Title: Emergence of Supergroups from Junctions of M-branes

Date: May 16, 2017 02:30 PM

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Abstract: <p>We study the boundary conditions in the topologically twisted Chern-Simons matter theories with the Lie 3-algebraic structure. We find that the supersymmetric boundary conditions and the gauge invariant boundary conditions can be unified as the complexified gauge invariant boundary conditions which lead to the supergroup WZW models. We examine the BPS indices of the supergroup WZW models which may describe certain junctions of M2-branes and M5-branes by identifying the vacuum configurations of the brane system with the weight diagrams of the associated Lie superalgebras. For the case arising from the BLG model an explicit expression can be given in terms of a mock modular form. We will also consider the alternative matrix quantum mechanical descriptions of the supergroup WZW models. The talk will be based on recent works arXiv:1512.06646, arXiv:1612.07565 and ongoing research.</p>

Perimeter Institute
2017 May 16

Emergence of Supergroups from Junctions of M-branes

Tadashi Okazaki
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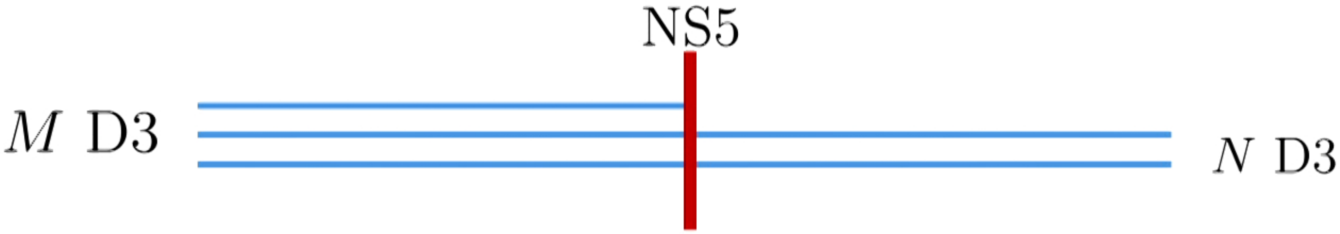
Joint work with Douglas J Smith
arXiv:1512.06646, arXiv:1612.07565, ongoing

Junctions of D3 and NS5

'14 Mikhaylov & Witten



Junctions of D3 and NS5

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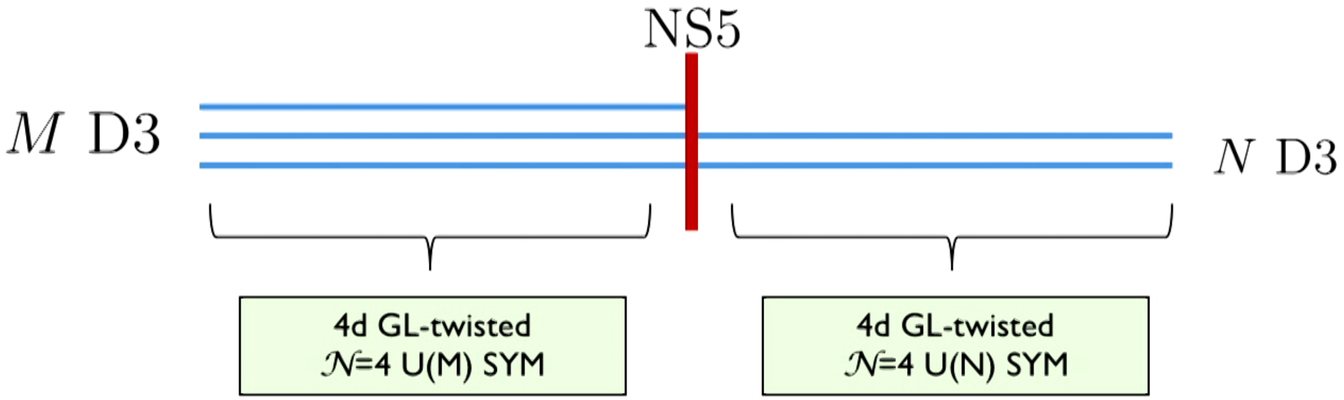

4d GL-twisted
$\mathcal{N}$=4 U(M) SYM

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Junctions of D3 and NS5

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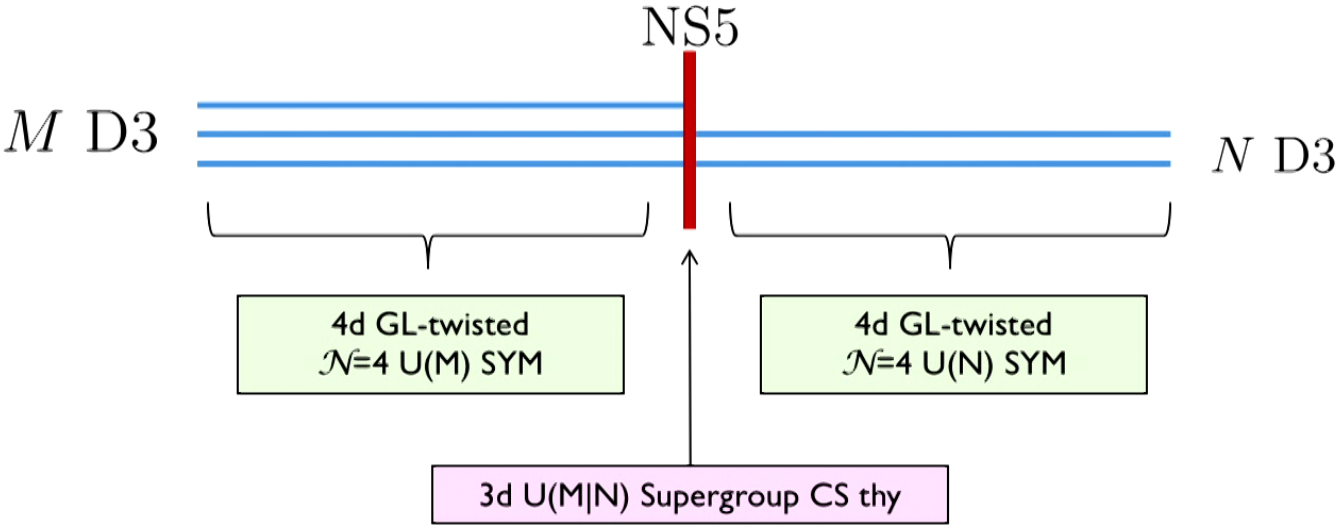


4d GL-twisted
$\mathcal{N}$=4 U(M) SYM

4d GL-twisted
$\mathcal{N}$=4 U(N) SYM

3d U(M|N) Supergroup CS thy

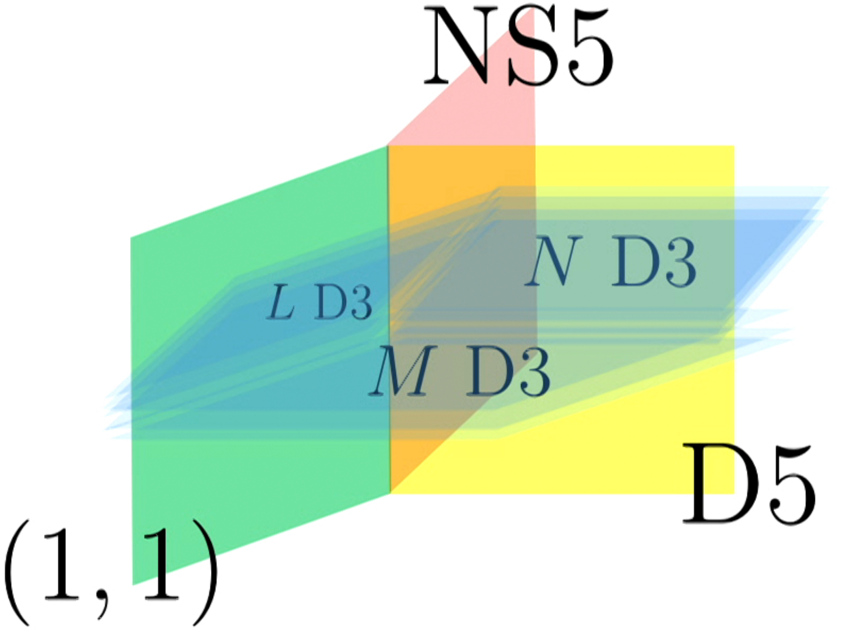
$$S = \{\mathcal{Q}, \cdots\} + \frac{i\mathcal{K}}{8\pi}\int \mathrm{Str}\left(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right)$$

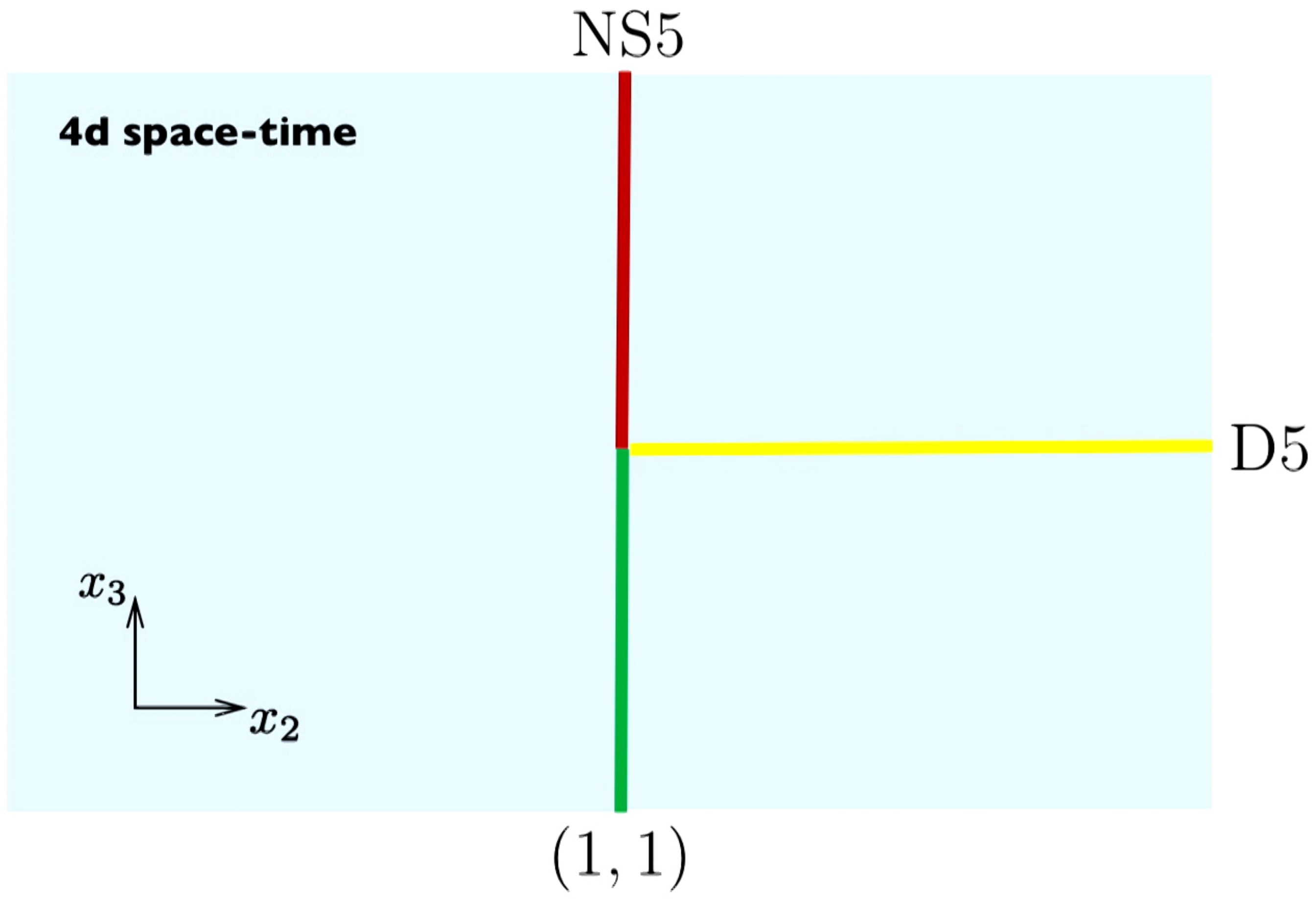
$$\Psi = \mathcal{K} + h\,\mathrm{sgn}(\mathcal{K}) \qquad \Psi = \frac{\theta_{\mathrm{YM}}}{2\pi} + \frac{4\pi i}{g^2}\frac{t-t^{-1}}{t+t^{-1}} \qquad \epsilon = u\epsilon_\ell + v\epsilon_r$$

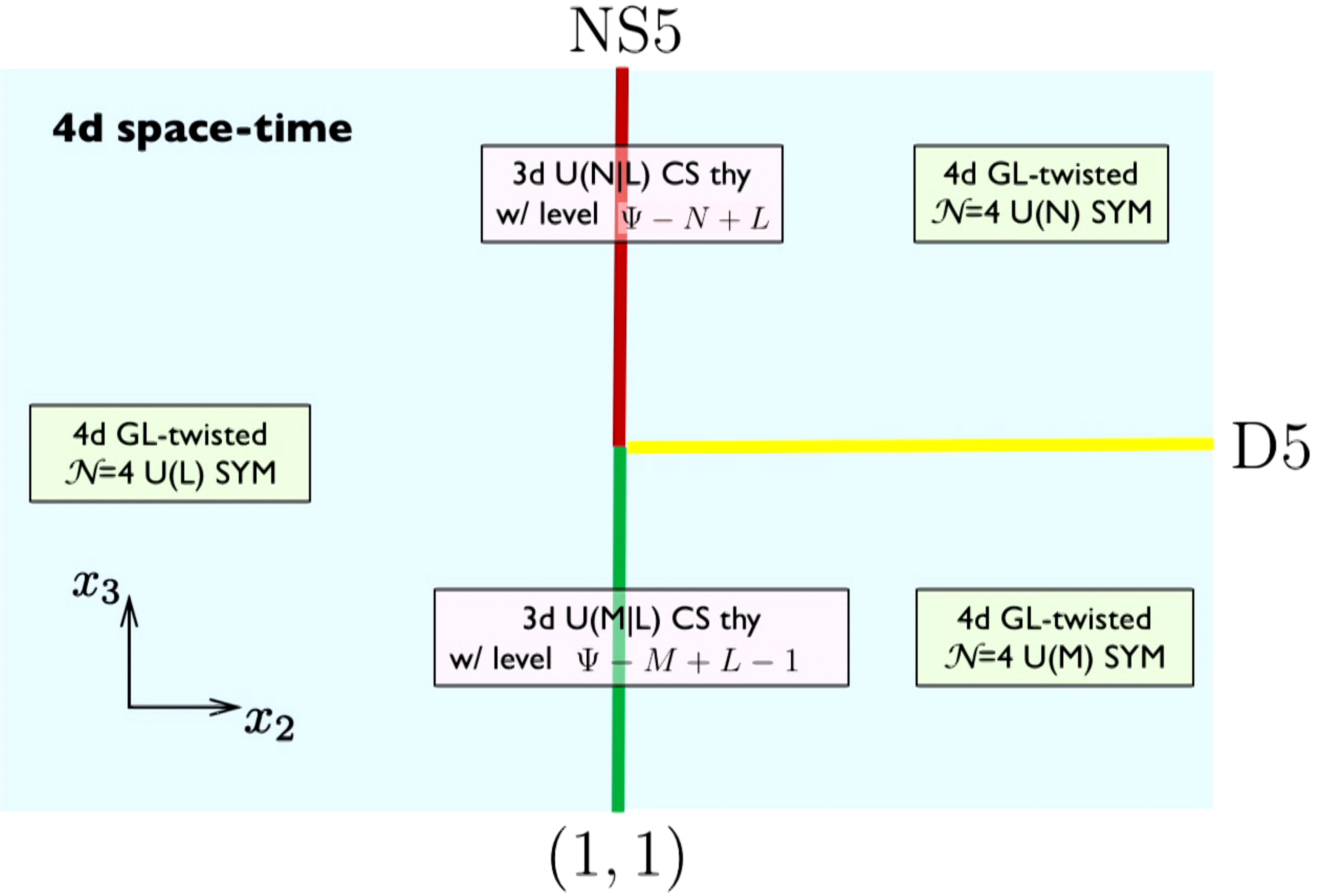
$$t = \frac{v}{u}$$

Junctions of D3 and five-branes

'17 Gaiotto & Rapcak




NS5
4d space-time
D5
x_3
x_2
$(1,1)$



4d space-time

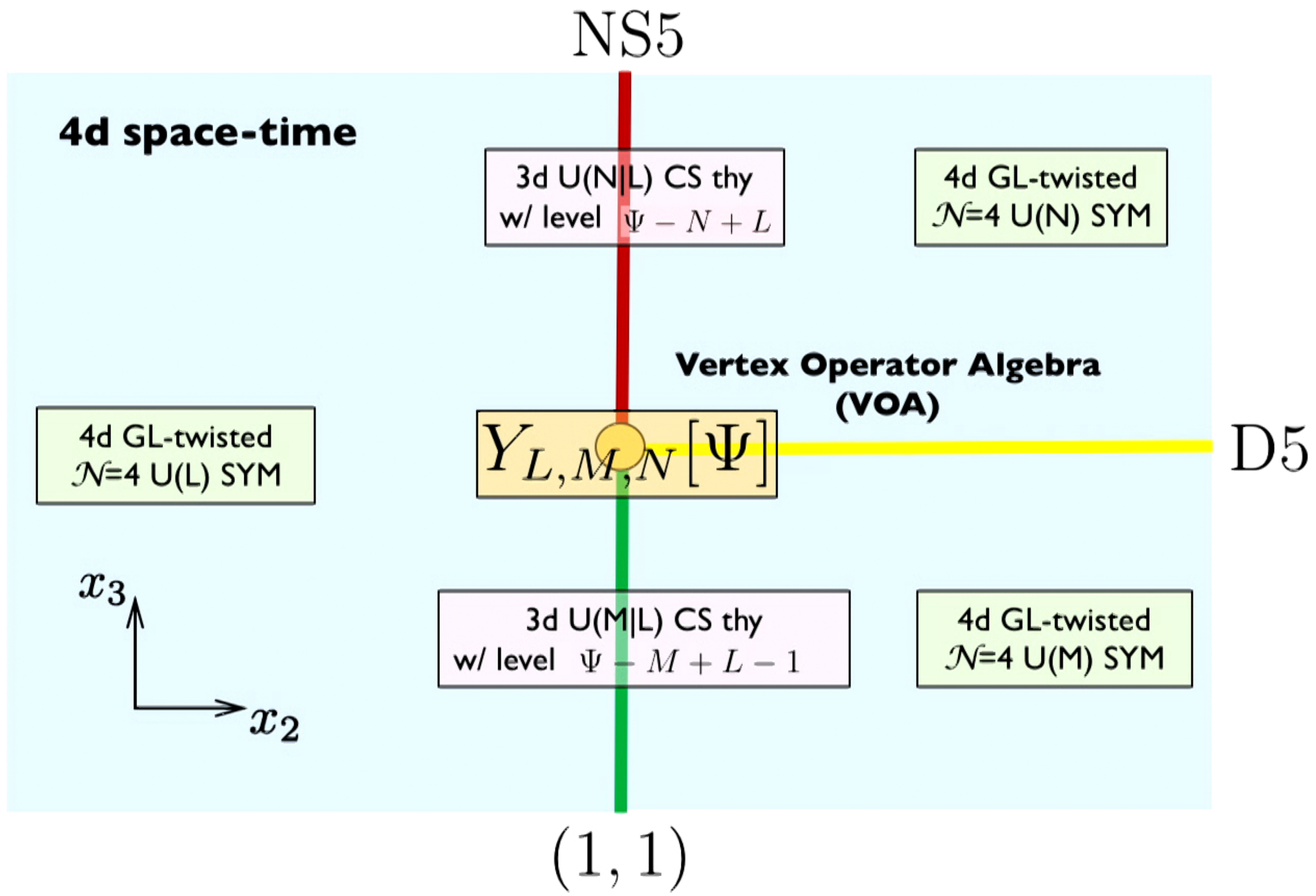
3d U(N|L) CS thy
w/ level $\Psi - N + L$

4d GL-twisted
$\mathcal{N}$=4 U(N) SYM

4d GL-twisted
$\mathcal{N}$=4 U(L) SYM

3d U(M|L) CS thy
w/ level $\Psi - M + L - 1$

4d GL-twisted
$\mathcal{N}$=4 U(M) SYM



VOA originates from ***WZW algebra*** *of*

U(N|L) x U(M|L) ***Lie supergroup*** *at interface !*

Q. What about in M-theory ?

Outline

I. Introduction

II. Topological Twist and SUSY boundary conditions

review

III. Topological M-Strings and SG WZW Models

IV. Mock Modular Index

recent work

V. Quantum Mechanical Matrix Model

ongoing work

II. Topological Twist
and
SUSY boundary conditions

Topological twist = Modification of the **Euclidean rotational sym. group** with the **R-sym group**

'88 Witten

Topologically Twisted Theories

(describing **topological sectors** of original ones)

Topological twist = Modification of the **Euclidean rotational sym. group** with the **R-sym group**

'88 Witten

Topologically Twisted Theories

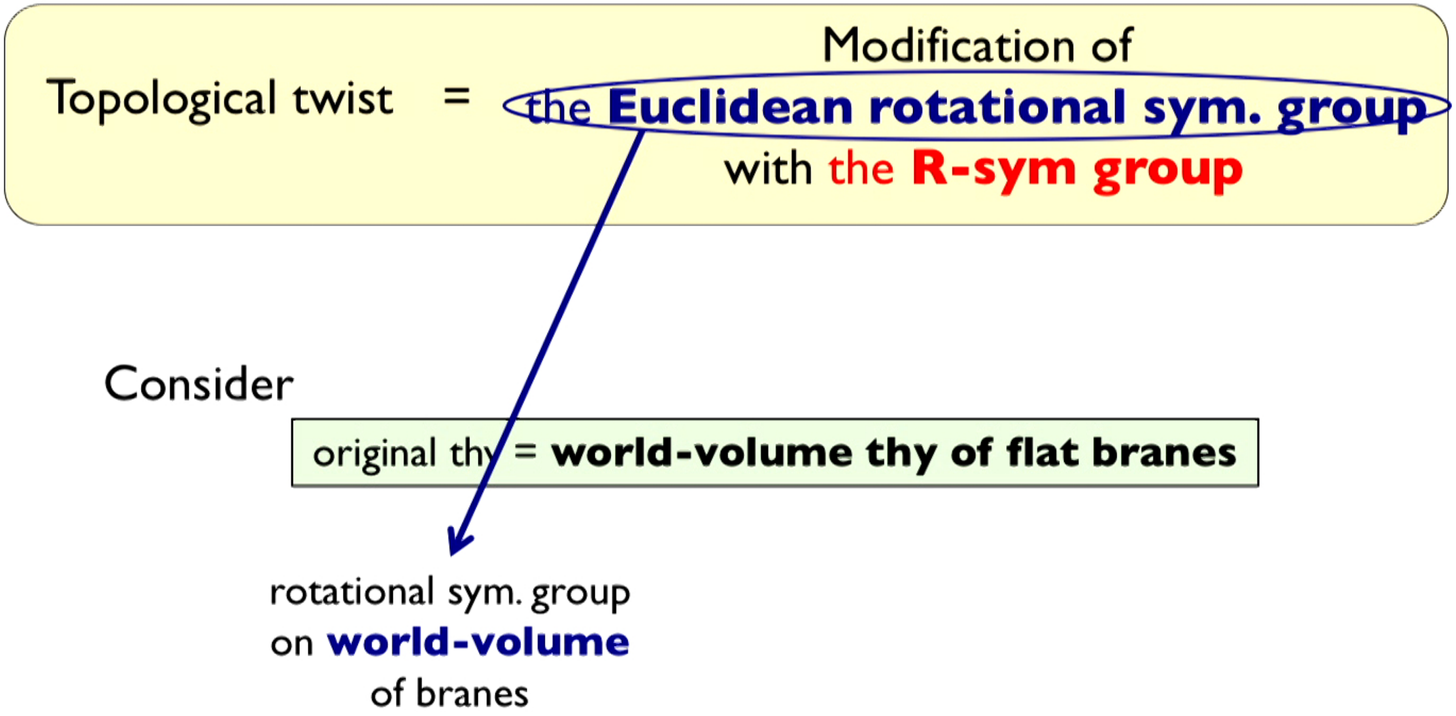
(describing **topological sectors** of original ones)

- defined on arbitrary curved mfd
- exact computation of pfn & index

Topological twist = Modification of the **Euclidean rotational sym. group** with the **R-sym group**

