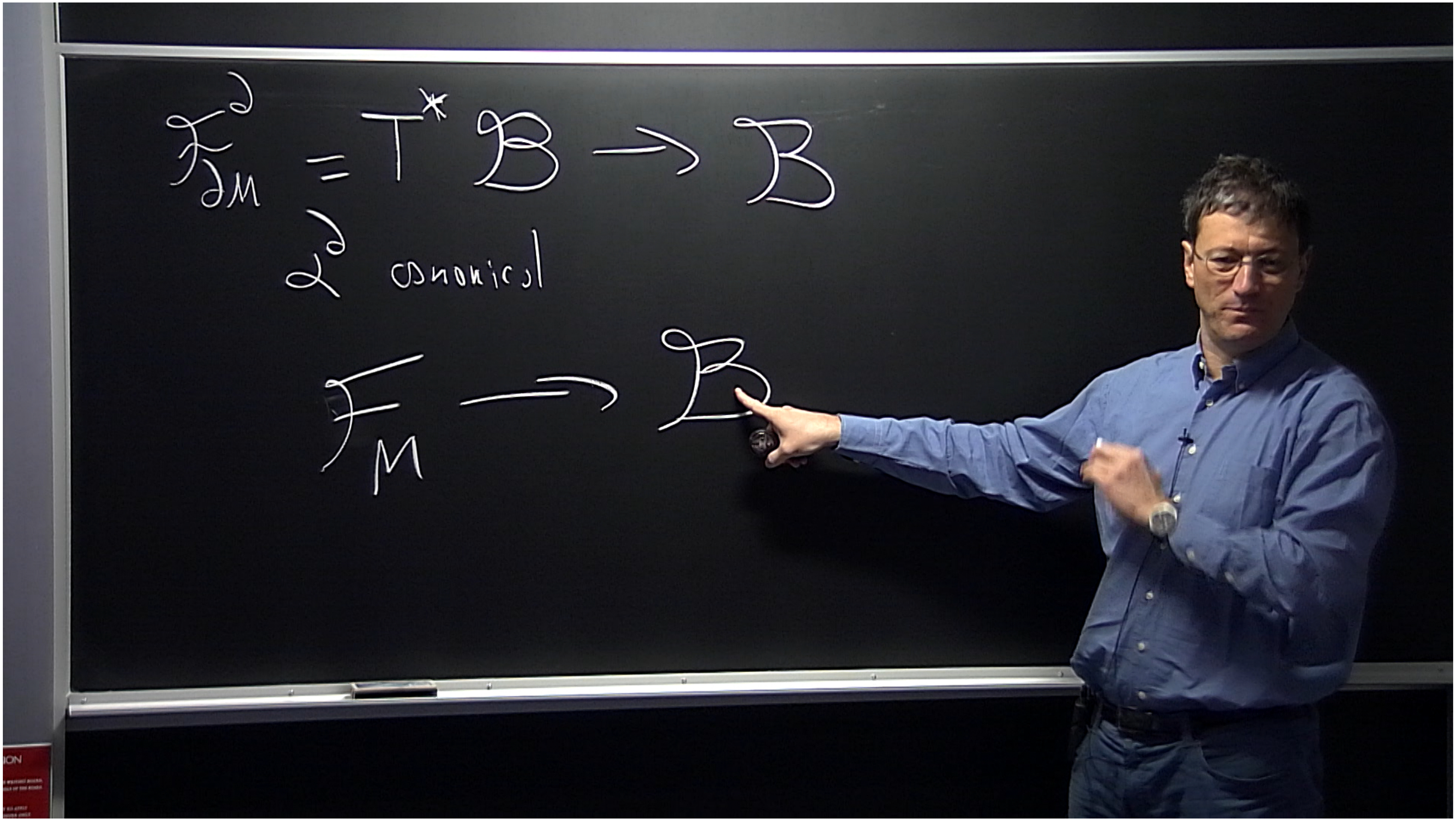
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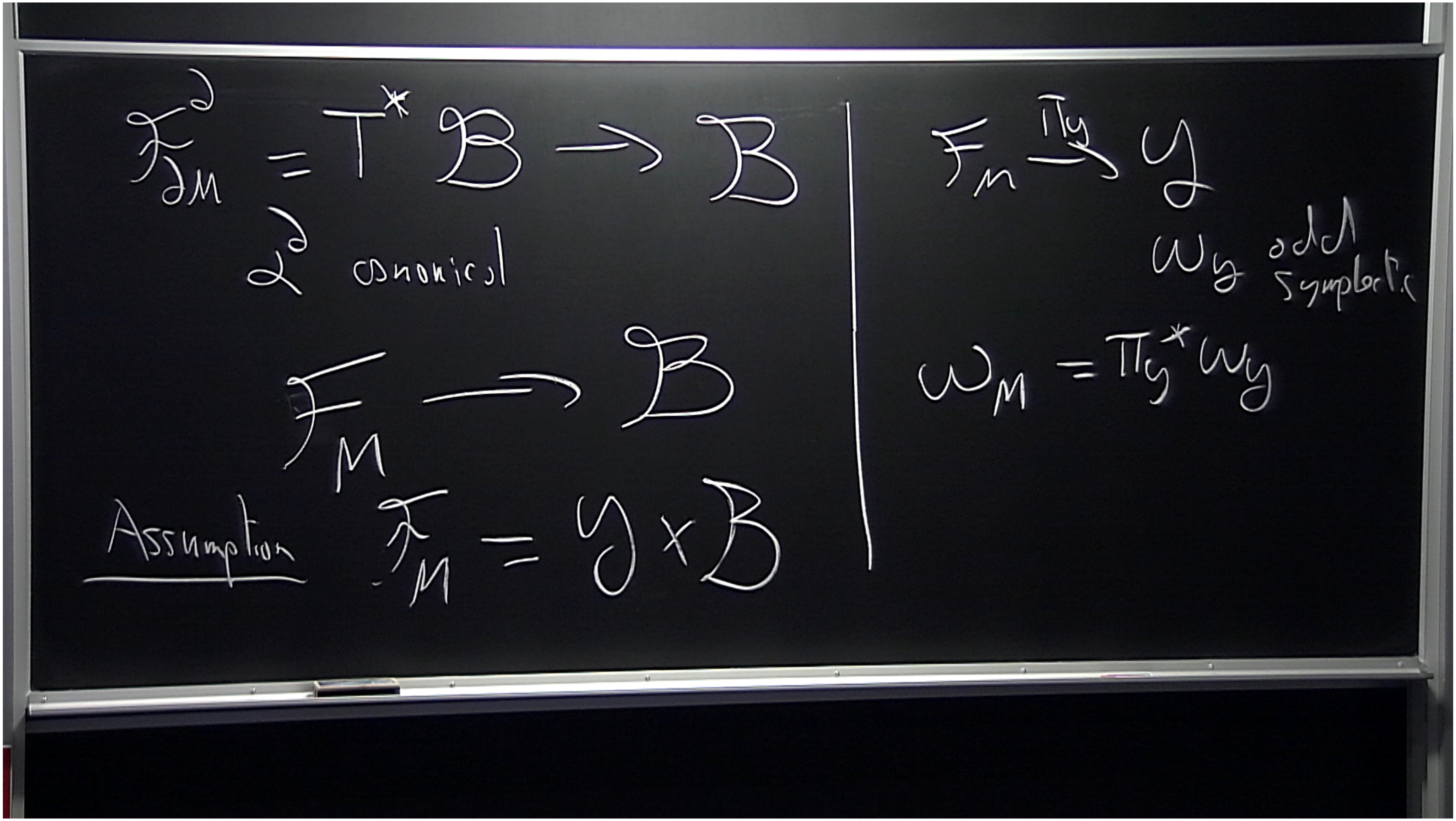
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Assumption $\mathcal{F}_M = \mathcal{Y} \times \mathcal{B}$
$\mathcal{F}_M \xrightarrow{\pi_{\mathcal{Y}}} \mathcal{Y}$
$\omega_{\mathcal{Y}}$ odd symplectic
$\omega_M = \pi_{\mathcal{Y}}^* \omega_{\mathcal{Y}}$

The modified quantum master equation I

Using the splitting, we rewrite the mCME as (we no longer write π^*)

$$\delta_{\mathcal{Y}} S = \iota_{Q_{\mathcal{Y}}} \omega$$
$$\delta_{\mathcal{B}} S = -\alpha^{\partial}$$

The two equations imply

$$\frac{1}{2}(S, S)_{\mathcal{Y}} = \frac{1}{2} \iota_{Q_{\mathcal{Y}}} \iota_{Q_{\mathcal{Y}}} \omega = S^{\partial} \qquad (*)$$

Now assume we have adapted Darboux coordinates (b, p) on $\mathcal{F}^{\partial}$ with b on $\mathcal{B}$, p on the leaves and $\alpha^{\partial} = -\sum p\,\delta b$. Then the second equation implies

$$\frac{\delta S}{\delta b} = p \qquad (**)$$

This means that, in this splitting, S is linear in the b's.

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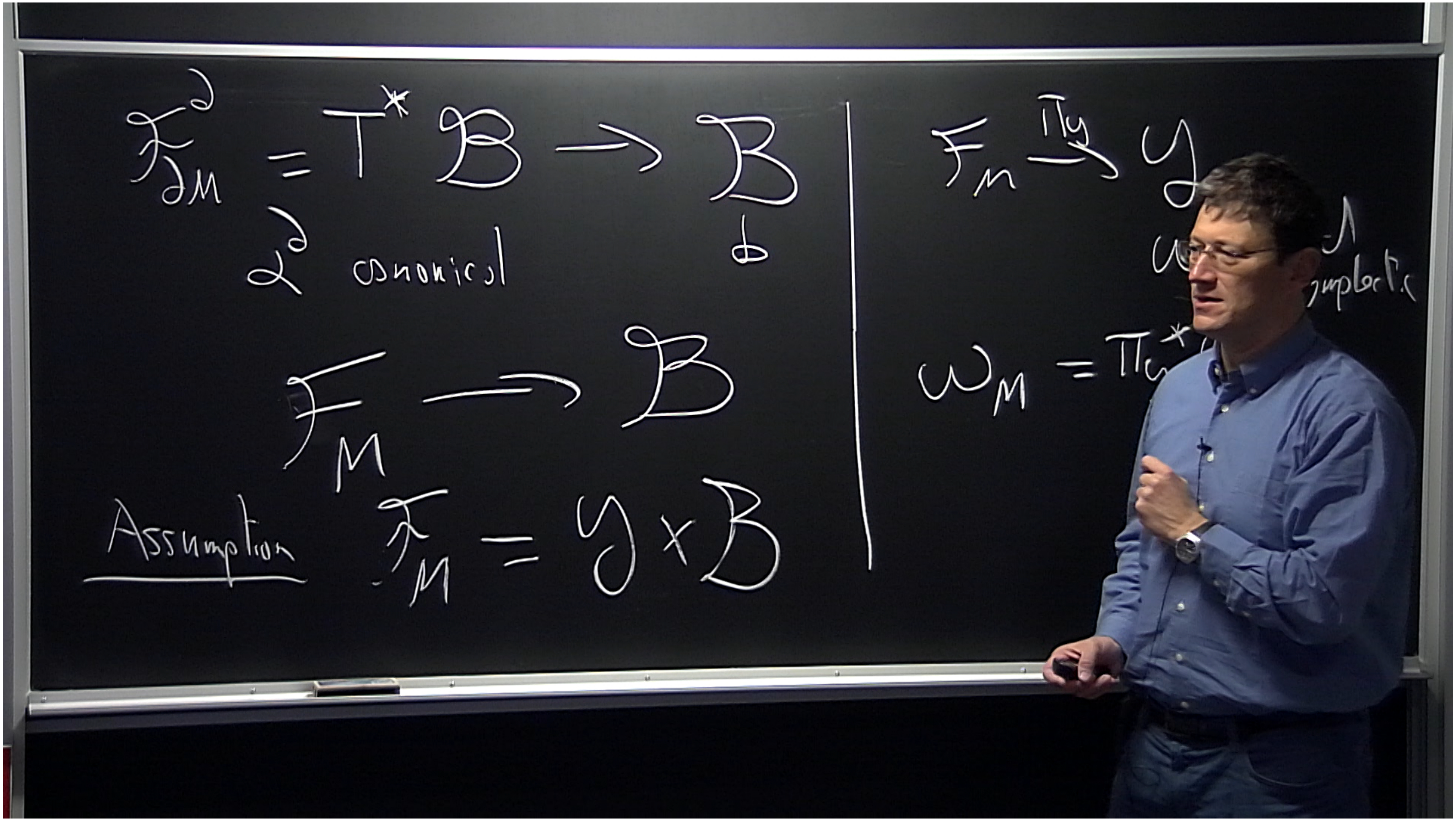
canonical
Assumption

$$\mathcal{F}^{\partial}_{\partial M} = T^*_p \mathcal{B} \longrightarrow \mathcal{B}_{\partial}$$

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The modified quantum master equation II

We now assume that S also solves the equation

$$\Delta_{\mathcal{Y}} S = 0$$

Remark

Without boundary this means that we assume that S solves both the classical and the quantum master equation. With boundary, Δ makes sense only on the $\mathcal{Y}$-factor. We will return on this.

