Title: Toward the geometry of double affine flag varieties and double affine Grassmannians

Date: Apr 28, 2017 02:00 PM

URL: http://pirsa.org/17040077

Abstract: <p>I will discuss some results on double loop groups that point to geometric phenomena about double affine flag varieties and double affine Grassmannian. One result of this study is a definition of double affine Kazhdan-Lusztig polynomials. </p>

§1. Def'n of double loop groups & convolution algebras
(p-adic)

§2 Towards geometry: Bruhat order

§3 Double affine Schubert slices & double affine Grassmannian slices
(Birkhoff decomp & KL polynomials)

§1

$\underline{G}$ - (affine) Kac-Moody group (including the loop rotation $\underline{\mathbb{G}_m}$)

$$\underline{G} \supset \underline{B}^-, \quad \underline{G} \supset \underline{B}^+$$

$$\underline{U}^- \subset \underline{B}^- \qquad \underline{B}^+ \supset \underline{U}^+$$

$$\underline{B}^- \supset \underline{A} \subset \underline{B}^+$$

$$K = k((\pi)) \qquad k = \mathbb{F}_q$$

$$\mathcal{O} = k[[\pi]] \qquad \left(\text{also take } K = \mathbb{Q}_p\right)$$

$$G = \underline{G}(K), \quad B^+ = \underline{B}^+(K), \ldots$$

$$K = \underline{G}(\mathcal{O})$$

$$\cup$$

$$I = \{ g \in K \mid g \in B^+(k) \bmod \pi \}$$

$$K_\infty = \underline{G}(k[\pi^{-1}])$$

$$\cup$$

$$I_\infty = \{ g \in K_\infty \mid g \in \underline{B}^-(k) \bmod \pi^{-1} \}$$

$$\cup$$

$$K_\infty^1 = \{ g \in K_\infty \mid g \equiv 1 \bmod \pi^{-1} \}$$

Λ = coweight lattice of $\underline{A}$

$$\Lambda \hookrightarrow A = \underline{A}(\mathcal{K})$$

$$\mu \longmapsto \pi^{\mu}$$

W – Weyl group

Λ^{++} – dominant weights

Q – coroot lattice

$\mathcal{T}$ – Tits cone

$= W \cdot \Lambda^{++}$

1st new phenomenon: Failure of Cartan decomp [Garland]

$$G^+ = \bigsqcup_{\lambda \in \Lambda^{++}} K \pi^\lambda K \subsetneq G$$

$$\not\ni$$

$$\pi^{-\lambda}$$

§1

$\underline{G}$ - (affine) Kac-Moody group (including the loop rotation $\underline{\mathbb{G}_m}$)

$$\underline{U}^- \subset \underline{B}^- \subset \underline{G} \supset \underline{B}^+ \supset \underline{U}^+$$

$$\underline{B}^- \supset \underline{A} \subset \underline{B}^+$$

$F = K = k((\pi))$ $\quad k = \mathbb{F}_q$

$\mathcal{O} = k[[\pi]]$ (also take $K = \mathbb{Q}_p$)

Spherical case:

$$\mathcal{H}_K = \mathbb{C}[K\backslash G^+/K]$$

Thm: [Braverman-Kazhdan, Gaussent-Rousseau]

$\mathcal{H}_K$ is an algebra & Satake isom

tensor

$$\mathcal{H}_K \cong \mathrm{Rep}\,\mathrm{Ring}(\underline{G}^\vee)$$

Iwahori case

$$\mathcal{H}_I = \mathbb{C}[I \backslash G^+ / I]$$

Thm: [Braverman-Kazhdan-Patnaik].

