Title: Symplectic realization of 4D Wall-Crossing Formula on HyperKahler Surfaces

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Abstract: <p>In this talk, we will discuss an open Gromov-Witten invariant on hyperKahler surfaces, including K3 surfaces and certain Hitchin moduli spaces. The invariant is defined via the Lagrangian Floer theory and satisfy the Kontsevich-Soibelman wall-crossing formula and are expect to recover the generalized Donaldson-Thomas invariants studied in the work of Gaiotto-Moore-Neitzke.</p>

Symplectic Realization of
4D Wall-Crossing Formula on HK Surfaces

'96 SYZ Conj.

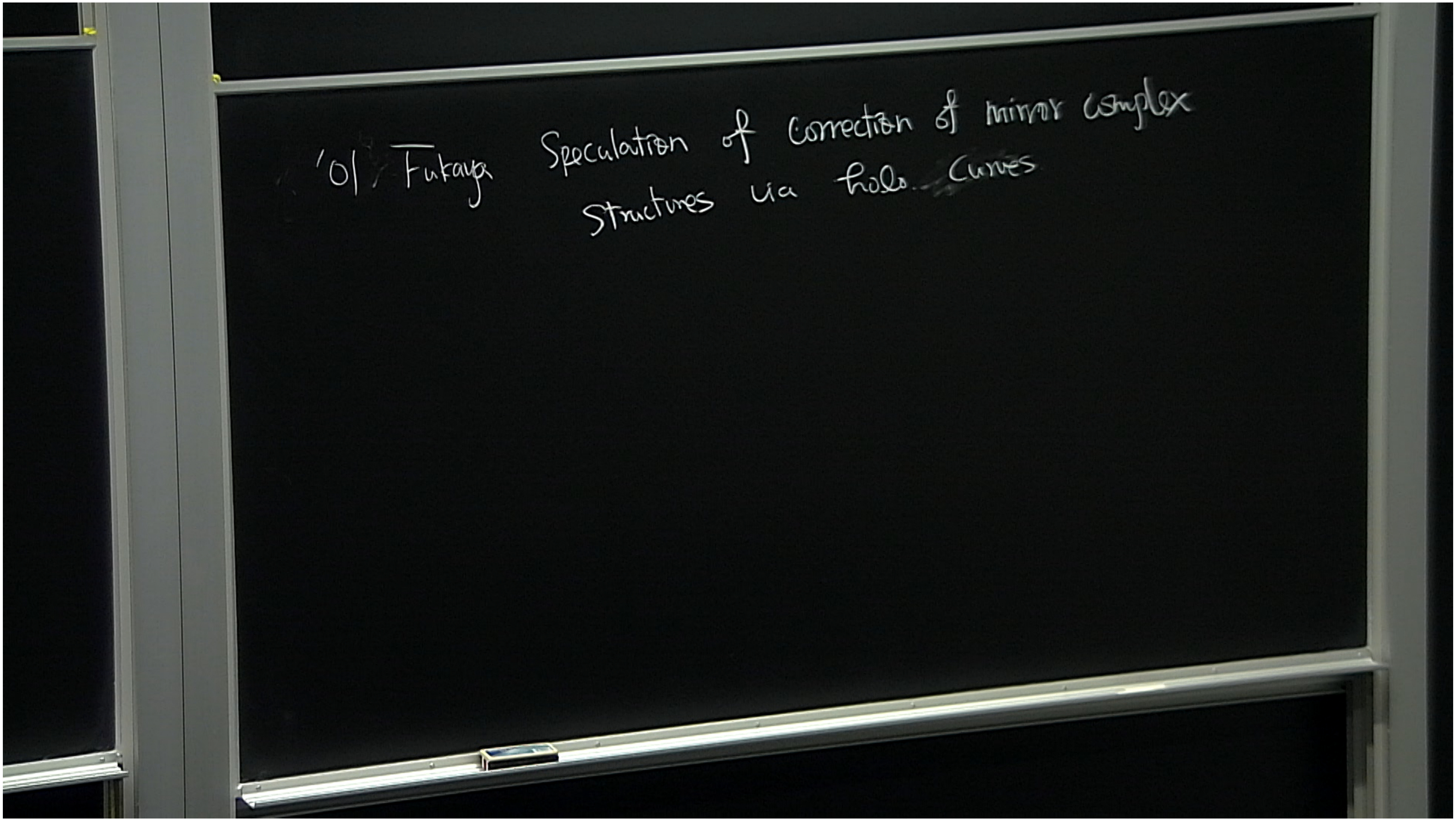
- CY mfd admits SLAG fibration

mirror $\longleftrightarrow$ dual fibration.

toy model

$$
\begin{tikzcd}
X = TB/\Lambda \arrow[rr, bend left, leftrightarrow, "\text{mirror}"] \arrow[dr] & & T^*B/\Lambda^* = \check{X} \arrow[dl] \\
& B &
\end{tikzcd}
$$

integral affine


'01 Fukaya Speculation of correction of mirror complex
structures via holo. curves

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$$\begin{tikzcd} X = TB/\Lambda \arrow[dr] & & T^{*}B/\Lambda^{*} = \check{X} \arrow[ll, bend right, "\text{mirror}"'] \arrow[dl] \\ & B & \end{tikzcd}$$

B integral affine

'04 Kontsevich–Soibelman

Construction of mirror K3 via rigid analytic geometry

'07 Gross-Siebert

generalize to all dimension CY.

'08 Gaiotto-Moore-Neitzke

Construction HK metric on Hitchin spaces.

'01 Fukaya Speculation of correction of mirror complex structures via holo. curves

'08 Auroux Some examples of the mirror ex. $\mathbb{C}^2$

'10 (Chan-Lau-Leung) toric CY

no scattering

'13 (Tu, Abouzaid, Fukaya) use family Floer theory to construct the mirror.