We then have

$$\Delta_{\mathcal{Y}} e^{\frac{i}{\hbar}S} = \left(\frac{i}{\hbar}\right)^2 \frac{1}{2}(S,S)_{\mathcal{Y}}\, e^{\frac{i}{\hbar}S}$$

and equation (*) implies

$$-\hbar^2 \Delta_{\mathcal{Y}} e^{\frac{i}{\hbar}S} = S^{\partial} e^{\frac{i}{\hbar}S} \qquad (\dagger)$$

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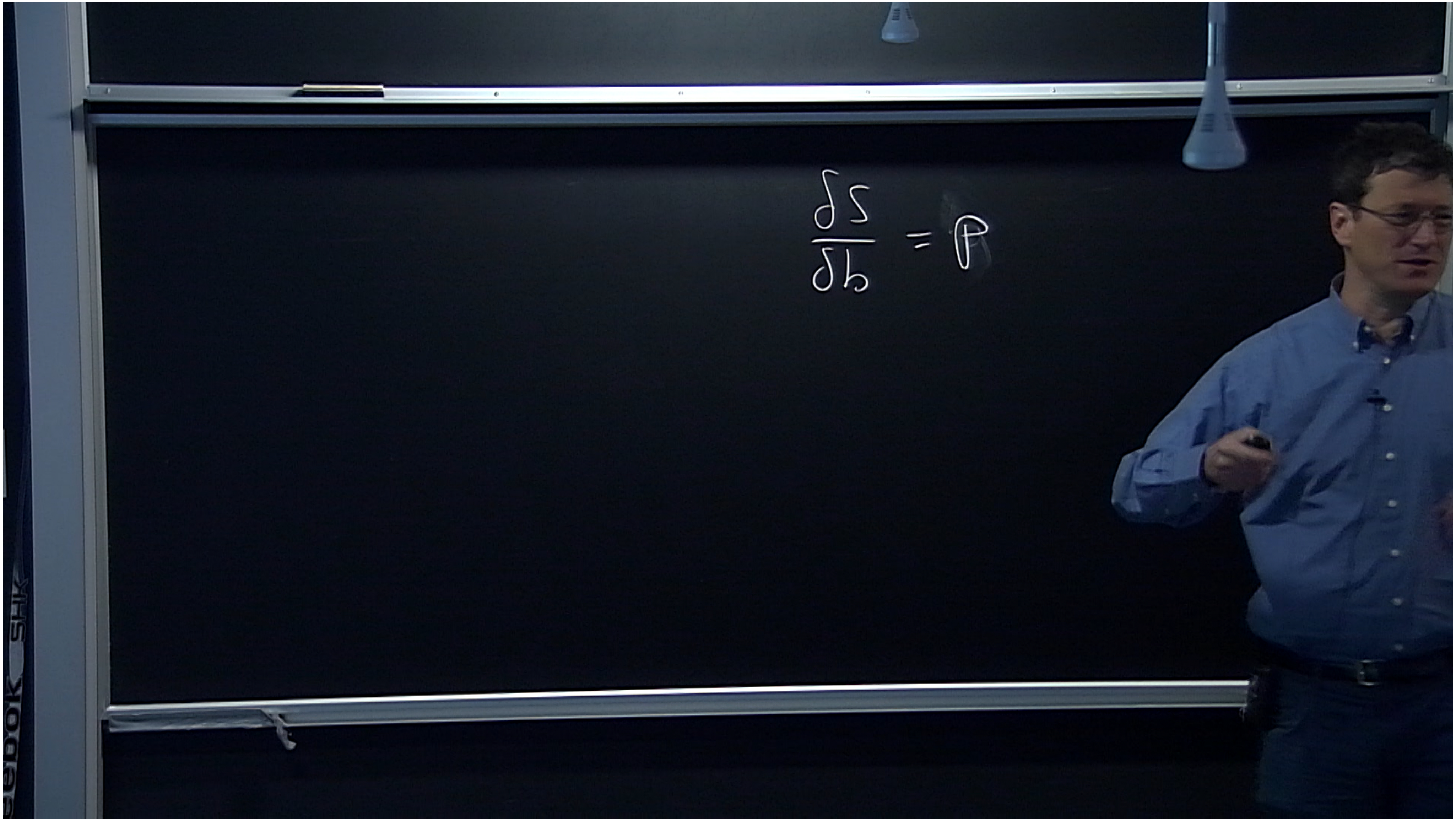
$\frac{\delta S}{\delta b} = P$

The modified quantum master equation III

We now move to the quantization. We take $\mathcal{H}$ to be an appropriate space of functions on $\mathcal{B}$.
Equation (**) essentially says that

$$\hat{p}S = -\mathrm{i}\hbar p \quad \text{with} \quad \hat{p} = -\mathrm{i}\hbar\frac{\delta}{\delta b}$$

Remark

Here S is an element of $\mathcal{H}$ parametrized by $\mathcal{Y}$. The p appearing in the equation is now an element of $\mathcal{Y}$.

If we quantize S^∂ by the Schrödinger prescription

$$\Omega := S^\partial\left(b, -\mathrm{i}\hbar\frac{\delta}{\delta b}\right)$$

with all derivatives placed to the right, we get

$$\Omega e^{\frac{\mathrm{i}}{\hbar}S} = S^\partial e^{\frac{\mathrm{i}}{\hbar}S} \qquad (\ddagger)$$

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The modified quantum master equation IV

Putting (†) and (‡) together we finally get the modified quantum master equation (mQME)

$$(\hbar^2\Delta_y + \Omega)e^{\frac{i}{\hbar}S} = 0$$

Remark

The assumption $\Delta_y S = 0$ is not really necessary (and is often not justified). More generally, we have

$$\Delta_y e^{\frac{i}{\hbar}S} = \left(\left(\frac{i}{\hbar}\right)\Delta_y S + \left(\frac{i}{\hbar}\right)^2 \frac{1}{2}(S,S)_y\right) e^{\frac{i}{\hbar}S}$$

If we define

$$S^\partial_\hbar := \frac{1}{2}(S,S)_y - i\hbar\Delta_y S = S^\partial + O(\hbar)$$

and Ω as the Schrödinger quantization of $S^\partial_\hbar$, we recover the mQME.

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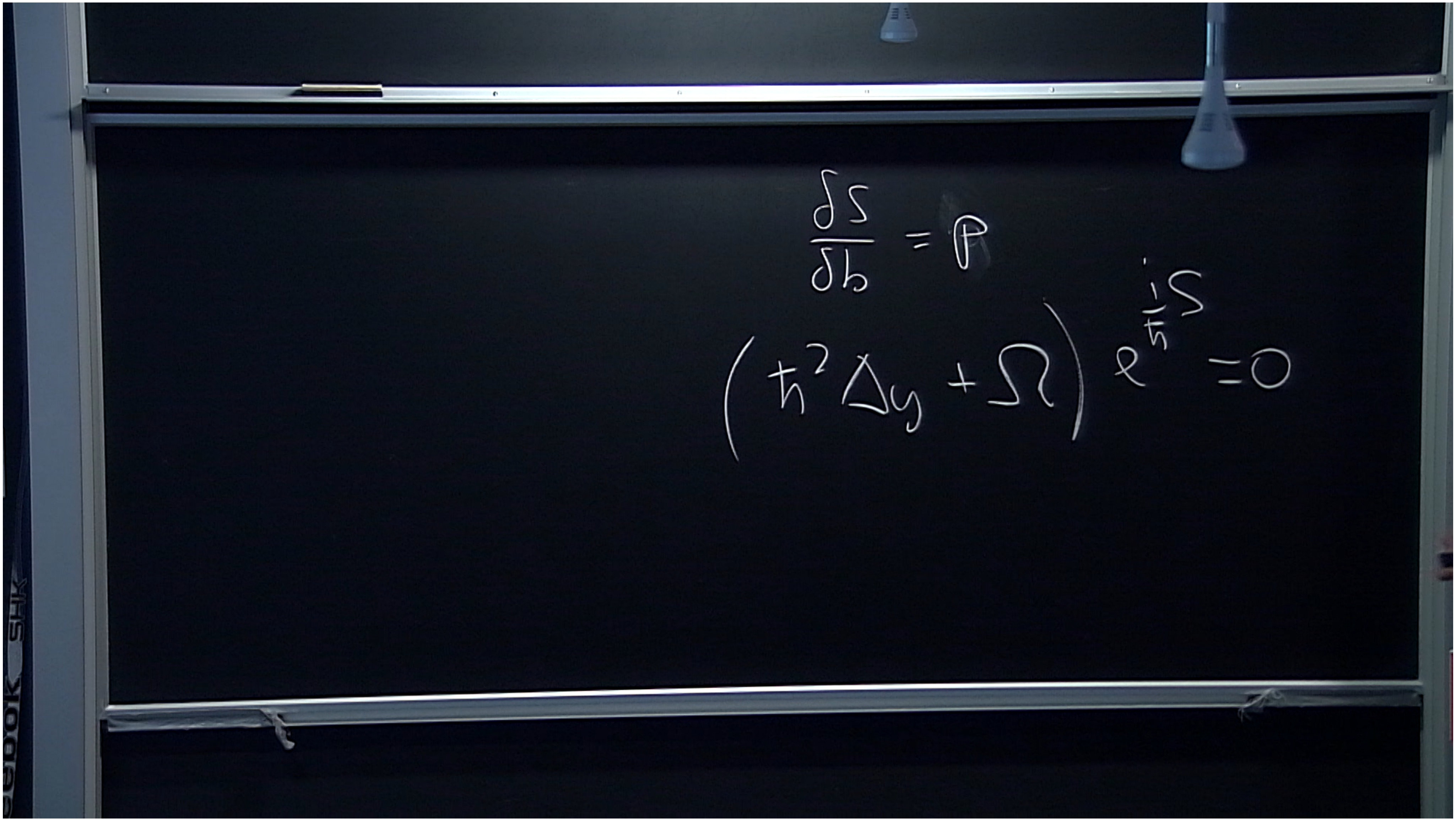

$\frac{\delta S}{\delta b} = P$
$\left(\hbar^2 \Delta_g + \Omega\right) e^{\frac{i}{\hbar} S} = 0$