Consider

original thy = **world-volume thy of flat branes**



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Consider

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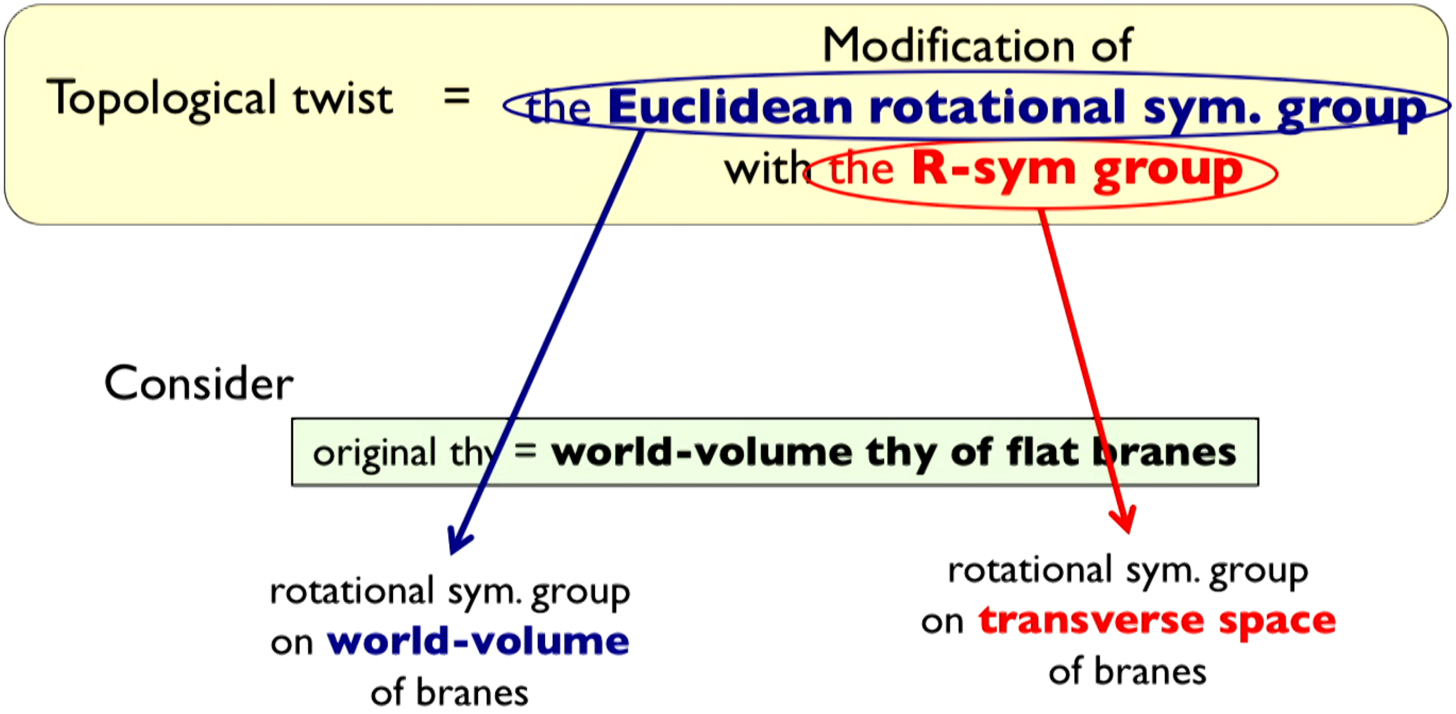
rotational sym. group on **world-volume** of branes



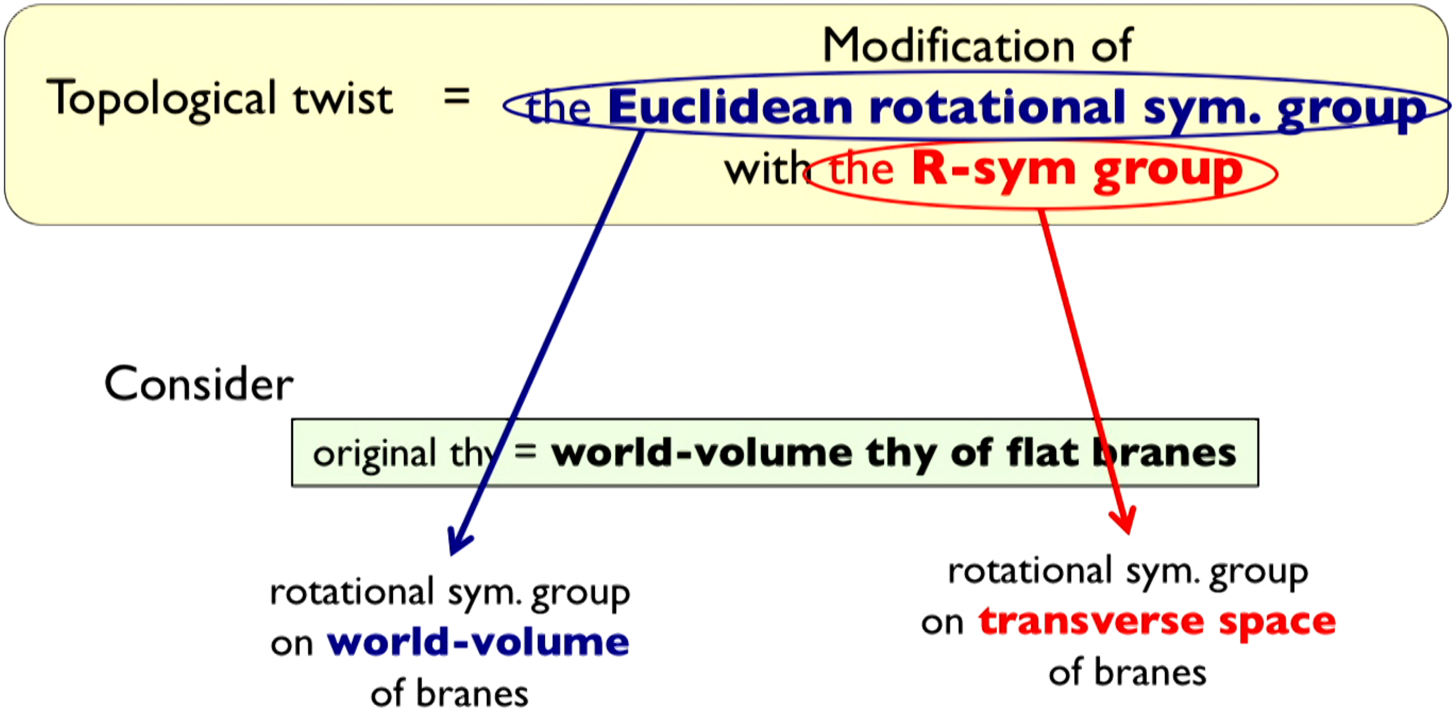
Topological twist = Modification of **the Euclidean rotational sym. group** with **the R-sym group**

Consider

original thy = **world-volume thy of flat branes**

rotational sym. group on **world-volume** of branes

rotational sym. group on **transverse space** of branes



Topological twist = Modification of **the Euclidean rotational sym. group** with **the R-sym group**

Consider

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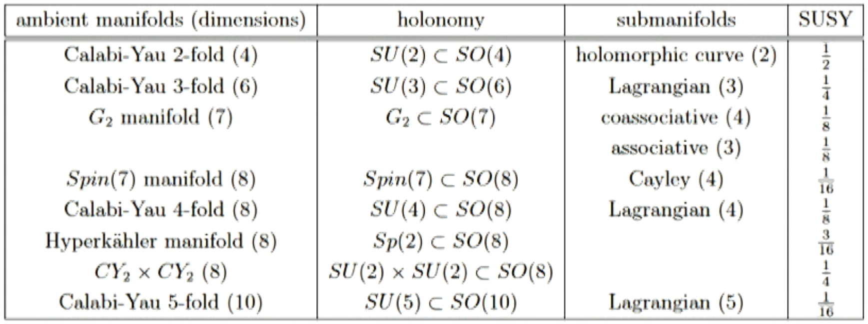
rotational sym. group
on **world-volume**
of branes

rotational sym. group
on **transverse space**
of branes

Topological twist describes curved branes !



ambient manifolds (dimensions)	holonomy	submanifolds	SUSY
Calabi-Yau 2-fold (4)	$SU(2) \subset SO(4)$	holomorphic curve (2)	$\frac{1}{2}$
Calabi-Yau 3-fold (6)	$SU(3) \subset SO(6)$	Lagrangian (3)	$\frac{1}{4}$
G_2 manifold (7)	$G_2 \subset SO(7)$	coassociative (4)	$\frac{1}{8}$
		associative (3)	$\frac{1}{8}$
$Spin(7)$ manifold (8)	$Spin(7) \subset SO(8)$	Cayley (4)	$\frac{1}{16}$
Calabi-Yau 4-fold (8)	$SU(4) \subset SO(8)$	Lagrangian (4)	$\frac{1}{8}$
Hyperkähler manifold (8)	$Sp(2) \subset SO(8)$		$\frac{3}{16}$
$CY_2 \times CY_2$ (8)	$SU(2) \times SU(2) \subset SO(8)$		$\frac{1}{4}$
Calabi-Yau 5-fold (10)	$SU(5) \subset SO(10)$	Lagrangian (5)	$\frac{1}{16}$

Curved BPS branes

on **calibrated submfd** (SUSY cycle) $\subset$ **ambient space**

decoupling limit

world-volume theory of **curved** BPS branes = **Topologically twisted theory**

'95 Bershadsky Sadov Vafa

Ex. Curved D3-branes

world-volume theory of flat N D3-branes = **4d $\mathcal{N}$=4 U(N) SYM thy**

$$\boxed{\mathcal{L} = \frac{1}{e^2}\mathrm{Tr}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\phi_I D^\mu\phi^I - \frac{1}{4}[\phi_I,\phi_I][\phi^I,\phi^I] \quad +\frac{1}{2}\overline{\psi}\Gamma^\mu D_\mu\psi - \frac{i}{2}\overline{\psi}\Gamma^I[\phi_I,\psi]\right)}$$

world-volume of D3-branes

transverse space of D3-branes

$$\mathbf{R}^4 \times \mathbf{R}^6$$

$$\mathrm{SO}(4)_E \times \mathrm{SO}(6)_R$$

$$\cong \mathrm{SU}(2)_l \times \mathrm{SU}(2)_r \times \mathrm{SU}(4)_R$$

$$\phi : (\mathbf{1},\mathbf{1},\mathbf{6})$$

$$\psi : (\mathbf{2},\mathbf{1},\overline{\mathbf{4}}) \oplus (\mathbf{1},\mathbf{2},\mathbf{4})$$

$$A_\mu : (\mathbf{2},\mathbf{2},\mathbf{1})$$

$$\boxed{SO(4)'_E = (1+\rho)SO(4)}$$

ρ: homomorphism (embedding) of $SO(4)_E$ into $SO(6)_R$

There are only 3 distinct topological twists

$$\mathbf{SO(6)_R = SU(4)_R} \longrightarrow \mathbf{SU(2) \times SU(2) \times U(1)}$$

$$4 \quad = \begin{cases} \text{i.} \quad (2,1)_+ + (1,2)_- & \text{GL twist} \\ \text{ii.} \quad (2,1)_+ + (2,1)_- & \text{VW twist} \\ \text{iii.} \quad (2,1)_0 + (1,1)_+ + (1,1)_- & \text{DW twist} \end{cases}$$

3 twisted 4d N=4 SYM

twist	$SO(4)'_E = SU(2)'_l \times SU(2)'_r$		dim X	SUSY
	$\phi : (\mathbf{1},\mathbf{1},\mathbf{6})$	$\psi : (\mathbf{2},\mathbf{1},\overline{\mathbf{4}}) \oplus (\mathbf{1},\mathbf{2},\mathbf{4})$		
GL	$(\mathbf{1},\mathbf{1})_2 \oplus (\mathbf{1},\mathbf{1})_{-2}$ $(\mathbf{2},\mathbf{2})_0$	$(\mathbf{1},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{1})_{-1}$ $(\mathbf{2},\mathbf{2})_1 \oplus (\mathbf{2},\mathbf{2})_1$ $(\mathbf{3},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{3})_{-1}$		
VW	$(\mathbf{1},\mathbf{1})_2 \oplus (\mathbf{1},\mathbf{1})_{-2}$ $\oplus (\mathbf{1},\mathbf{1})_0$ $(\mathbf{1},\mathbf{3})_0$	$(\mathbf{1},\mathbf{1})_+ \oplus (\mathbf{1},\mathbf{1})_-$ $(\mathbf{2},\mathbf{2})_- \oplus (\mathbf{2},\mathbf{2})_+$ $(\mathbf{1},\mathbf{3})_- \oplus (\mathbf{1},\mathbf{3})_+$		
DW	$(\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1})_0$ $(\mathbf{1},\mathbf{2})_1 \oplus (\mathbf{1},\mathbf{2})_{-1}$	$(\mathbf{1},\mathbf{1})_0$ $(\mathbf{2},\mathbf{1})_{-1} \oplus (\mathbf{2},\mathbf{1})_1$ $(\mathbf{1},\mathbf{2})_1 \oplus (\mathbf{1},\mathbf{2})_{-1}$ $(\mathbf{1},\mathbf{3})_0$ $(\mathbf{2},\mathbf{2})_0$		

3 twisted 4d N=4 SYM

twist	SO(4)'$_E$ = SU(2)'$_l$ × SU(2)'$_r$		dim X	SUSY
	$\phi : (\mathbf{1},\mathbf{1},\mathbf{6})$	$\psi : (\mathbf{2},\mathbf{1},\overline{\mathbf{4}}) \oplus (\mathbf{1},\mathbf{2},\mathbf{4})$		
GL	$(\mathbf{1},\mathbf{1})_2 \oplus (\mathbf{1},\mathbf{1})_{-2}$ $(\mathbf{2},\mathbf{2})_0$	$(\mathbf{1},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{1})_{-1}$ $(\mathbf{2},\mathbf{2})_1 \oplus (\mathbf{2},\mathbf{2})_1$ $(\mathbf{3},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{3})_{-1}$		
VW	$(\mathbf{1},\mathbf{1})_2 \oplus (\mathbf{1},\mathbf{1})_{-2}$ $\oplus\, (\mathbf{1},\mathbf{1})_0$ $(\mathbf{1},\mathbf{3})_0$	$(\mathbf{1},\mathbf{1})_+ \oplus (\mathbf{1},\mathbf{1})_-$ $(\mathbf{2},\mathbf{2})_- \oplus (\mathbf{2},\mathbf{2})_+$ $(\mathbf{1},\mathbf{3})_- \oplus (\mathbf{1},\mathbf{3})_+$		
DW	$(\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1})_0$ $(\mathbf{1},\mathbf{2})_1 \oplus (\mathbf{1},\mathbf{2})_{-1}$	$(\mathbf{1},\mathbf{1})_0$ $(\mathbf{2},\mathbf{1})_{-1} \oplus (\mathbf{2},\mathbf{1})_1$ $(\mathbf{1},\mathbf{2})_1 \oplus (\mathbf{1},\mathbf{2})_{-1}$ $(\mathbf{1},\mathbf{3})_0$ $(\mathbf{2},\mathbf{2})_0$		

3 twisted 4d N=4 SYM

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DW	$(\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1})_0$ $(\mathbf{1},\mathbf{2})_1 \oplus (\mathbf{1},\mathbf{2})_{-1}$	$(\mathbf{1},\mathbf{1})_0$ $(\mathbf{2},\mathbf{1})_{-1} \oplus (\mathbf{2},\mathbf{1})_1$ $(\mathbf{1},\mathbf{2})_1 \oplus (\mathbf{1},\mathbf{2})_{-1}$ $(\mathbf{1},\mathbf{3})_0$ $(\mathbf{2},\mathbf{2})_0$	8	

There exist 3 known SUSY 4-cycle $M_4 \subset$ ambient space X

submanifold (dimension)	ambient manifold (dimension)	SUSY
Lagrangian (4)	Calabi-Yau 4-fold (8)	$\frac{16}{8} = 2$
coassociative (4)	G_2 manifold (7)	$\frac{16}{8} = 2$
Cayley (4)	$Spin(7)$ manifold (8)	$\frac{16}{16} = 1$

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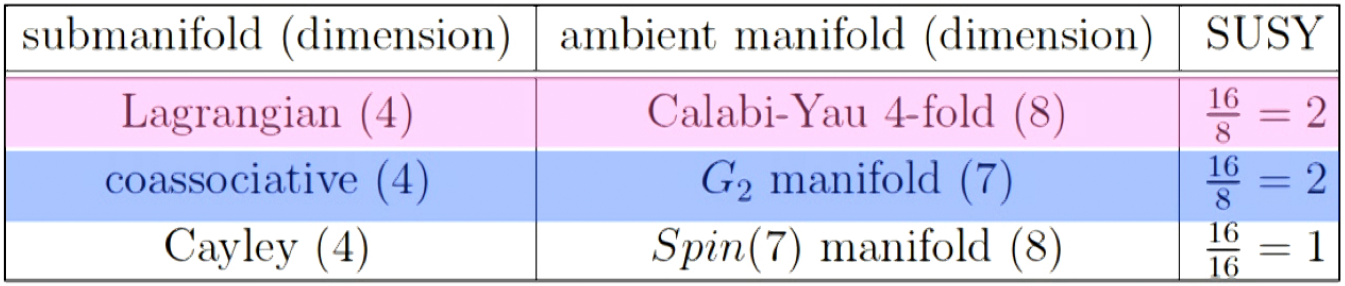

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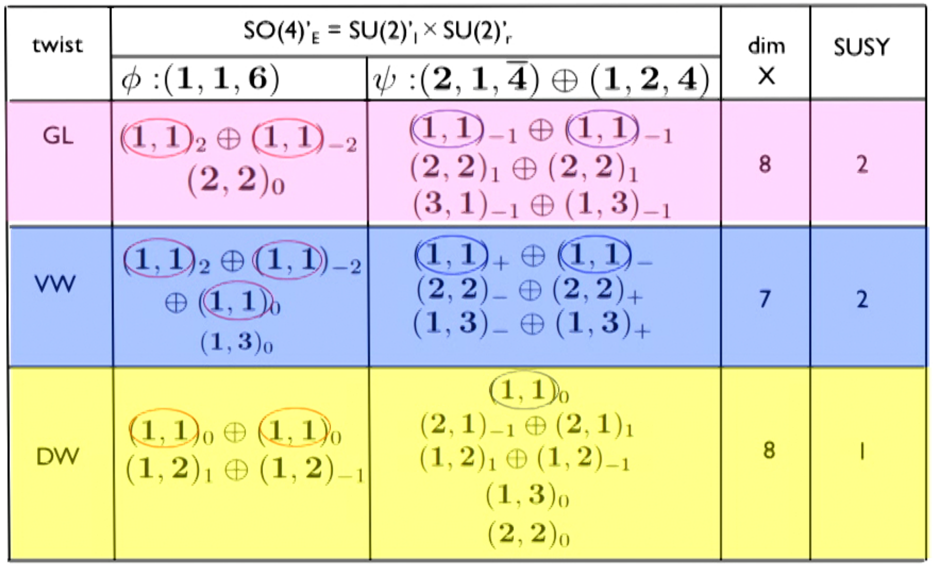

twist	$SO(4)'_E = SU(2)'_l \times SU(2)'_r$		dim X	SUSY
	$\phi : (\mathbf{1}, \mathbf{1}, \mathbf{6})$	$\psi : (\mathbf{2}, \mathbf{1}, \overline{\mathbf{4}}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{4})$		
GL	$(\mathbf{1}, \mathbf{1})_2 \oplus (\mathbf{1}, \mathbf{1})_{-2}$ $(\mathbf{2}, \mathbf{2})_0$	$(\mathbf{1}, \mathbf{1})_{-1} \oplus (\mathbf{1}, \mathbf{1})_{-1}$ $(\mathbf{2}, \mathbf{2})_1 \oplus (\mathbf{2}, \mathbf{2})_1$ $(\mathbf{3}, \mathbf{1})_{-1} \oplus (\mathbf{1}, \mathbf{3})_{-1}$	8	2
VW	$(\mathbf{1}, \mathbf{1})_2 \oplus (\mathbf{1}, \mathbf{1})_{-2}$ $\oplus (\mathbf{1}, \mathbf{1})_0$ $(\mathbf{1}, \mathbf{3})_0$	$(\mathbf{1}, \mathbf{1})_+ \oplus (\mathbf{1}, \mathbf{1})_-$ $(\mathbf{2}, \mathbf{2})_- \oplus (\mathbf{2}, \mathbf{2})_+$ $(\mathbf{1}, \mathbf{3})_- \oplus (\mathbf{1}, \mathbf{3})_+$	7	2
DW	$(\mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{1})_0$ $(\mathbf{1}, \mathbf{2})_1 \oplus (\mathbf{1}, \mathbf{2})_{-1}$	$(\mathbf{1}, \mathbf{1})_0$ $(\mathbf{2}, \mathbf{1})_{-1} \oplus (\mathbf{2}, \mathbf{1})_1$ $(\mathbf{1}, \mathbf{2})_1 \oplus (\mathbf{1}, \mathbf{2})_{-1}$ $(\mathbf{1}, \mathbf{3})_0$ $(\mathbf{2}, \mathbf{2})_0$	8	1

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GL	$(\mathbf{1}, \mathbf{1})_2 \oplus (\mathbf{1}, \mathbf{1})_{-2}$ $(\mathbf{2}, \mathbf{2})_0$	$(\mathbf{1}, \mathbf{1})_{-1} \oplus (\mathbf{1}, \mathbf{1})_{-1}$ $(\mathbf{2}, \mathbf{2})_1 \oplus (\mathbf{2}, \mathbf{2})_1$ $(\mathbf{3}, \mathbf{1})_{-1} \oplus (\mathbf{1}, \mathbf{3})_{-1}$	8	2
VW	$(\mathbf{1}, \mathbf{1})_2 \oplus (\mathbf{1}, \mathbf{1})_{-2}$ $\oplus (\mathbf{1}, \mathbf{1})_0$ $(\mathbf{1}, \mathbf{3})_0$	$(\mathbf{1}, \mathbf{1})_+ \oplus (\mathbf{1}, \mathbf{1})_-$ $(\mathbf{2}, \mathbf{2})_- \oplus (\mathbf{2}, \mathbf{2})_+$ $(\mathbf{1}, \mathbf{3})_- \oplus (\mathbf{1}, \mathbf{3})_+$	7	2
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Beautiful interpretation of twisted 4d $\mathcal{N}$=4 SYM as curved D3-branes !

twist	$SO(4)'_E = SU(2)'_l \times SU(2)'_r$		dim X	SUSY
	[illegible]	[illegible]		
GL	$(1,1)_2 \oplus (1,1)_{-2}$ $(2,2)_0$	$(1,1)_{-1} \oplus (1,1)_1$ $(2,2)_{-1} \oplus (2,2)_1$ [illegible]	8	2
VW	$(1,1)_2 \oplus (1,1)_{-2}$ $\oplus (1,1)_0$ $(1,3)_0$	$(1,1)_{-1} \oplus (1,1)_1$ $(2,2)_{-1} \oplus (2,2)_1$ $(1,3)_{-1} \oplus (1,3)_1$	7	2
DW	$(1,1)_0 \oplus (1,1)_0$ $(1,2)_1 \oplus (1,2)_{-1}$	$(1,1)_0$ $(2,1)_{-1} \oplus (2,1)_1$ $(1,2)_1 \oplus (1,2)_{-1}$ $(1,3)_0$ $(2,2)_0$	8	1

'08 BLG

world-volume theory
of **flat** 2 M2-branes = **3d $\mathcal{N}$=8 BLG-model**

$$
\begin{aligned}
\mathcal{L} = &-\frac{1}{2}(D_\mu X^I, D^\mu X^I) + \frac{i}{2}(\overline{\Psi}, \Gamma^\mu D_\mu \Psi) \\
&+ \frac{i}{4}\left(\overline{\Psi}\Gamma^{IJ}[X^I, X^J, \Psi]\right) - \frac{1}{12}\left([X^I, X^J, X^K], [X^I, X^J, X^K]\right) \\
&+ \frac{1}{2}\epsilon^{\mu\nu\lambda}\left[\mathrm{Tr}\left(A_{\mu ab}\partial_\nu \tilde{A}_\lambda^{ab}\right) + \frac{2}{3}\mathrm{Tr}\left(A_{\mu ab}\tilde{A}^a_{\nu g}\tilde{A}^b_{\lambda g}\right)\right]
\end{aligned}
$$

world-volume
of M2-branes
$\mathbf{R}^{1,2}$ × $\mathbf{R}^8$
transverse space
of M2-branes

$SO(1,2) \times SO(8)_R$

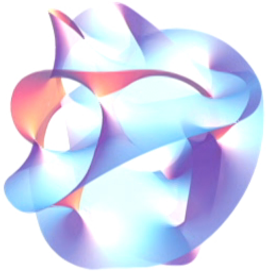
X^I	$\mathbf{8}_v$
Ψ	$\mathbf{8}_c$
A_μ	$\mathbf{1}$