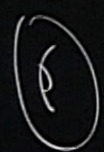
'07 Gross-Siebert

generalize to all dimension CY

'08 Gaiotto-Moore-Neitzke

Construction HK metric on Hitchin spaces

GMN $SL_2(\mathbb{C})$-Hitchin moduli space



$u \in$ space of quadratic differentials on C

① flat metric on C w/ punctures

② Spectral curve

$$\Sigma_u = \{x^2 = u\} \xrightarrow{2:1} C$$

$$\Sigma_u \subset T^*C$$

Charge lattice

$\gamma \in \Gamma$ = Subquotient of $H_1(\Sigma_u, \mathbb{Z})$

$Z_\gamma := \int_\gamma \lambda$ → Seiberg-Witten differential

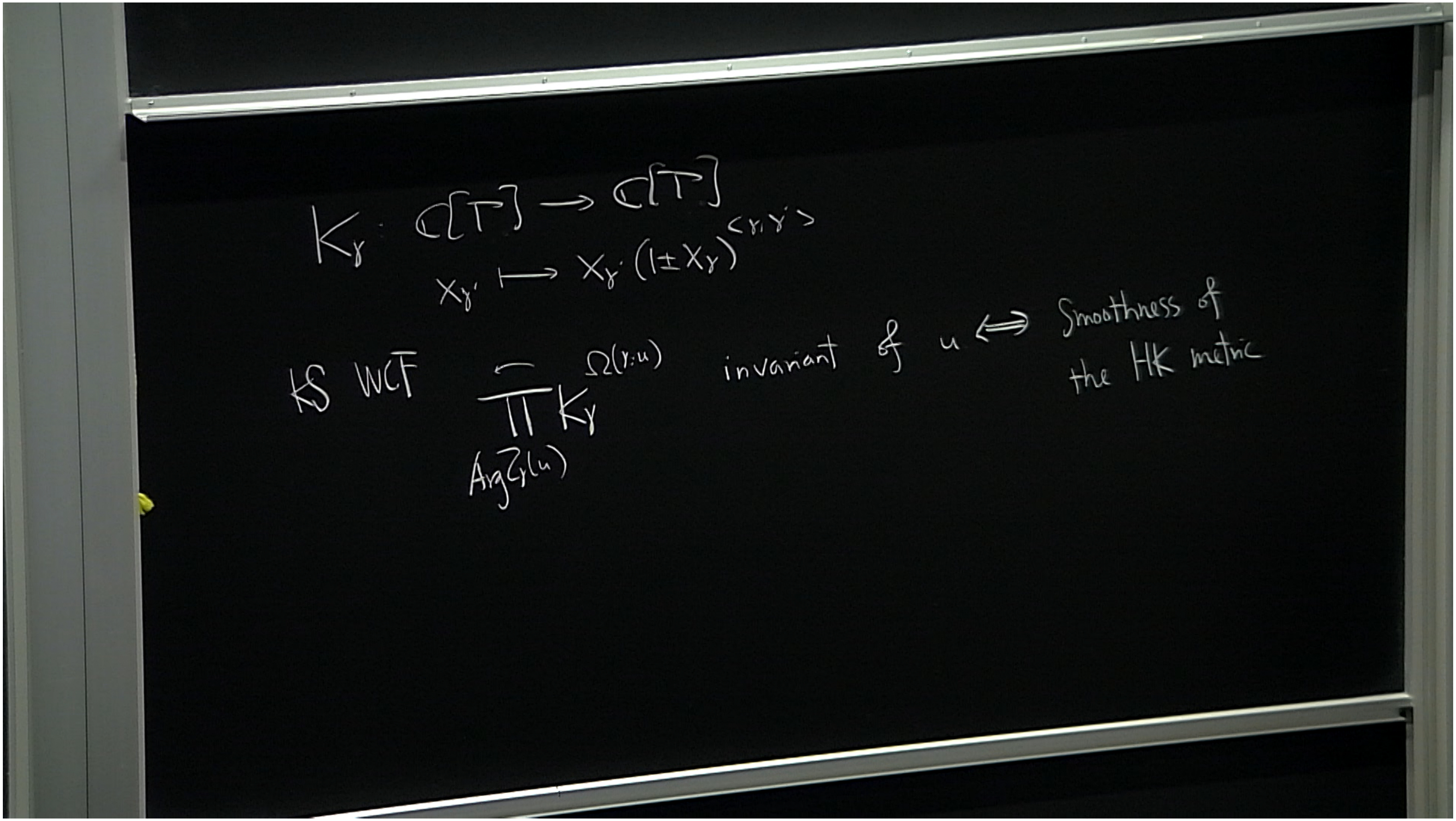
$\Omega(\gamma; u)$ = Counting of closed geodesics on Σ_u

classified by Strebel

① Saddle connection $+1$

zero zero

② 1-parameter family -2


$K_\gamma : \mathbb{C}[\Gamma] \longrightarrow \mathbb{C}[\Gamma]$
$X_{\gamma'} \longmapsto X_{\gamma'} (1 \pm X_\gamma)^{\langle \gamma, \gamma' \rangle}$
KS WCF $\overleftarrow{\prod_{\mathrm{Arg} Z_\gamma(u)}} K_\gamma^{\Omega(\gamma;u)}$ invariant of u $\Longleftrightarrow$ Smoothness of the HK metric

Symplectic Realization of 4D Wall-Crossing Formula on HK Surfaces

$$\begin{array}{ccc} L_u \subseteq & X & \text{HK surface } (\overset{1,1}{\omega}, \overset{2,0}{\Omega}) \\ & & \text{elliptic fibration} \\ \downarrow & \downarrow & \\ u \in & B & \supseteq B_0 \text{ parametrized sm. elliptic curves} \end{array}$$