The modified quantum master equation V

By construction we have

$$\Delta_{\mathcal{Y}}^2 = 0 \qquad [\Delta_{\mathcal{Y}}, \Omega] = 0$$

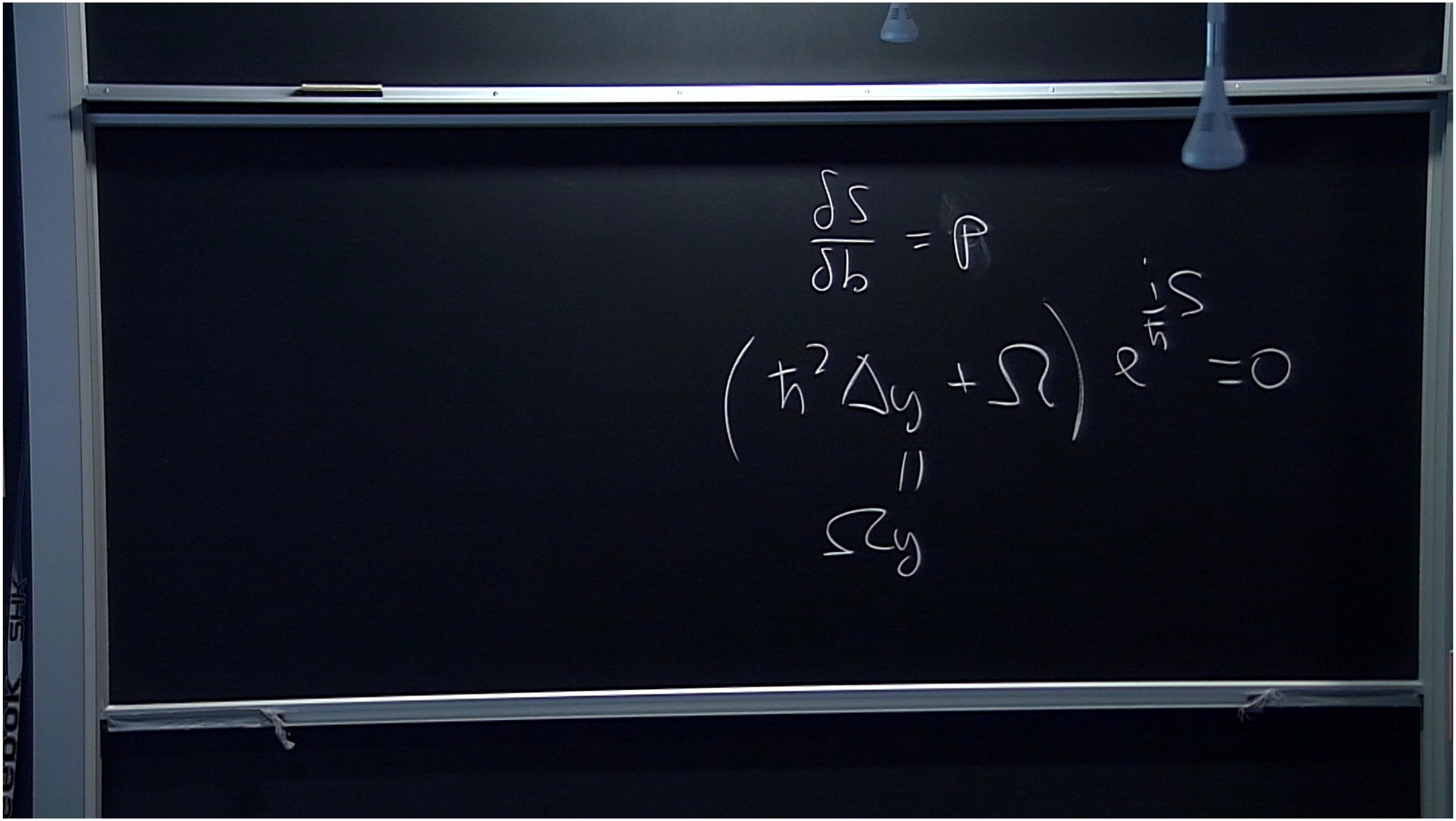
The operator

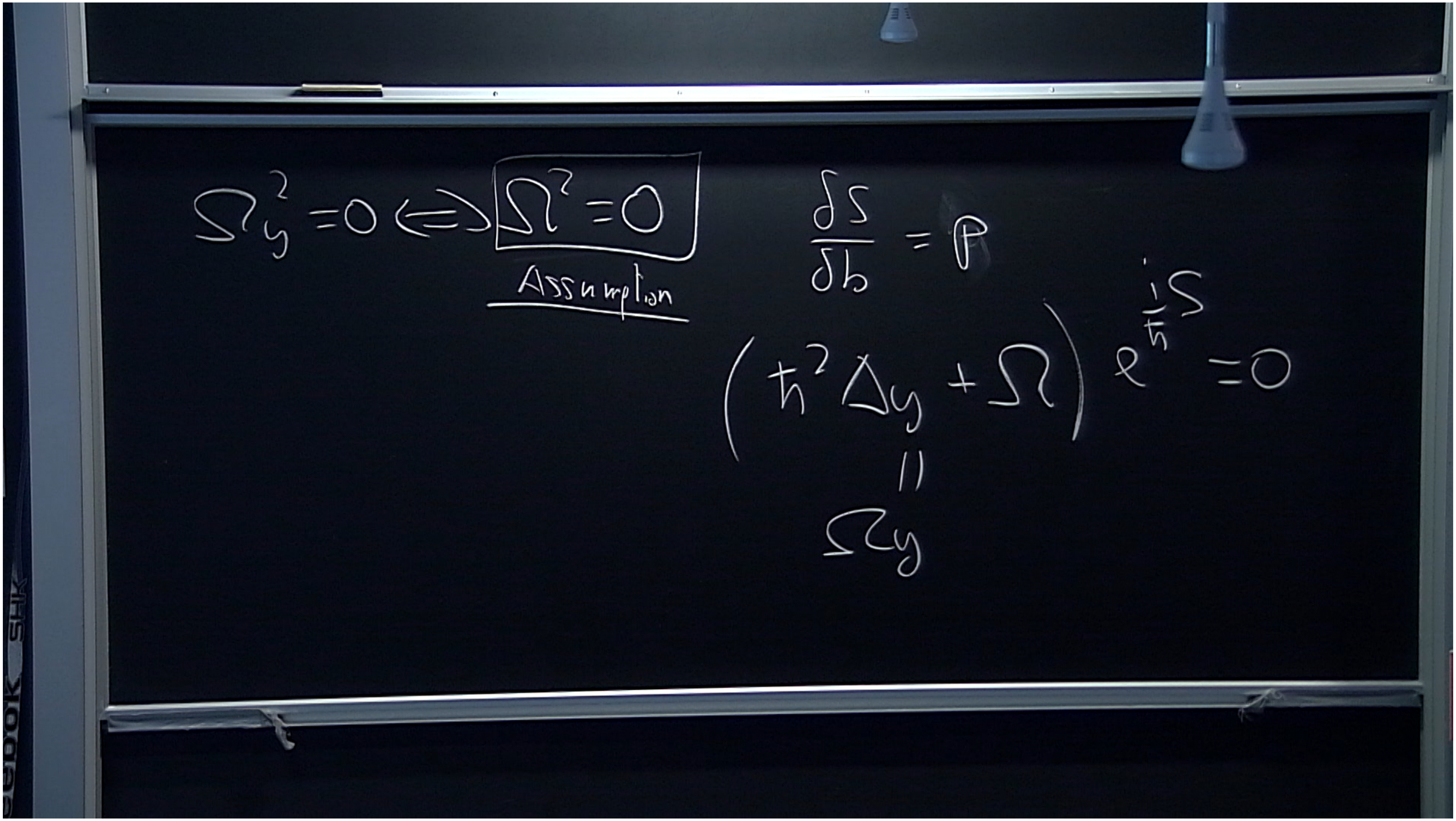
$$\Omega_{\mathcal{Y}} := \hbar^2 \Delta_{\mathcal{Y}} + \Omega$$

appearing in the mQME then squares to zero iff

$$\Omega^2 = 0$$

The existence of a splitting such that this holds is a fundamental condition (absence of anomalies) which allows passing to the $\Omega_{\mathcal{Y}}$-cohomology. Cohomology in degree zero describes $\mathcal{Y}$-parametrized physical states.


$\frac{\delta S}{\delta b} = P$
$\left(\hbar^2 \Delta_y + \Omega\right) e^{\frac{i}{\hbar} S} = 0$
$\parallel$
Ω_y


$\Omega_y^2 = 0 \iff \Omega^2 = 0$
Assumption
$\frac{\delta S}{\delta b} = P$
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$\hbar^2 \Delta_y = \Omega_y$

The quantum state

- Assume the mQME

$$\Omega_{\mathcal{Y}} e^{\frac{i}{\hbar} S} = 0$$

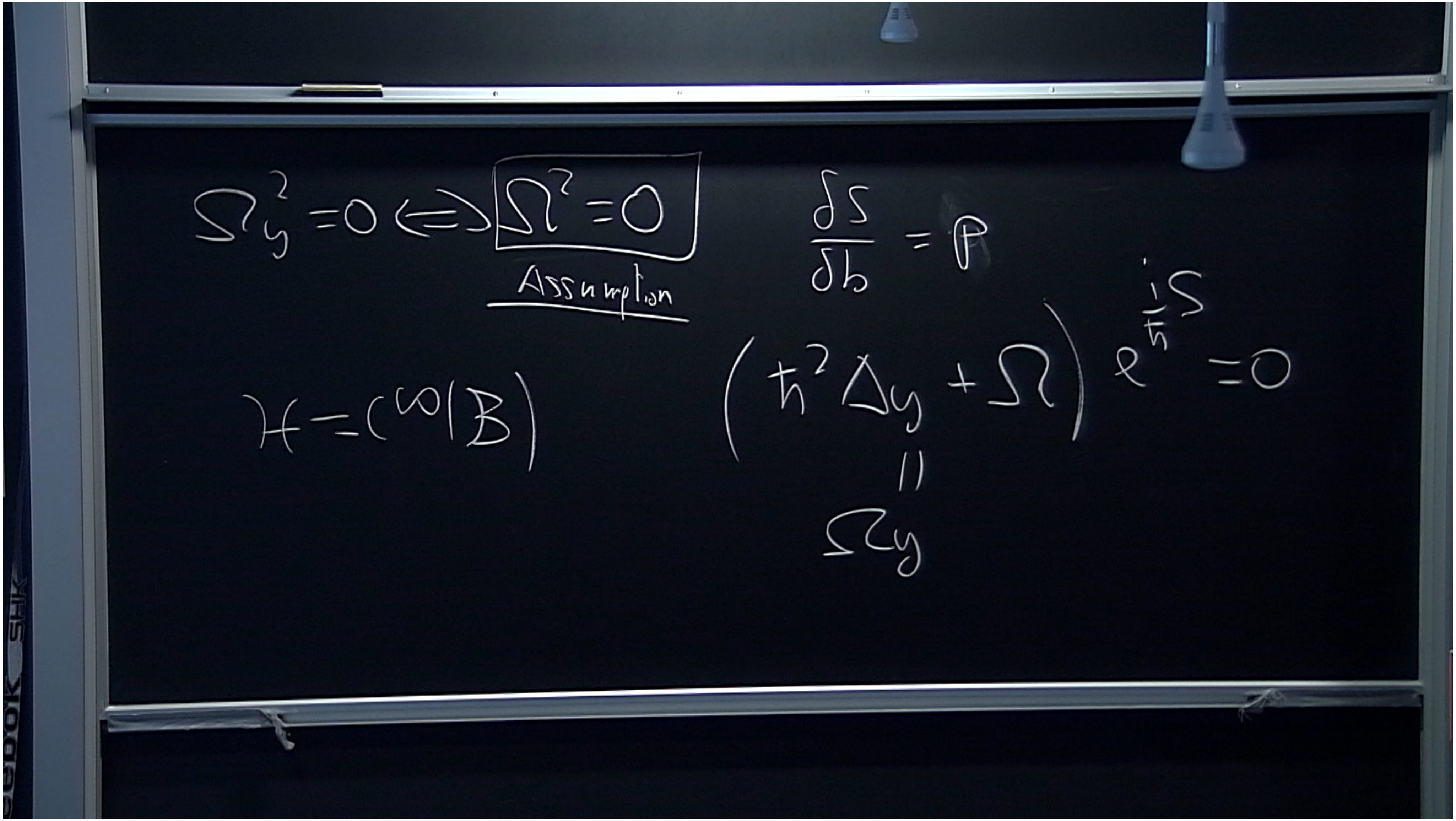
- Suppose $\mathcal{Y} = \mathcal{Y}' \times \mathcal{Y}''$ (possibly $\mathcal{Y}'$ a point).
- Pick a Lagrangian submanifold $\mathcal{L}$ of $\mathcal{Y}''$.
- Define

$$\psi := \int_{\mathcal{L}} e^{\frac{i}{\hbar} S} \in \mathcal{H} \otimes C^\infty(\mathcal{Y}')$$

- Then
  - We have the induced mQME

$$\Omega_{\mathcal{Y}'} \psi = 0$$

  - Changing the "gauge fixing" $\mathcal{L}$ changes ψ by an $\Omega_{\mathcal{Y}'}$-exact term.
- Hence ψ defines a $\Omega_{\mathcal{Y}'}$-cohomology class (of degree 0).
- We might iterate this procedure ("Wilson renormalization with boundary") and eventually arrive at $\mathcal{Y}'$ a point. In this case, ψ will be an Ω-cohomology class of degree zero on $\mathcal{H}$: a physical state.


$\Omega_y^2 = 0 \Leftrightarrow \Omega^2 = 0$
Assumption
$\frac{\delta S}{\delta b} = P$
$\left(\hbar^2 \Delta_y + \Omega\right) e^{\frac{i}{\hbar} S} = 0$
Ω_y
$\mathcal{H} = (\omega | B)$

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Axiomatics

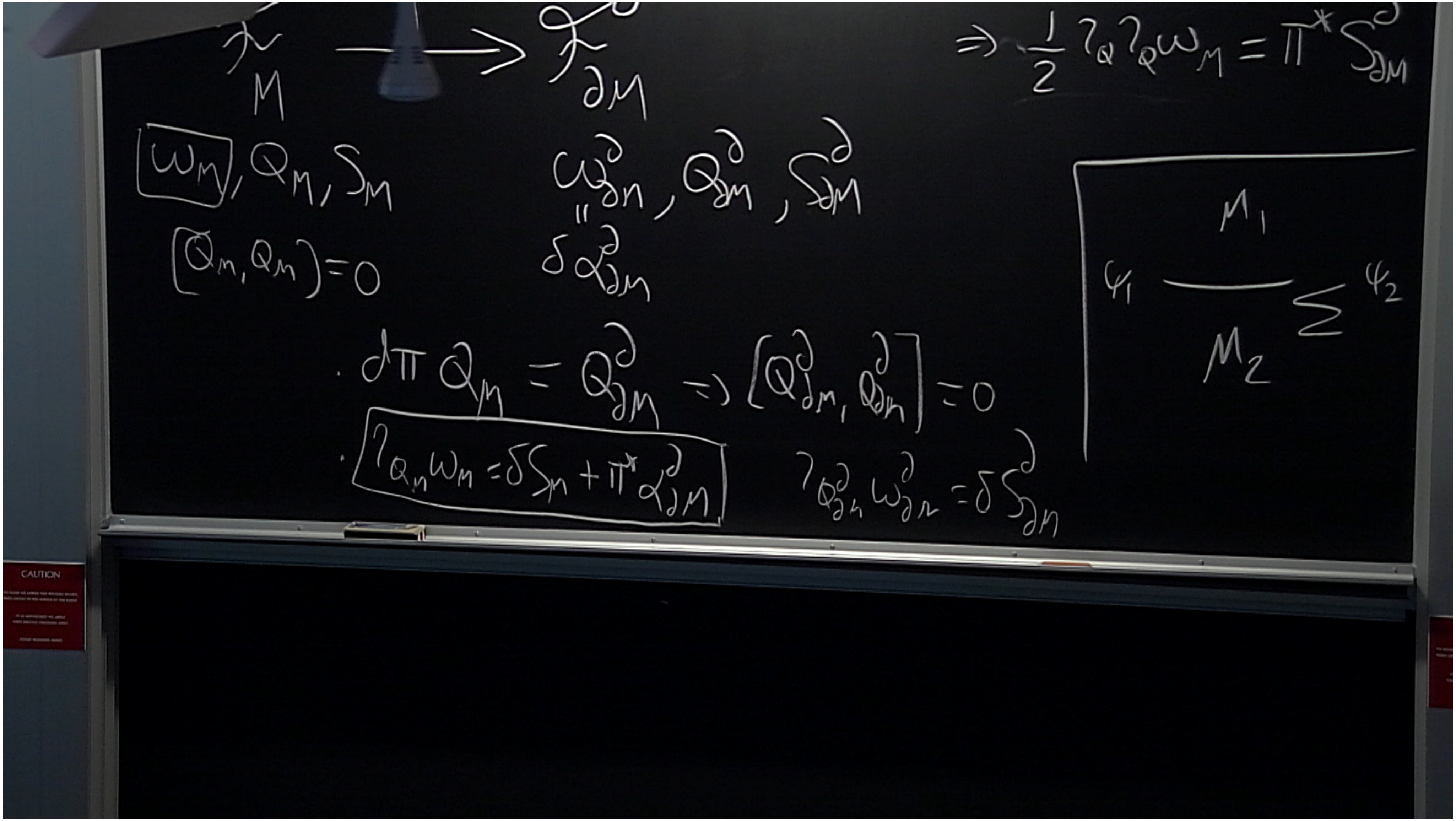
- To each $(d-1)$-manifold Σ we associate a complex $(\mathcal{H}_\Sigma, \Omega_\Sigma)$.
- To each d-manifold M we as associate a state ψ_M satisfying the mQME.
- Plus functorial properties.
  In particular, gluing is given by pairing states and doing a BV-pushforward

$$\mathcal{Y}'_{M_1} \times \mathcal{Y}'_{M_2} \to \mathcal{Y}'_{M_1 \cup_\Sigma M_2}$$

