Title: Perturbative BV-BFV theories on manifolds with boundary

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Abstract:

An Introduction to the BV Formalism

Perturbative BV-BFV theories on manifolds with boundary

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Introduction

- Lift Atiyah–Segal's axioms to the perturbative QFTs
  boundaries $\rightsquigarrow$ vector spaces
  manifolds (with boundaries) $\rightsquigarrow$ states/operators
- Do it for general Lagrangian theories (including gauge theories)

  Lift Atiyah–Segal to the cochain level

Quantization of regular Lagrangian field theories

- In a regular Lagrangian field theory on a manifold M with action S_M, one may "define" a state by the functional integral

$$\psi_M(\varphi) = \int_{\Phi \in \pi_M^{-1}(p_{\partial M}^{-1}(\varphi))} e^{\frac{i}{\hbar} S_M(\Phi)} [D\Phi], \qquad \varphi \in B_{\partial M}$$

- Here we assume we have a projection $\pi_M \colon F_M \to F^{\partial}_{\partial M}$ from the space of bulk fields to the symplectic space of boundary fields.
- For simplicity we assume $F^{\partial}_{\partial M} = T^* B_{\partial M}$, with $p_{\partial M}$ the projection.
- If the Lagrangian is degenerate we have to resort to BRST or BV for gauge fixing.

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The local, finite-dimensional BV formalism

The Batalin–Vilkovisky (BV) formalism is used to gauge fix gauge theories and check gauge-fixing independence.

We start with a local, finite-dimensional version.

- Consider super coordinates q^i, p_i and the symplectic form $\omega = \sum_i \mathrm{d}p_i \mathrm{d}q^i$.
- Functions are ordinary smooth functions of the even coordinates tensor the Grassmann algebra generated by the odd coordinates. Here p_i has parity opposite to q^i.
- The BV Laplacian is defined as

$$\Delta = \sum_i (-1)^{|q_i|} \frac{\partial^2}{\partial q^i \partial p_i}.$$

Equivalently, $\Delta f = -\frac{1}{2} \operatorname{div} X_f$.

Lemma

$$\Delta^2 = 0, \qquad \Delta(fg) = \Delta f\, g \pm f \Delta g \pm (f, g).$$

Here $(\ ,\ )$ denotes the BV bracket (odd Poisson bracket given by ω).

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Here (,) denotes the BV bracket (odd Poisson bracket given by ω).

Let f be a function of the p, qs and ψ an odd function of the qs only. One defines the BV integral

$$\int_{\mathcal{L}_\psi} f := \int f(q, p_i = \partial_i \psi)\, dq^1 \dots dq^n$$