$\Gamma = \bigcup_{u \in B_0} H_2(X, L_u; \mathbb{Z})$

central charge

$Z : \Gamma \longrightarrow \mathbb{C}$

$\gamma_u \longmapsto \int_{\gamma_u} \overset{2,0}{\Omega} =: Z_\gamma(u)$

Symplectic Realization of 4D Wall-Crossing Formula on HK Surfaces

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$\downarrow \quad \downarrow$

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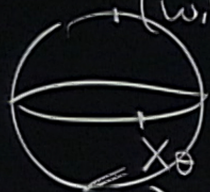
$\gamma_u \mapsto \int_{\gamma_u} \overset{2,0}{\Omega} =: Z_\gamma(u)$

$\tilde{\Omega}(\gamma; u) \overset{?}{=}$ Counting of SLAG w/ bdd on L_u

twistor sphere

$(\omega, \Omega) = X$



X_θ

$(\mathrm{Re}(e^{-i\theta}\Omega), \omega + i\,\mathrm{Im}(e^{-i\theta}\Omega))$

elliptic fibration

"count SLAG" (of phase θ)

HK rotation

SLAG fibration

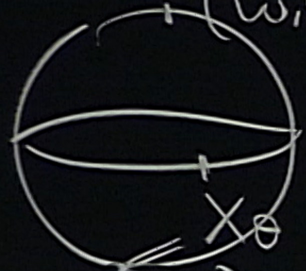
count holo discs in X_θ

Twistor sphere

elliptic fibration

$(\omega, \Omega) = X$

= "count SLAG" (of phase



HK rotation

X_θ

SLAG fibration

$(\underset{\omega_\theta}{\underset{\shortparallel}{\mathrm{Re}(e^{-i\theta}\Omega)}}, \underset{\Omega_\theta}{\underset{\shortparallel}{\omega + i\,\mathrm{Im}(e^{-i\theta}\Omega)}})$ count holo discs in X_θ.

If $\gamma \in H_2(X, L)$ represented by holo. disc

then $Z_\gamma(u) = e^{i\theta} \int_\gamma e^{-i\theta}\Omega = e^{i\theta} \int \underset{\omega_\theta}{\underset{\shortparallel}{\mathrm{Re}(e^{-i\theta}\Omega)}}$

HK rotation

X_θ SLAG fibration

$\operatorname{Im}(e^{-i\theta}\Omega)$ count holo discs in X_θ

Ω_θ

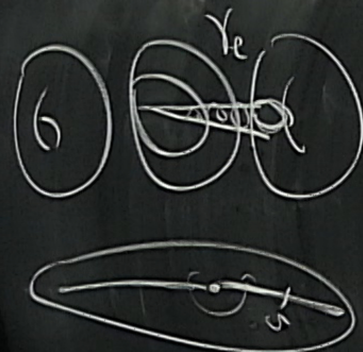
$\gamma \in H_2(X, L)$ represented by holo. disc

then $Z_\gamma(u) = e^{i\theta} \int_\gamma e^{-i\theta}\Omega = e^{i\theta} \underbrace{\int \operatorname{Re}(e^{-i\theta}\Omega)}_{\text{symplectic area}} \in e^{i\theta}\mathbb{R}_{>0}$

$\operatorname{Re}(e^{-i\theta}\Omega) = \omega_\theta$

A local model Ooguri–Vafa space

Fix θ



(Chan)