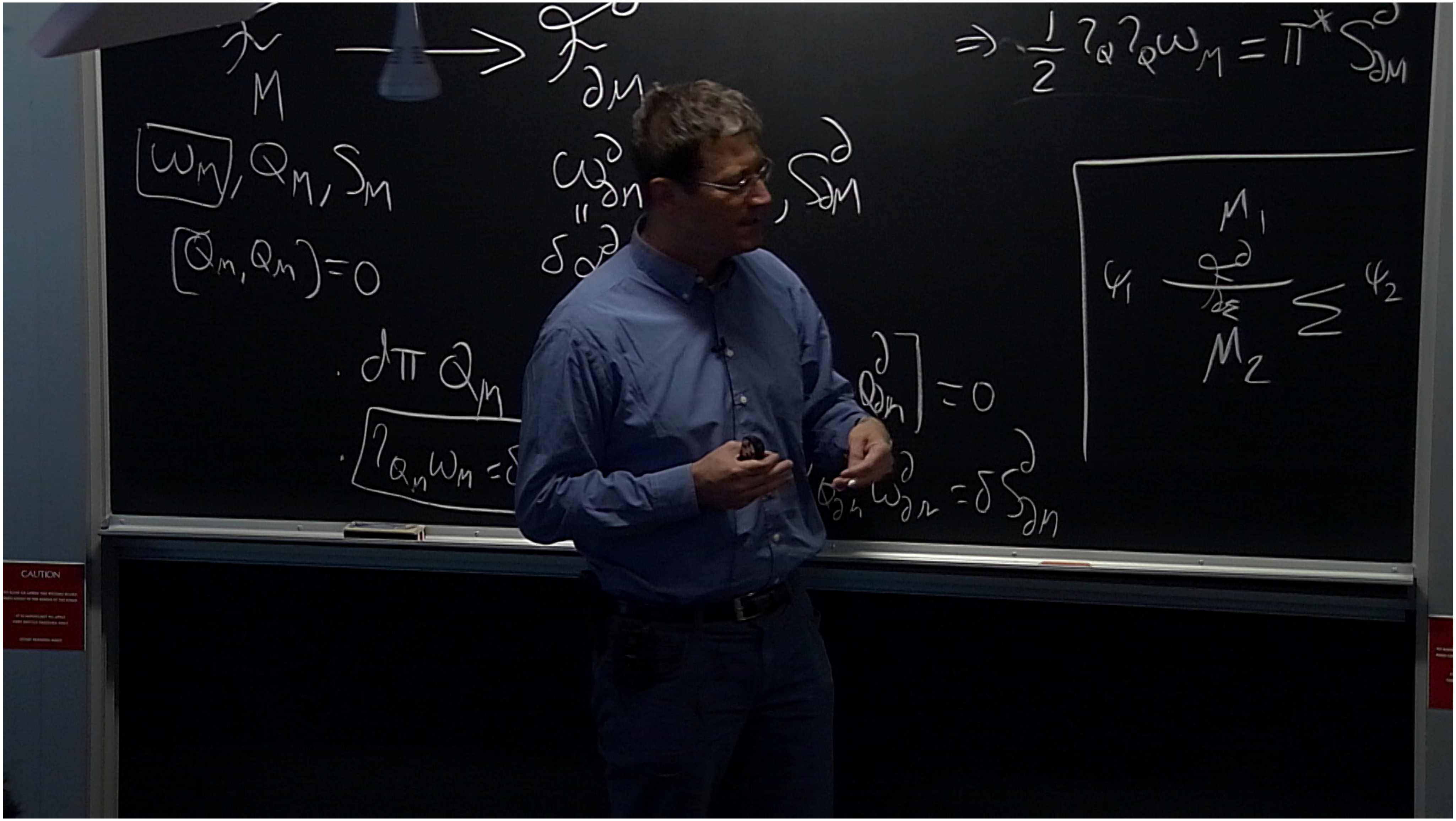
This could provide some new insight for physical theories.
In TFTs it yields a perturbative version of Atiyah's axioms.

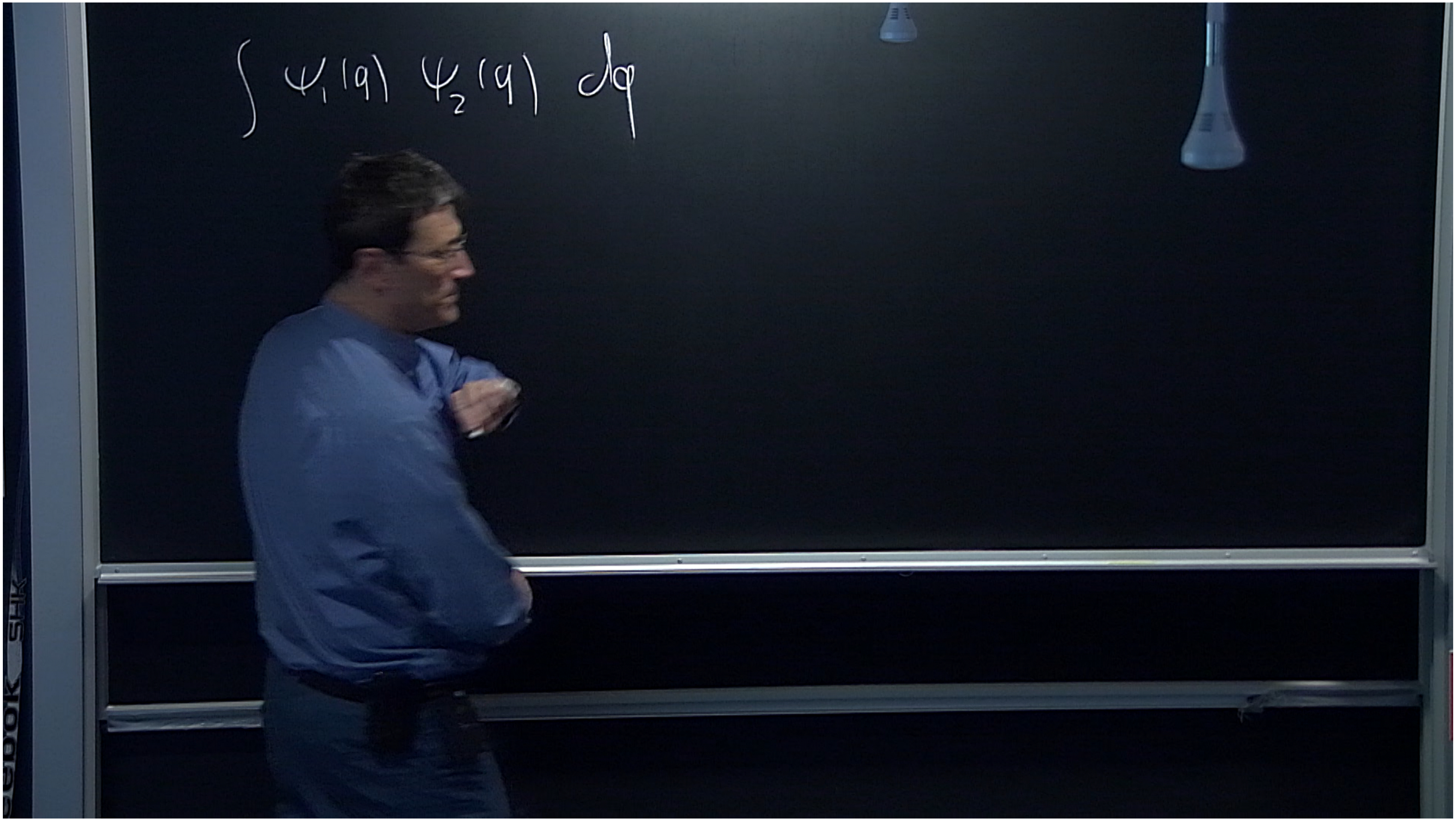
The full power of this approach is as follows:

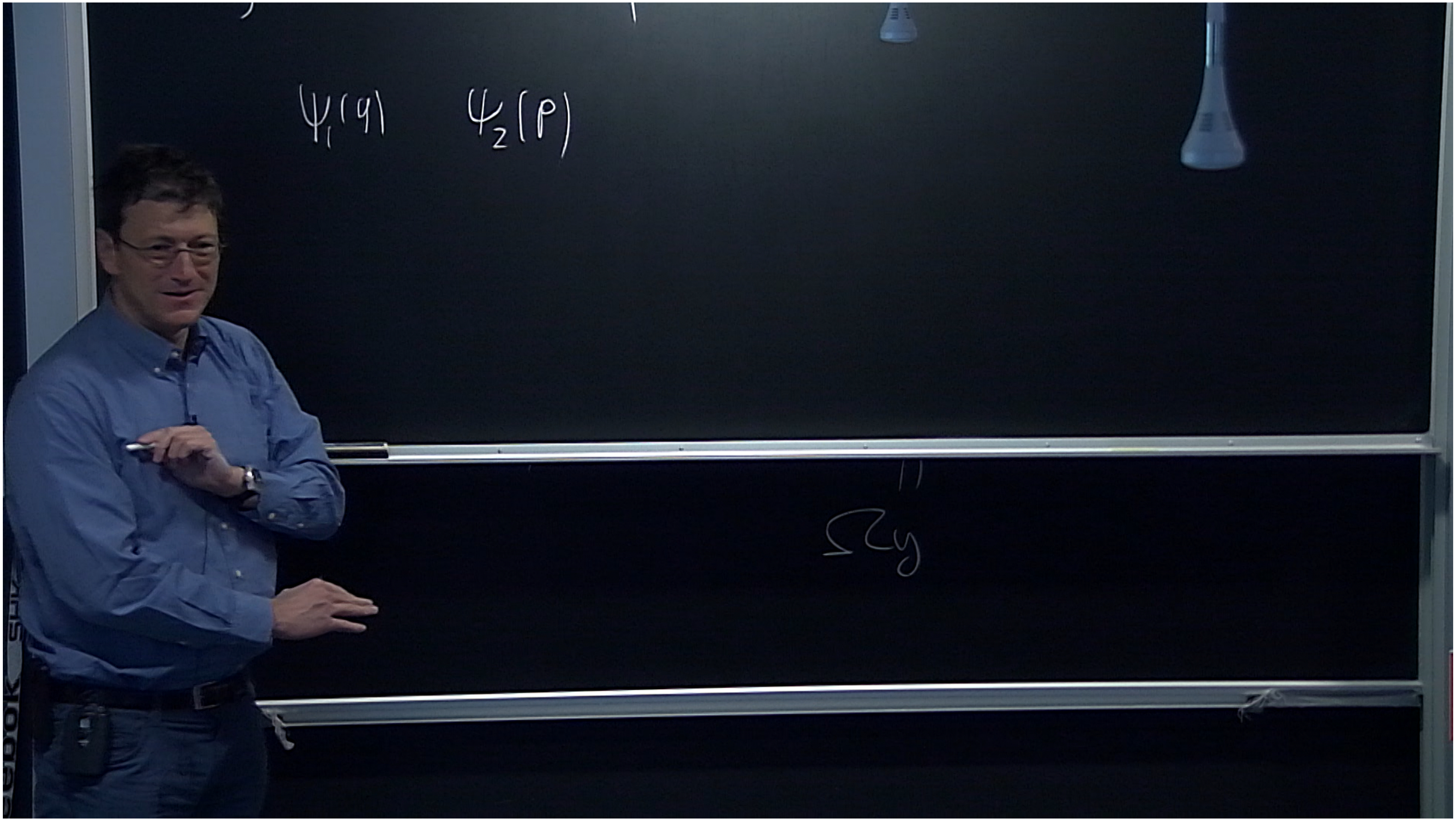
- Cut the original manifold M into topologically simple or metrically tiny pieces
- Do the perturbative quantization on the pieces
- Glue and reduce