to be intended as the integral of f on the Lagrangian submanifold

$$\mathcal{L}_\psi = \text{graph}\, \mathrm{d}\psi.$$

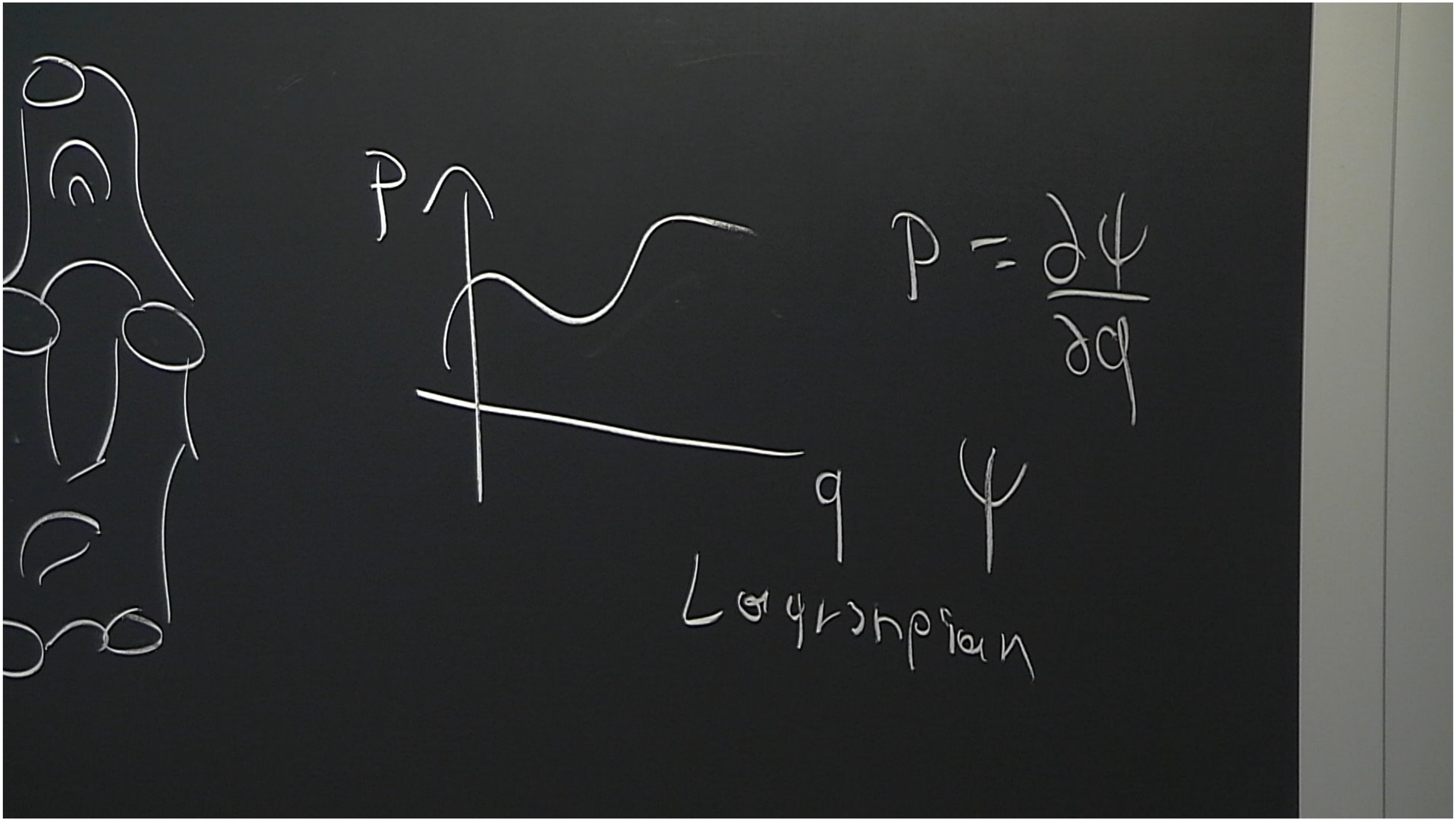
Remark

$dq^1 \dots dq^n$ denotes integration: In the even coordinates it is the standard Lebesgue integration; in the odd coordinates it is the Berezin integration, i.e., just the selection of the top coefficient in the Grassmann algebra (with a choice of orientation).

Lemma

Assume that the integrals converge. Then:

- *If $f = \Delta g$, then $\int_{\mathcal{L}_\psi} f = 0$.*
- *If $\Delta f = 0$, then $\int_{\mathcal{L}_\psi} f$ is invariant under deformations of ψ.*


P
P = ∂ψ/∂q
q
ψ
Lagrangian

The main application

- Suppose $\int_{\mathcal{L}_0} f$ is ill defined but $\Delta f = 0$. Then we may replace the ill-defined integral by a well-defined one $\int_{\mathcal{L}_\psi} f$ and the above Lemma says that it does not matter which ψ we choose (as long as the integral converges). This procedure is called gauge fixing.
- In view of applications to path integrals, we write $f = e^{\frac{i}{\hbar}S}$. Then $\Delta f = 0$ corresponds to the Quantum Master Equation (QME)

$$\frac{1}{2}(S, S) - i\hbar\Delta S = 0$$

The central idea is to allow S to depend on the (possibly formal) parameter $\hbar$ and solve the QME order by order (if possible). The lowest order term is the Classical Master Equation (CME)

$$(S, S) = 0$$

- The main point here is that the CME may be defined on infinite dimensional manifolds (needed in field theory). Integration together with the actual definition of Δ are deferred to a second step (e.g., perturbative path integral quantization).