Curved M2-branes

wrapped M2-branes
around Riemann surface Σ_g

$\cup$

holomorphic curve
(SUSY 2-cycle) Σ_g $\subset$



CY mfd X

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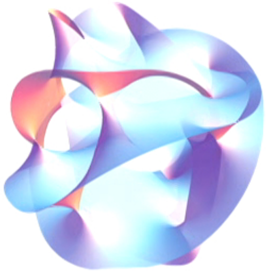


CY mfd X

⬇ decoupling limit

Q. What is the world-volume theory

Consider

$$X = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3 \oplus \mathcal{L}_4 \to \Sigma_g$$

Consider

$$X = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3 \oplus \mathcal{L}_4 \to \Sigma_g$$

$\mathrm{SO}(8)_R \to \mathrm{SO}(2)_1 \times \mathrm{SO}(2)_2 \times \mathrm{SO}(2)_3 \times \mathrm{SO}(2)_4$

$$\boxed{s' := \overset{\mathrm{SO}(2)_E}{s} - \sum_{i=1}^{n} a_i \overset{\text{Cartan generator of } \mathrm{SO}(8)_R}{T_i}}$$

Consider

$$X = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3 \oplus \mathcal{L}_4 \to \Sigma_g$$

$$SO(8)_R \to SO(2)_1 \times SO(2)_2 \times SO(2)_3 \times SO(2)_4$$

twisted $SO(2)'_E$

$SO(2)_E$

Cartan generator of $SO(8)_R$

$$s' := s - \sum_{i=1}^{n} a_i T_i$$

Consider

$$X = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3 \oplus \mathcal{L}_4 \rightarrow \Sigma_g$$

$$SO(8)_R \rightarrow SO(2)_1 \times SO(2)_2 \times SO(2)_3 \times SO(2)_4$$

$SO(2)_E$

Cartan generator of $SO(8)_R$

twisted $SO(2)'_E$

$$s' := s - \sum_{i=1}^{n} a_i T_i$$

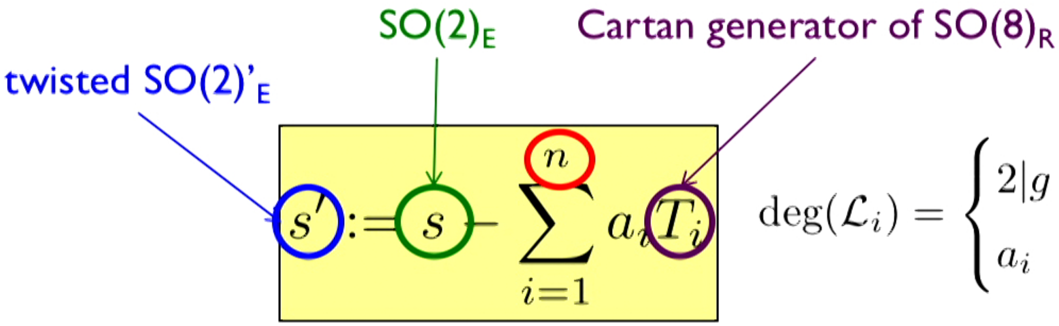
$$\deg(\mathcal{L}_i) = \begin{cases} 2|g-1|a_i & (g \neq 1) \\ a_i & (g = 1) \end{cases}$$

$$\sum_{i=1}^{n} a_i = \begin{cases} -1 & (g = 0) \\ 0 & (g = 1) \\ 1 & (g > 1) \end{cases}$$

Consider

$$X = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3 \oplus \mathcal{L}_4 \to \Sigma_g$$

$SO(8)_R \to SO(2)_1 \times SO(2)_2 \times SO(2)_3 \times SO(2)_4$



$SO(2)_E$ Cartan generator of $SO(8)_R$

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$n=1$ → CY_2 $\mathcal{N}=8$

$n=2$ → CY_3 $\mathcal{N}=4$

$n=3$ → CY_4 $\mathcal{N}=2$

$n=4$ → CY_5 $\mathcal{N}=2$

SUSY Boundary Condition

'08 Gaiotto & Witten

EOM

$$\delta S = \int d^d x \partial_\mu \left[\frac{\delta \mathcal{L}}{\delta \partial_\mu \Phi} \delta \Phi \right] = 0$$

SUSY Boundary Condition

'08 Gaiotto & Witten

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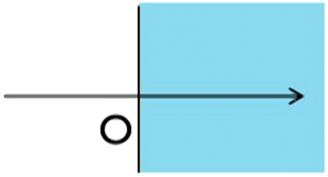
$$\frac{\delta \mathcal{L}}{\delta \partial_\perp \Phi} \delta \Phi \Big| = 0$$

required for QFT



SUSY

$$\delta_\epsilon S = \int d^d x \partial_\mu \mathcal{M}^\mu_\epsilon = 0$$

$$\mathcal{M}^\perp_\epsilon \Big| = 0$$

required for SUSY

SUSY Boundary Condition

'08 Gaiotto & Witten

EOM

$$\delta S = \int d^d x \partial_\mu \left[\frac{\delta \mathcal{L}}{\delta \partial_\mu \Phi} \delta \Phi \right] = 0$$

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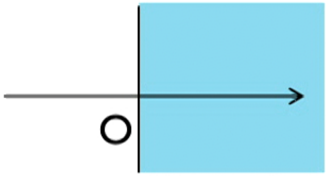
required for QFT



SUSY

$$\delta_\epsilon S = \int d^d x \partial_\mu \mathcal{M}^\mu_\epsilon = 0$$

$$\mathcal{M}^\perp_\epsilon | = 0$$

required for SUSY

Supercurrent

Lagrangian & SUSY transf => $J^\mu = \frac{\delta \mathcal{L}}{\delta \partial_\mu \Phi} \delta_\epsilon \Phi - \mathcal{M}^\mu_\epsilon = 0$

SUSY BC $\boxed{\epsilon J^\perp | = 0}$

Ex. Open D3-branes

	0	1	2	3	4	5	6	7	8	9
D3	○	○	○	—	—	—	○	—	—	—
NS5	○	○	○	○	○	○	—	—	—	—
D5	○	○	○	—	—	—	—	○	○	○

’96 Hanany & Witten
’08 Gaiotto & Witten

D3-NS5-D5 brane system

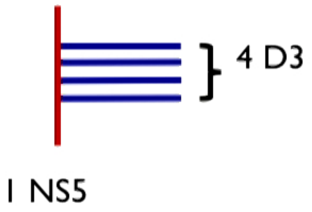
$$\mathrm{SO}(6)_R \longrightarrow \mathrm{SO}(1,2) \times \mathrm{SO}(3)_Y \times \mathrm{SO}(3)_X$$

$$\vec{\phi} \to \left\{ \vec{Y}, \vec{X} \right\}$$

$$
\begin{aligned}
0 &= \bar{\varepsilon} \left(\Gamma^{\mu\nu} F_{\mu\nu} + 2\Gamma^{3\mu} F_{3\mu} \right) \Psi' \\
0 &= \sum_{\mu=0,1,2} \bar{\varepsilon} \left(\Gamma^{\mu a} D_\mu X_a \right) \Psi' \\
0 &= \sum_{\mu=0,1,2} \bar{\varepsilon} \left(\Gamma^{\mu m} D_\mu Y_m \right) \Psi' \\
0 &= \bar{\varepsilon} \Gamma^{am} [X_a, Y_m] \Psi' \\
0 &= \bar{\varepsilon} \left(2\Gamma^{3a} D_3 X_a + \Gamma^{ab} [X_a, X_b] \right) \Psi' \\
0 &= \bar{\varepsilon} \left(2\Gamma^{3m} D_3 Y_a + \Gamma^{mn} [Y_m, Y_n] \right) \Psi'.
\end{aligned}
$$

NS5-like BC

'08 Gaiotto & Witten

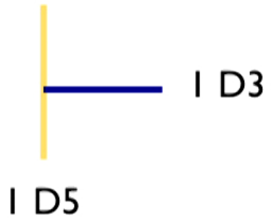


$$F_{3\mu}| = 0$$

$$D_3 Y| = 0$$

Neumann B.C. of U(4) thy.

D5-like BC



$$F_{\mu\nu}| = 0$$

$$D_3 X| = 0$$

Dirichlet B.C. of U(1) thy.

Ex. Open M2-branes

world-volume theory
of **flat** 2 M2-branes = **3d $\mathcal{N}$=8 BLG-model**

$$
\begin{aligned}
\mathcal{L} = & -\frac{1}{2}(D_\mu X^I, D^\mu X^I) + \frac{i}{2}(\overline{\Psi}, \Gamma^\mu D_\mu \Psi) \\
& + \frac{i}{4}\left(\overline{\Psi}\Gamma^{IJ}[X^I, X^J, \Psi]\right) - \frac{1}{12}\left([X^I, X^J, X^K], [X^I, X^J, X^K]\right) \\
& + \frac{1}{2}\epsilon^{\mu\nu\lambda}\left[\mathrm{Tr}\left(A_{\mu ab}\partial_\nu \tilde{A}_\lambda^{ab}\right) + \frac{2}{3}\mathrm{Tr}\left(A_{\mu ab}\tilde{A}^a_{\nu g}\tilde{A}^b_{\lambda g}\right)\right]
\end{aligned}
$$

world-volume of M2-branes $\mathbf{R}^{1,2}$ × **transverse space** of M2-branes $\mathbf{R}^{8}$

$SO(1,2) \times SO(8)_R$

X^I	$\mathbf{8}_v$
Ψ	$\mathbf{8}_c$
A_μ	$\mathbf{1}$

$$\boxed{J^\mu = -D_\nu X^I_a \Gamma^\nu \Gamma^I \Gamma^\mu \Psi - \frac{1}{6} X^I_a X^J_b X^K_c f^{abcd} \Gamma^{IJK} \Gamma^\mu \Psi}$$

Q. How to solve $\boxed{\epsilon J^2|_{\rm bdy} = 0}$?

Bdy must break some of SUSY

SO(1,2) × SO(8) => SO(1,1) × H

$$\boxed{J^\mu = -D_\nu X_a^I \Gamma^\nu \Gamma^I \Gamma^\mu \Psi - \frac{1}{6} X_a^I X_b^J X_c^K f^{abcd} \Gamma^{IJK} \Gamma^\mu \Psi}$$

Q. How to solve $\boxed{\epsilon J^2|_{\rm bdy} = 0}$?

Bdy must break some of SUSY

SO(1,2) × SO(8) => SO(1,1) × H

Reduced R-sym

=> Fermionic BC

=> Bosonic BC

$$\boxed{J^\mu = -D_\nu X^I_a \Gamma^\nu \Gamma^I \Gamma^\mu \Psi - \frac{1}{6} X^I_a X^J_b X^K_c f^{abcd} \Gamma^{IJK} \Gamma^\mu \Psi}$$

Q. How to solve $\boxed{\epsilon J^2|_{\mathrm{bdy}} = 0}$?

Bdy must break some of SUSY

SO(1,2) × SO(8) => SO(1,1) × H

Reduced R-sym

=> Fermionic BC

=> Bosonic BC

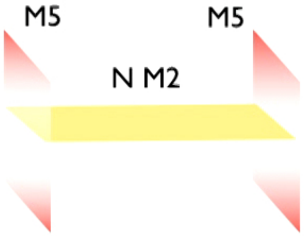
rotational sym. group on **world-volume** of M2-branes

rotational sym. group on **transverse space** of M2-branes

reduced by **additional branes**

'08 Berman etal.

	0	1	2	3	4	5	6	7	8	9	10
M2	○	○	○	×	×	×	×	×	×	×	×
M5	○	○	×	○	○	×	×	×	×	○	○



M-Strings

$SO(8)_R \longrightarrow SO(4)_{34910} \times SO(4)_{5678}$

$$X^I \to \left\{X^i, Y^{\hat{i}}\right\}$$

$$\begin{aligned}
0 &= \bar{\epsilon} J^2 \Big|_{\mathrm{bdy}} \\
&= D_\alpha X^i \left(\bar{\epsilon}\Gamma^{i\alpha 2}\Psi\right) + D_\alpha Y^{\hat{i}} \left(\bar{\epsilon}\Gamma^{\hat{i}\alpha 2}\Psi\right) \\
&+ D_2 X^i \left(\bar{\epsilon}\Gamma^{i}\Psi\right) + D_2 Y^{\hat{i}} \left(\bar{\epsilon}\Gamma^{\hat{i}}\Psi\right) \\
&- \frac{1}{6}\left[X^i, X^j, X^k\right]\left(\bar{\epsilon}\Gamma^{ijk}\Gamma^2\Psi\right) - \frac{1}{6}\left[Y^{\hat{i}}, Y^{\hat{j}}, Y^{\hat{k}}\right]\left(\bar{\epsilon}\Gamma^{\hat{i}\hat{j}\hat{k}}\Gamma^2\Psi\right) \\
&- \frac{1}{2}\left[X^i, X^j, Y^{\hat{k}}\right]\left(\bar{\epsilon}\Gamma^{ij\hat{k}}\Gamma^2\Psi\right) - \frac{1}{2}\left[X^i, Y^{\hat{j}}, Y^{\hat{k}}\right]\left(\bar{\epsilon}\Gamma^{i\hat{j}\hat{k}}\Gamma^2\Psi\right)\Big|_{\mathrm{bdy}}
\end{aligned}$$

Basu-Harvey equations (tangent to M5)

'04 Basu Harvey

$$D_2 X^i + \frac{1}{6}\epsilon^{ijkl}[X^j, X^k, X^l]\Big|_{\rm bdy} = 0$$

$\Rightarrow$ Matter fields are non-commutative

Dirichlet conditions (normal to M5)

$$D_\alpha Y^{\hat{i}}\Big|_{\rm bdy} = 0$$

$\Rightarrow$ Matter fields are fixed

Basu-Harvey equations (tangent to M5)

'04 Basu Harvey

$$D_2 X^i + \frac{1}{6}\epsilon^{ijkl}[X^j, X^k, X^l]\Big|_{\text{bdy}} = 0$$

$\Rightarrow$ Matter fields are non-commutative

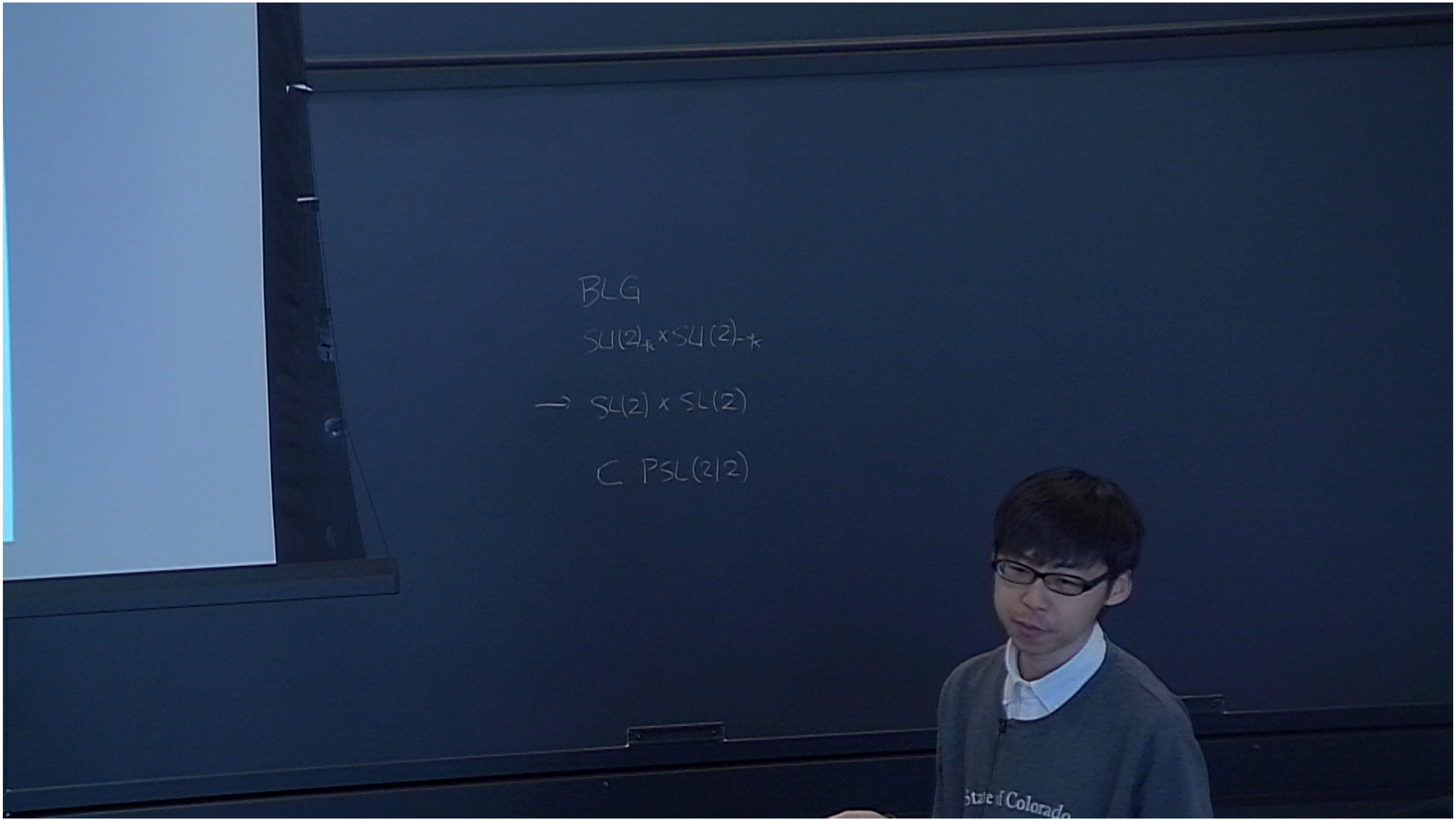
$\Rightarrow$ Matter fields satisfy fermionic Jacobi id for **psl**(2|2)

'08 de Medeiros etal.
'15 Bielawski

Dirichlet conditions (normal to M5)

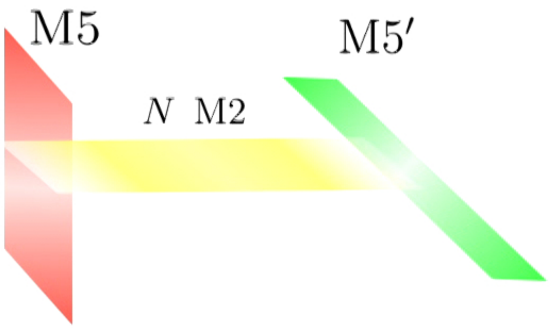
$$D_\alpha Y^{\hat{i}}\Big|_{\text{bdy}} = 0$$

$\Rightarrow$ Matter fields are fixed


BLG
$SU(2)_k \times SU(2)_{-k}$
$\rightarrow SL(2) \times SL(2)$
$\subset PSL(2|2)$
State of Colorado

III. Topological M-Strings
&
SG WZW Models

Topological M-Strings



K3

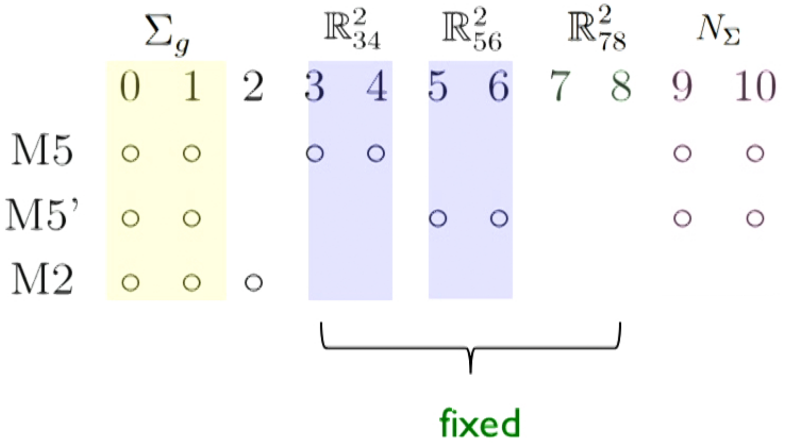
32/2=16

supercharges

M2 projection

⇒ 16/2=8

supercharges



	Σ_g			$\mathbb{R}^2_{34}$		$\mathbb{R}^2_{56}$		$\mathbb{R}^2_{78}$		N_Σ	
	0	1	2	3	4	5	6	7	8	9	10
M5	○	○		○	○					○	○
M5’	○	○				○	○			○	○
M2	○	○	○								

$\underbrace{\quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad}_{\text{fixed}}$

M5-M5’ projection

⇒ $8/2^2=2$

supercharges

Our Approach

Step1

Topologically twists of BLG (ABJM) dsescribing curved M2 wrapped on Σ_g in K3.

M2 projection
$\Rightarrow$ 16/2=8
supercharges

Step2

Boundary conditions of the topologically twisted BLG (ABJM) describing "Topological M-Strings"

M5-M5' projection
$\Rightarrow$ $8/2^2=2$
supercharges

Topological twisted BLG

$$SO(8)_R \longrightarrow SO(2)_R \times SO(6)_R$$

X^I	$\mathbf{8}_v$	$\mathbf{6}_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_{-2}$ $\phi^I \quad \Phi_z \quad \Phi_{\overline{z}}$
Ψ	$\mathbf{8}_{c+} \oplus \mathbf{8}_{c-}$	$\mathbf{4}_2 \oplus \overline{\mathbf{4}}_0 \oplus \mathbf{4}_0 \oplus \overline{\mathbf{4}}_{-2}$ $\Psi_z \quad \tilde{\lambda} \quad \psi \quad \tilde{\Psi}_{\overline{z}}$
ϵ	$\mathbf{8}_{s+} \oplus \mathbf{8}_{s-}$	$\mathbf{4}_0 \oplus \overline{\mathbf{4}}_2 \oplus \mathbf{4}_{-2} \oplus \overline{\mathbf{4}}_0$ $\epsilon \quad \tilde{\epsilon}_z \quad \epsilon_{\overline{z}} \quad \tilde{\epsilon}$

Topological twisted BLG

$$\mathbf{SO(8)_R} \longrightarrow \mathbf{SO(2)_R} \times \mathbf{SO(6)_R}$$

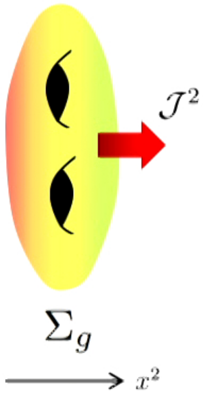
X^I	$\mathbf{8}_v$	$\mathbf{6}_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_{-2}$ $\phi^I \quad \Phi_z \quad \Phi_{\overline{z}}$ **flat space** $\quad$ **N$_\Sigma$**
Ψ	$\mathbf{8}_{c+} \oplus \mathbf{8}_{c-}$	$\mathbf{4}_2 \oplus \overline{\mathbf{4}}_0 \oplus \mathbf{4}_0 \oplus \overline{\mathbf{4}}_{-2}$ $\Psi_z \quad \tilde{\lambda} \quad \psi \quad \tilde{\Psi}_{\overline{z}}$
ϵ	$\mathbf{8}_{s+} \oplus \mathbf{8}_{s-}$	$\mathbf{4}_0 \oplus \overline{\mathbf{4}}_2 \oplus \mathbf{4}_{-2} \oplus \overline{\mathbf{4}}_0$ $\epsilon \quad \tilde{\epsilon}_z \quad \epsilon_{\overline{z}} \quad \tilde{\epsilon}$

8 supercharges

SUSY BC in twisted BLG

$$\epsilon_A^{\alpha\beta} = \frac{i}{\sqrt{2}}\tilde{\xi}_A(\gamma_+\epsilon^{-1})^{\alpha\beta} + i\epsilon_{\bar{z}A}(\gamma^{\bar{z}}\epsilon^{-1})^{\alpha\beta} - \frac{i}{\sqrt{2}}\xi_A(\gamma_-\epsilon^{-1})^{\alpha\beta} - i\tilde{\epsilon}_{zA}(\gamma^z\epsilon^{-1})^{\alpha\beta}$$

$$\mathbf{4}_0 \oplus \overline{\mathbf{4}}_2 \oplus \mathbf{4}_{-2} \oplus \overline{\mathbf{4}}_0$$



$$\begin{aligned}
0 =& \bar{\xi}\mathcal{J}^2 - \bar{\tilde{\xi}}\tilde{\mathcal{J}}^2\Big|_{\text{bdy}} \\
=& -\bar{\xi}\left[D_2\phi^I\hat{\Gamma}^I - \frac{1}{6}[\phi^I,\phi^J,\phi^K]\hat{\Gamma}^{IJK} - [\Phi_z,\Phi_{\bar{z}},\phi^I]\hat{\Gamma}^I\right]\psi \\
& -\bar{\xi}\left[iD_{\bar{z}}\phi^I\hat{\Gamma}^I\right]\Psi_z - \bar{\xi}\left[2iD_{\bar{z}}\Phi_z\right]\tilde{\lambda} - \bar{\xi}\left[D_2\Phi_z + \frac{1}{2}[\phi^I,\phi^J,\Phi_z]\hat{\Gamma}^{IJ}\right]\tilde{\Psi}_{\bar{z}} \\
& +\bar{\tilde{\xi}}\left[D_2\phi^I\hat{\Gamma}^I - \frac{1}{6}[\phi^I,\phi^J,\phi^K]\hat{\Gamma}^{IJK} - [\Phi_z,\Phi_{\bar{z}},\phi^I]\hat{\Gamma}^I\right]\tilde{\lambda} \\
& +\bar{\tilde{\xi}}\left[iD_z\phi^I\hat{\Gamma}^I\right]\tilde{\Psi}_{\bar{z}} + \bar{\tilde{\xi}}\left[2iD_z\Phi_{\bar{z}}\right]\psi + \bar{\tilde{\xi}}\left[-D_2\Phi_{\bar{z}} - \frac{1}{2}[\phi^I,\phi^J,\Phi_{\bar{z}}]\hat{\Gamma}^{IJ}\right]\Psi_z\Bigg|_{\text{bdy}}
\end{aligned}$$