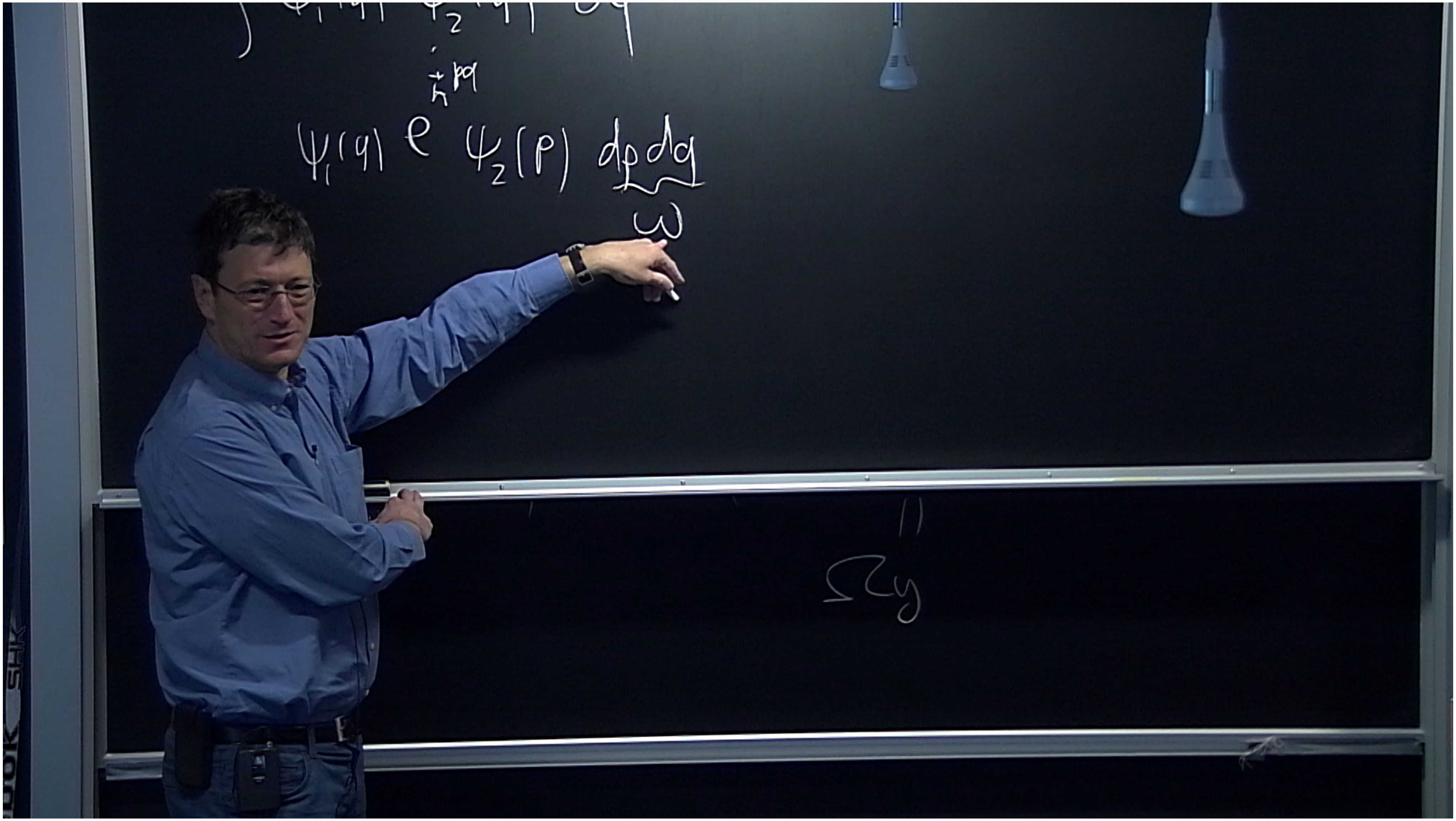


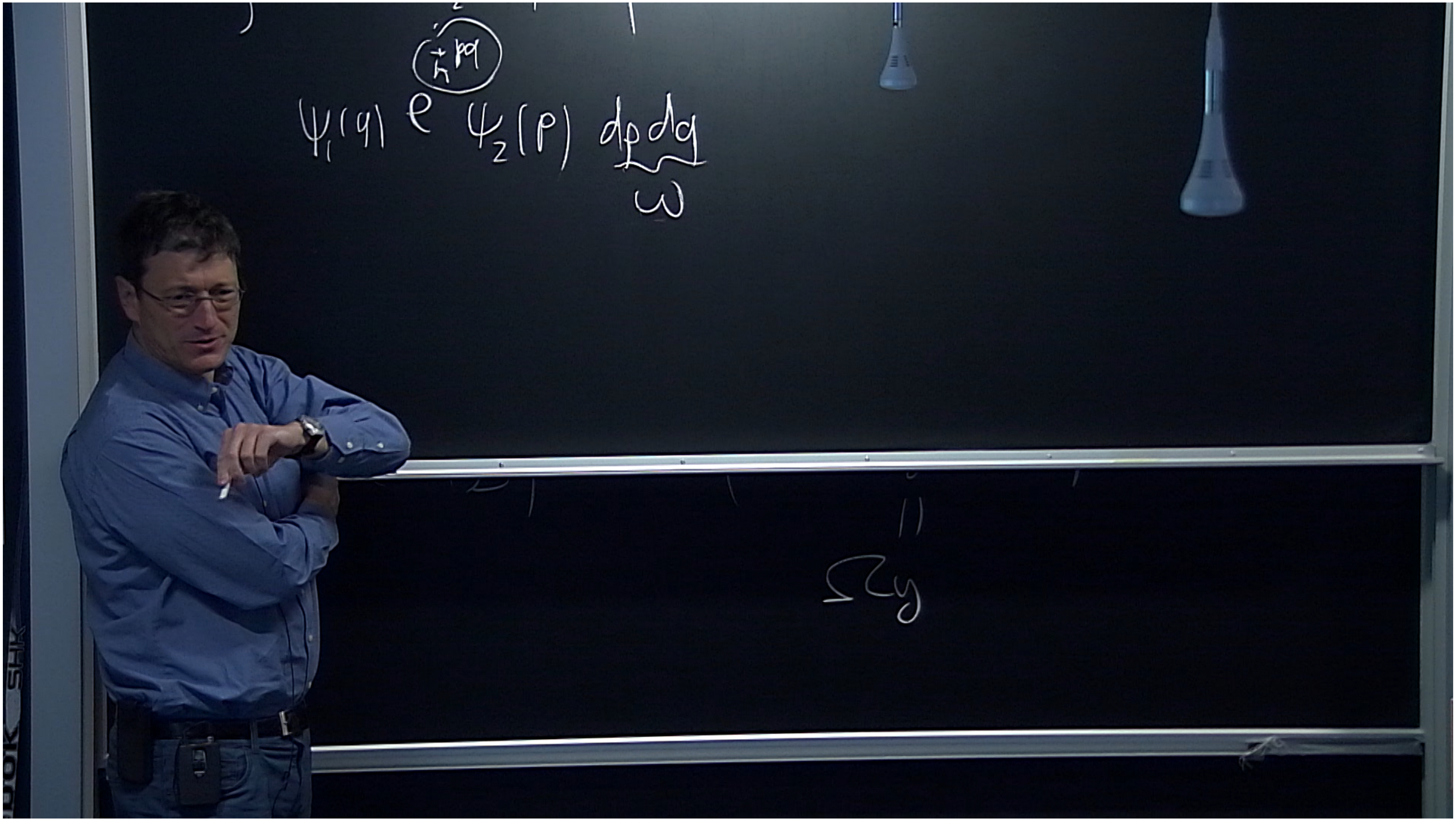
M → ∂M
⇒ ½ ι_Q ι_Q ω_M = π* S_∂M
ω_M, Q_M, S_M
ω_∂M, Q_∂M, S_∂M
[Q_M, Q_M] = 0
δ α_∂M
dπ Q_M = Q_∂M ⇒ [Q_∂M, Q_∂M] = 0
ι_{Q_M} ω_M = δS_M + π* α_∂M
ι_{Q_∂M} ω_∂M = δS_∂M
M_1
M_2
φ_1
φ_2
Σ
CAUTION