(1) $G^+ = \bigsqcup I \pi^\mu w I$

$\pi^\mu w \in \mathfrak{I} \ltimes W$

$\{T_x = \mathbb{1}_{IxI}\}$ is a basis of $\mathcal{H}_I$

(2) $\mathcal{H}_{\underline{I}}$ is an al ... ction)

(3) $\mathcal{H}_{\underline{I}}$ has a Bernstein presentation (essentially a DAHA)

(4) Compute Satake transform (correction factor)

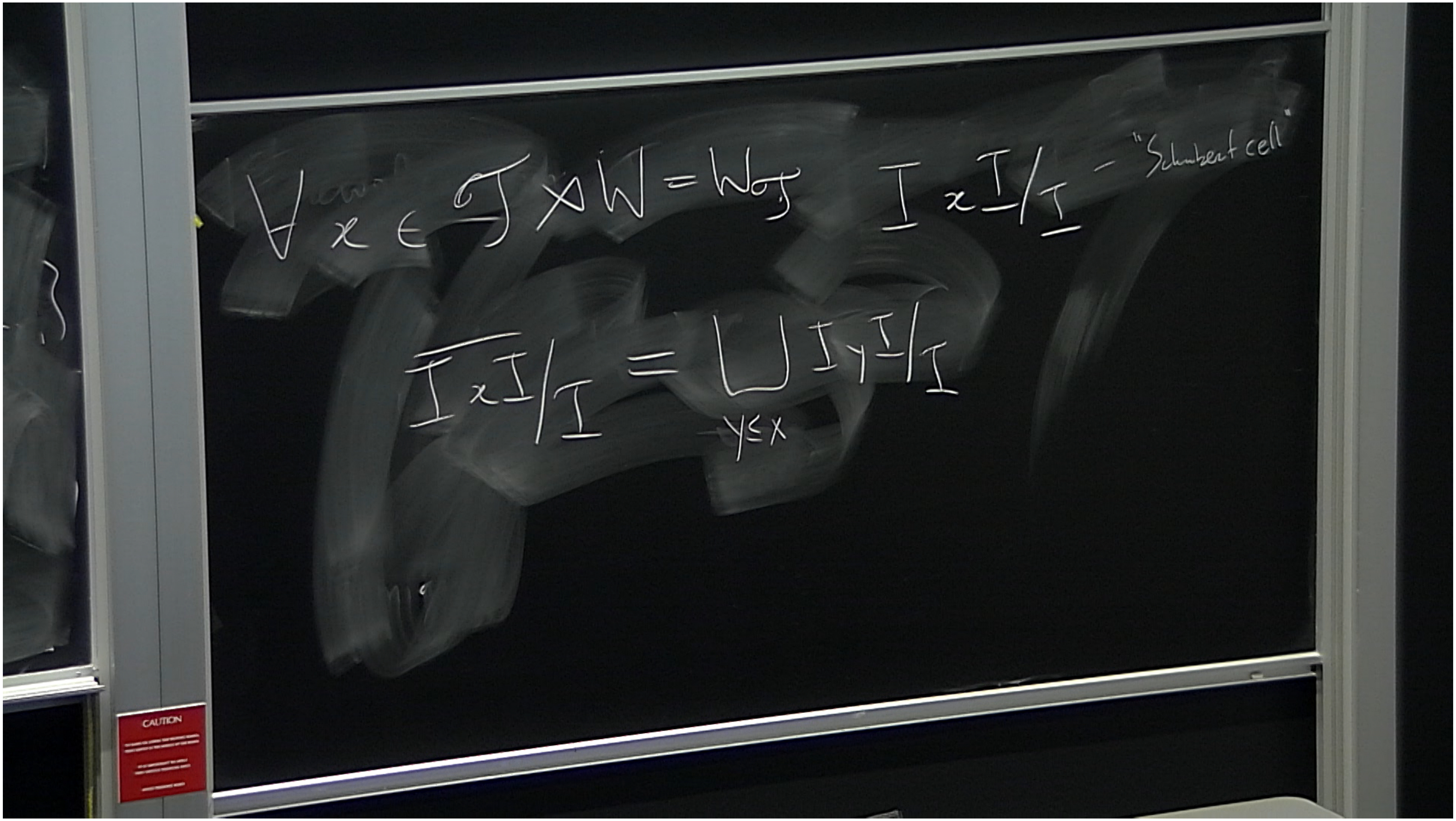
Thm: [M, Gaussent + Rousseau + Bardy-Panse]

The structure coeffs of $\{T_x\}$ are given by evaluating some universal polynomials at $q = \#k$.