SUSY BC preserving 2 supercharges

$$D_2\varphi^{\hat{i}} - [\Phi_z, \Phi_{\overline{z}}, \varphi^{\hat{i}}]\Big|_{\mathrm{bdy}} = 0 \quad D_z\varphi^{\hat{i}}\Big|_{\mathrm{bdy}} = 0 \quad D_{\overline{z}}\varphi^{\hat{i}}\Big|_{\mathrm{bdy}} = 0$$

$$D_2\varsigma^{\hat{l}} - [\Phi_z, \Phi_{\overline{z}}, \varsigma^{\hat{l}}]\Big|_{\mathrm{bdy}} = 0 \quad D_z\varsigma^{\hat{l}}\Big|_{\mathrm{bdy}} = 0 \quad D_{\overline{z}}\varsigma^{\hat{l}}\Big|_{\mathrm{bdy}} = 0$$

$$D_z\varrho^{\hat{p}}\Big|_{\mathrm{bdy}} = 0 \quad D_{\overline{z}}\varrho^{\hat{p}}\Big|_{\mathrm{bdy}} = 0$$

	Σ_g			$\mathbb{R}^2_{34}$		$\mathbb{R}^2_{56}$		$\mathbb{R}^2_{78}$		N_Σ	
	0	1	2	3	4	5	6	7	8	9	10
M5	○	○		○	○					○	○
M5’	○	○				○	○			○	○
M2	○	○	○								

$$D_2\Phi_z + \frac{1}{2}[\varrho^1, \varrho^2, \Phi_z]\Big|_{\mathrm{bdy}} = 0 \quad D_{\overline{z}}\Phi_z\Big|_{\mathrm{bdy}} = 0$$

$$D_2\Phi_{\overline{z}} + \frac{1}{2}[\varrho^1, \varrho^2, \Phi_{\overline{z}}]\Big|_{\mathrm{bdy}} = 0 \quad D_z\Phi_{\overline{z}}\Big|_{\mathrm{bdy}} = 0$$

SUSY BC preserving 2 supercharges

$$\underbrace{\begin{array}{c} D_2\varphi^{\hat{i}} - [\Phi_z, \Phi_{\overline{z}}, \varphi^{\hat{i}}]\Big|_{\mathrm{bdy}} = 0 \\[2em] D_2\varsigma^{\hat{l}} - [\Phi_z, \Phi_{\overline{z}}, \varsigma^{\hat{l}}]\Big|_{\mathrm{bdy}} = 0 \end{array}}_{\text{Basu-Harvey like eq}} \quad \underbrace{\begin{array}{cc} D_z\varphi^{\hat{i}}\Big|_{\mathrm{bdy}} = 0 & D_{\overline{z}}\varphi^{\hat{i}}\Big|_{\mathrm{bdy}} = 0 \\[2em] D_z\varsigma^{\hat{l}}\Big|_{\mathrm{bdy}} = 0 & D_{\overline{z}}\varsigma^{\hat{l}}\Big|_{\mathrm{bdy}} = 0 \end{array}}_{\text{Dirichlet BC}}$$

$$D_z\varrho^{\hat{p}}\Big|_{\mathrm{bdy}} = 0 \quad D_{\overline{z}}\varrho^{\hat{p}}\Big|_{\mathrm{bdy}} = 0$$

				φ		ς					
	Σ_g			$\mathbb{R}^2_{34}$		$\mathbb{R}^2_{56}$		$\mathbb{R}^2_{78}$		N_Σ	
	0	1	2	3	4	5	6	7	8	9	10
M5	○	○		○	○					○	○
M5’	○	○				○	○			○	○
M2	○	○	○								

$$D_2\Phi_z + \frac{1}{2}[\varrho^1, \varrho^2, \Phi_z]\Big|_{\mathrm{bdy}} = 0 \quad D_{\overline{z}}\Phi_z\Big|_{\mathrm{bdy}} = 0$$

$$D_2\Phi_{\overline{z}} + \frac{1}{2}[\varrho^1, \varrho^2, \Phi_{\overline{z}}]\Big|_{\mathrm{bdy}} = 0 \quad D_z\Phi_{\overline{z}}\Big|_{\mathrm{bdy}} = 0$$

SUSY BC preserving 2 supercharges

$$\underbrace{\begin{array}{l} D_2\varphi^{\hat{i}} - [\Phi_z, \Phi_{\bar{z}}, \varphi^{\hat{i}}]\Big|_{\text{bdy}} = 0 \\[2ex] D_2\varsigma^{\hat{l}} - [\Phi_z, \Phi_{\bar{z}}, \varsigma^{\hat{l}}]\Big|_{\text{bdy}} = 0 \end{array}}_{\text{Basu-Harvey like eq}} \quad \underbrace{\begin{array}{l} D_z\varphi^{\hat{i}}\Big|_{\text{bdy}} = 0 \quad D_{\bar{z}}\varphi^{\hat{i}}\Big|_{\text{bdy}} = 0 \\[2ex] D_z\varsigma^{\hat{l}}\Big|_{\text{bdy}} = 0 \quad D_{\bar{z}}\varsigma^{\hat{l}}\Big|_{\text{bdy}} = 0 \end{array}}_{\text{Dirichlet BC}}$$

$$\underbrace{D_z\varrho^{\hat{p}}\Big|_{\text{bdy}} = 0 \quad D_{\bar{z}}\varrho^{\hat{p}}\Big|_{\text{bdy}} = 0}_{\text{Dirichlet BC}}$$

		φ	ς	ϱ	

	Σ_g			$\mathbb{R}^2_{34}$		$\mathbb{R}^2_{56}$		$\mathbb{R}^2_{78}$		N_Σ	
	0	1	2	3	4	5	6	7	8	9	10
M5	○	○		○	○					○	○
M5’	○	○				○	○			○	○
M2	○	○	○								

$$D_2\Phi_z + \frac{1}{2}[\varrho^1, \varrho^2, \Phi_z]\Big|_{\text{bdy}} = 0 \quad D_{\bar{z}}\Phi_z\Big|_{\text{bdy}} = 0$$

$$D_2\Phi_{\bar{z}} + \frac{1}{2}[\varrho^1, \varrho^2, \Phi_{\bar{z}}]\Big|_{\text{bdy}} = 0 \quad D_z\Phi_{\bar{z}}\Big|_{\text{bdy}} = 0$$

SUSY BC preserving 2 supercharges

$$\underbrace{D_2\varphi^{\hat{i}} - [\Phi_z, \Phi_{\bar{z}}, \varphi^{\hat{i}}]\Big|_{\rm bdy} = 0 \quad D_2\varsigma^{\hat{l}} - [\Phi_z, \Phi_{\bar{z}}, \varsigma^{\hat{l}}]\Big|_{\rm bdy} = 0}_{\text{Basu-Harvey like eq}} \quad \underbrace{D_z\varphi^{\hat{i}}\Big|_{\rm bdy} = 0 \quad D_{\bar{z}}\varphi^{\hat{i}}\Big|_{\rm bdy} = 0 \quad D_z\varsigma^{\hat{l}}\Big|_{\rm bdy} = 0 \quad D_{\bar{z}}\varsigma^{\hat{l}}\Big|_{\rm bdy} = 0}_{\text{Dirichlet BC}}$$

$$\underbrace{D_z\varrho^{\hat{p}}\Big|_{\rm bdy} = 0 \quad D_{\bar{z}}\varrho^{\hat{p}}\Big|_{\rm bdy} = 0}_{\text{Dirichlet BC}}$$

$$\underbrace{D_2\Phi_z + \frac{1}{2}[\varrho^1, \varrho^2, \Phi_z]\Big|_{\rm bdy} = 0 \quad D_2\Phi_{\bar{z}} + \frac{1}{2}[\varrho^1, \varrho^2, \Phi_{\bar{z}}]\Big|_{\rm bdy} = 0}_{\text{Basu-Harvey like eq}} \quad \underbrace{D_{\bar{z}}\Phi_z\Big|_{\rm bdy} = 0 \quad D_z\Phi_{\bar{z}}\Big|_{\rm bdy} = 0}_{\text{holomorphic BC}}$$

				φ		ς		ϱ		Φ_α	
	Σ_g			$\mathbb{R}^2_{34}$		$\mathbb{R}^2_{56}$		$\mathbb{R}^2_{78}$		N_Σ	
	0	1	2	3	4	5	6	7	8	9	10
M5	○	○		○	○					○	○
M5'	○	○				○	○			○	○
M2	○	○	○								
				fixed							

$$\underbrace{\begin{array}{c} D_2\Phi_z + \frac{1}{2}[\varrho^1, \varrho^2, \Phi_z]\Big|_{\mathrm{bdy}} = 0 \\ \\ D_2\Phi_{\bar{z}} + \frac{1}{2}[\varrho^1, \varrho^2, \Phi_{\bar{z}}]\Big|_{\mathrm{bdy}} = 0 \end{array}}_{\text{Basu-Harvey like eq}} \quad \underbrace{\begin{array}{c} D_{\bar{z}}\Phi_z\Big|_{\mathrm{bdy}} = 0 \\ \\ D_z\Phi_{\bar{z}}\Big|_{\mathrm{bdy}} = 0 \end{array}}_{\text{holomorphic BC}}$$

One-form fields Φ_α only appear describing topological feature in N_Σ

fermionic Jacobi id. for **psl(2|2)** '08 de Medeiros etal.
'15 Bielawski

$$\underbrace{\begin{aligned} &D_2\Phi_z + \frac{1}{2}[\varrho^1, \varrho^2, \Phi_z]\Big|_{\rm bdy} = 0 \\ &D_2\Phi_{\bar z} + \frac{1}{2}[\varrho^1, \varrho^2, \Phi_{\bar z}]\Big|_{\rm bdy} = 0 \end{aligned}}_{\text{Basu-Harvey like eq}} \quad \underbrace{\begin{aligned} &D_{\bar z}\Phi_z\Big|_{\rm bdy} = 0 \\ &D_z\Phi_{\bar z}\Big|_{\rm bdy} = 0 \end{aligned}}_{\text{holomorphic BC}}$$

One-form fields Φ_α only appear describing topological feature in N_Σ

fermionic Jacobi id. for **psl(2|2)** '08 de Medeiros etal. '15 Bielawski

choice of holomorphic curve in K3

$$\underbrace{\begin{matrix} D_2\Phi_z + \frac{1}{2}[\varrho^1, \varrho^2, \Phi_z]\Big|_{\mathrm{bdy}} = 0 \\[2em] D_2\Phi_{\bar{z}} + \frac{1}{2}[\varrho^1, \varrho^2, \Phi_{\bar{z}}]\Big|_{\mathrm{bdy}} = 0 \end{matrix}}_{\text{Basu-Harvey like eq}} \qquad \underbrace{\begin{matrix} D_{\bar{z}}\Phi_z\Big|_{\mathrm{bdy}} = 0 \\[2em] D_z\Phi_{\bar{z}}\Big|_{\mathrm{bdy}} = 0 \end{matrix}}_{\text{holomorphic BC}}$$

One-form fields Φ_α only appear describing topological feature in N_Σ

Gauge inv BC

SUSY BC cannot fix BC on gauge fields (CS term)

$$S_{\rm CS} = \frac{k}{4\pi}\int_M d^3x \epsilon^{\mu\nu\lambda}{\rm Tr}\left(A_\mu \partial_\nu A_\lambda + \frac{2i}{3}A_\mu A_\nu A_\lambda\right)$$

$$\delta S_{\rm CS} = \frac{k}{4\pi}\int_M d^3x \epsilon^{\mu\nu\lambda}{\rm Tr}\left(\delta A_\mu F_{\nu\lambda}\right) + \frac{k}{4\pi}\int_{\partial M} d^2x \epsilon^{\alpha\beta}{\rm Tr}\left(\delta A_\alpha A_\beta\right)$$

Gauge inv BC

SUSY BC cannot fix BC on gauge fields (CS term)

$$S_{\mathrm{CS}} = \frac{k}{4\pi}\int_M d^3x\,\epsilon^{\mu\nu\lambda}\mathrm{Tr}\left(A_\mu \partial_\nu A_\lambda + \frac{2i}{3}A_\mu A_\nu A_\lambda\right)$$

$$\delta S_{\mathrm{CS}} = \frac{k}{4\pi}\int_M d^3x\,\epsilon^{\mu\nu\lambda}\mathrm{Tr}\left(\delta A_\mu F_{\nu\lambda}\right) + \frac{k}{4\pi}\int_{\partial M} d^2x\,\underset{\substack{\| \\ 0}}{\epsilon^{\alpha\beta}\mathrm{Tr}\left(\delta A_\alpha A_\beta\right)}$$

=> One of the components of gauge fields vanish at bdy

$$\boxed{A_{\bar{z}}\Big|_{\mathrm{bdy}} = 0}$$

Gauge inv BC '89 Elitzur etal.

=> Lagrange multiplier requiring the flatness cond. $\boxed{F_{2z} = 0}$

For simplicity assume that it can be solved via pure gauge

$$A_2 = g^{-1}\partial_2 g \qquad A_z = g^{-1}\partial_z g$$

=> Lagrange multiplier requiring the flatness cond. $\boxed{F_{2z} = 0}$

For simplicity assume that it can be solved via pure gauge

$$A_2 = g^{-1}\partial_2 g \qquad A_z = g^{-1}\partial_z g$$

⬇

$$\boxed{S_{WZW} = -\frac{k}{8\pi}\int_\Sigma d^2x \mathrm{Tr}\left(g^{-1}\partial_\alpha g\right)^2 - \frac{ik}{12\pi}\int_M d^3x \epsilon^{\mu\nu\lambda}\mathrm{Tr}\left(g^{-1}\partial_\mu g \cdot g^{-1}\partial_\nu g \cdot g^{-1}\partial_\lambda g\right)}$$

WZW model ’89 Elitzur etal.

BLG model = SU(2)$_k$ × SU(2)$_{-k}$ quiver CS matter theory

'08 Raamsdonk

$$S_{CS} = \frac{k}{4\pi}\int_M d^3x\epsilon^{\mu\nu\lambda}\left[\mathrm{Tr}\left(A_\mu\partial_\nu A_\lambda + \frac{2i}{3}A_\mu A_\nu A_\lambda\right) - \mathrm{Tr}\left(\hat{A}_\mu\partial_\nu\hat{A}_\lambda + \frac{2i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda\right)\right]$$

BLG model = $SU(2)_k$ × $SU(2)_{-k}$ quiver CS matter theory

'08 Raamsdonk

$$S_{CS}=\frac{k}{4\pi}\int_M d^3x\epsilon^{\mu\nu\lambda}\left[\mathrm{Tr}\left(A_\mu\partial_\nu A_\lambda+\frac{2i}{3}A_\mu A_\nu A_\lambda\right)-\mathrm{Tr}\left(\hat{A}_\mu\partial_\nu\hat{A}_\lambda+\frac{2i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda\right)\right]$$

$$\begin{aligned}&S_{SU(2)_k\mathrm{WZW}}[g]+S_{SU(2)_{-k}\mathrm{WZW}}[\hat{g}]\\=&-\frac{k}{8\pi}\int_\Sigma d^2x\mathrm{Tr}\left(g^{-1}\partial_\alpha g\right)^2-\frac{ik}{12\pi}\int_M d^3x\epsilon^{\mu\nu\lambda}\mathrm{Tr}\left(g^{-1}\partial_\mu g\cdot g^{-1}\partial_\nu g\cdot g^{-1}\partial_\lambda g\right)\\&+\frac{k}{8\pi}\int_\Sigma d^2x\mathrm{Tr}\left(\hat{g}^{-1}\partial_\alpha\hat{g}\right)^2+\frac{ik}{12\pi}\int_M d^3x\epsilon^{\mu\nu\lambda}\mathrm{Tr}\left(\hat{g}^{-1}\partial_\mu\hat{g}\cdot\hat{g}^{-1}\partial_\nu\hat{g}\cdot\hat{g}^{-1}\partial_\lambda\hat{g}\right)\end{aligned}$$

$SU(2)_k$ × $SU(2)_{-k}$ WZW model

BCs of topological M-Strings in twisted BLG-model

SUSY BC on twisted matter fields

$$\underbrace{\begin{array}{c} D_2\varphi^{\hat{i}} - [\Phi_z, \Phi_{\bar{z}}, \varphi^{\hat{i}}]\Big|_{\mathrm{bdy}} = 0 \\ D_2\varsigma^{\hat{l}} - [\Phi_z, \Phi_{\bar{z}}, \varsigma^{\hat{l}}]\Big|_{\mathrm{bdy}} = 0 \end{array}}_{\text{Basu-Harvey like eq}} \quad \underbrace{\begin{array}{cc} D_z\varphi^{\hat{i}}\Big|_{\mathrm{bdy}} = 0 & D_{\bar{z}}\varphi^{\hat{i}}\Big|_{\mathrm{bdy}} = 0 \\ D_z\varsigma^{\hat{l}}\Big|_{\mathrm{bdy}} = 0 & D_{\bar{z}}\varsigma^{\hat{l}}\Big|_{\mathrm{bdy}} = 0 \end{array}}_{\text{Dirichlet BC}}$$

$$\underbrace{\begin{array}{cc} D_z\varrho^{\hat{p}}\Big|_{\mathrm{bdy}} = 0 & D_{\bar{z}}\varrho^{\hat{p}}\Big|_{\mathrm{bdy}} = 0 \end{array}}_{\text{Dirichlet BC}}$$

$$\underbrace{\begin{array}{c} D_2\Phi_z + \frac{1}{2}[\varrho^1, \varrho^2, \Phi_z]\Big|_{\mathrm{bdy}} = 0 \\ D_2\Phi_{\bar{z}} + \frac{1}{2}[\varrho^1, \varrho^2, \Phi_{\bar{z}}]\Big|_{\mathrm{bdy}} = 0 \end{array}}_{\text{Basu-Harvey like eq}} \quad \underbrace{\begin{array}{c} D_{\bar{z}}\Phi_z\Big|_{\mathrm{bdy}} = 0 \\ D_z\Phi_{\bar{z}}\Big|_{\mathrm{bdy}} = 0 \end{array}}_{\text{holomorphic BC}}$$

Gauge inv BC on gauge fields

$$\underbrace{\tilde{A}_z\Big|_{\mathrm{bdy}} = 0}_{\text{holomorphic BC}} \quad \Rightarrow \quad \underbrace{\tilde{F}_{2\bar{z}} = 0}_{\text{flatness condition}}$$

BCs of topological M-Strings in twisted BLG-model

SUSY BC on twisted matter fields

$$\underbrace{\begin{array}{c} D_2\varphi^{\hat{i}} - [\Phi_z, \Phi_{\bar{z}}, \varphi^{\hat{i}}]\Big|_{\text{bdy}} = 0 \\[2ex] D_2\varsigma^{\hat{l}} - [\Phi_z, \Phi_{\bar{z}}, \varsigma^{\hat{l}}]\Big|_{\text{bdy}} = 0 \end{array}}_{\text{Basu-Harvey like eq}} \quad \underbrace{\begin{array}{cc} D_z\varphi^{\hat{i}}\Big|_{\text{bdy}} = 0 & D_{\bar{z}}\varphi^{\hat{i}}\Big|_{\text{bdy}} = 0 \\[2ex] D_z\varsigma^{\hat{l}}\Big|_{\text{bdy}} = 0 & D_{\bar{z}}\varsigma^{\hat{l}}\Big|_{\text{bdy}} = 0 \end{array}}_{\text{Dirichlet BC}}$$

$$\underbrace{\begin{array}{c} D_2\Phi_z + \frac{1}{2}[\varrho^1, \varrho^2, \Phi_z]\Big|_{\text{bdy}} = 0 \\[2ex] D_2\Phi_{\bar{z}} + \frac{1}{2}[\varrho^1, \varrho^2, \Phi_{\bar{z}}]\Big|_{\text{bdy}} = 0 \end{array}}_{\text{Basu-Harvey like eq}} \quad \underbrace{\begin{array}{c} D_{\bar{z}}\Phi_z\Big|_{\text{bdy}} = 0 \\[2ex] D_z\Phi_{\bar{z}}\Big|_{\text{bdy}} = 0 \end{array}}_{\text{holomorphic BC}}$$

$$\underbrace{D_z\varrho^{\hat{p}}\Big|_{\text{bdy}} = 0 \quad D_{\bar{z}}\varrho^{\hat{p}}\Big|_{\text{bdy}} = 0}_{\text{Dirichlet BC}}$$

Gauge inv BC on gauge fields

$$\tilde{A}_z\Big|_{\text{bdy}} = 0 \quad \Rightarrow \quad \tilde{F}_{2\bar{z}} = 0$$

holomorphic BC   flatness condition

Consider the **complexfied gauge fields**

$$\tilde{\mathcal{A}}_z{}^b{}_a = \tilde{A}_z{}^b{}_a + f^{cdb}{}_a\Phi_{\bar{z}c}\varphi^*_d + f^{cdb}{}_a\Phi_{\bar{z}c}\varsigma^*_d$$
$$\tilde{\mathcal{A}}_{\bar{z}}{}^b{}_a = \tilde{A}_{\bar{z}}{}^b{}_a + f^{cdb}{}_a\Phi_{zc}\varphi_d + f^{cdb}{}_a\Phi_{zc}\varsigma_d$$
$$\tilde{\mathcal{A}}_2{}^b{}_a = \tilde{A}_2{}^b{}_a + f^{cdb}{}_a\Phi_{\bar{z}c}\Phi_{zd} + f^{cdb}{}_a\varphi_c\varsigma_d + f^{cdb}{}_a\varrho_c\varrho^*_d$$

$$\text{e.g. } \varphi := \tfrac{1}{\sqrt{2}}(\varphi^1 - i\varphi^2)$$

BCs of topological M-Strings in twisted BLG-model

SUSY BC on twisted matter fields

$$\underbrace{D_2\varphi^{\hat{i}} - [\Phi_z, \Phi_{\bar{z}}, \varphi^{\hat{i}}]\Big|_{\mathrm{bdy}} = 0 \quad D_2\varsigma^{\hat{l}} - [\Phi_z, \Phi_{\bar{z}}, \varsigma^{\hat{l}}]\Big|_{\mathrm{bdy}} = 0}_{\text{Basu-Harvey like eq}} \qquad \underbrace{D_z\varphi^{\hat{i}}\Big|_{\mathrm{bdy}} = 0 \quad D_{\bar{z}}\varphi^{\hat{i}}\Big|_{\mathrm{bdy}} = 0 \quad D_z\varsigma^{\hat{l}}\Big|_{\mathrm{bdy}} = 0 \quad D_{\bar{z}}\varsigma^{\hat{l}}\Big|_{\mathrm{bdy}} = 0}_{\text{Dirichlet BC}}$$

$$\underbrace{D_z\varrho^{\hat{p}}\Big|_{\mathrm{bdy}} = 0 \quad D_{\bar{z}}\varrho^{\hat{p}}\Big|_{\mathrm{bdy}} = 0}_{\text{Dirichlet BC}}$$

$$\underbrace{D_2\Phi_z + \frac{1}{2}[\varrho^1, \varrho^2, \Phi_z]\Big|_{\mathrm{bdy}} = 0 \quad D_2\Phi_{\bar{z}} + \frac{1}{2}[\varrho^1, \varrho^2, \Phi_{\bar{z}}]\Big|_{\mathrm{bdy}} = 0}_{\text{Basu-Harvey like eq}} \qquad \underbrace{D_{\bar{z}}\Phi_z\Big|_{\mathrm{bdy}} = 0 \quad D_z\Phi_{\bar{z}}\Big|_{\mathrm{bdy}} = 0}_{\text{holomorphic BC}}$$