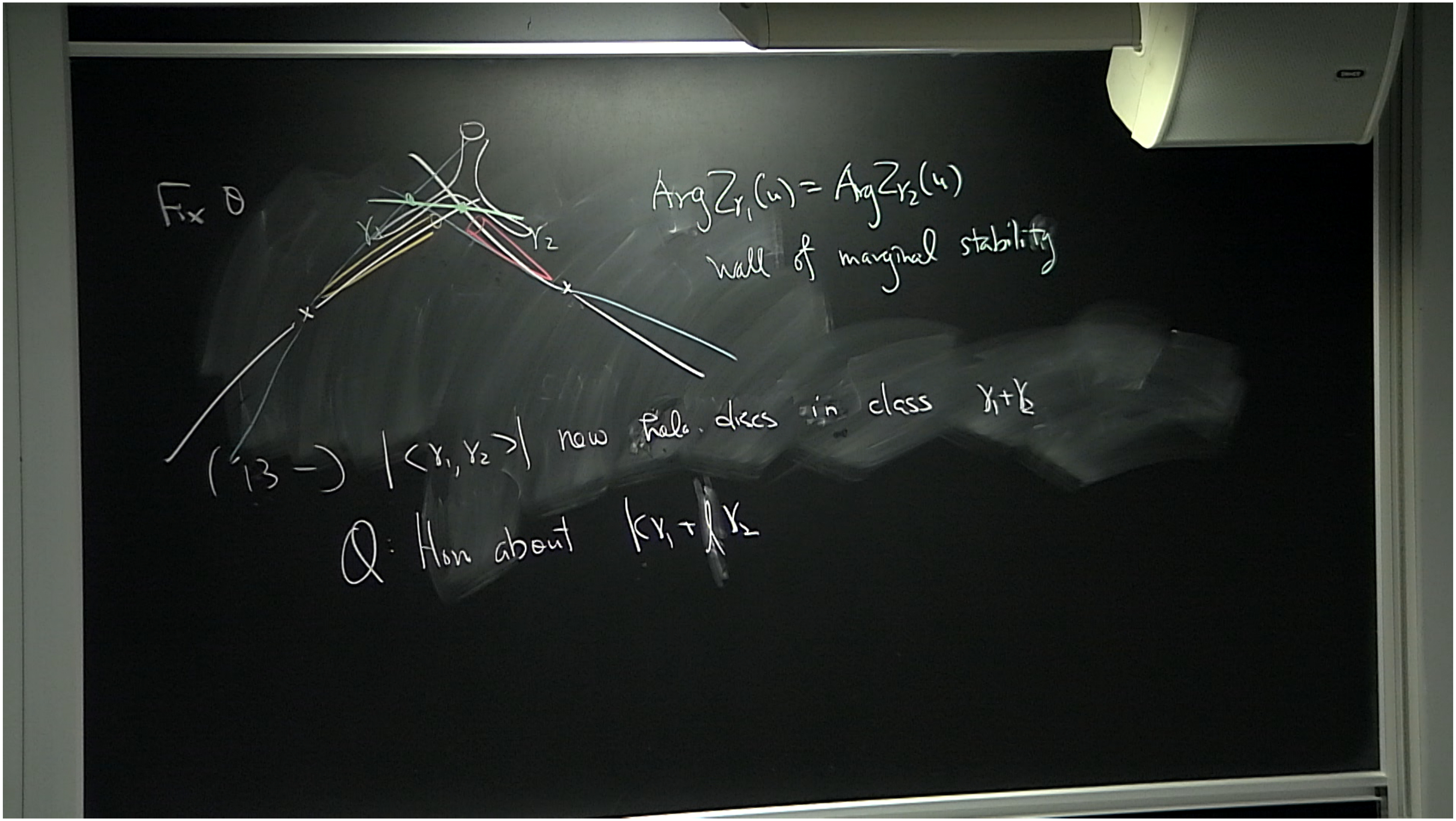
① Fix θ, only torus fibre above ——•—— (unique simple discs)
can bound holo. discs in X_θ

② If θ goes around S^1
then point is swept once

monodromy $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$

$$\tilde{\Omega}(\gamma_e, u) = 1$$

$$\tilde{\Omega}(d\gamma_e; u) = \frac{(-1)^{d-1}}{d^2}$$



Fix θ
γ_1
γ_2
$\mathrm{Arg}\, Z_{\gamma_1}(u) = \mathrm{Arg}\, Z_{\gamma_2}(u)$
wall of marginal stability
('13 –) $|\langle \gamma_1, \gamma_2 \rangle|$ new hol. discs in class $\gamma_1+\gamma_2$
Q: How about $k\gamma_1 + l\gamma_2$