§2:

G^+/I — k-points of "double affine flag variety"

G^+/K — " — Grassmannian


∀ x ∈ T, xW = W_T
I x I/I – "Schubert cell"
I x I/I = ⊔_{y≤x} I y I/I

Defn [BKP]

$\beta \in \underline{\Phi}_{real}$ — real roots of $\underline{G}$

$\{\beta + n\pi \mid \beta \in \underline{\Phi}_{real},\ n \in \mathbb{Z}\}$ = double affine real roots.

$\beta + n\pi > 0 \iff n > 0$ or $n = 0$ & $\beta > 0$

$\beta \in \Phi^+_{real}$

$n \in \mathbb{Z}$

$|\beta + n\pi|$

$\|$

$\beta[n]$

$\rightsquigarrow s_{\beta[n]} = \pi^{n\beta^\vee} s_\beta \in Q \rtimes W$

$x \in Q \rtimes W \qquad x s_{\beta[n]} > x \iff x(\beta[n]) > 0$

"double affine Bruhat preorder"

M-Theory is wrong-ish

Thm {M}

$<$ is a partial order

$\exists\ l_\varepsilon : W_\sigma \longrightarrow \underline{\mathbb{Z} \oplus \mathbb{Z}_\varepsilon}$ ordered lex

l_ε is strictly compatible w/ $<$

$\ell : W_J \xrightarrow{\ell_\varepsilon} \mathbb{Z} \oplus \mathbb{Z}\varepsilon \xrightarrow[\varepsilon=1]{} \mathbb{Z}$

Thm: {M-Orr}

ℓ is strictly compatible w/ $<$

$x < y$

Thm: {M-Orr}

In Affine ADE, ℓ classifies covers.

$\ell(x) =$ "vdim $I x I / I$" $\simeq \mathrm{Inv}(x^{-1})$

However: $\ell(\pi^{\Lambda_0}) \neq \ell(\pi^{\Lambda_0 + \delta})$

§3

Conjecture

open piece in a transverse slice to

$$\overline{IyI}/I \subseteq \overline{IxI}/I = U$$

$$K_\infty = \underline{G}(k[\pi^{-1}])$$

$$I_\infty = \{ g \in K_\infty \mid g \in \underline{B}^-(k) \bmod \pi^{-1} \}$$

$$(I_\infty y I \cap I x I)/I$$

$$I_x I / I = \bigcup$$

$$I_\infty = \left\{ g \in K_\infty \;\middle|\; \begin{array}{l} g \in \underline{B}^-(k) \\ \bmod \pi^{-1} \end{array} \right\}$$