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The BV pushforward

One may generalize the BV integral to a partial integration. Assume a splitting of coordinates $(p, q) = (p', p'', q', q'')$ with $\omega = \omega' + \omega''$ and $\Delta = \Delta' + \Delta''$. If f is a function of all coordinates and ψ an odd function of the q''s, one defines the BV pushforward

$$\int_{\mathcal{L}_\psi} f := \int f|_{p_i''=\partial_i\psi}\, dq''$$

One can then prove that

$$\Delta' \int_{\mathcal{L}_\psi} f = \int_{\mathcal{L}_\psi} \Delta f$$

and that, if $\Delta f = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{L}_\psi(t)} f = \Delta'(\cdots)$$

Application in field theory: Integrate out "fast" fields $\to$ effective field theory.

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The global BV formalism

- Let $\mathcal{F}$ be an odd finite dimensional symplectic manifold. By results of Batalin–Vilkovisky, Manin, Witten, Schwarz, Khudaverdian, Ševera...:
  1. Half densities on $\mathcal{F}$ naturally define densities on Lagrangian submanifolds of $\mathcal{F}$, so they can be integrated.
  2. There is a canonically defined operator Δ on half densities satisfying $\Delta^2 = 0$.
  3. $\int_{\mathcal{L}} \Delta\rho = 0$, for $\mathcal{L}$ Lagrangian, ρ half density.
  4. $\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{L}_t} \sigma = 0$ if $\Delta\sigma = 0$ and $\mathcal{L}_t$ a family of Lagrangian submanifolds.
- Main example: $\mathcal{F} = \Pi T^*M$, M manifold.
  Then half densities on $\mathcal{F}$ = differential forms on M,
  $\int_{\Pi N^*C} \rho = \int_C \rho, \qquad \Delta = \mathrm{d}.$

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BV on functions

- Fix a nowhere vanishing half density ρ. On functions define Δ by

$$\Delta f := \frac{\Delta(f\rho)}{\rho}.$$

We still have $\Delta^2 = 0$.

- If $\Delta\rho = 0$ (always possible), then we have, like in the local case,

$$\Delta(fg) = \Delta f\, g \pm f\Delta g \pm (f, g).$$

- Again, $\Delta e^{\frac{i}{\hbar}S} = 0$ is equivalent to the Quantum Master Equation (QME)

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Further remarks

- It is often convenient to introduce a $\mathbb{Z}$-grading (ghost number).
- One assigns degrees so that S has degree zero and its Hamiltonian vector field Q (the BRST operator)

$$\iota_Q \omega = \delta S$$

has degree 1. This forces ω to have degree -1.
- The CME for S is equivalent to

$$[Q, Q] = 0$$

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Example: gauge theories

- Suppose we have a gauge theory with space of fields F_M, a local action functional S^0_M and a gauge group acting on it.
- We denote by ϕ the fields, by c the ghosts and by δ_{BRST} the BRST operator. We assign degrees by $|\phi| = 0$ and $|c| = 1$.
- We introduce antifields ϕ^+, c^+ with opposite nature than the field, opposite parity and degrees given by $|\phi^+| = -1$ and $|c^+| = -2$.
- We denote by $\mathcal{F}_M$ the space of all the (ϕ, c, ϕ^+, c^+)'s and we set

$$S_M = S^0_M(\phi) + \int_M (\phi^+ \, \delta_{\text{BRST}}\phi + c^+ \, \delta_{\text{BRST}} c),$$

$$\omega_M = \int_M (\delta\phi^+ \delta\phi + \delta c^+ \delta c)$$

- Then, if $\partial M = \emptyset$, the BV action S satisfies the CME.
- We usually consider these "BRST-like" theories, but there are more general examples.