Conjecture. $\tilde{\Omega}(d\gamma) = \sum_{k|d} \underbrace{c\left(\frac{d}{k}\gamma\right)}_{\pm 1}{}^{d} \frac{\Omega\left(\frac{d}{k}\gamma\right)}{k^2} \quad \in \mathbb{Z}$

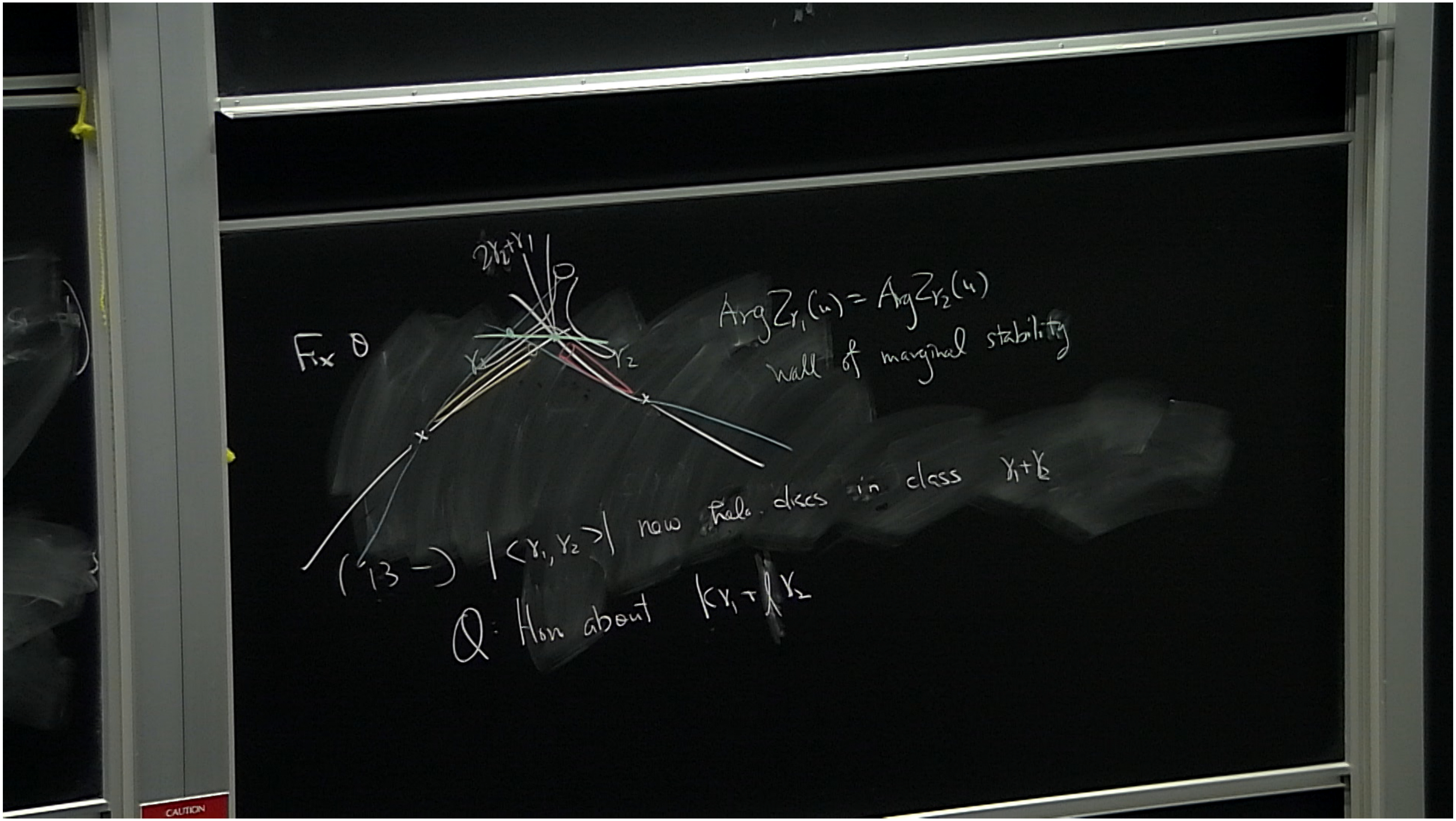
$$\forall \gamma \in H_2(X, L_u) \rightsquigarrow \mathbb{C}[[H_1(L_u;\mathbb{Z})]] \xrightarrow{\tilde{K}_\gamma(u)} \mathbb{C}[[H_1(L_u,\mathbb{Z})]]$$