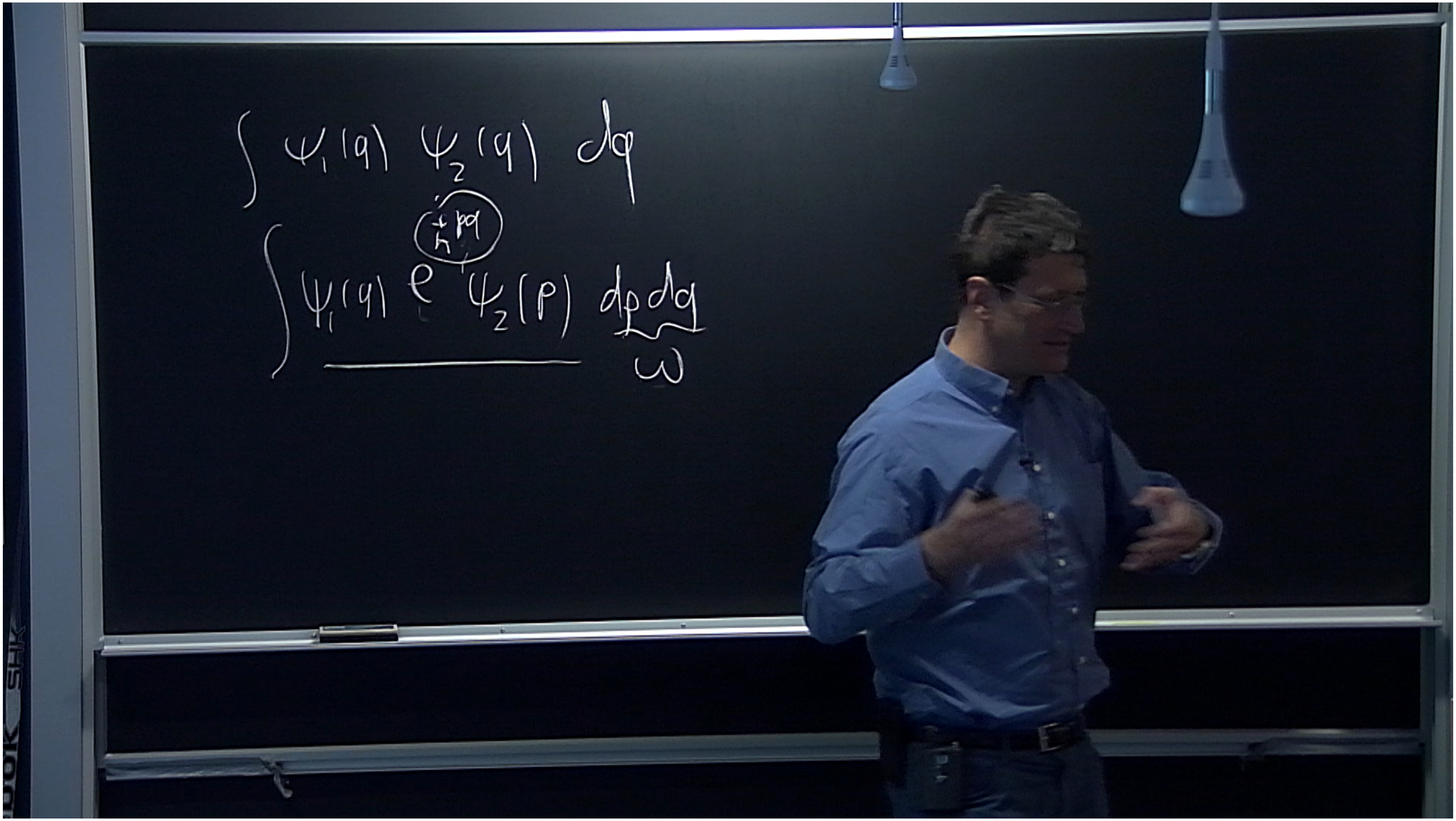


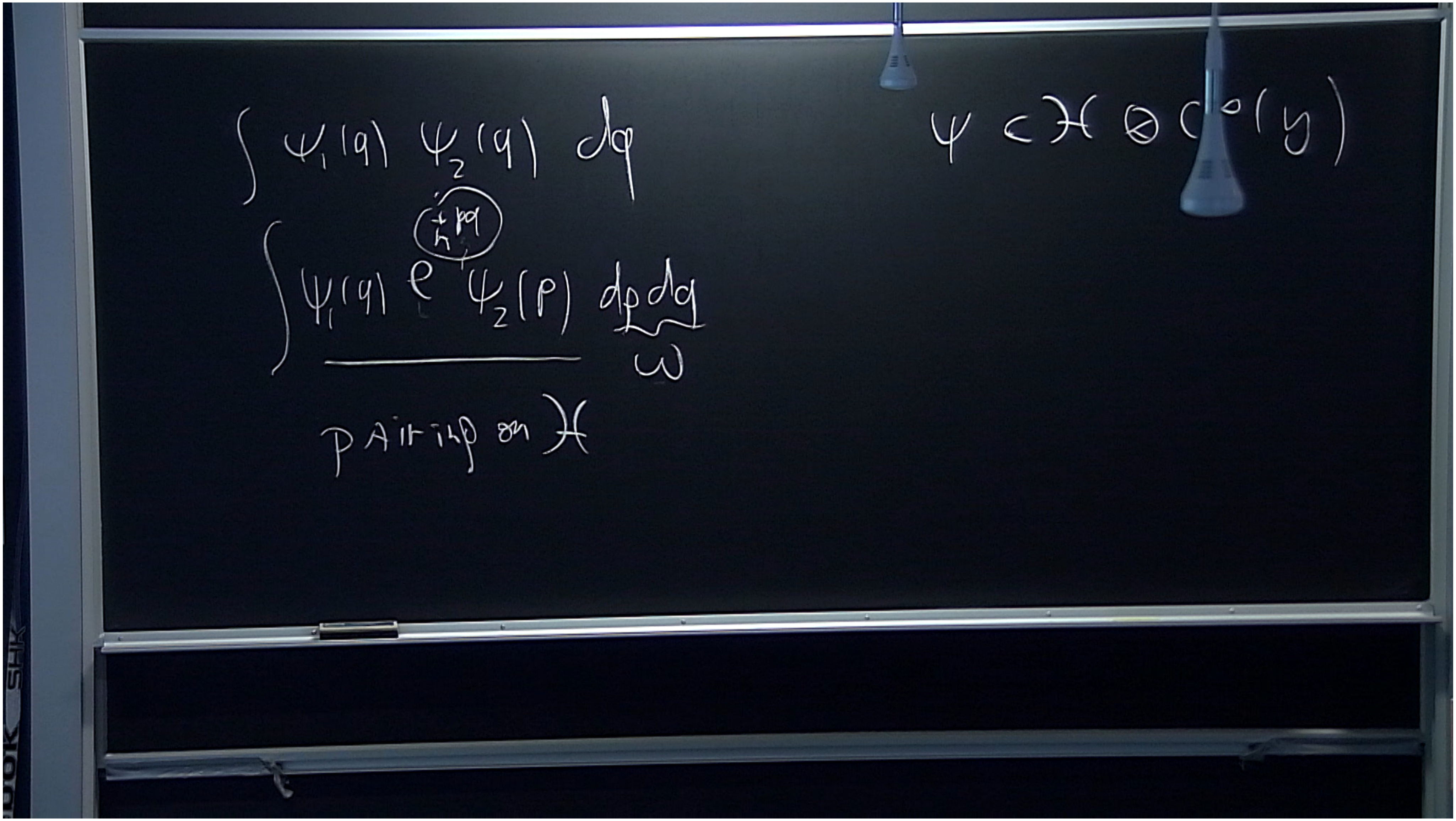

∫ ψ1(q) ψ2(q) dq

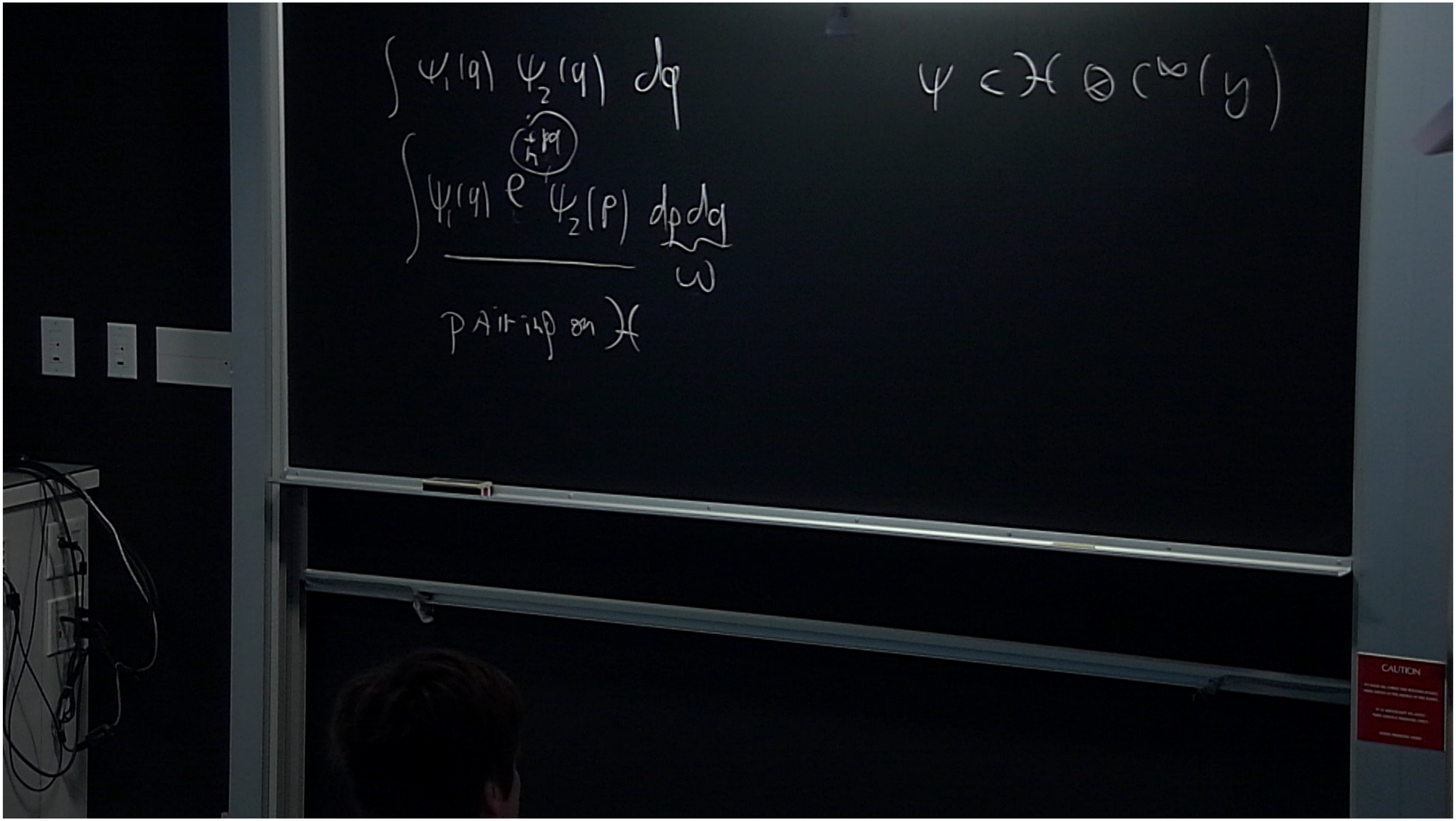

$\psi_1(q)$ $\psi_2(p)$




$\psi_1(q)\, e^{\frac{i}{\hbar}pq}\, \psi_2(p)\, dp\, dq$
ω
Ω_y


∫ ψ₁(q) ψ₂(q) dq
∫ ψ₁(q) e^{(i/ħ)pq} ψ₂(p) dp dq
ω


$\int \psi_1(q)\,\psi_2(q)\,dq$
$\int \psi_1(q)\, e^{\frac{i}{\hbar}pq}\, \psi_2(p)\, dp\, dq$
ω
pairing on $\mathcal{H}$
$\psi \in \mathcal{H} \otimes C^\infty(y)$


pairing on H

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Abelian *BF* theories: The classical and the BV formalism

Fix a dimension d. Then, for a d-manifold M, one defines

$$F_M = \Omega^1(M) \oplus \Omega^{d-2}(M) \ni (A_1, B_{d-2})$$

and

$$S^0_M = \int_M B_{d-2}\, dA_1$$

In the BV version

$$\mathcal{F}_M = \Omega^\bullet(M)[1] \oplus \Omega^\bullet(M)[d-2] \ni (\mathsf{A}, \mathsf{B})$$

and

$$S_M = \int_M \mathsf{B}\, d\mathsf{A}$$

The convention means that

$$\mathsf{A} = \sum_{i=0}^{d} A_i, \qquad \mathsf{B} = \sum_{i=0}^{d} B_i$$

with A_i an i-form of ghost number $1-i$, and B_i an i-form of ghost number $d-2-i$. (This theory contains "ghosts for ghosts" if $d > 3$.)

Abelian *BF* theories: The classical and the BV formalism

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and

$$S_M^0 = \int_M B_{d-2}\, \mathrm{d}A_1$$

In the BV version

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The convention means that

$$\mathsf{A} = \sum_{i=0}^{d} A_i, \qquad \mathsf{B} = \sum_{i=0}^{d} B_i$$

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The BV-BFV structure

We have

$$Q_M = (-1)^d \int_M \mathrm{dB}\,\frac{\delta}{\delta \mathrm{B}} + \mathrm{dA}\,\frac{\delta}{\delta \mathrm{A}}$$

$$\mathcal{F}^{\partial}_{\partial M} = \Omega^\bullet(\partial M)[1] \oplus \Omega^\bullet(\partial M)[d-2] \ni (\mathrm{A}, \mathrm{B})$$

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The polarization

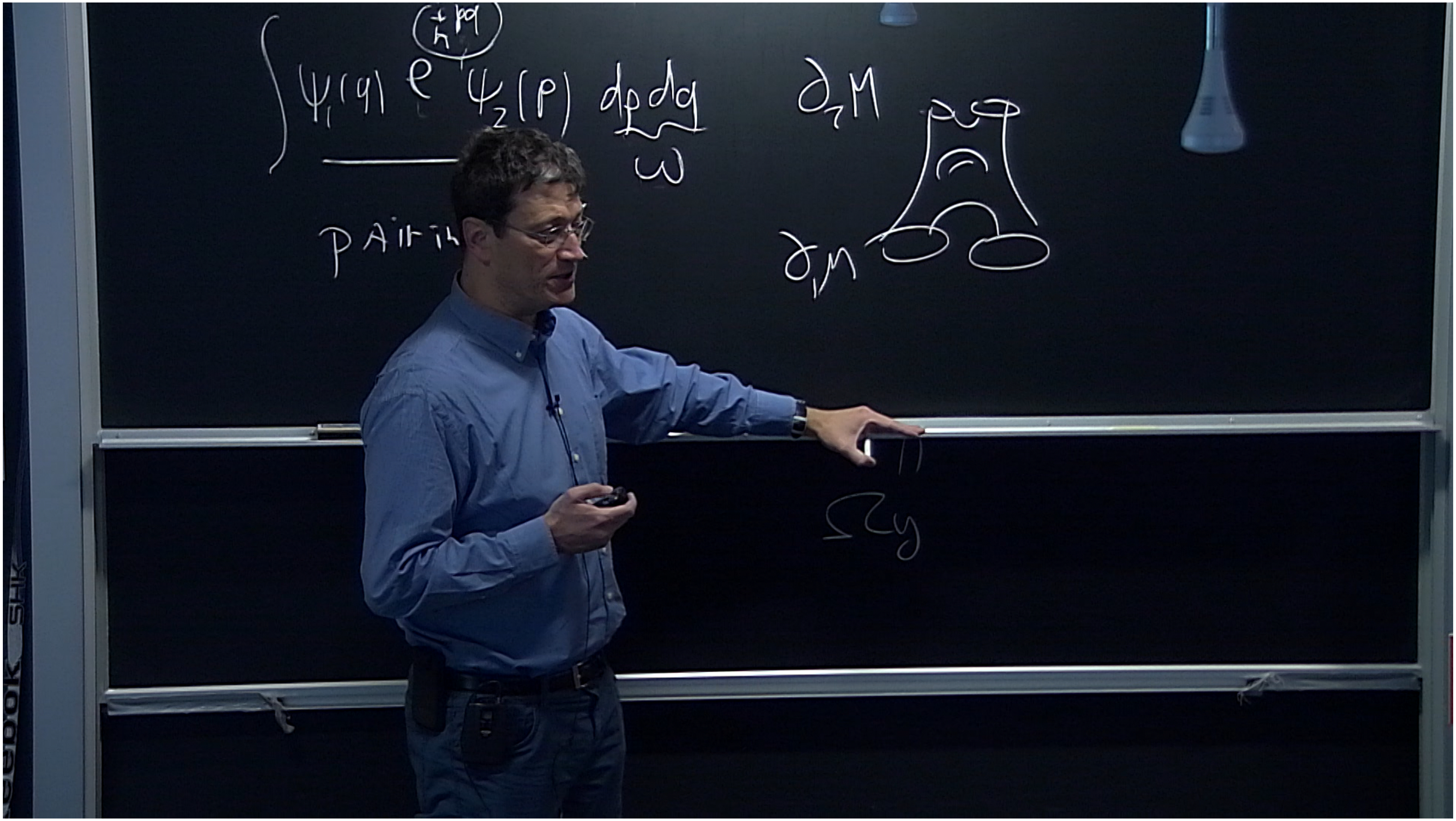
- Write ∂M as a disjoint union $\partial_1 M \cup \partial_2 M$ (possibly empty).
- Choose the polarization $\mathcal{P}$ given by the $\frac{\delta}{\delta \mathsf{B}}$-distribution on $\partial_1 M$ and by the $\frac{\delta}{\delta \mathsf{A}}$-distribution on $\partial_2 M$.
- To make this compatible with the boundary 1-form we make the gauge transformation generated by $f = (-1)^{d-1} \int_{\partial_2 M} \mathsf{B}\mathsf{A}$