$$(I_\infty y I \cap I x I)/I \neq \phi \iff y < x$$

$$\dim (I_\infty y I \cap I x I)/I = \ell(x) - \ell(y)$$

$$\#($$

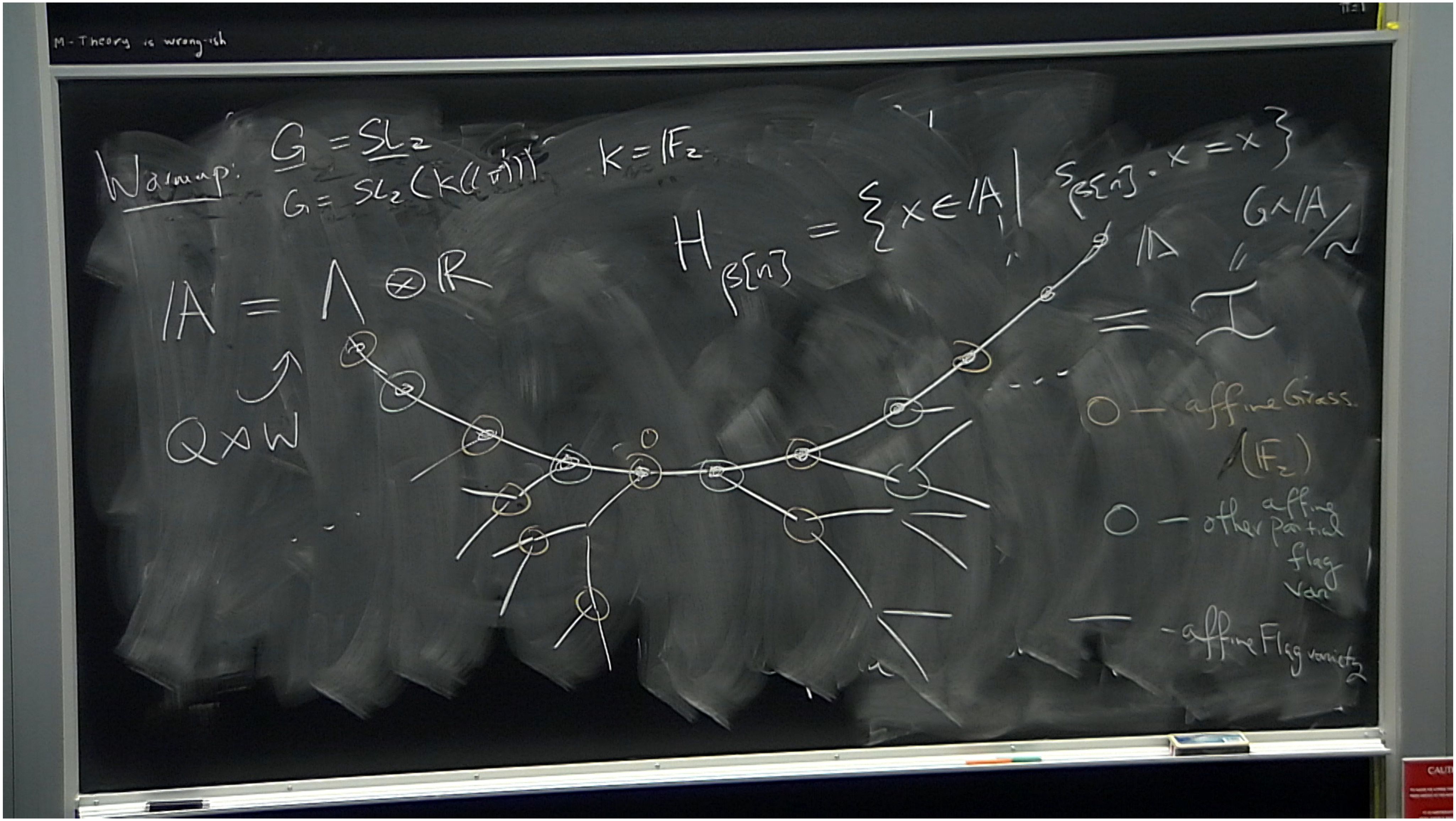
$$R_{y,x}(q) = \#(I_\infty y I \cap I x I)/I$$