Gauge inv BC on gauge fields

$$\tilde{A}_z\Big|_{\mathrm{bdy}} = 0 \;\Rightarrow\; \tilde{F}_{2\bar{z}} = 0$$

holomorphic BC

flatness condition

$$\tilde{\mathcal{F}}_{2\bar{z}} = \partial_\nu \tilde{\mathcal{A}}_{\mu}{}^{b}{}_{a} - \partial_\mu \tilde{\mathcal{A}}_{\nu}{}^{b}{}_{a} - \tilde{\mathcal{A}}_{\mu}{}^{b}{}_{c}\tilde{\mathcal{A}}_{\nu}{}^{c}{}_{a} + \tilde{\mathcal{A}}_{\nu}{}^{b}{}_{c}\tilde{\mathcal{A}}_{\mu}{}^{c}{}_{a}$$

$$= \tilde{F}_{2\bar{z}} - \Big[D_2\Phi_z + \tfrac{1}{2}[\varrho, \varrho^*, \Phi_z], \varphi, \ \Big] - \Big[D_2\Phi_z + \tfrac{1}{2}[\varrho, \varrho^*, \Phi_z], \varsigma, \ \Big] - \Big[\Phi_z, D_2\varphi - [\Phi_{\bar{z}}, \Phi_{\bar{z}}, \varphi], \ \Big] - \Big[\Phi_{\bar{z}}, D_2\varsigma - [\Phi_z, \Phi_{\bar{z}}, \varsigma], \ \Big]$$

$$+ \Big[D_{\bar{z}}\Phi_{\bar{z}}, \Phi_z, \ \Big] + \Big[\Phi_{\bar{z}}, D_{\bar{z}}\Phi_z, \ \Big] + \Big[D_{\bar{z}}\varphi, \varsigma, \ \Big] + \Big[\varphi, D_{\bar{z}}\varsigma, \ \Big] + \Big[D_{\bar{z}}\varrho, \varrho^*, \ \Big] + \Big[\varrho, D_{\bar{z}}\varrho^*, \ \Big]$$

Conformally inv topologically twisted BLG on $\Sigma_g \times I$

$$\boxed{S_{\text{bosonic TBLG}} = \int_{\Sigma_g \times I} d^3x \left[\frac{1}{2} \epsilon^{\mu\nu\lambda} \left(f^{abcd} \mathcal{A}_{\mu ab} \partial_\nu \mathcal{A}_{\lambda cd} + \frac{2}{3} f^{cda}{}_g f^{efgb} \mathcal{A}_{\mu ab} \mathcal{A}_{\nu cd} \mathcal{A}_{\lambda ef} \right) \right]}$$

$$\tilde{\mathcal{A}}_z \Big|_{\text{bdy}} = 0 \quad ➡ \quad \tilde{\mathcal{F}}_{2\bar{z}} = 0$$

holomorphic BC
on complexified gauge fields

flatness condition
on complexified gauge fields

BCs of topological M-Strings in twisted BLG-model

SUSY BC on twisted matter fields

$$\underbrace{\begin{matrix} D_2\varphi^{\hat{i}} - [\Phi_z, \Phi_{\bar{z}}, \varphi^{\hat{i}}]\Big|_{\rm bdy} = 0 \\ D_2\varsigma^{\hat{l}} - [\Phi_z, \Phi_{\bar{z}}, \varsigma^{\hat{l}}]\Big|_{\rm bdy} = 0 \end{matrix}}_{\text{Basu-Harvey like eq}} \quad \underbrace{\begin{matrix} D_z\varphi^{\hat{i}}\Big|_{\rm bdy} = 0 \quad D_{\bar{z}}\varphi^{\hat{i}}\Big|_{\rm bdy} = 0 \\ D_z\varsigma^{\hat{l}}\Big|_{\rm bdy} = 0 \quad D_{\bar{z}}\varsigma^{\hat{l}}\Big|_{\rm bdy} = 0 \end{matrix}}_{\text{Dirichlet BC}}$$

$$\underbrace{D_z\varrho^{\hat{p}}\Big|_{\rm bdy} = 0 \quad D_{\bar{z}}\varrho^{\hat{p}}\Big|_{\rm bdy} = 0}_{\text{Dirichlet BC}}$$

$$\underbrace{\begin{matrix} D_2\Phi_z + \frac{1}{2}[\varrho^1, \varrho^2, \Phi_z]\Big|_{\rm bdy} = 0 \\ D_2\Phi_{\bar{z}} + \frac{1}{2}[\varrho^1, \varrho^2, \Phi_{\bar{z}}]\Big|_{\rm bdy} = 0 \end{matrix}}_{\text{Basu-Harvey like eq}} \quad \underbrace{\begin{matrix} D_{\bar{z}}\Phi_z\Big|_{\rm bdy} = 0 \\ D_z\Phi_{\bar{z}}\Big|_{\rm bdy} = 0 \end{matrix}}_{\text{holomorphic BC}}$$

Gauge inv BC on gauge fields

$$\boxed{\tilde{A}_z\Big|_{\rm bdy} = 0} \Rightarrow \boxed{\tilde{F}_{2\bar{z}} = 0}$$

holomorphic BC

flatness condition

$\Updownarrow$

Gauge inv BC on complexified gauge fields

$$\tilde{\mathcal{A}}_z\Big|_{\rm bdy} = 0 \Rightarrow \tilde{\mathcal{F}}_{2\bar{z}} = 0$$

holomorphic BC
on complexified gauge fields

flatness condition
on complexified gauge fields

For simplicity assume that it can be solved via pure gauge

$$
\begin{aligned}
S_{\text{bosonic}} =& S_{SL(2,\mathbb{C})_k \text{WZW}}[g] + S_{SL(2,\mathbb{C})_{-k} \text{WZW}}[\hat{g}] \\
=& -\frac{k}{8\pi}\int_{\Sigma_g} d^2x \text{Tr}\left(g^{-1}\partial_\alpha g\right)^2 - \frac{ik}{12\pi}\int_M d^3x \epsilon^{\mu\nu\lambda}\text{Tr}\left(g^{-1}\partial_\mu g \cdot g^{-1}\partial_\nu g \cdot g^{-1}\partial_\lambda g\right) \\
& + \frac{k}{8\pi}\int_{\Sigma_g} d^2x \text{Tr}\left(\hat{g}^{-1}\partial_\alpha \hat{g}\right)^2 + \frac{ik}{12\pi}\int_M d^3x \epsilon^{\mu\nu\lambda}\text{Tr}\left(\hat{g}^{-1}\partial_\mu \hat{g} \cdot \hat{g}^{-1}\partial_\nu \hat{g} \cdot \hat{g}^{-1}\partial_\lambda \hat{g}\right)
\end{aligned}
$$

SL(2,C)$_k$ × SL(2,C)$_{-k}$ WZW model

Cf. D3 wrapped on Σ_g in K3

Hitchin's eq as BPS eq on Σ_g

$$\begin{cases} F_{z\bar{z}} + i[\Phi_z, \Phi_{\bar{z}}] = 0 \\ D_{\bar{z}}\Phi_z = 0 \quad D_z\Phi_{\bar{z}} = 0 \end{cases}$$

$\Downarrow$ $\mathcal{A}_z := A_z - i\Phi_z$

Flatness cond on complexified gauge fields

$$\mathcal{F}_{z\bar{z}} = 0$$

Such phenomena of complexification is similar to our results!

Fermionic action

$$\mathbf{SO(8)_R} \longrightarrow \mathbf{SO(2)_R} \times \mathbf{SO(6)_R}$$

$$\begin{array}{lll} \Psi & \mathbf{8}_{c+} \oplus \mathbf{8}_{c-} & \mathbf{4}_2 \oplus \overline{\mathbf{4}}_0 \oplus \mathbf{4}_0 \oplus \overline{\mathbf{4}}_{-2} \\ & & \Psi_z \quad \tilde{\lambda} \quad \psi \quad \tilde{\Psi}_{\overline{z}} \end{array}$$

Fermionic action

$$\mathbf{SO(8)_R} \longrightarrow \mathbf{SO(2)_R} \times \mathbf{SO(6)_R}$$

$$\Psi \qquad \mathbf{8}_{c+} \oplus \mathbf{8}_{c-} \qquad \underset{\Psi_z}{\mathbf{4}_2} \oplus \underset{\tilde{\lambda}}{\overline{\mathbf{4}}_0} \oplus \underset{\psi}{\mathbf{4}_0} \oplus \underset{\tilde{\Psi}_{\overline{z}}}{\overline{\mathbf{4}}_{-2}}$$

$$\xrightarrow{\text{M5}} \mathbf{SO(2)_R} \times \mathbf{SO(2)_{34}} \times \mathbf{SO(4)_{5678}}$$

$$\xrightarrow{\text{M5'}} \mathbf{SO(2)_R} \times \mathbf{SO(2)_{34}} \times \mathbf{SO(2)_{56}} \times \mathbf{SO(2)_{78}}$$

$$\underbrace{\theta \qquad \overline{\theta}}_{\text{(anti)holomorphic scalars}} \qquad \underbrace{p_z \qquad \overline{p}_{\overline{z}}}_{\text{(anti)holomorphic one-form fields}}$$

i) kinetic terms of fermions in twisted BLG

=> kinetic terms of symplectic fermions $p_z\partial_{\overline{z}}\overline{\theta} \quad \overline{p}_{\overline{z}}\partial_z\theta$

ii) interaction terms in twisted BLG

=> interaction terms $\partial_z\theta\hat{g}^{-1}\partial_{\overline{z}}\overline{\theta}g \qquad \partial_z\theta g^{-1}\partial_{\overline{z}}\overline{\theta}\hat{g}$

$$p_z g^{-1}\overline{p}_{\overline{z}}\hat{g} \qquad p_z\hat{g}^{-1}\overline{p}_{\overline{z}}g$$

i) kinetic terms of fermions in twisted BLG

=> kinetic terms of symplectic fermions $p_z\partial_{\bar{z}}\bar{\theta} \quad \bar{p}_{\bar{z}}\partial_z\theta$

ii) interaction terms in twisted BLG

=> interaction terms $\partial_z\theta\hat{g}^{-1}\partial_{\bar{z}}\bar{\theta}g \quad \partial_z\theta g^{-1}\partial_{\bar{z}}\bar{\theta}\hat{g}$

$$p_z g^{-1}\bar{p}_{\bar{z}}\hat{g} \qquad p_z\hat{g}^{-1}\bar{p}_{\bar{z}}g$$

integrate out auxiliary one-form fields

$$\boxed{S_{\text{int}} = \int_{\Sigma_g} d^2x \mathrm{Tr}\left(\kappa_A\partial_z\theta\hat{g}^{-1}\partial_{\bar{z}}\bar{\theta}g + \kappa_B\partial_z\theta g^{-1}\partial_{\bar{z}}\bar{\theta}\hat{g}\right)}$$

$$\begin{aligned} S_{\text{bosonic}} =& S_{SL(2,\mathbb{C})_k \text{WZW}}[g] + S_{SL(2,\mathbb{C})_{-k}\text{WZW}}[\hat{g}] \\ =& -\frac{k}{8\pi}\int_{\Sigma_g} d^2x \text{Tr}\left(g^{-1}\partial_\alpha g\right)^2 - \frac{ik}{12\pi}\int_M d^3x \epsilon^{\mu\nu\lambda}\text{Tr}\left(g^{-1}\partial_\mu g \cdot g^{-1}\partial_\nu g \cdot g^{-1}\partial_\lambda g\right) \\ &+ \frac{k}{8\pi}\int_{\Sigma_g} d^2x \text{Tr}\left(\hat{g}^{-1}\partial_\alpha \hat{g}\right)^2 + \frac{ik}{12\pi}\int_M d^3x \epsilon^{\mu\nu\lambda}\text{Tr}\left(\hat{g}^{-1}\partial_\mu \hat{g} \cdot \hat{g}^{-1}\partial_\nu \hat{g} \cdot \hat{g}^{-1}\partial_\lambda \hat{g}\right) \end{aligned}$$

$$+$$

$$S_{\text{int}} = \int_{\Sigma_g} d^2x \text{Tr}\left(\kappa_A \partial_z \theta \hat{g}^{-1} \partial_{\bar{z}} \overline{\theta} g + \kappa_B \partial_z \theta g^{-1} \partial_{\bar{z}} \overline{\theta} \hat{g}\right)$$

Q. How to determine coeff. κ_A & κ_B ?

(integration of auxiliary fields may produce quantum effects, e.g. level shift)

Observe that there would be a str. of **psl(2|2)** and consider the PSL(2|2)$_k$ WZW models

$$S_{PSL(2|2)_k}[s] \;= S_{SL(2|2)_k}[s]$$

$$= -\frac{k}{8\pi}\int_{\Sigma_g} d^2x \mathrm{Str}\left(s^{-1}\partial_\alpha s\right)^2 - \frac{ik}{12\pi}\int_M d^3x \epsilon^{\mu\nu\lambda}\mathrm{Str}\left(s^{-1}\partial_\mu s\cdot s^{-1}\partial_\nu s\cdot s^{-1}\partial_\lambda s\right)$$

Observe that there would be a str. of **psl(2|2)** and consider the PSL(2|2)$_k$ WZW models

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Polyakov-Wiegmann id

$$S[s_1 s_2] = S[s_1] + S[s_2] + \frac{k}{2\pi}\int d^2x \mathrm{Str}\left(s_1^{-1}\partial_{\bar z} s_1 \partial_z s_2 s_2^{-1}\right)$$

Gauss decomposition

$$s = \exp(u)\begin{pmatrix} \mathbb{I} & 0 \\ \bar\theta & \mathbb{I}\end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & \hat g\end{pmatrix}\begin{pmatrix} \mathbb{I} & \theta \\ 0 & \mathbb{I}\end{pmatrix}$$

$$= S_{PSL(2|2)_k}\left[\begin{pmatrix} \mathbb{I} & 0 \\ \bar\theta & \mathbb{I}\end{pmatrix}\right] + S_{PSL(2|2)_k}\left[\begin{pmatrix} g & 0 \\ 0 & \hat g\end{pmatrix}\right] + S_{PSL(2|2)_k}\left[\begin{pmatrix} \mathbb{I} & \theta \\ 0 & \mathbb{I}\end{pmatrix}\right]$$

$$+ \frac{k}{2\pi}\int_{\Sigma_g} d^2x \mathrm{Str}\begin{pmatrix} 0 & g^{-1}\partial_{\bar z} g \partial_z \theta \\ 0 & \hat g^{-1}\partial_{\bar z}\bar\theta g \partial_z \theta\end{pmatrix}$$

$$\boxed{= S_{SL(2,\mathbb{C})_k \mathrm{WZW}}[g] + S_{SL(2,\mathbb{C})_{-k} \mathrm{WZW}}[\hat{g}] - \frac{k}{2\pi} \int_{\Sigma_g} d^2x \mathrm{Tr}\left(\hat{g}^{-1} \partial_{\bar{z}} \bar{\theta} g \partial_z \theta\right)}$$

$$\begin{cases} \kappa_A = -\frac{k}{2\pi} \\ \kappa_B = 0 \end{cases}$$

$$= S_{SL(2,\mathbb{C})_k \mathrm{WZW}}[g] + S_{SL(2,\mathbb{C})_{-k} \mathrm{WZW}}[\hat{g}] - \frac{k}{2\pi}\int_{\Sigma_g} d^2x \,\mathrm{Tr}\left(\hat{g}^{-1}\partial_{\bar{z}}\bar{\theta} g \partial_z \theta\right)$$

$$\begin{cases} \kappa_A = -\frac{k}{2\pi} \\ \kappa_B = 0 \end{cases}$$

Our Proposal

Effective action of 2 Topological M-strings as
PSL(2|2) WZW model

$$= S_{SL(2,\mathbb{C})_k \mathrm{WZW}}[g] + S_{SL(2,\mathbb{C})_{-k} \mathrm{WZW}}[\hat{g}] - \frac{k}{2\pi} \int_{\Sigma_g} d^2x \mathrm{Tr}\left(\hat{g}^{-1} \partial_{\bar{z}} \bar{\theta} g \partial_z \theta\right)$$

$$\begin{cases} \kappa_A = -\dfrac{k}{2\pi} \\ \kappa_B = 0 \end{cases}$$

Our Proposal

Effective action of 2 Topological M-strings as
PSL(2|2) WZW model

Supergroup ⇔ 3d highly extended SUSY as CS-matter thy

$$= S_{SL(2,\mathbb{C})_k}\text{WZW}[g] + S_{SL(2,\mathbb{C})_{-k}}\text{WZW}[\hat{g}] - \frac{k}{2\pi}\int_{\Sigma_g} d^2x \,\text{Tr}\left(\hat{g}^{-1}\partial_{\bar{z}}\bar{\theta} g \partial_z \theta\right)$$

$$\begin{cases} \kappa_A = -\dfrac{k}{2\pi} \\ \kappa_B = 0 \end{cases}$$

Our Proposal

Effective action of 2 Topological M-strings as
PSL(2|2) WZW model

Supergroup ⇔ 3d highly extended SUSY as CS-matter thy

Str of Supergroup ⇔ **opposite CS levels** of CS-matter thy

Topological twisted ABJM

$\mathbf{SU(4)_R} \longrightarrow \mathbf{SU(3)_R \times U(1)_R}$

Y^A	$\mathbf{4}_0$
$Y^\dagger_A$	$\overline{\mathbf{4}}_0$
ψ_A	$\overline{\mathbf{4}}_+ \oplus \overline{\mathbf{4}}_-$
$\psi^{\dagger A}$	$\mathbf{4}_+ \oplus \mathbf{4}_-$
ω	$\mathbf{6}_+ \oplus \mathbf{6}_-$

Topological twisted ABJM

SU(4)

$\mathrm{SU}(4)_R \longrightarrow \mathrm{SU}(3)_R \times \mathrm{U}(1)_R$

$\mathrm{SU}(4)_R \Rightarrow \mathrm{SO}(8)_R$

w/ $\mathrm{U}(1)_B$ for k=1,2 '08 ABJM

Y^A	$\mathbf{4}_0$
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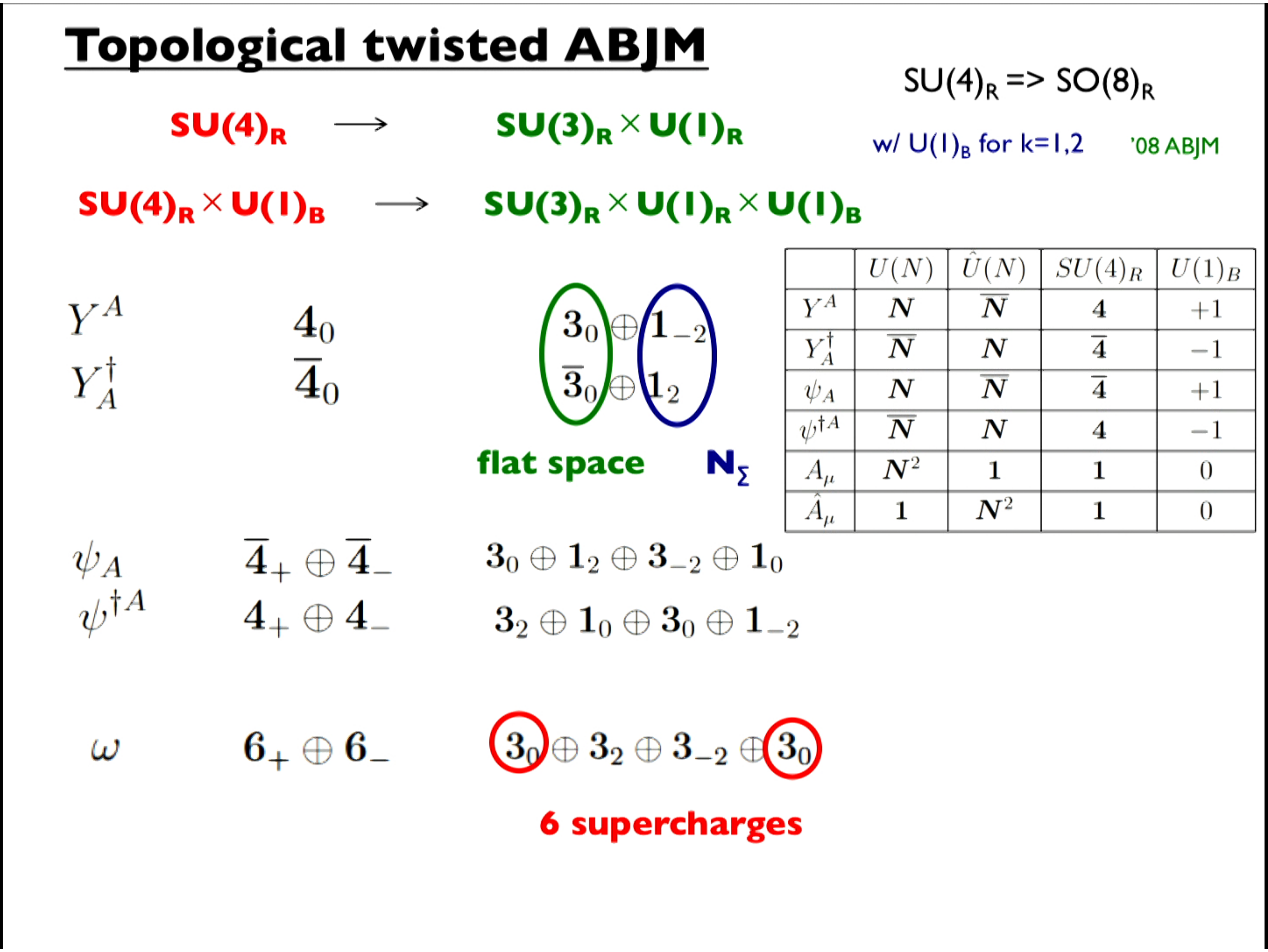

Topological twisted ABJM

$SU(4)_R \Rightarrow SO(8)_R$

w/ $U(1)_B$ for k=1,2 '08 ABJM

$SU(4)_R \longrightarrow SU(3)_R \times U(1)_R$

$SU(4)_R \times U(1)_B \longrightarrow SU(3)_R \times U(1)_R \times U(1)_B$

	$U(N)$	$\hat{U}(N)$	$SU(4)_R$	$U(1)_B$
Y^A	$\boldsymbol{N}$	$\overline{\boldsymbol{N}}$	$\mathbf{4}$	$+1$
$Y^\dagger_A$	$\overline{\boldsymbol{N}}$	$\boldsymbol{N}$	$\bar{\mathbf{4}}$	-1
ψ_A	$\boldsymbol{N}$	$\overline{\boldsymbol{N}}$	$\bar{\mathbf{4}}$	$+1$
$\psi^{\dagger A}$	$\overline{\boldsymbol{N}}$	$\boldsymbol{N}$	$\mathbf{4}$	-1
A_μ	$\boldsymbol{N}^2$	$\mathbf{1}$	$\mathbf{1}$	0
$\hat{A}_\mu$	$\mathbf{1}$	$\boldsymbol{N}^2$	$\mathbf{1}$	0

Y^A $\mathbf{4}_0$ $\mathbf{3}_0 \oplus \mathbf{1}_{-2}$

$Y^\dagger_A$ $\bar{\mathbf{4}}_0$ $\bar{\mathbf{3}}_0 \oplus \mathbf{1}_2$

flat space N_Σ

ψ_A $\bar{\mathbf{4}}_+ \oplus \bar{\mathbf{4}}_-$ $\mathbf{3}_0 \oplus \mathbf{1}_2 \oplus \mathbf{3}_{-2} \oplus \mathbf{1}_0$

$\psi^{\dagger A}$ $\mathbf{4}_+ \oplus \mathbf{4}_-$ $\mathbf{3}_2 \oplus \mathbf{1}_0 \oplus \mathbf{3}_0 \oplus \mathbf{1}_{-2}$

ω $\mathbf{6}_+ \oplus \mathbf{6}_-$ $\mathbf{3}_0 \oplus \mathbf{3}_2 \oplus \mathbf{3}_{-2} \oplus \mathbf{3}_0$

6 supercharges

Exactly same field contents as twisted BLG !

Basu-Harvey-Terashima type BPS BC => **gl(N|N)**

'15 Bielawski

$$\boxed{S_{GL(N|N)_k}[s] = -\frac{k}{8\pi}\int_{\Sigma_g} d^2x \left\langle s^{-1}\partial_\alpha s, s^{-1}\partial^\alpha s\right\rangle - \frac{ik}{24\pi}\int_M d^3x \epsilon^{\mu\nu\lambda}\left\langle s^{-1}\partial_\mu s, [s^{-1}\partial_\nu s, s^{-1}\partial_\lambda s]\right\rangle}$$

Exactly same field contents as twisted BLG !