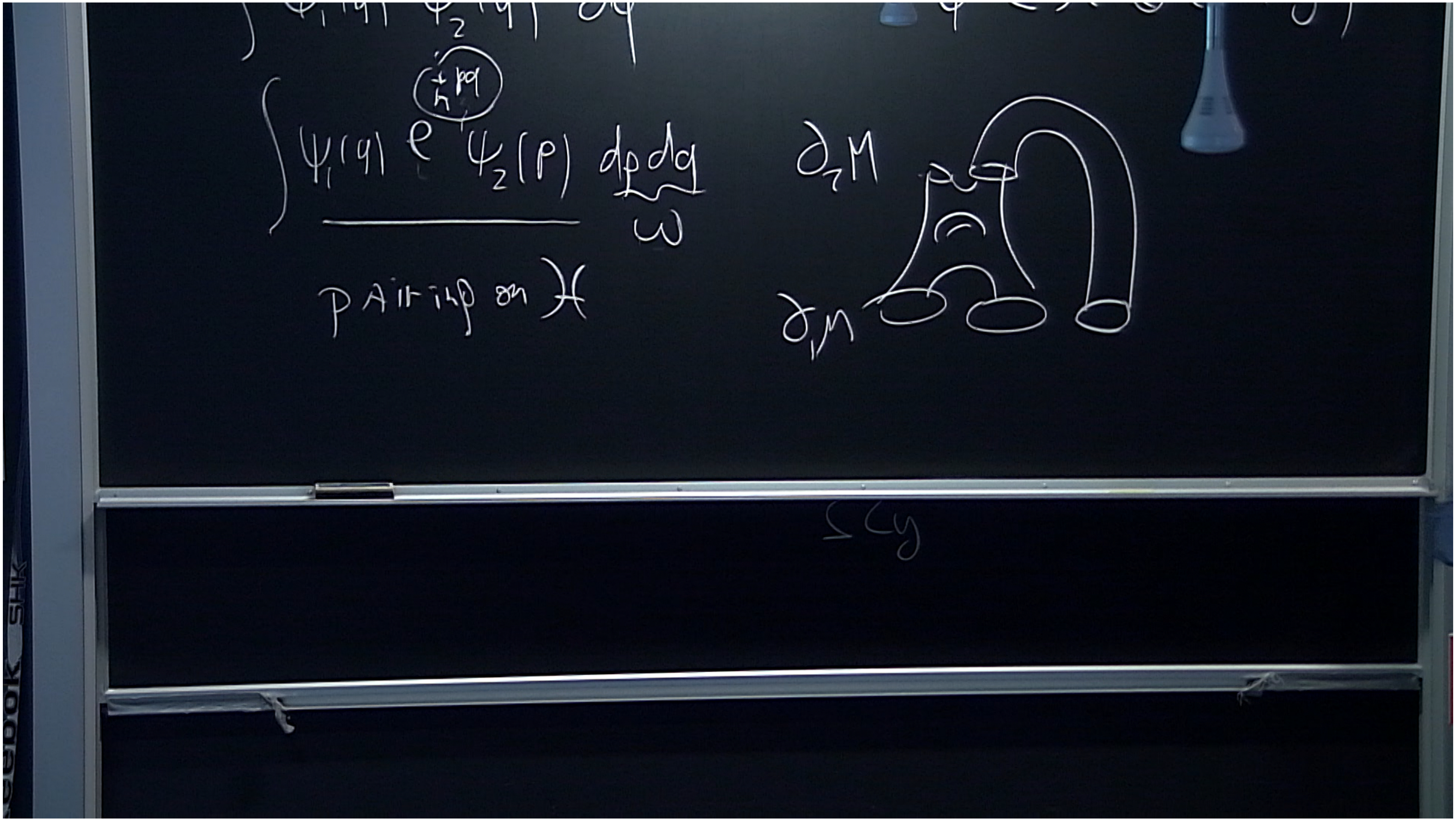
$$\alpha^{\partial}_{\partial M} \mapsto (-1)^d \int_{\partial_1 M} \mathsf{B}\, \delta \mathsf{A} - \int_{\partial_2 M} \delta \mathsf{B}\, \mathsf{A}$$

$$S_M \mapsto \int_M \mathsf{B}\, \mathrm{d}\mathsf{A} + (-1)^{d-1} \int_{\partial_2 M} \mathsf{B}\mathsf{A}$$

- We then have

$$\mathcal{B} = \Omega^\bullet(\partial_1 M)[1] \oplus \Omega^\bullet(\partial_2 M)[d-2] \ni (\mathbb{A}, \mathbb{B})$$




∫ ψ₁(q) e^(i/ħ pq) ψ₂(p) dp dq
ω
pairing on H
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∂₁M

The splitting

- We write

$$\mathsf{A} = \mathbb{A} + \hat{\mathsf{A}},$$
$$\mathsf{B} = \mathbb{B} + \hat{\mathsf{B}},$$

where $\mathbb{A}$ and $\mathbb{B}$ now denote the extension by zero of $\mathbb{A}$ and $\mathbb{B}$ to the bulk, and $\hat{\mathsf{A}}$ and $\hat{\mathsf{B}}$ are the coordinates of $\mathcal{Y}$ ("the fluctuations").

- By construction, the restriction of $\hat{\mathsf{A}}$ to $\partial_1 M$ vanishes, and the restriction of $\hat{\mathsf{B}}$ to $\partial_2 M$ vanishes.
- After appropriately integrating by parts in order to avoid deriving the discontinuous fields $\mathbb{A}$ and $\mathbb{B}$, we get

$$S_M = \int_M \hat{\mathsf{B}}\, \mathrm{d}\hat{\mathsf{A}} + (-1)^{d-1}\left(\int_{\partial_2 M} \mathbb{B}\hat{\mathsf{A}} - \int_{\partial_1 M} \hat{\mathsf{B}}\mathbb{A}\right)$$

This is now a Gaussian theory in $\hat{\mathsf{A}}$ and $\hat{\mathsf{B}}$ and linear sources on the boundary coupled to $\mathbb{A}$ and $\mathbb{B}$. To "invert" d we have to separate cohomology and to fix the gauge.

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Residual fields and gauge fixing

For this final step we choose a metric on M and write

$$\hat{\mathsf{A}} = \mathsf{a} + \alpha$$
$$\hat{\mathsf{B}} = \mathsf{b} + \beta$$

with a and b harmonic forms (with the appropriate boundary conditions) and α, β in the orthogonal complement:

$$(\mathsf{a},\mathsf{b}) \in \mathcal{Y}' \simeq H^\bullet(M, \partial_1 M)[1] \oplus H^\bullet(M, \partial_2 M)[d-2], \quad (\alpha, \beta) \in \mathcal{Y}'' = (\mathcal{Y}')^\perp$$

Notice that $\mathcal{Y}'$ is a finite dimensional odd symplectic space (by pairing in cohomology).

As a gauge fixing we may choose $\mathcal{L} = \{\mathrm{Im}\, \mathrm{d}^* : \mathcal{Y}'' \to \mathcal{Y}''\}$.

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With the above choices, the state of abelian *BF* theory reads

$$\psi_M = T_M \, e^{\frac{i}{\hbar} S_M^{\text{eff}}}$$

where:

1. $T_M = \xi\,\tau(M, \partial_1 M) = \xi\,\tau(M, \partial_2 M)$ with ξ a phase (which can be computed, e.g., by discretizing) and τ the Reidemeister torsion
2. $S_M^{\text{eff}} = (-1)^{d-1}\left(\int_{\partial_2 M} \mathbb{B}\mathsf{a} - \int_{\partial_1 M} \mathsf{b}\mathbb{A}\right) - \int_{\partial_2 M \times \partial_1 M} \pi_1^* \mathbb{B}\,\eta\,\pi_2^* \mathbb{A}$ where η is the propagator.

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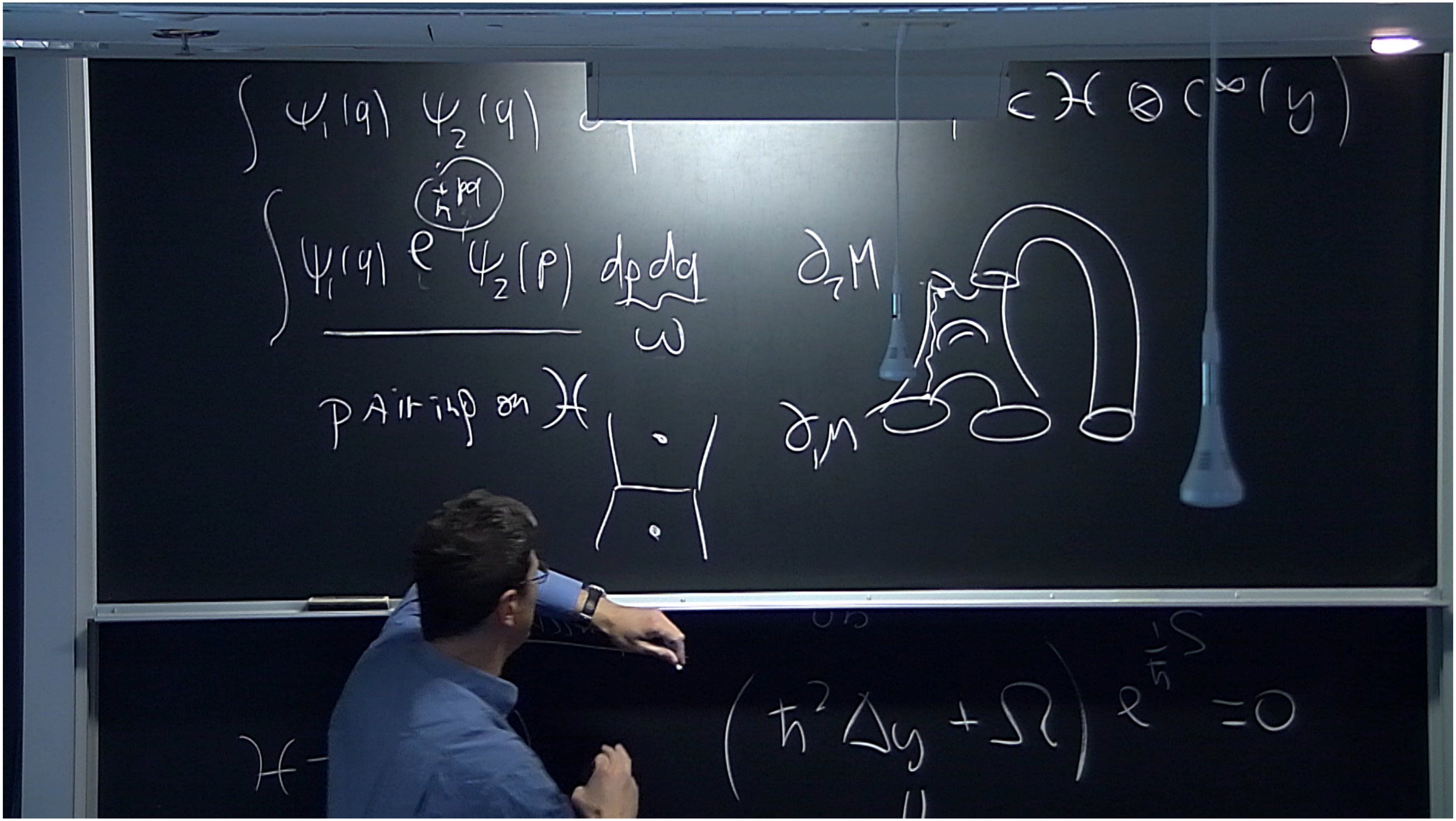
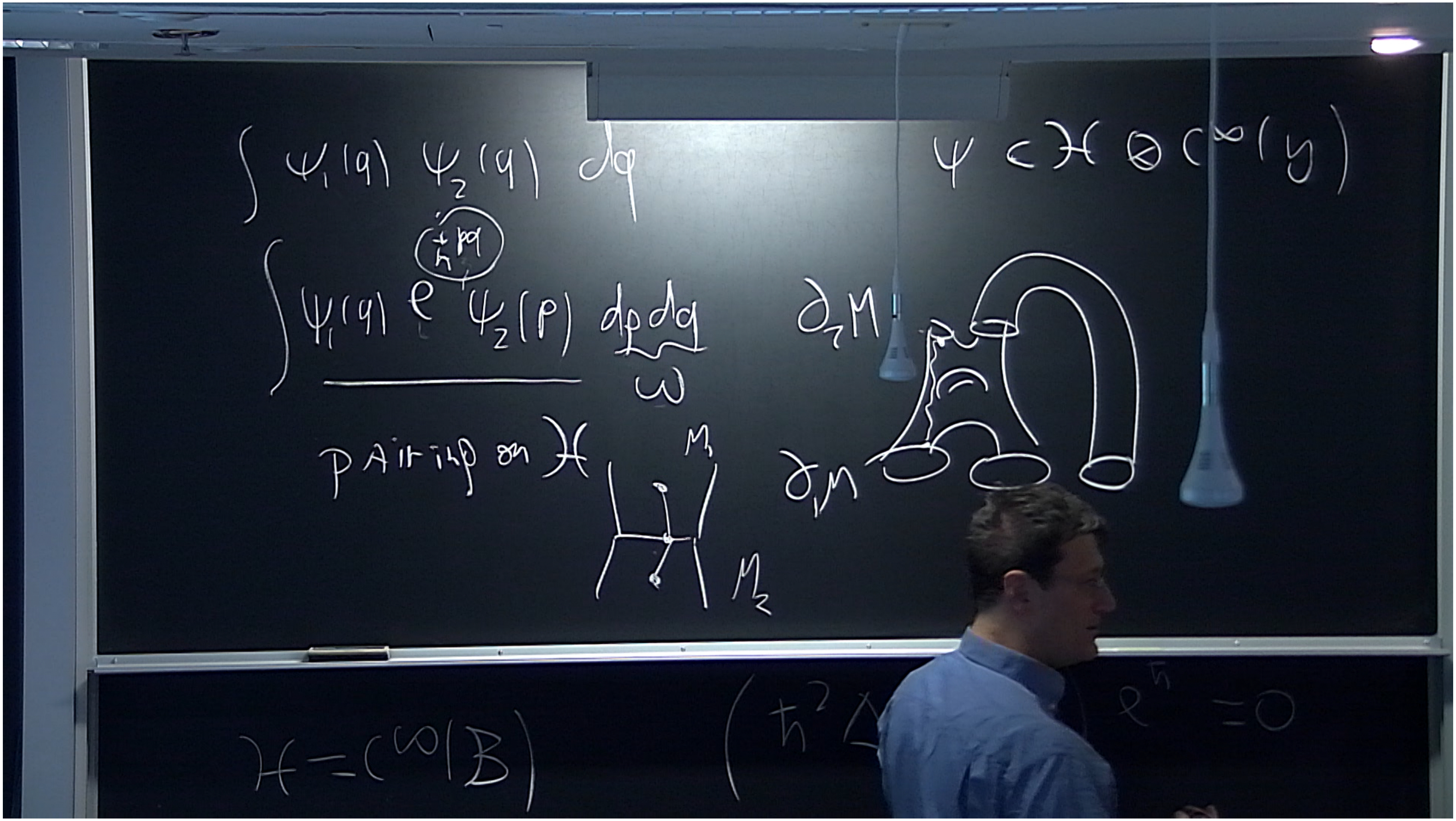
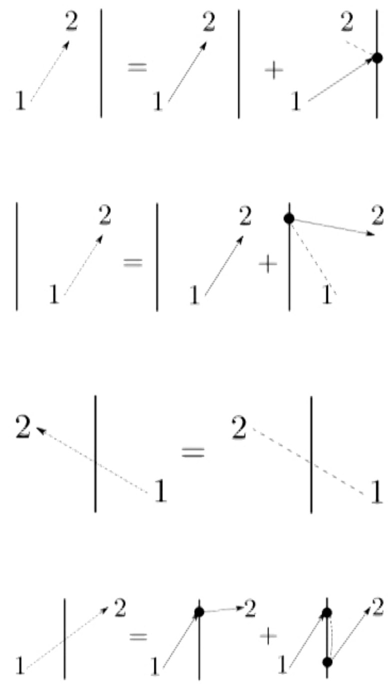