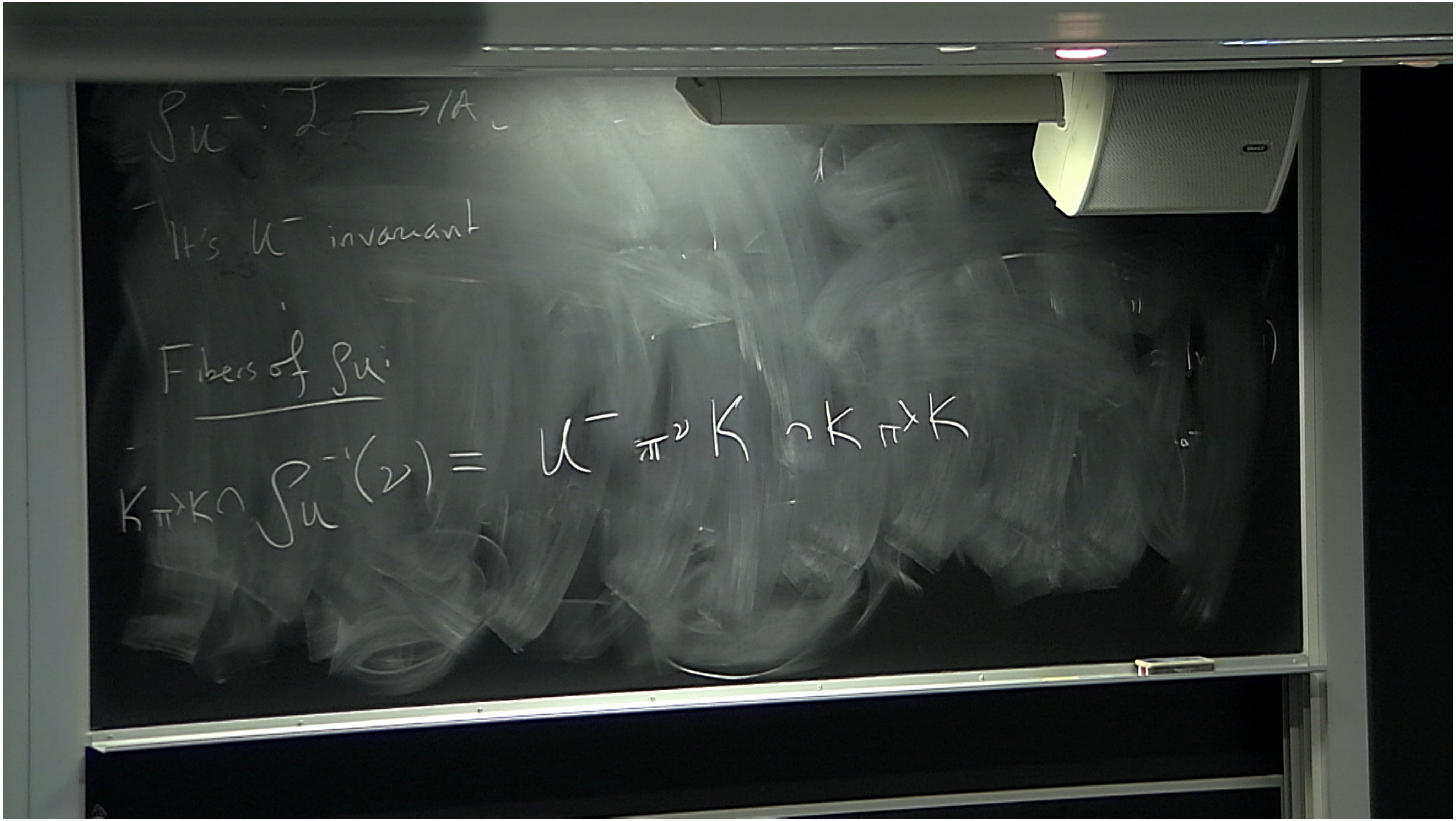
KL R-polynomial

$$P_{y,x} - q^{\ell(x)-\ell(y)}\,\overline{P_{y,x}} = \sum_{y<z\leq x} R_{y,z} P_{z,x}$$

- **Masures** (a.k.a. 'hovels') [Gaussent-Rousseau]

"double affine building"


M-Theory is wrong-ish
Warmup:
G = SL2
G = SL2(k((t)))
k = F2
H β[n] = {x ∈ A | β[n] · x = x}
A = Λ ⊗ R
Q ⋊ W
affine Grass.
other affine partial flag var
affine Flag variety