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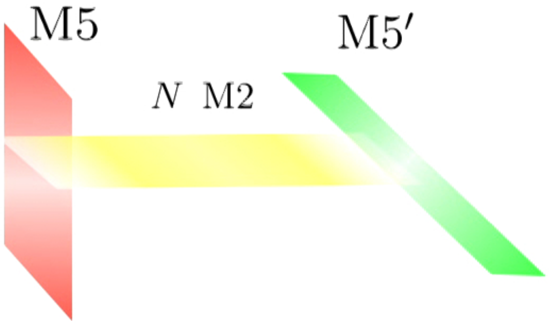
'15 Bielawski

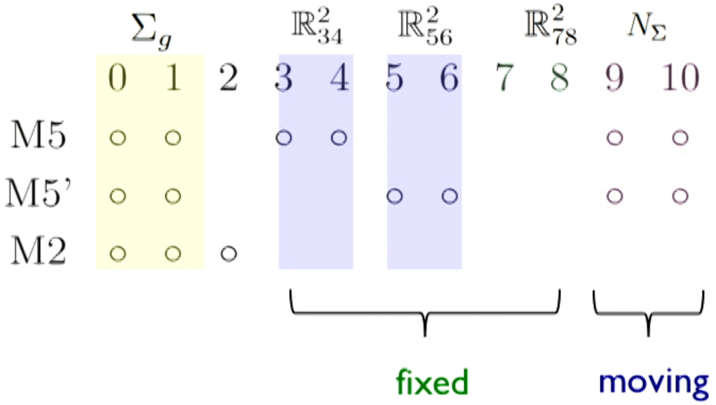
$$S_{GL(N|N)_k}[s] = -\frac{k}{8\pi}\int_{\Sigma_g} d^2x \left\langle s^{-1}\partial_\alpha s, s^{-1}\partial^\alpha s\right\rangle - \frac{ik}{24\pi}\int_M d^3x\,\epsilon^{\mu\nu\lambda}\left\langle s^{-1}\partial_\mu s, [s^{-1}\partial_\nu s, s^{-1}\partial_\lambda s]\right\rangle$$

Effective action of N Topological M-string as
GL(N|N) WZW model

M2-M5 system

$\mathrm{K3} \times \mathbb{R}^7$





	Σ_g			$\mathbb{R}^2_{34}$		$\mathbb{R}^2_{56}$		$\mathbb{R}^2_{78}$		N_Σ	
	0	1	2	3	4	5	6	7	8	9	10
M5	○	○		○	○					○	○
M5’	○	○				○	○			○	○
M2	○	○	○								

fixed

moving

GL(N|N) WZW model

arising from **twisted U(N)xU(N) ABJM model**

$$S[s] = -\frac{k}{8\pi}\int_\Sigma d^2x \left(s^{-1}\partial^\alpha s, s^{-1}\partial_\alpha s\right) - \frac{ik}{24\pi}\int_M d^3x \epsilon^{\mu\nu\lambda}\left(s^{-1}\partial_\mu s, [s^{-1}\partial_\nu \lambda, s^{-1}\partial_\lambda s]\right)$$

III. Mock Modular Index

Current algebra

$$\boxed{S[s] = -\frac{k}{8\pi}\int_{\Sigma} d^2x \left(s^{-1}\partial^{\alpha}s, s^{-1}\partial_{\alpha}s\right) - \frac{ik}{24\pi}\int_{M} d^3x\epsilon^{\mu\nu\lambda}\left(s^{-1}\partial_{\mu}s, [s^{-1}\partial_{\nu}\lambda, s^{-1}\partial_{\lambda}s]\right)}$$

$$s(z,\overline{z}) \to \Omega(z)s(z,\overline{z})\overline{\Omega}^{-1}(\overline{z})$$

Noether thm.

$$J(z) = J^a(z)T_a = -k\partial_z s s^{-1}$$

Laurent expansion $J^a(z) = \sum_{n=-\infty}^{\infty} \frac{J_n^a}{z^{n+1}}$

$$\boxed{[J_n^a, J_m^b] = [T^a, T^b]_c J_{n+m}^c + m(T^a, T^b)\delta_{n+m,0}k}$$

Affine Lie superalgebra

$$\widehat{\mathfrak{sg}} = (\mathbb{C}[t, t^{-1}] \otimes \mathfrak{sg}) \oplus \mathbb{C}K \oplus \mathbb{C}d$$

$$[at^n, bt^m] = [a,b]t^{n+m} + m\delta_{m+n,0}(a,b)K \qquad (at^n, bt^m) = \delta_{m+n,0}(a,b)$$

$$[d, at^n] = nat^n \qquad (K, d) = 1$$

$a, b \in \mathfrak{sg},\ m, n \in \mathbb{Z}.$

Affine Lie superalgebra

$$\widehat{\mathfrak{sg}} = \underbrace{(\mathbb{C}[t,t^{-1}] \otimes \mathfrak{sg})}_{\text{loop algebra}} \oplus \underbrace{\mathbb{C}K}_{\text{level}} \oplus \underbrace{\mathbb{C}d}_{\text{derivation}}$$

$$d = t\tfrac{d}{dt} = -L_0$$

$$[at^n, bt^m] = [a,b]t^{n+m} + m\delta_{m+n,0}(a,b)K \qquad (at^n, bt^m) = \delta_{m+n,0}(a,b)$$

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Cartan subalg.

$$\widehat{\mathfrak{h}} = \mathfrak{h} + \mathbb{C}d + \mathbb{C}K$$

Affine Lie superalgebra

$$\widehat{\mathfrak{sg}} = (\mathbb{C}[t, t^{-1}] \otimes \mathfrak{sg}) \oplus \mathbb{C}K \oplus \mathbb{C}d$$

loop algebra level derivation

$$d = t\tfrac{d}{dt} = -L_0$$

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Cartan subalg.

coordinate

$$\widehat{\mathfrak{h}} = \mathfrak{h} + \mathbb{C}d + \mathbb{C}K \qquad h = (\tau, z, t) := 2\pi i(-\tau d + z + tK)$$

$$\tau, t \in \mathbb{C}, \; z \in \mathfrak{h}$$

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$$\widehat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda_0$$

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root space

$\widehat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda_0$ basis of $\mathfrak{h}^*$ $\{\epsilon_1, \cdots, \epsilon_N; \delta_1, \cdots, \delta_M\}$

$$(\epsilon_i, \epsilon_j) = -(\delta_i, \delta_j) = \delta_{ij}$$

$$(\delta, \Lambda_0) = 1$$

Affine Lie superalgebra

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basis of $\mathfrak{h}^*$ $\{\epsilon_1, \cdots, \epsilon_N; \delta_1, \cdots, \delta_M\}$

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weight

$$\Lambda = (k, \lambda, n)$$

Highest weight representation $L(\Lambda)$

$$hv_\Lambda = \Lambda(h)v_\Lambda \qquad h \in \widehat{\mathfrak{h}}$$

$$\mathfrak{n}^+ v_\Lambda = 0$$

$$(t^n \otimes \mathfrak{sg})\, v_\Lambda = 0 \qquad n > 0$$

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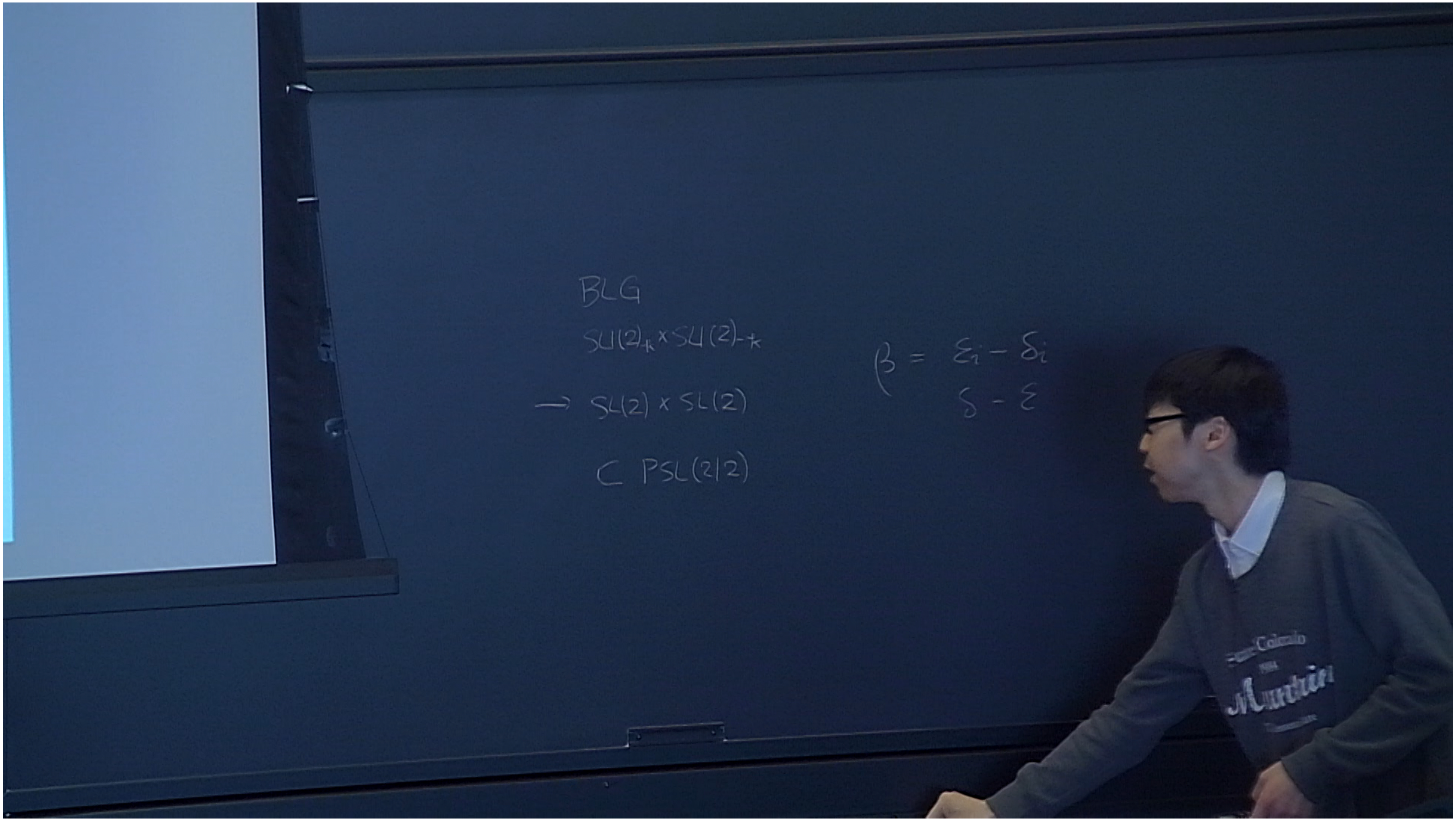
Here is a novel feature in Lie superalg.

Atypicality

$$(\lambda + \rho, \beta_i) = 0 \qquad (\beta_i, \beta_j) = 0$$


BLG
$SU(2)_k \times SU(2)_{-k}$
$\longrightarrow SL(2) \times SL(2)$
$\subset PSL(2|2)$
$\beta = \varepsilon_i - \delta_i$
$\delta - \varepsilon$

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isotropic root $\in \Delta_{\bar{1}} = \{\epsilon_i - \delta_k | 1 \leq i \leq N, 1 \leq k \leq M\}$

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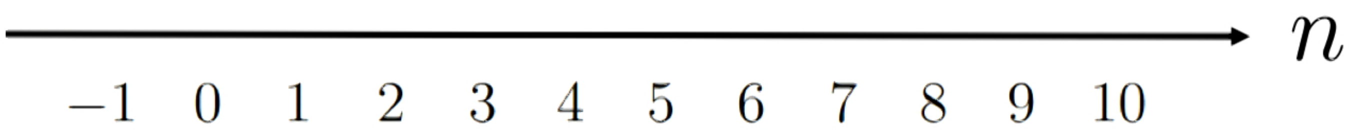
Via combinatorial ***weight diagram,***
detailed description of rep has been constructed !

Weight diagram

'11 Brundan & Stroppel

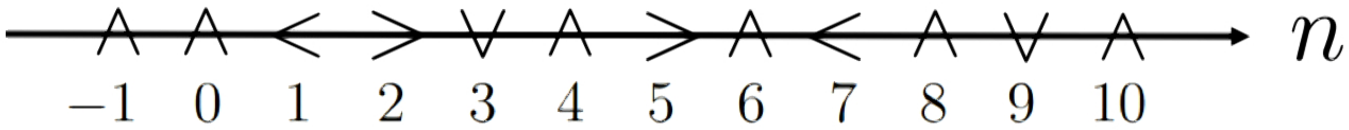
$$\vee \ \wedge \ > <$$



Weight diagram

'11 Brundan & Stroppel

$$\vee \ \wedge \ > <$$



$$\begin{cases} \vee & \text{if } n \in \{x_i\} \cap \{y_k\} \\ > & \text{if } n \in \{x_i\} \setminus \{y_k\} \\ < & \text{if } n \in \{y_k \setminus \{x_i\} \\ \wedge & \text{if } n \notin \{x_i \cup \{y_k\} \end{cases}$$

Weight diagram

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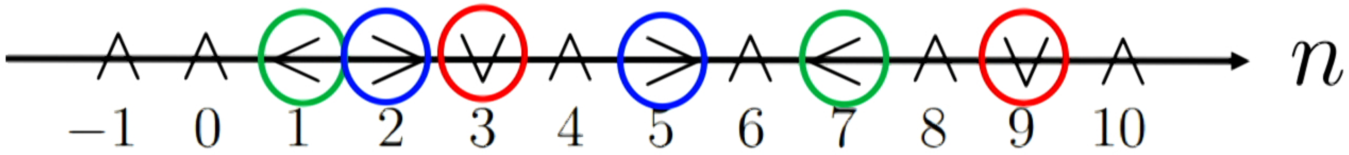

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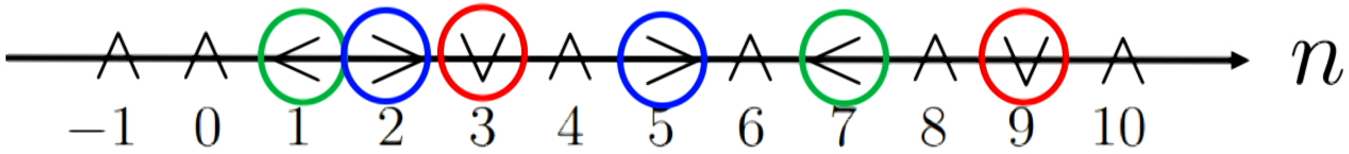

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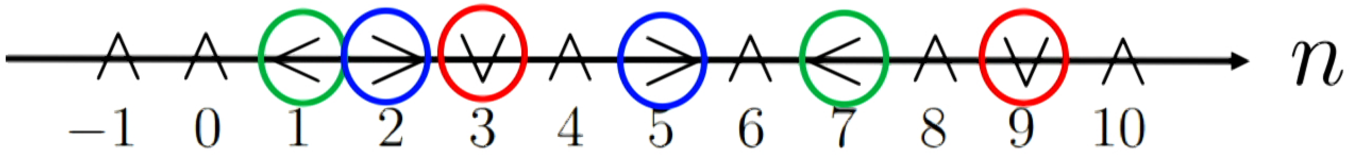
$$\lambda + \rho = 9\epsilon_1 + 5\epsilon_2 + 3\epsilon_3 + 2\epsilon_4 - \delta_1 - 3\delta_2 - 7\delta_3 - 9\delta_4$$

Weight diagram

'11 Brundan & Stroppel

$$\vee \ \wedge \ > <$$



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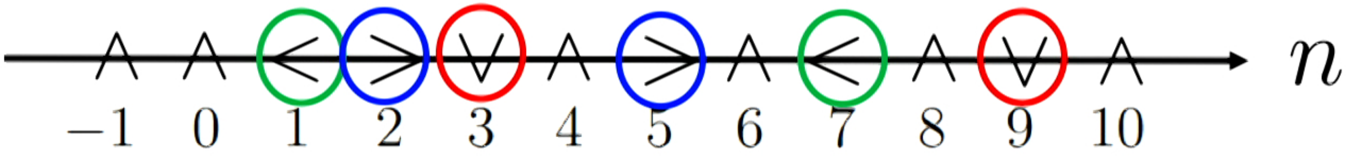

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$$N = \#(>) + \#(\vee) \qquad M = \#(<) + \#(\vee) \qquad \mathrm{aty}(\lambda) = \#(\vee)$$

Branes & weight diagram

M2-brane charge

holomorphic U(1) bdls over Σ

$$C = \sum_i A^i \wedge \alpha_i$$

Branes & weight diagram

M2-brane charge

holomorphic U(1) bdls over Σ

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$$\Big\Downarrow \quad \text{two M5 \& M5'}$$

$$x_i \in \mathbb{Z} \qquad y_k \in \mathbb{Z}$$

$$i = 1, \cdots, N$$
$$k = 1, \cdots, N$$

Branes & weight diagram

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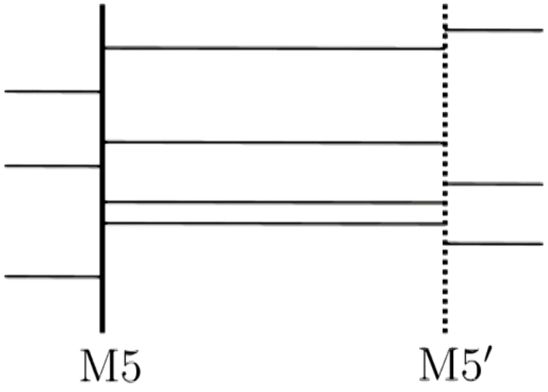

Branes & weight diagram

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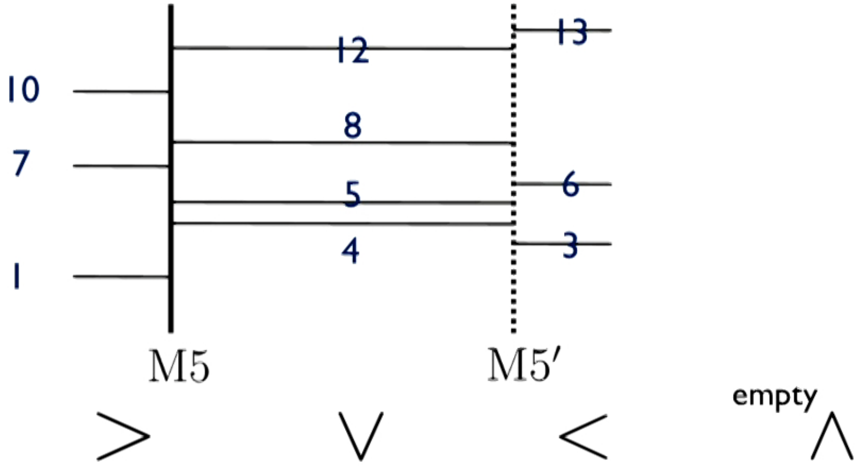

Branes & weight diagram

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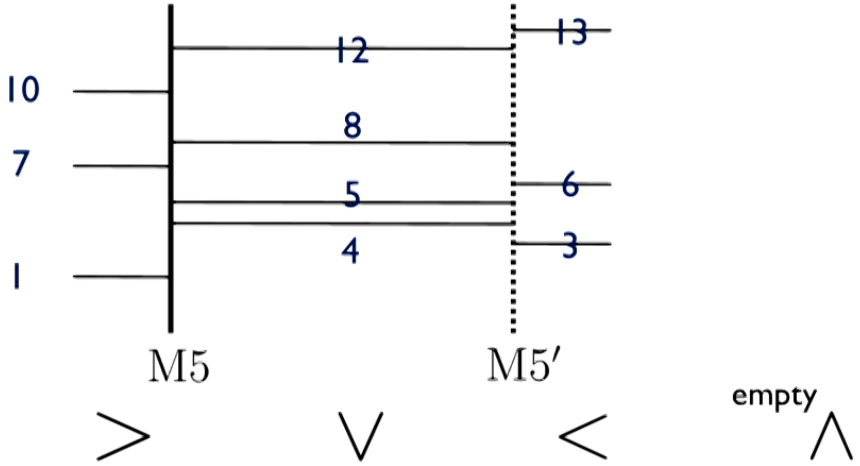

$\wedge$	$>$	$\wedge$	$<$	$\vee$	$\vee$	$<$	$>$	$\vee$	$\wedge$	$>$	$\wedge$	$\vee$	$<$
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Branes & weight diagram

M2-brane charge

holomorphic U(1) bdls over Σ

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⬇ two M5 & M5’

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$$i = 1, \cdots, N$$
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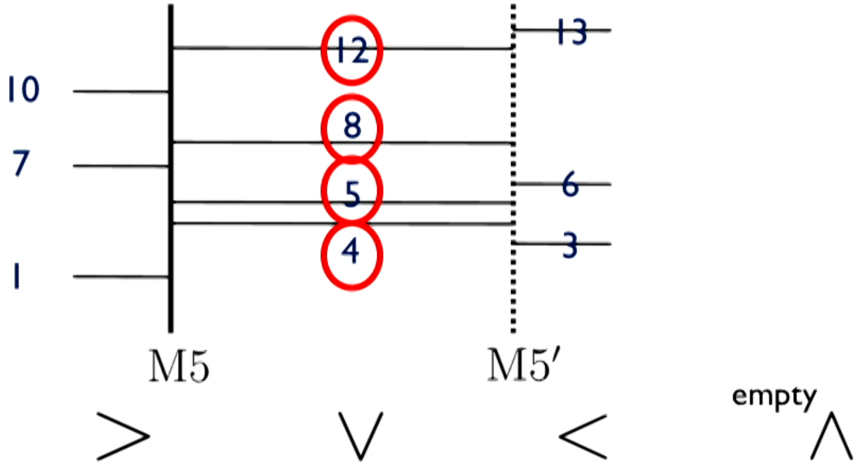

$$\begin{array}{cccccccccccccc} \wedge & > & \wedge & < & \enclose{circle}{\vee} & \enclose{circle}{\vee} & < & > & \enclose{circle}{\vee} & \wedge & > & \wedge & \enclose{circle}{\vee} & < \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{array}$$

$$\lambda + \rho = \enclose{circle}{12}\epsilon_1 + 10\epsilon_2 + \enclose{circle}{8}\epsilon_3 + 7\epsilon_4 + \enclose{circle}{5}\epsilon_5 + \enclose{circle}{4}\epsilon_6 + \epsilon_7$$
$$- 3\delta_1 - \enclose{circle}{4}\delta_2 - \enclose{circle}{5}\delta_3 - 6\delta_4 - \enclose{circle}{8}\delta_5 - \enclose{circle}{12}\delta_6 - 13\delta_7$$

Branes & weight diagram

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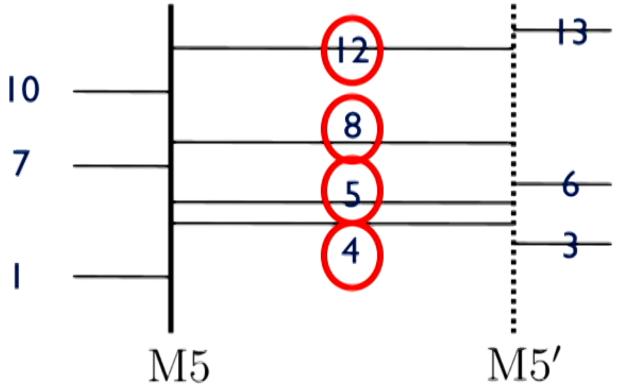
holomorphic U(1) bdls over Σ

$$C = \sum_i A^i \wedge \alpha_i$$

two M5 & M5'

$$x_i \in \mathbb{Z} \qquad y_k \in \mathbb{Z}$$

$$i = 1, \cdots, N$$
$$k = 1, \cdots, N$$



$>$ $\vee$ $<$ empty $\wedge$

$$\begin{array}{cccccccccccccc} \wedge & > & \wedge & < & \enclose{circle}{\vee} & \enclose{circle}{\vee} & < & > & \enclose{circle}{\vee} & \wedge & > & \wedge & \enclose{circle}{\vee} & < \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{array}$$

$$\lambda + \rho = \enclose{circle}{12\epsilon_1} + 10\epsilon_2 + \enclose{circle}{8\epsilon_3} + 7\epsilon_4 + \enclose{circle}{5\epsilon_5} + \enclose{circle}{4\epsilon_6} + \epsilon_7$$
$$- 3\delta_1 - \enclose{circle}{4\delta_2} - \enclose{circle}{5\delta_3} - 6\delta_4 - \enclose{circle}{8\delta_5} - \enclose{circle}{12\delta_6} - 13\delta_7$$

aty(λ) = #(stretched M2)

Supergroup WZW Index

$$\mathcal{I}(\tau, z) := \mathrm{Tr}_{\mathcal{H}}(-1)^F q^{L_0 - \frac{c}{24}} \prod_a x_a^{F_a}$$

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L_0	energy of stretched M2 = winding # around time circle of stretched M2
F_a	M2-brane charge of stretched M2

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M2-M5 BPS states counting !