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Example: gauge theories II

Mathematical description:

- Start with a manifold F (the space of classical fields), a Lie algebra $\mathfrak{g}$ acting on it (the symmetries) and a $\mathfrak{g}$-invariant function s (the classical action).
- Note that the Chevalley–Eilenberg differential for the $\mathfrak{g}$-module $C^\infty(F)$ may be interpreted as an odd vector field δ on $F_\mathfrak{g} := F \times \Pi\mathfrak{g}$ which satisfies $[\delta, \delta] = 0$.
- Set $\mathcal{F} := T^*[1]F_\mathfrak{g}$ with its canonical symplectic structure.
- Denote by $\hat{\delta}$ the fiber linear function on $\mathcal{F}$ canonically associated to δ and set
$$S := p^*s + \hat{\delta},$$
where p is the composition of the projections $T^*[1]F_\mathfrak{g} \to F_\mathfrak{g}$ and $F_\mathfrak{g} \to F$.
- The classical master equation $(S, S) = 0$ follows.
- The quantum master equation requires choosing an invariant measure on F.

The modified classical master equation

Let us go back to the classical master equation on some odd symplectic manifold $(\mathcal{F}, \omega)$:

$$(S, S) = 0$$

Its Hamiltonian vector field $Q := (S, \ )$, the "BRST operator," squares to zero. We may also write

$$\iota_Q \omega = \delta S, \qquad [Q, Q] = 0. \qquad (*)$$

These two equations are equivalent to the CME.
We now modify this setting by weakening the first equation allowing Q not to be Hamiltonian. We also do no longer insist on ω being non degenerate (this relaxation is important for finite-dimensional examples). Define

$$\breve{\alpha} := \iota_Q \omega - \delta S$$

Note that $\breve{\alpha}$ is an even 1-form of degree 0.

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Properties

Define

$$\check{\omega} := \delta\check{\alpha}$$

Note that $\check{\omega}$ is an even, closed 2-form of degree 0.

It turns out that Q preserves $\check{\omega}$. By degree reasons it is actually Hamiltonian.

Lemma

There is a uniquely defined odd function $\check{S}$, of degree 1, such that

$$\iota_Q\check{\omega} = \delta\check{S}$$

Moreover,

$$\text{“}\frac{1}{2}(S,S)\text{”} := \frac{1}{2}\iota_Q\iota_Q\omega = \check{S}$$

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Symplectic reduction

The 2-form $\check{\omega} := \delta\check{\alpha}$ is usually degenerate. Its kernel $\mathcal{K}$ is an involutive distribution. We make the following

crucial assumption $\mathcal{K}$ has constant rank and $\mathcal{F}^\partial := \mathcal{F}/\mathcal{K}$ is smooth

simplifying assumption $\check{\alpha}$ is $\mathcal{K}$-basic

This means that we have a uniquely defined 1-form α^∂ on $\mathcal{F}^\partial$ such that

$$\check{\alpha} = \pi^*\alpha^\partial, \qquad \check{\omega} = \pi^*\omega^\partial$$

with $\omega^\partial = \delta\alpha^\partial$ and $\pi\colon \mathcal{F} \to \mathcal{F}^\partial$ the natural projection.
It also turns out that Q is $\mathcal{K}$-projectable, so we have a uniquely defined cohomological vector field Q^∂ on $\mathcal{F}^\partial$ such that

$$Q\pi^* f = \pi^* Q^\partial f, \qquad \forall f \in \mathcal{F}^\partial.$$

Finally, there is a uniquely defined odd function S^∂ such that

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The classical (exact) BV-BFV formalism

We may summarize as follows:

1. We have a quadruple $(\mathcal{F}^\partial, \omega^\partial, S^\partial, Q^\partial)$ where $\omega^\partial = \delta\alpha^\partial$ is an even symplectic form of degree zero such that

$$\iota_{Q^\partial}\omega^\partial = \delta S^\partial, \qquad [Q^\partial, Q^\partial] = 0.$$

2. We have a quintuple $(\mathcal{F}, \omega, S, Q, \pi)$ where ω is an odd "symplectic" form of degree -1, $\pi\colon \mathcal{F} \to \mathcal{F}^\partial$ is a surjective submersion such that Q is π-projectable to Q^∂ and the mCME holds

$$\iota_Q\omega = \delta S + \pi^*\alpha^\partial, \qquad [Q, Q] = 0.$$

- The structure (1) is the well-known BFV formalism (used to give a cohomological description of the reduction of coisotropic submanifolds)
- The structure (2) is what we call the BV-BFV formalism. The crucial point is the mCME that establishes a relation between the broken bulk BV theory and the boundary BFV theory.

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Gauge transformations

The theory allows for “gauge transformations”. Given $f \in \mathcal{F}^{\partial}$, one has the transformations

$$\alpha^{\partial} \mapsto \alpha^{\partial} + \delta f, \qquad S \mapsto S - \pi^{*} f.$$

which leave all relations unchanged. This can be used to choose a “better” α^{∂} (e.g., for quantization). It also shows that the natural framework is in terms of sections of line bundles and connection 1-forms.

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which leave all relations unchanged. This can be used to choose a "better" α^∂ (e.g., for quantization). It also shows that the natural framework is in terms of sections of line bundles and connection 1-forms.

Example: local field theories on manifolds with boundary

- Let $(\mathcal{F}_M, \omega_M, S_M)$ be the BV data for a local field theory (i.e., both ω_M and S_M are local) with $\partial M = \emptyset$. Then also $Q_M := (S_M,)$ is local.
- By locality, we may define $\mathcal{F}_M$, ω_M, S_M and Q_M also on a manifold with boundary.
- The equation $\iota_{Q_M}\omega_M = \delta S_M$ is satisfied up to boundary terms. Hence $\mathcal{F}^\partial$ only depends on boundary data ("boundary fields"): we write $\mathcal{F}^\partial_{\partial M}$.
- Notice the self similarity (up to degree shift) of the structure. In local field theory, it often happens that $(\mathcal{F}^\partial, \omega^\partial, S^\partial, Q^\partial)$ still consists of local data. One can thus iterate the construction to higher codimensional boundary strata of manifolds with corners.
- In the definition of $\mathcal{F}_M$ one may incorporate some boundary conditions. One must choose $\mathcal{F}_M$ in such a way that the boundary BFV manifold $\mathcal{F}^\partial_{\partial M}$ describes the correct reduced phase space. Very often, the naif choice (i.e. no boundary conditions) is the right one.

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Example: Electromagnetism

- Maxwell's equations: $\mathrm{d}^*\mathrm{d}A = 0$, A connection 1-form.
- First-order formalism: $S_M^{cl} = \int_M B\,\mathrm{d}A + \frac{1}{2}B * B$
  B a $(d-2)$-form. Then $EL = \{*B = \mathrm{d}A,\ \mathrm{d}B = 0\}$.
- BV: $S_M = \int_M B\,\mathrm{d}A + \frac{1}{2}B * B + A^+\,\mathrm{d}c$
  A^+: $(d-1)$-form, ghost number -1; c: 0-form, ghost number 1.
  $\omega_M = \int_M \delta A\,\delta A^+ + \delta B\,\delta B^+ + \delta c\,\delta c^+$,
  B^+ and c^+ do not show up in the action.
  $QA = \mathrm{d}c,\ QA^+ = \mathrm{d}B,\ QB^+ = *B + \mathrm{d}A,\ Qc^+ = \mathrm{d}A^+$.
- Boundary fields: A, B, A^+, c,
  $S^\partial_\Sigma = \int_\Sigma c\,\mathrm{d}B$,
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  Interpretation:
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