$\rho_{U^-} : \mathcal{L} \longrightarrow \mathbb{A}$
It's U^- invariant
Fibers of ρ_U:
$K_{\pi^\lambda}K \cap \rho_U^{-1}(\nu) = U^-_{\pi^\nu}K \cap K_{\pi^\lambda}K$

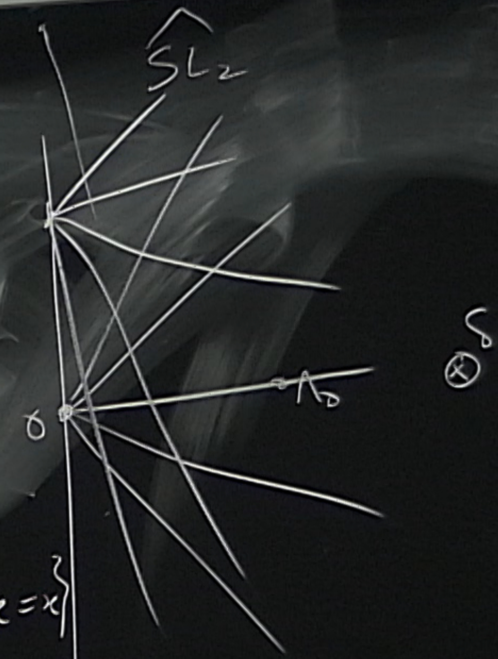
Measures

$G(k((\pi)))$

$\mathbb{A}$ $\{H_{\beta\geq[n]}\}$ — not locally finite

Gluing procedure:

For each $x \in \mathbb{A}$ $W_x = \{\pi^\mu w \in Q \rtimes W \mid \pi^\mu w . x = x\}$



$$P_x = U^-_x U^+_x A_{\mathcal{O}} W_x \qquad e_\beta(f) \quad val(f) \geq \langle x, \beta \rangle$$

Masures (a.k.a. 'hovels') [Gaussent-Rousseau]

"double affine building"

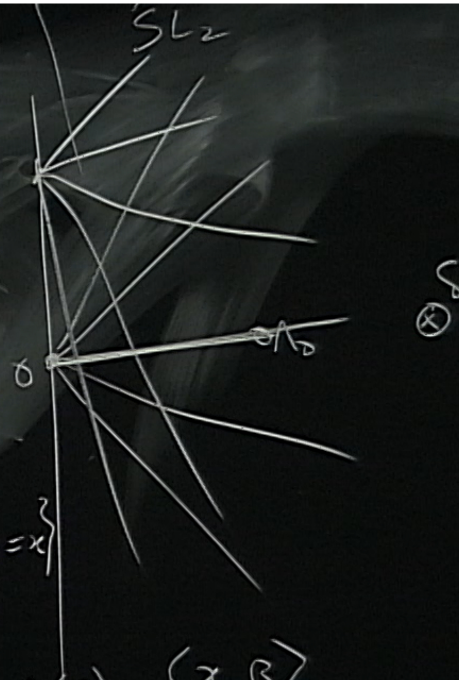
$$\mathcal{I} = G \times \mathbb{A} / \sim$$

$$(P_x, x) = (1, x)$$

$$(\pi^{\lambda^\vee} w, x) = (1, \pi^{\lambda^\vee} w \cdot x)$$

Measures $\underline{G}(k((\pi)))$

$\mathbb{A}$ $\{H_{\beta\{n\}}\}$ — not locally finite



Gluing procedure:

For each $x \in \mathbb{A}$ $\quad W_x = \{\pi^\mu w \in Q \rtimes W \mid \pi^\mu w . x = x\}$

$$P_x = U^-_x U^+_x A_\theta W_x \qquad e_\beta(f) \quad val(f) \geq \langle x, \beta \rangle$$

from 0 to $\lambda \in \Lambda^{++}$

$\varphi \in K \cdot [0,\lambda] \longleftarrow$ the set of segments in $\mathcal{I}$ starting at 0 & of shape λ