Kac-Wakimoto formula

$$\mathrm{sch}_{L(\Lambda)} := \sum_{h \in \widehat{\mathfrak{h}}} \mathrm{sdim} L(\Lambda) e^{h}$$

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$$\mathrm{sch}_{L(\Lambda)} := \sum_{h\in\widehat{\mathfrak{h}}} \mathrm{sdim} L(\Lambda) e^{h} \quad = \mathrm{Str}_{L(\Lambda)} e^{2\pi i(-\tau d + z + tK)}$$

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’01 Kac Wakimoto

$$\boxed{e^{\widehat{\rho}}\widehat{R}^{-}\mathrm{sch}_{L(\Lambda)} = \sum_{w\in\widehat{W}^{\sharp}} \mathrm{sgn}^{-}(w)\frac{e^{w(\Lambda+\widehat{\rho})}}{\prod_{\beta\in S}(1-e^{-w(\beta)})}}$$

affine superdenominator $\widehat{R}^{-} = \dfrac{\prod_{\alpha\in\widehat{\Delta}^{+}_{\bar{0}}}(1-e^{-\alpha})}{\prod_{\alpha\in\widehat{\Delta}^{+}_{\bar{1}}}(1-e^{-\alpha})}$

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⬇ L_0 contribution $m_\Lambda = h_\Delta - \frac{c}{24}$

$$\mathrm{sch}_{\Lambda} = q^{m_\Lambda}\mathrm{sch}_{L(\Lambda)}(\tau, z, t)$$

Index formula

$$\mathcal{I}(\tau, z) \;=\; \frac{\sum_{w\in W^\sharp} \mathrm{sgn}^-(w)\Theta^{L^\sharp,-}_{\Lambda+\widehat{\rho},S}}{q^{\frac{\mathrm{sdim}\mathfrak{g}}{24}}\widehat{R}^-}$$

Ramanujan's mock theta function

$$\Theta^{\pm}_{\Lambda,T}(\tau,z,t) = e^{2\pi i K t} \sum_{\gamma\in L^{\#}+K^{-1}\overline{\Lambda}} \epsilon_{\pm}(t_\gamma)\frac{q^{\frac{K\langle\gamma,\gamma\rangle}{2}}e^{2\pi i K\gamma(z)}}{\prod_{\beta\in T}(1\pm q^{-\langle\gamma,\beta\rangle}e^{-2\pi i\beta(z)})}$$

Mock modular index !

Q. What does mock modularity mean ?

Q. What does mock modularity mean ?

⬇

1. Non-holomorphic torus partition function
2. Wall-crossring
3. Applications in physics and math

Mock modular form

$$h(\tau) : \mathbb{H} \to \mathbb{R}$$

$\exists$ shadow $g(\tau) \in M_{2-k}$

Modular completion

$$\boxed{\widehat{h}(\tau) = h(\tau) + g^*(\tau)}$$

Euchler integral

$$g^* = \left(\frac{i}{2\pi}\right)^{k-1} \int_{-\overline{\tau}}^{\infty} (z+\tau)^{-k} \overline{g(-\overline{z})} dz$$

'08 Zwegers

Non-holomorphic modular functions

Torus pfn of SG WZW

$$\mathcal{Z}(\tau,\overline{\tau},z) := \mathrm{Tr}_{\mathcal{H}}(-1)^F q^{L_0-\frac{c}{24}}\overline{q}^{\overline{L}_0-\frac{c}{24}}\prod_a x_a^{F_a}$$

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holomorphic factorized Hilbert space $\mathcal{H} = \bigoplus_\mu \mathcal{L}_\mu \oplus \mathcal{L}_\mu^*$

discrete spectrum

SUSY

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⬇ But for SG WZW model…

Hilbert space $\mathcal{H} = \bigoplus_{\mu\in\text{typical}} \mathcal{L}_\mu \otimes \mathcal{L}_\mu^* \oplus \bigoplus_{\nu\in\text{atypical}} \mathcal{H}_\nu$

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$\Downarrow$

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non-holomorphic atypical sector

$$\boxed{\mathcal{Z}(\tau,\overline{\tau},z) = \widehat{\mathcal{I}}_{(}\tau,\overline{\tau},z) + (\text{holomorphic modular function})}$$

non-holomorphic modular form

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non-holomorphic modular form

PSL(2|2)

We get

$$\mathcal{I}(\tau, z_1, z_2) = \frac{1}{\eta(\tau)^4} \frac{\vartheta_{11}(\tau, z_1)^2 \vartheta_{11}(\tau, z_2)^2}{\vartheta_{11}(\tau, z_1 - z_2)\vartheta_{11}(\tau, z_1 + z_2)}$$

$$\times \ \left(\mathcal{A}_{2,1}(\tau, z_1, z_1 + z_2) - \mathcal{A}_{2,1}(\tau, -z_2, -z_1 - z_2)\right)$$

multi-variable Appell-Lerch sum of order 2 and index 1

$$\boxed{\mathcal{A}_{2,1}(\tau, u, v) = U \sum_{n \in \mathbb{Z}} \frac{q^{n(n+1)} V^n}{(1 - U q^n)^2}}$$

$$U = \exp(2\pi i u) \quad V = \exp(2\pi i v)$$

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Appell-Lerch sum of order 2 and index m

$$A_{2,m}(\tau, z) = \sum_{n \in \mathbb{Z}} \frac{q^{mn^2 + n} x^{2mn+1}}{(1 - q^n x)^2}$$

Modular completion

We get

$$\boxed{\hat{\mathcal{I}} = \frac{1}{\eta^4(\tau)} \frac{\theta_{11}^2(z_1;\tau)\theta_{11}^2(z_2;\tau)}{\theta_{11}(z_1-z_2;\tau)\theta_{11}(z_1+z_2;\tau)} \left(\hat{\mathcal{A}}_{2,1}(\tau, z_1, z_1+z_2) - \hat{\mathcal{A}}_{2,1}(\tau, -z_2, -z_1-z_2) \right)}$$

$$\boxed{\hat{\mathcal{A}}_{2,1}(\tau,u,v) = \left(\mathcal{D} + \frac{\Im(v)}{\tau_2} - 2\frac{\Im(u)}{\tau_2} \right) \hat{\mathcal{A}}_{1,1}(\tau,u,v)}$$

$$\mathcal{D} = \frac{1}{2\pi i}\frac{\partial}{\partial u} \qquad \tau_2 = \Im(\tau)$$

$$\hat{\mathcal{A}}_{1,k}(\tau,u,v) = \mathcal{A}_{1,k}(\tau,u,v) + \mathcal{R}_{1,k}(\tau,u,v)$$

Appell-Lerch sum

$$A_{2,m}(\tau, z) = \sum_{n\in\mathbb{Z}} \frac{q^{mn^2+n} x^{2mn+1}}{(1-q^n x)^2} \sim \frac{x}{(1-x)^2}$$

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Averaging operation

$$\boxed{\mathrm{Av}^{(m)}\left[f(x)\right] := \sum_{\lambda\in\mathbb{Z}} q^{m\lambda^2} x^{2m\lambda} f(q^\lambda x)}$$

creating ***Jacobi form*** *of index m from f(x)*

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Consier the Fourier expansion

$$\frac{x}{(1-x)^2}$$

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Consier the Fourier expansion

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \cdots \qquad |x| < 1$$

Appell-Lerch sum

$$A_{2,m}(\tau,z)=\sum_{n\in\mathbb{Z}}\frac{q^{mn^2+n}x^{2mn+1}}{(1-q^nx)^2}\quad\sim\quad\frac{x}{(1-x)^2}$$

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$$\frac{x}{(1-x)^2}\quad=x+2x^2+3x^3+\cdots\qquad |x|<1$$

✗ $|x|>1$

Namely Fourier coefficients depend on the variables !
Wall Crossing *happen !*

Consider the Fourier expansion

$$A_{2,m}(\tau, z) = \sum_{n\in\mathbb{Z}} \frac{q^{mn^2+n}x^{2mn+1}}{(1-q^nx)^2}$$

$$= \sum_{r\geq l>0,\ r\equiv l \bmod 2m} l q^{\frac{r^2-l^2}{4m}} x^r \qquad 0 < \mathrm{Im}\, z < \mathrm{Im}\, \tau$$

✗ $$n < \frac{\mathrm{Im} z}{\mathrm{Im} \tau} < n+1$$

Again ***wall crossing*** *happen !*

‘10 Dabholkar, Murthy, Zagier

Thm

meromorphic Jacobi form of weight 2 has a decomposition

$$\varphi_m(\tau, z) = \varphi_m^F(\tau, z) + \varphi_m^P(\tau, z)$$

$$z = z_s = \alpha\tau + \beta$$

$$\varphi_m^F(\tau, z) = \sum_{l \in \mathbb{Z}\backslash 2m\mathbb{Z}} h_l(\tau)\vartheta_{m,l}(\tau, z)$$

$$\varphi_m^P(\tau, z) = \sum_{s \in S\backslash \mathbb{Z}^2} \big(D_s(\tau)A_{1,m}^s(\tau, z) + E_s(\tau)A_{2,m}^s(\tau, z)\big)$$

$$e^{2\pi i m\alpha z_s}\varphi(\tau, z_s + \epsilon) = \frac{E_s(\tau)}{(2\pi i\epsilon)^2} + \frac{D_s(\tau) - 2m\alpha E_s(\tau)}{2\pi i\epsilon} + \mathcal{O}(1)$$

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Appell-Lerch sum of order 2

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Appell-Lerch sum of order 2

= *Fourier coefficients (polar part) of* ***meromorphic Jacobi form of weight 2***

'10 Dabholkar, Murthy, Zagier

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$$\mathcal{Z} = \frac{1}{\Phi_{10}(\Omega)} = \sum_{m=-1}^{\infty} \varphi_m(\tau,z) y^m$$

meromorphic Jacobi form of weight 2

Pfn of ¼ BPS dyon

‘10 Dabholkar, Murthy, Zagier

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meromorphic Jacobi form of weight 2

Pfn of ¼ BPS dyon

$$\varphi_m(\tau,z) = \varphi_m^F(\tau,z) + \frac{p_{24}(m+1)}{\Delta(\tau)} A_{2,m}(\tau,z)$$

(immortal BH)

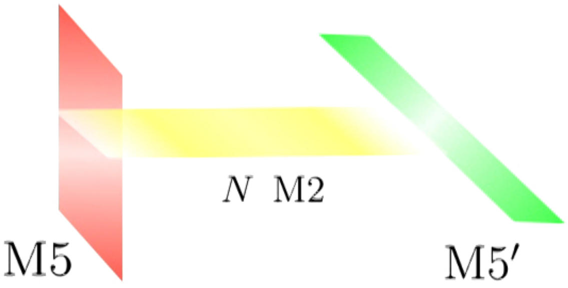
(multi-centered BH)

wall-crossing

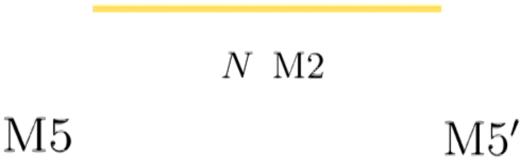
Q. What can we learn from our
***multi-variable** Appell-Lerch sum of order 2 ?*

(indicative of wall-crossing in M2-M5 system and BH counting)

IV. Quantum Mechanical Matrix Model



Compactifying the direction along which M2-branes are stretched, then index would be obtained from a certain QM.



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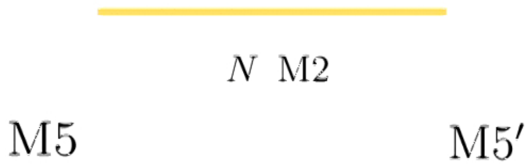

Compactifying the direction along which M2-branes are stretched, then index would be obtained from a certain QM.

Q. What about 1d QM description ?

M2-M2 intersection as vortex

'97 Gauntlett, Gomis, Townsend

	0	1	2	3	4	5	6	7	8	9	10
M2	○	○	○								
$M2'$			○	○	○						

M2-M2 intersection as vortex

'97 Gauntlett, Gomis, Townsend

	0	1	2	3	4	5	6	7	8	9	10
M2	○	○	○								
M2′			○	○	○						

$$Q = \int d^2x \left(-\Gamma^\mu \Gamma^I \Gamma^t D_\mu X^{Ia} \Psi_a - \frac{1}{6} \Gamma^{IJK} \Gamma^t X^I_a X^J_b X^K_c \Psi_d f^{abcd} \right)$$

M2-M2 intersection as vortex

'97 Gauntlett, Gomis, Townsend

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$$\boxed{\{Q_\alpha, Q_\beta\} = -2P_\mu \left(\Gamma^\mu \Gamma^t\right)^{\alpha\beta} + Z_{IJ} \left(\Gamma^{IJ} \Gamma^t\right)^{\alpha\beta} + Z_{iIJKL} \left(\Gamma^{IJKL} \Gamma^i \Gamma^t\right)^{\alpha\beta}}$$

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0-form central charge

$$Z_{IJ} = -\int d^2x \mathrm{Tr}\left(D_i X^I D_j X^J \epsilon^{ij} - D_t X^K F^{KIJ}\right)$$

1-form central charge

$$Z_{iIJKL} = \frac{1}{3} \int d^2x \mathrm{Tr}\left(D_j X^{[I} F^{JKL]} \epsilon^{ij}\right)$$

M2-M2 intersection as vortex

'97 Gauntlett, Gomis, Townsend

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$$\mathcal{H} = P_0 = \frac{1}{2}\left(D_t X^I, D_t X^I\right) + \frac{1}{2}\left(D_i X^I, D_i X^I\right) + V(X) = \frac{Z_{34}}{2} + 8\mathrm{Tr}(D_{\bar{z}}\Phi, D_z\overline{\Phi}) \geq \frac{Z_{34}}{2}$$

M2-M2 intersection as vortex

'97 Gauntlett, Gomis, Townsend

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$$\boxed{\{Q_\alpha, Q_\beta\} = -2P_\mu \left(\Gamma^\mu \Gamma^t\right)^{\alpha\beta} + Z_{IJ} \left(\Gamma^{IJ}\Gamma^t\right)^{\alpha\beta} + Z_{iIJKL} \left(\Gamma^{IJKL}\Gamma^i\Gamma^t\right)^{\alpha\beta}}$$

0-form central charge

$$Z_{IJ} = -\int d^2x \mathrm{Tr}\left(D_i X^I D_j X^J \epsilon^{ij} - D_t X^K F^{KIJ}\right)$$

1-form central charge

$$Z_{iIJKL} = \frac{1}{3}\int d^2x \mathrm{Tr}\left(D_j X^{[I} F^{JKL]} \epsilon^{ij}\right)$$

$$\mathcal{H} = P_0 = \frac{1}{2}\left(D_t X^I, D_t X^I\right) + \frac{1}{2}\left(D_i X^I, D_i X^I\right) + V(X) = \frac{Z_{34}}{2} + 8\mathrm{Tr}(D_{\bar{z}}\Phi, D_z\overline{\Phi}) \geq \frac{Z_{34}}{2}$$

$$\boxed{D_{\bar{z}}\Phi = 0} \qquad \boxed{\mathcal{H} = \frac{Z_{34}}{2}}$$

holomorphic curve
(vortex)

M2-M2 intersection as vortex

'97 Gauntlett, Gomis, Townsend

fixed (5–10)

	0	1	2	3	4	5	6	7	8	9	10
M2	○	○	○								
M2′			○	○	○						

$$\boxed{\Gamma^{0134}\epsilon = \pm\epsilon}$$

$$Q = \int d^2x \left(-\Gamma^\mu \Gamma^I \Gamma^t D_\mu X^{Ia} \Psi_a - \frac{1}{6} \Gamma^{IJK} \Gamma^t X^I_a X^J_b X^K_c \Psi_d f^{abcd} \right)$$

$$\boxed{\{Q_\alpha, Q_\beta\} = -2P_\mu \left(\Gamma^\mu \Gamma^t\right)^{\alpha\beta} + Z_{IJ} \left(\Gamma^{IJ}\Gamma^t\right)^{\alpha\beta} + Z_{iIJKL} \left(\Gamma^{IJKL}\Gamma^i\Gamma^t\right)^{\alpha\beta}}$$

0-form central charge

$$Z_{IJ} = -\int d^2x \mathrm{Tr}\left(D_i X^I D_j X^J \epsilon^{ij} - D_t X^K F^{KIJ}\right)$$

1-form central charge

$$Z_{iIJKL} = \frac{1}{3}\int d^2x \mathrm{Tr}\left(D_j X^{[I} F^{JKL]} \epsilon^{ij}\right)$$

$$\mathcal{H} = P_0 = \frac{1}{2}\left(D_t X^I, D_t X^I\right) + \frac{1}{2}\left(D_i X^I, D_i X^I\right) + V(X) = \frac{Z_{34}}{2} + 8\mathrm{Tr}(D_{\bar{z}}\Phi, D_z\overline{\Phi}) \geq \frac{Z_{34}}{2}$$

$$\boxed{D_{\bar{z}}\Phi = 0} \qquad \boxed{\mathcal{H} = \frac{Z_{34}}{2}}$$

holomorphic curve
(vortex)

M2-M2 intersection as vortex

	Σ_g		t			fixed					
	0	1	2	3	4	5	6	7	8	9	10
M2	○	○	○	Φ							
M2′			○	○	○						

'97 Gauntlett, Gomis, Townsend

$$\boxed{\Gamma^{0134}\epsilon = \pm\epsilon}$$

$$Q = \int d^2x \left(-\Gamma^\mu \Gamma^I \Gamma^t D_\mu X^{Ia} \Psi_a - \frac{1}{6} \Gamma^{IJK} \Gamma^t X^I_a X^J_b X^K_c \Psi_d f^{abcd} \right)$$

$$\boxed{\{Q_\alpha, Q_\beta\} = -2P_\mu \left(\Gamma^\mu \Gamma^t\right)^{\alpha\beta} + Z_{IJ} \left(\Gamma^{IJ}\Gamma^t\right)^{\alpha\beta} + Z_{iIJKL} \left(\Gamma^{IJKL}\Gamma^i\Gamma^t\right)^{\alpha\beta}}$$

0-form central charge

$$Z_{IJ} = -\int d^2x \mathrm{Tr} \left(D_i X^I D_j X^J \epsilon^{ij} - D_t X^K F^{KIJ} \right)$$

1-form central charge

$$Z_{iIJKL} = \frac{1}{3} \int d^2x \mathrm{Tr} \left(D_j X^{[I} F^{JKL]} \epsilon^{ij} \right)$$

$$\mathcal{H} = P_0 = \frac{1}{2}\left(D_t X^I, D_t X^I\right) + \frac{1}{2}\left(D_i X^I, D_i X^I\right) + V(X) = \frac{Z_{34}}{2} + 8\mathrm{Tr}(D_{\bar{z}}\Phi, D_z\overline{\Phi}) \geq \frac{Z_{34}}{2}$$

$$\boxed{D_{\bar{z}}\Phi = 0} \qquad \boxed{\mathcal{H} = \frac{Z_{34}}{2}}$$

holomorphic curve
(vortex)

CS Vortex QM Matrix Model

'00 Polychronakos '16 Dorey, Tong, Turner

$$\boxed{S = \int dt \left[i \,\mathrm{Tr}\left(Z^\dagger D_t Z\right) + i \sum_{i=1}^{p} \phi_i^\dagger D_t \phi_i - (k+p)\,\mathrm{Tr}\,\alpha - \omega\,\mathrm{Tr}\,Z^\dagger Z \right]}$$

$$D_t Z = \dot{Z} - i[\alpha, Z] \quad D_t \phi_i = \dot{\phi}_i - i\alpha\phi_i$$

EOM

$$iDZ = \omega Z \qquad\qquad D\phi = 0$$

Gauss law

$$Z \to U Z U^\dagger$$

$$\phi_i \to U \phi_i \qquad U \in U(N)$$

$$\boxed{[Z, Z^\dagger] + \sum_{i=1}^{p} \varphi_i \varphi_i^\dagger = (k+p)\mathbf{1}_N}$$

classical graound state

$$p = 1$$

$$Z = \sqrt{k+p}\begin{pmatrix} 0 & 1 & & & & \\ & 0 & \sqrt{2} & & & \\ & & \ddots & \ddots & & \\ & & & & \sqrt{N-2} & \\ & & & & 0 & \sqrt{N-1} \\ & & & & & 0 \end{pmatrix} \qquad \phi = \sqrt{k+p}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \sqrt{N} \end{pmatrix}$$

$$\alpha = \omega \ \mathrm{diag}(N-1, N-2, \cdots, 1, 0)$$

classical graound state

$$N \equiv 0 \mod p$$

$$Z = \sqrt{k+p}\begin{pmatrix} 0 & 1 & & & & \\ & 0 & \sqrt{2} & & & \\ & & \ddots & \ddots & & \\ & & & & \sqrt{N-2} & \\ & & & & 0 & \sqrt{N-1} \\ & & & & & 0 \end{pmatrix} \qquad \phi = \sqrt{k+p}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \sqrt{N} \end{pmatrix}$$

$$\alpha = \omega \ \mathrm{diag}(N-1, N-2, \cdots, 1, 0)$$

classical graound state

$$N \equiv 0 \mod p$$

$$Z = \sqrt{k+p}\begin{pmatrix} 0 & 1 & & & & \\ & 0 & \sqrt{2} & & & \\ & & \ddots & \ddots & & \\ & & & & \sqrt{N-2} & \\ & & & & 0 & \sqrt{N-1} \\ & & & & & 0 \end{pmatrix} \otimes \mathbf{1}_p \qquad \phi = \sqrt{k+p}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \sqrt{N} \end{pmatrix} \otimes \mathbf{1}_p$$

$$\alpha = \omega \ \mathrm{diag}(N-1, N-2, \cdots, 1, 0)$$

current

$$J^m_{ij} = \left(\frac{(k+p)N}{p}\right)^{-\frac{m}{2}} \widetilde{J}^m_{ij} \qquad \widetilde{J}^m_{ij} = i\left(\phi^\dagger_i Z^m \phi_j - \frac{1}{p}\delta_{ij}\phi^\dagger_k Z^m \phi_k\right) \qquad \widetilde{J}^m_{ij} = \widetilde{J}^{|m|\dagger}_{ji}$$

classical graound state

$$N \equiv 0 \mod p$$

$$Z = \sqrt{k+p}\begin{pmatrix} 0 & 1 & & & & \\ & 0 & \sqrt{2} & & & \\ & & \ddots & \ddots & & \\ & & & & \sqrt{N-2} & \\ & & & & 0 & \sqrt{N-1} \\ & & & & & 0 \end{pmatrix} \otimes \mathbf{1}_p \qquad \phi = \sqrt{k+p}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \sqrt{N} \end{pmatrix} \otimes \mathbf{1}_p$$

$$\alpha = \omega\ \mathrm{diag}(N-1, N-2, \cdots, 1, 0)$$

current

$$J^m_{ij} = \left(\frac{(k+p)N}{p}\right)^{-\frac{m}{2}} \widetilde{J}^m_{ij} \qquad \widetilde{J}^m_{ij} = i\left(\phi^\dagger_i Z^m \phi_j - \frac{1}{p}\delta_{ij}\phi^\dagger_k Z^m \phi_k\right) \qquad \widetilde{J}^m_{ij} = \widetilde{J}^{|m|\dagger}_{ji}$$

⬇ classical ground state & large N limit

$$\boxed{\left[J^m_{ij}, J^n_{kl}\right] \sim i\left(\delta_{il}J^{m+n}_{kj} - \delta_{kj}J^{m+n}_{il}\right) + km\left(\delta_{jk}\delta_{il} - \frac{1}{p}\delta_{ij}\delta_{kl}\right)}$$

Kac Moody algebra $\widehat{\mathfrak{su}(p)}$

CS Vortex QM Matrix Model

'00 Polychronakos '16 Dorey, Tong, Turner

$$\boxed{S = \int dt \left[i \operatorname{Tr} \left(Z^\dagger D_t Z \right) + i \sum_{i=1}^{p} \phi_i^\dagger D_t \phi_i - (k+p) \operatorname{Tr} \alpha - \omega \operatorname{Tr} Z^\dagger Z \right]}$$

$$D_t Z = \dot{Z} - i[\alpha, Z] \quad D_t \phi_i = \dot{\phi}_i - i \alpha \phi_i$$