∫ ψ₁(q) ψ₂(q)
⊗ H ⊗ C^∞(y)
∫ ψ₁(q) e^{i/ħ pq} ψ₂(p) dp dq
ω
pairing on H
∂₂M
∂₁M
(ħ² Δy + Ω) e^{i/ħ S} = 0


∫ ψ₁(q) ψ₂(q) dq
ψ ∈ H ⊗ C∞(y)
∫ ψ₁(q) e^{(i/ħ) pq} ψ₂(p) dp dq
ω
∂₂M
∂₁M
pairing on H
M₁
M₂
H = C∞(B)

Gluing for propagators

The propagator for a glued manifold can be obtained from the propagators and the 1-point functions of the components.



More on *BF*

- One can extend the previous construction to non-abelian *BF* theory:

$$S_M^0 = \int_M \left\langle B, \mathrm{d}A + \frac{1}{2}[A, A] \right\rangle, \qquad A \in \Omega(M, \mathfrak{g}), \ B \in \Omega(M, \mathfrak{g}^*)$$

In perturbation theory we get

Figure: $\frac{\delta}{\delta B}$-polarization on the left and $\frac{\delta}{\delta A}$-polarization on the right

- One can also define an exact, discretized version of (non-abelian) *BF* theory.

Perturbation of *BF* theory

- By perturbing *BF* theory, one can extend the above construction to other theories like
  1. Quantum mechanics
  2. Split Chern–Simons theory
  3. Poisson sigma model
  4. 2D Yang–Mills theory
- Other theories like scalar field, spinor field, Yang–Mills can be treated alike (but one has to take renormalization into account).
- GR (in the Einstein–Hilbert formulation) has a nice classical BV-BFV description if we restrict to metrics that do not have light like directions on the boundary.

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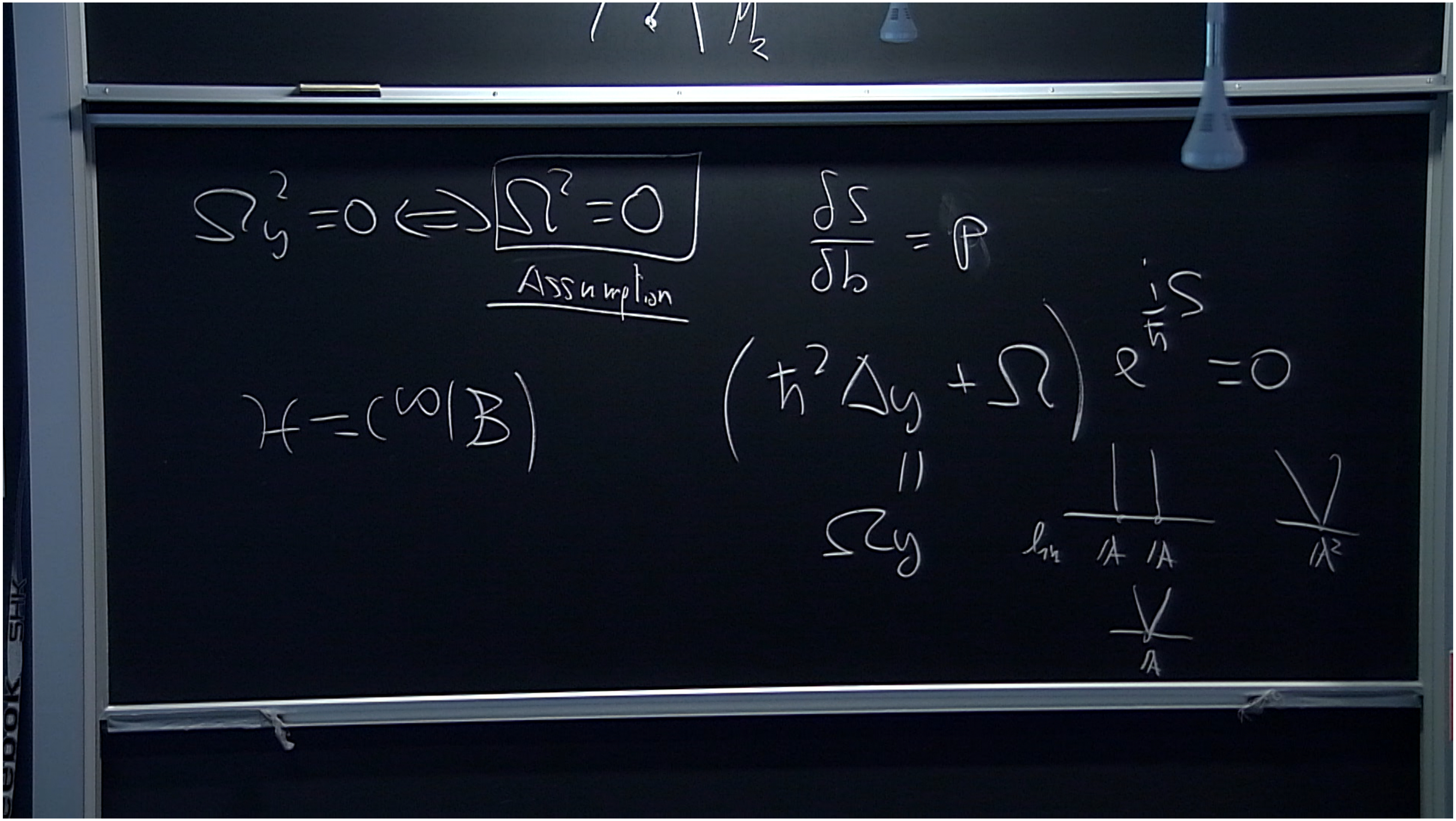
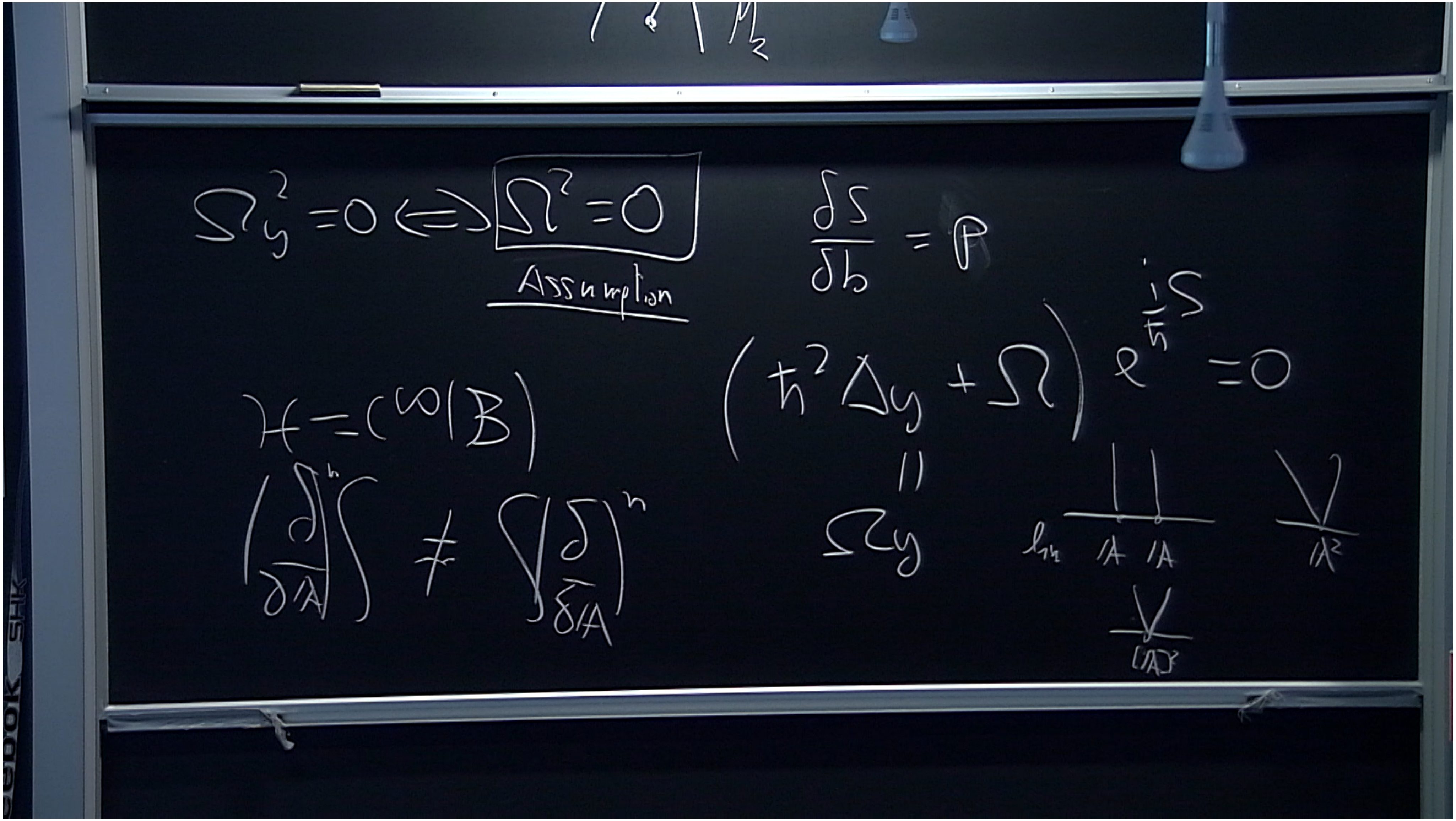
Main references:

1. A. S. Cattaneo, P. Mnëv and N. Reshetikhin, "Classical and quantum Lagrangian field theories with boundary," PoS(CORFU2011)044
2. A. S. Cattaneo, P. Mnëv and N. Reshetikhin, "Classical BV theories on manifolds with boundaries," Commun. Math. Phys. **332**, 535–603 (2014)
3. A. S. Cattaneo, P. Mnëv and N. Reshetikhin, "Perturbative quantum gauge theories on manifolds with boundary," arXiv:1507.01221, Commun. Math. Phys.

Thanks


Ω_y^2 = 0 ⟺ Ω^2 = 0
Assumption
H = (ω|B)
δS/δb = p
(ħ^2 Δ_y + Ω) e^{iS/ħ} = 0
Ω_y
/A /A
/A^2
/A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