$\downarrow \qquad \cong$

$\varphi(1) \qquad K\pi^{\lambda}K/K$

$[0,\lambda]$ = straight line path
$[0,1] \longrightarrow \mathbb{A}$
from 0 to $\lambda \in \Lambda^{++}$

$\varphi \in K \cdot [0,\lambda] \leftarrow$ the set of segments in $\mathcal{I}$
starting at 0 & of shape λ

$\downarrow \qquad \cong$

$\varphi(1) \quad K\pi^{\lambda}K/K$

$\{G\text{-}R\}$ (

<u>Define</u> $\mathcal{S}_{U^-}$: $\mathbb{T} \longrightarrow \mathbb{A}$

$\mathcal{S}_{U^-}(K.[0,\lambda]) \Longleftarrow$ paths in $\mathbb{A}$

$\cup$

τ

" "Hecke paths of shape λ"

$= \mathcal{H}^{\lambda}_{U^-}$

$\tau \in \mathcal{H}^{\lambda}_{U^-}$ $\quad \tau(1) \overset{?}{=} \nu$

$\mathcal{H}^{\lambda}_{U^-,\nu} = \{\tau \in \mathcal{H}^{\lambda}_{U^-} \mid \tau(1) = \nu\}$

Thm: [Gaussent–Rousseau]

(i) $u^- \mathcal{H}^\lambda_\nu$ is finite.

(ii) $\{ \rho_{u^-}(\tau) \cap K_\cdot [a,\lambda] \}$ is finite (explicit formula for its cardinality)