EOM

$$iDZ = \omega Z \qquad D\phi = 0$$

Gauss law

$$Z \to U Z U^\dagger$$
$$\phi_i \to U \phi_i \qquad U \in U(N)$$

$$\boxed{[Z, Z^\dagger] + \sum_{i=1}^{p} \varphi_i \varphi_i^\dagger = (k+p) \mathbf{1}_N}$$

classical graound state

$$N \equiv 0 \mod p$$

$$Z = \sqrt{k+p}\begin{pmatrix} 0 & 1 & & & \\ & 0 & \sqrt{2} & & \\ & & \ddots & \ddots & \\ & & & \sqrt{N-2} & \\ & & & 0 & \sqrt{N-1} \\ & & & & 0 \end{pmatrix} \otimes \mathbf{1}_p \qquad \phi = \sqrt{k+p}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \sqrt{N} \end{pmatrix} \otimes \mathbf{1}_p$$

$$\alpha = \omega \ \mathrm{diag}(N-1, N-2, \cdots, 1, 0)$$

current

$$J^m_{ij} = \left(\frac{(k+p)N}{p}\right)^{-\frac{m}{2}} \tilde{J}^m_{ij} \qquad \tilde{J}^m_{ij} = i\left(\phi^\dagger_i Z^m \phi_j - \frac{1}{p}\delta_{ij}\phi^\dagger_k Z^m \phi_k\right) \qquad \tilde{J}^m_{ij} = \tilde{J}^{|m|\dagger}_{ji}$$

classical ground state & large N limit

$$\boxed{\left[J^m_{ij}, J^n_{kl}\right] \sim i\left(\delta_{il}J^{m+n}_{kj} - \delta_{kj}J^{m+n}_{il}\right) + km\left(\delta_{jk}\delta_{il} - \frac{1}{p}\delta_{ij}\delta_{kl}\right)}$$

Kac Moody algebra $\widehat{\mathfrak{su}(p)}$

Junctions of M2-branes

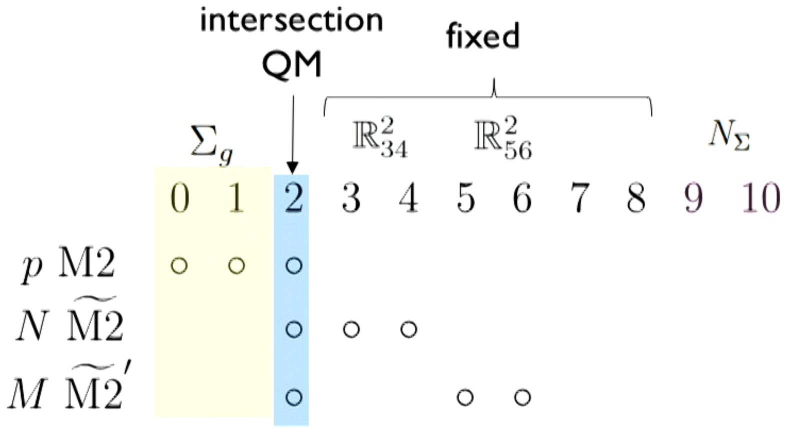
	Σ_g			$\mathbb{R}^2_{34}$		$\mathbb{R}^2_{56}$				N_Σ	
	0	1	2	3	4	5	6	7	8	9	10
p M2	○	○	○								
N $\widetilde{\text{M2}}$			○	○	○						
M $\widetilde{\text{M2}}'$			○			○	○				

Junctions of M2-branes

intersection
QM

	Σ_g			$\mathbb{R}^2_{34}$		$\mathbb{R}^2_{56}$		N_Σ			
	0	1	2	3	4	5	6	7	8	9	10
p M2	○	○	○								
N $\widetilde{\mathrm{M2}}$			○	○	○						
M $\widetilde{\mathrm{M2}}'$			○			○	○				

Junctions of M2-branes



intersection QM

fixed

	Σ_g			$\mathbb{R}^2_{34}$		$\mathbb{R}^2_{56}$				N_Σ	
	0	1	2	3	4	5	6	7	8	9	10
p M2	○	○	○								
$N\ \widetilde{\mathrm{M2}}$			○	○	○						
$M\ \widetilde{\mathrm{M2}}'$			○			○	○				

Junctions of M2-branes

intersection QM

fixed

	Σ_g			$\mathbb{R}^2_{34}$		$\mathbb{R}^2_{56}$				N_Σ	
	0	1	2	3	4	5	6	7	8	9	10
p M2	○	○	○								
N $\widetilde{\mathrm{M2}}$	Z_{ab}		○	○	○						
M $\widetilde{\mathrm{M2}}'$	$\widetilde{Z}_{\alpha\beta}$		○			○	○				

Junctions of M2-branes

intersection QM

fixed

		Σ_g			$\mathbb{R}^2_{34}$		$\mathbb{R}^2_{56}$				N_Σ		
		0	1	2	3	4	5	6	7	8	9	10	
heavier	p M2	○	○	○									$U(p)$ global
	N $\widetilde{\mathrm{M2}}$	Z_{ab}		○	○	○							$U(N)$ gauge
	M $\widetilde{\mathrm{M2}}'$	$\widetilde{Z}_{\alpha\beta}$		○			○	○					$U(M)$ gauge

Junctions of M2-branes

				intersection QM	fixed									
		Σ_g			$\mathbb{R}^2_{34}$		$\mathbb{R}^2_{56}$				N_Σ			
		0	1	2	3	4	5	6	7	8	9	10		
heavier	p M2	○	○	○									$U(p)$ global	
	N $\widetilde{\mathrm{M2}}$	Z_{ab}		○	○	○					ϕ_{ia}		$U(N)$ gauge	α_{ab}
	M $\widetilde{\mathrm{M2}}'$	$\widetilde{Z}_{\alpha\beta}$		○			○	○			$\widetilde{\phi}_{k\alpha}$		$U(M)$ gauge	$\widetilde{\alpha}_{\alpha\beta}$

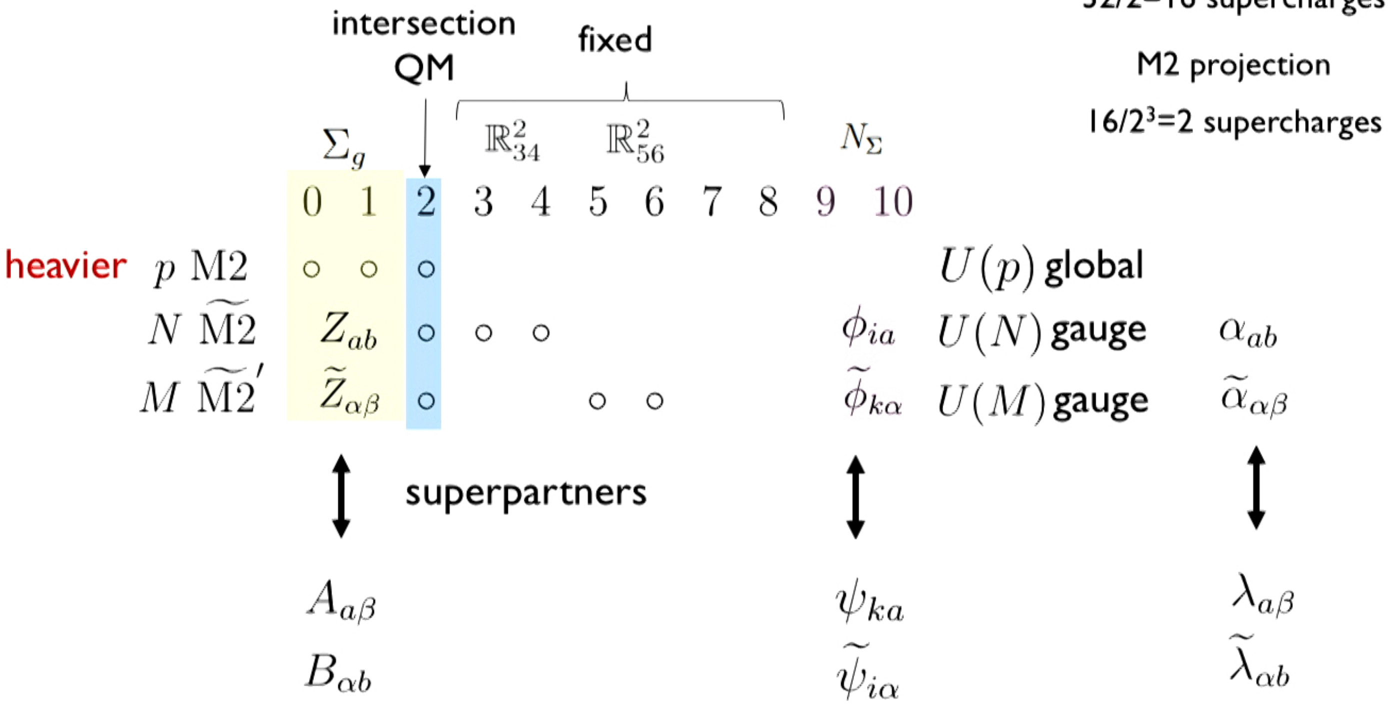
Junctions of M2-branes

K3

32/2=16 supercharges

M2 projection

$16/2^3=2$ supercharges



intersection QM

fixed

		Σ_g			$\mathbb{R}^2_{34}$		$\mathbb{R}^2_{56}$				N_Σ				
		0	1	2	3	4	5	6	7	8	9	10			
heavier	p M2	○	○	○										$U(p)$ global	
	N $\widetilde{\mathrm{M2}}$	Z_{ab}		○	○	○							ϕ_{ia}	$U(N)$ gauge	α_{ab}
	M $\widetilde{\mathrm{M2}}'$	$\widetilde{Z}_{\alpha\beta}$		○			○	○					$\widetilde{\phi}_{k\alpha}$	$U(M)$ gauge	$\widetilde{\alpha}_{\alpha\beta}$

$\updownarrow$ superpartners $\updownarrow$ $\updownarrow$

$A_{a\beta}$	ψ_{ka}	$\lambda_{a\beta}$
$B_{\alpha b}$	$\widetilde{\psi}_{i\alpha}$	$\widetilde{\lambda}_{\alpha b}$

U(N|M) gauged QM Matrix Model

$$S = \int dt \left[i \, \mathrm{Str} \left(\widehat{Z}^\dagger D_t \widehat{Z} \right) + i \sum_{I,A} \widehat{\Phi}^\dagger_{IA} D_t \widehat{\Phi}_{IA} - \kappa \, \mathrm{Str} \widehat{\alpha} - \omega \, \mathrm{Str} \widehat{Z}^\dagger \widehat{Z} \right]$$

$$D_t \widehat{Z} = \dot{\widehat{Z}} - i[\widehat{\alpha}, \widehat{Z}]$$
$$D_t \widehat{\Phi} = \dot{\widehat{\Phi}} - i \widehat{\alpha} \widehat{\Phi}$$

U(N|M) gauged QM Matrix Model

topological term for $\widehat{Z}$

$$S = \int dt \left[i \, \mathrm{Str}\left(\widehat{Z}^\dagger D_t \widehat{Z}\right) + i \sum_{I,A} \widehat{\Phi}^\dagger_{IA} D_t \widehat{\Phi}_{IA} - \kappa \ \mathrm{Str}\widehat{\alpha} - \omega \ \mathrm{Str}\widehat{Z}^\dagger \widehat{Z} \right]$$

$$= \int dt \Big[i \, \mathrm{Tr}\Big(Z^\dagger DZ + B^\dagger DB - iZ^\dagger \left(\lambda B - A\widetilde{\lambda}\right) - iB^\dagger \left(\widetilde{\lambda} Z - \widetilde{Z}\widetilde{\lambda}\right)$$

$$- \widetilde{Z}^\dagger D\widetilde{Z} - A^\dagger DA + i\widetilde{Z}^\dagger \left(\widetilde{\lambda} A - B\lambda\right) + iA^\dagger \left(\lambda \widetilde{Z} - Z\lambda\right) \Big)$$

$$+ i \sum \left(\phi^\dagger D\phi - i\phi^\dagger \lambda \widetilde{\psi} + \psi^\dagger D\psi - i\psi^\dagger \lambda \widetilde{\phi} + \widetilde{\phi}^\dagger D\widetilde{\phi} - i\widetilde{\phi}^\dagger \widetilde{\lambda}\psi + \widetilde{\psi}^\dagger D\widetilde{\psi} - i\widetilde{\psi}^\dagger \widetilde{\lambda}\phi \right)$$

$$- \kappa \, \mathrm{Tr}\left(\alpha - \widetilde{\alpha}\right) - \omega \, \mathrm{Tr}\left(Z^\dagger Z + B^\dagger B - \widetilde{Z}^\dagger \widetilde{Z} - A^\dagger A \right) \Big]$$

$$D_t \widehat{Z} = \dot{\widehat{Z}} - i[\widehat{\alpha}, \widehat{Z}]$$

$$D_t \widehat{\Phi} = \dot{\widehat{\Phi}} - i\widehat{\alpha}\widehat{\Phi}$$

U(N|M) gauged QM Matrix Model

topological term for $\widehat{Z}$

topological term for $\widehat{\Phi}$

topological term for $\widehat{\alpha}$

spatial regulator

$$\boxed{S = \int dt \left[i\,\mathrm{Str}\left(\widehat{Z}^\dagger D_t \widehat{Z}\right) + i \sum_{I,A} \widehat{\Phi}^\dagger_{IA} D_t \widehat{\Phi}_{IA} - \kappa\, \mathrm{Str}\widehat{\alpha} - \omega\, \mathrm{Str}\widehat{Z}^\dagger \widehat{Z} \right]}$$

$$\begin{aligned}
&= \int dt \Big[i\,\mathrm{Tr}\Big(Z^\dagger D Z + B^\dagger D B - i Z^\dagger \left(\lambda B - A\widetilde{\lambda}\right) - i B^\dagger \left(\widetilde{\lambda} Z - \widetilde{Z}\widetilde{\lambda}\right) \\
&\qquad - \widetilde{Z}^\dagger D \widetilde{Z} - A^\dagger D A + i\widetilde{Z}^\dagger \left(\widetilde{\lambda} A - B\lambda\right) + i A^\dagger \left(\lambda \widetilde{Z} - Z\lambda\right)\Big) \\
&\quad + i \sum \left(\phi^\dagger D\phi - i\phi^\dagger \lambda \widetilde{\psi} + \psi^\dagger D\psi - i\psi^\dagger \lambda \widetilde{\phi} + \widetilde{\phi}^\dagger D\widetilde{\phi} - i\widetilde{\phi}^\dagger \widetilde{\lambda}\psi + \widetilde{\psi}^\dagger D\widetilde{\psi} - i\widetilde{\psi}^\dagger \widetilde{\lambda}\phi \right) \\
&\quad - \kappa\, \mathrm{Tr}\left(\alpha - \widetilde{\alpha}\right) - \omega\, \mathrm{Tr}\left(Z^\dagger Z + B^\dagger B - \widetilde{Z}^\dagger \widetilde{Z} - A^\dagger A \right) \Big]
\end{aligned}$$

$$D_t \widehat{Z} = \dot{\widehat{Z}} - i[\widehat{\alpha}, \widehat{Z}]$$
$$D_t \widehat{\Phi} = \dot{\widehat{\Phi}} - i\widehat{\alpha}\widehat{\Phi}$$

Gauss law

$$\widehat{Z}_{AB} \to \widehat{U}_{AC} \widehat{Z}_{CD} \widehat{U}^\dagger_{DB},$$
$$\widehat{\Phi}_{IA} \to \widehat{U}_{AB} \widehat{\Phi}_{IB},$$
$$\widehat{\alpha}_{AB} \to \widehat{U}_{AC} \widehat{\alpha}_{CD} \widehat{U}^\dagger_{DB} + i \widehat{U}_{AC} \dot{\widehat{U}}^\dagger_{CB}$$

$$\boxed{\begin{aligned}
[Z, Z^\dagger] + AA^\dagger - B^\dagger B + \phi_i \phi_i^\dagger - \psi_k \psi_k^\dagger - \kappa \mathbb{I} &= 0, \\
[\widetilde{Z}, \widetilde{Z}^\dagger] + BB^\dagger - A^\dagger A - \widetilde{\phi}_k \widetilde{\phi}_k^\dagger + \widetilde{\psi}_i \widetilde{\psi}_i^\dagger - \kappa \mathbb{I} &= 0, \\
BZ^\dagger - \widetilde{Z}^\dagger B + \widetilde{Z} A^\dagger - A^\dagger Z + \widetilde{\psi}\phi^\dagger - \widetilde{\phi}\psi^\dagger &= 0, \\
ZB^\dagger - B^\dagger \widetilde{Z} - Z^\dagger A + A\widetilde{Z}^\dagger + \phi\widetilde{\psi}^\dagger - \psi\widetilde{\phi}^\dagger &= 0.
\end{aligned}}$$

Q. What is large N & M limit?

Condensate of M2 as M5

'08 Ho & Matsuo

'08 Bandos & Townsend

Lie 3-alegbra

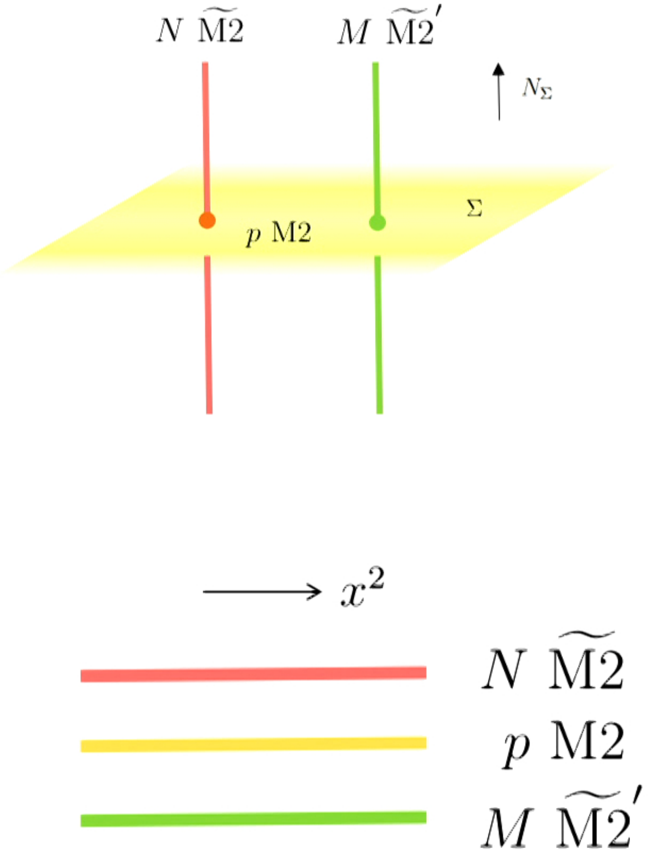
$$[T^a, T^b, T^c] = f^{abc}{}_d T^d$$
$$h^{ab} = \mathrm{Tr}(T^a, T^b)$$

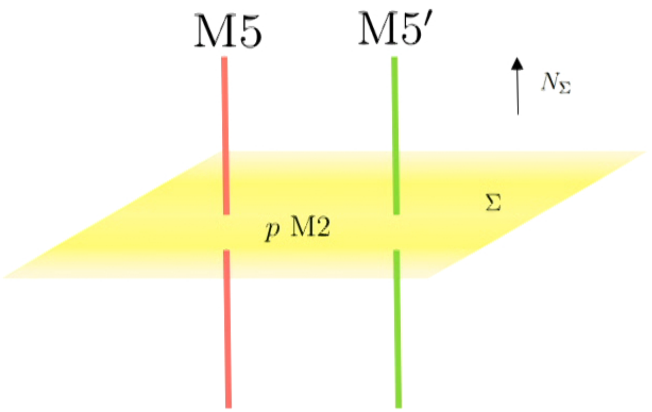
⬇ Large N & M limit

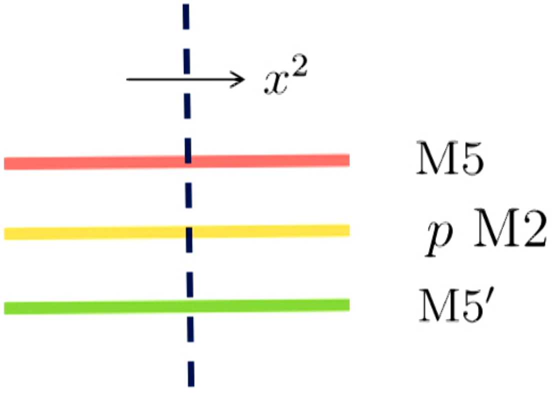
Nambu-Poisson bracket

$$\{\chi^a, \chi^b, \chi^c\} := \sum_{\dot\mu,\dot\nu,\dot\lambda} \epsilon_{\dot\mu\dot\nu\dot\lambda} \partial_{\dot\mu}\chi^a \partial_{\dot\nu}\chi^b \partial_{\dot\lambda}\chi^c = \sum_d f^{abc}{}_d \chi^d(y)$$
$$h^{ab} = (\chi^a, \chi^b)$$

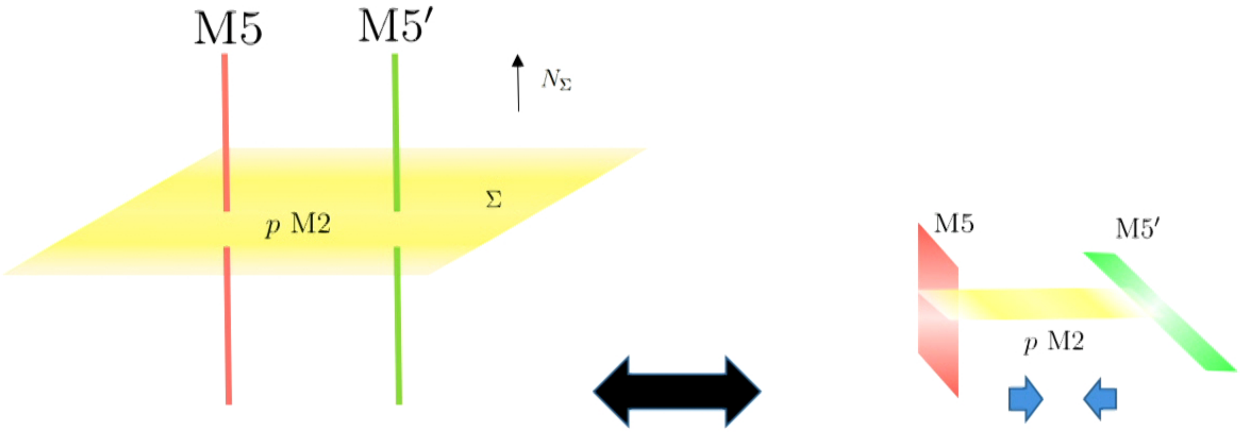
M5 appears as condensate of M2


$N\ \widetilde{\mathrm{M2}}$
$M\ \widetilde{\mathrm{M2}}'$
N_Σ
p M2
Σ
x^2
$N\ \widetilde{\mathrm{M2}}$
p M2
$M\ \widetilde{\mathrm{M2}}'$

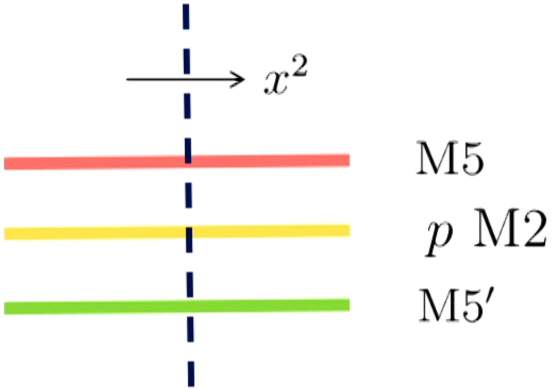




classical ground state



	Σ_g			$\mathbb{R}^2_{34}$		$\mathbb{R}^2_{56}$		$\mathbb{R}^2_{78}$		N_Σ	
	0	1	2	3	4	5	6	7	8	9	10
M5	○	○		○	○					○	○
M5’	○	○				○	○			○	○
M2	○	○	○								



classical ground state

$$\widehat{J}^m_{IJ} = i\left(\widehat{\Phi}^\dagger_I \widehat{Z}^m \widehat{\Phi}_J - \frac{1}{p}\delta_{IJ}\widehat{\Phi}^\dagger_K \widehat{Z}^m \widehat{\Phi}_K\right) =: \begin{pmatrix} \widehat{J}^m_{ij} & \widehat{J}^m_{il} \\ \widehat{J}^m_{kj} & \widehat{J}^m_{kl} \end{pmatrix}$$

⬇ **Large N & M limit**

$N \equiv 0,\ M \equiv 0 \mod p$

$$[\widehat{J}^m_{IJ}, \widehat{J}^n_{KL}] \sim \left(\delta_{IL}\widehat{J}^{m+n}_{KJ} - \delta_{KJ}\widehat{J}^{m+n}_{IL}\right) + \kappa m \left(\delta_{JK}\delta_{IL} - \frac{1}{p}\delta_{IJ}\delta_{KL}\right)$$

affine Lie superalg.

?

Discussion on Junctions of Branes

BPS Junctions of Branes

in string theory and M-theory involve many interesting problems !

Gauge theory

- Defects in gauge theory arising from Junctions of Branes
  => **dualities** in gauge theory

Gravity

- BH counting from Junctions of Branes (quantum gravity)
- Gravity dual of Junctions of Branes (holography)

Mathematics

- mock modular form and indefinite theta series (number theory)
- Generalized DT inv & ramified VW inv (algebraic geometry)
- character of affine Lie superalg (representation theory)
  => **Geometric Langlands program**