$$z^{\partial\gamma'} \longmapsto z^{\partial\gamma'} f_\gamma(u)^{\langle \gamma, \gamma' \rangle}$$

$$\log f_\gamma(u) := \sum_{d \geq 1} d\, \tilde{\Omega}(d\gamma; u) \left(z^{\partial\gamma}\right)^d$$

Theorem ('16–) $\displaystyle \tilde{\prod_{\operatorname{Arg} Z_\gamma(u) \nearrow}} \tilde{K}_\gamma(u) = \tilde{\prod_{\operatorname{Arg} Z_\gamma(u')}} \tilde{K}_\gamma(u')$

We can use WCF to compute $\tilde{\Omega}(\gamma;u)$ recursively w.r.t filtration of symplectic area


$2\gamma_2+\gamma_1$
Fix θ
γ_1
γ_2
$\mathrm{Arg}\, Z_{\gamma_1}(u) = \mathrm{Arg}\, Z_{\gamma_2}(u)$
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Application in Tropical Geometry

x $\tilde{\Omega}(\gamma;u) \neq 0$ $\Rightarrow$ $\exists\, l_\gamma$ affine line through u s.t. $\mathrm{Arg}\, Z_\gamma(u) = \text{const}$ along l_γ

① $\tilde{\Omega}(\gamma;u)$ invariant the line l

gradient estimate $\Rightarrow$ γ parallel transport of Lefschetz thimble

Conjecture: $\Omega(d\gamma) = \sum_{k|d} \underbrace{\quad}_{\mathbb{Z}_1} \quad k^-$

Application in Tropical Geometry

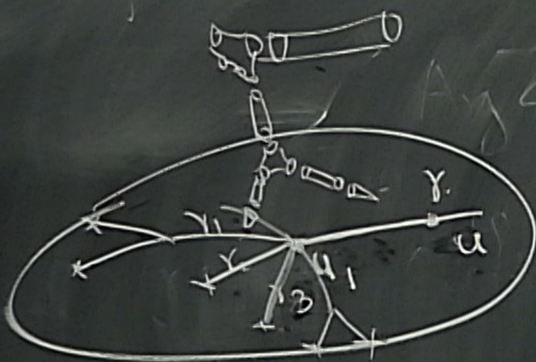
$x \quad \tilde{\Omega}(\gamma;u) \neq 0$

$\Rightarrow \exists\, l_\gamma$ affine line through u s.t. $\operatorname{Arg} Z_\gamma(u) = $ const along l_γ (attractor flow)

① $\tilde{\Omega}(\gamma;u)$ invariant the line l

gradient estimate $\Rightarrow$ γ parallel transport of Lefschetz thimble $\frac{(-1)^{d-1}}{d^2}$

② $\tilde{\Omega}(\gamma;u)$ jump at u_p



$\Rightarrow \ell$ hit a wall of marginal stability

at u_1 (split attractor flows)

$\gamma = \sum \gamma_i$ balancing condition

Theorem ('13 –) Union of ℓ_γ gives a tropical disc

Theorem ('16 –) For each diagram $\Gamma \rightsquigarrow \omega_\Gamma \in \mathbb{Q}$

$$\Omega(\gamma;u) = \sum_{\Gamma} \omega_\Gamma$$

admissible trivalent ones