$\tau \in \mathcal{H}^\lambda_\nu$

(iii) τ w/ largest dim $= \{$Littelman paths$\}$

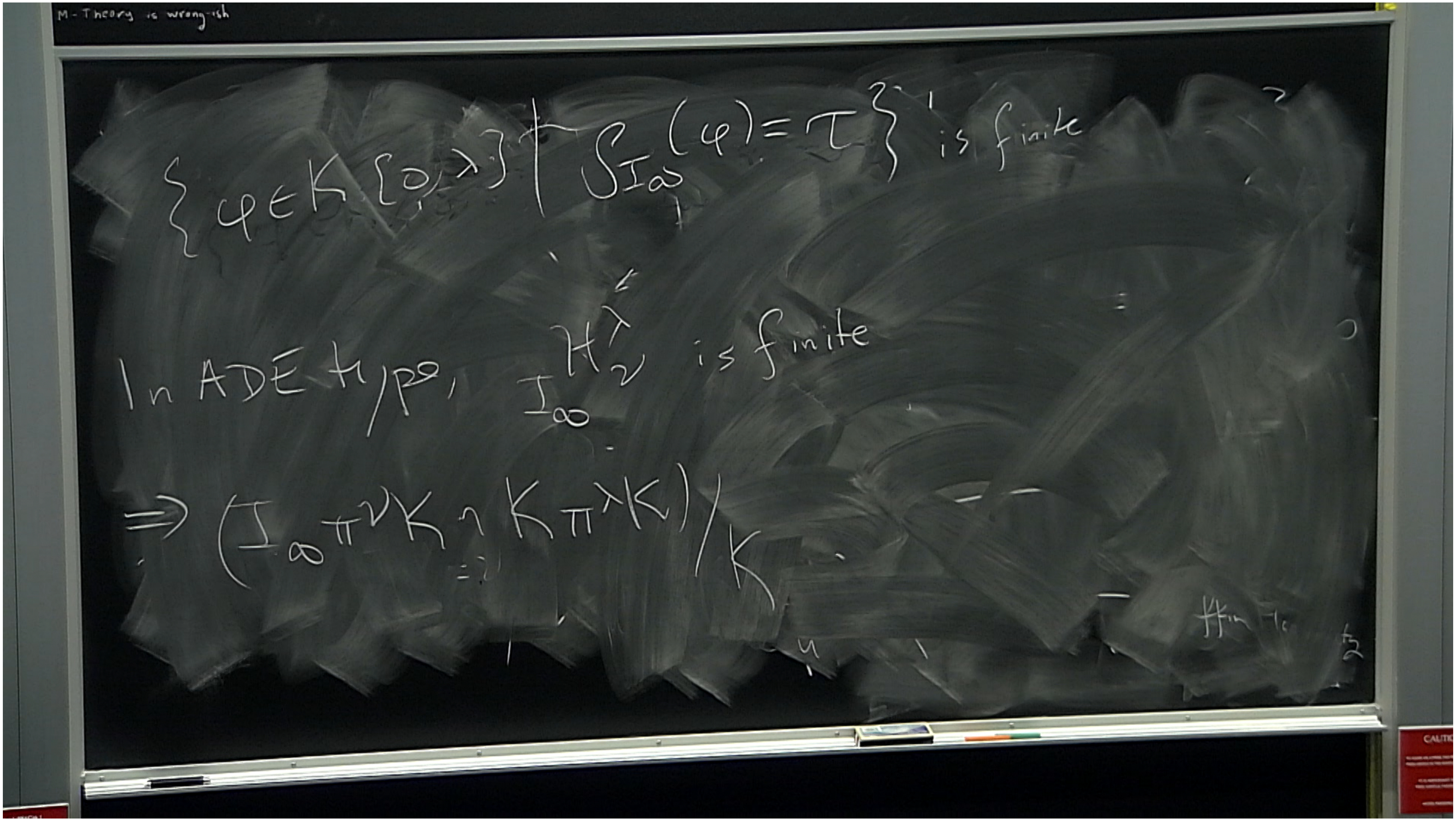
Retraction along I_∞

Let $\mathcal{I}^+ = G^+ \cdot \sigma_{\mathbb{R}}$

$\mathbb{R}$

$\cap|$

$/\mathbb{A}$


M - Theory is wrong-ish
$\{\varphi \in K[0,\lambda] \mid \int_{I_\infty}(\varphi) = \tau\}$ is finite
In ADE type, $I_\infty \mathcal{H}^{\lambda}_{\nu}$ is finite
$\Rightarrow (I_\infty \pi^\nu K \cap K \pi^\lambda K)/K$

$$\prod_{k=0}^{N}\Bigg(\underbrace{\big((x_{k-1}^{-1} I_\infty\, x_{k-1}) \cap P_{t_k\lambda}\big)\, x_k\, P_{[t_k,\, t_k+\varepsilon)\lambda}}_{\text{traslate of a Schubert cell}} \cap \underbrace{P_{(t_k-\varepsilon,\, t_k]\lambda}\, P_{[t_k,\, t_k+\varepsilon)\lambda}}_{\text{big cell}}\Bigg) \Big/ P_{[t_k,\, t_k+\varepsilon)\lambda}$$

$$\cap$$

$$P_{t_k\lambda} \big/ P_{[t_k,\, t_k+\varepsilon)\lambda}$$

partial flag variety of a KM group $/K$

$$\pi^{\lambda} \geq \underset{W_{t_0\lambda}}{\pi^{\lambda} x_0} > \underset{W_{t_1\lambda}}{\pi^{\lambda} x_1} > \cdots > \underset{W_{t_N\lambda}}{\pi^{\lambda} x_N} = \pi^{\nu} w$$

for some $w \in W$

$$\ell(\pi^{\lambda}) > \ell(\pi^{\nu} w)$$

Formula:

$R_{y,x}$

Also: Birkhoff decomp

$$G^+ \subseteq K_\infty \Lambda^{++} K$$

$$\left(\overline{K\pi^\lambda K} \cap I_\infty \pi^\nu K\right)/K$$

$$\|$$

$$\bigsqcup_{\nu \le \mu \le \lambda} \left(K\pi^\mu K \cap I_\infty \pi^\nu K\right)/K \times \text{Extra}$$