Title: Percolation transition vs. erasure thresholds for surface codes on graphs

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Abstract: <p>For a family of finite rate stabilizer codes, one can define two distinct error correction thresholds: the usual "block" threshold for the entire code, and the single-qubit threshold, where we only care about the stability of a single encoded qubit corresponding to a randomly chosen conjugate pair of logical X and Z operators. Our main result is that in the case of erasures, for hyperbolic surface codes related to a {p,q} tiling of the hyperbolic plane, it is the latter threshold that coincides exactly with the infinite-graph edge percolation transition. I will also discuss likely generalizations to more general codes and other error models. This is joint work with Nicolas Delfosse.</p>

Percolation vs. erasure thresholds for hyperbolic surface codes

Leonid Pryadko
UC, Riverside

April 26, 2017

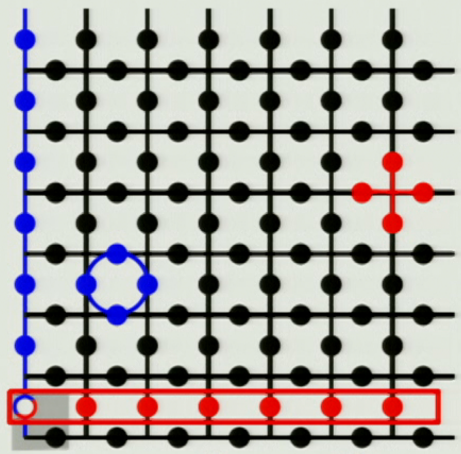
- Introduction: From quantum codes to spin models
- Percolation on infinite graphs
- Inequalities for erasure and homological thresholds
- Conjectured extensions to other error models

Alexey Kovalev (UNL)
Kathleen Hamilton (ORNL)
N. Delfosse (UCR+IQIM)

Surface codes

Family of codes invented by Alexey Kitaev (orig: *toric* codes)

Stabilizer generators: plaquette $A_\square = ZZZZ$ and vertex $B_+ = XXXX$ operators



toric code [[98, 2, 7]]

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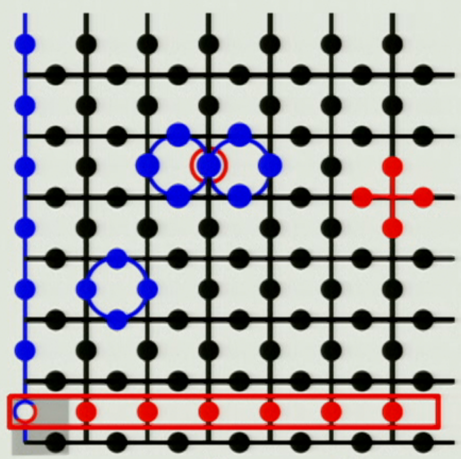
Code: common $+1$ eigenspace of all A and B operators

Detectable errors: open Z chains along the original lattice or open X chains on the dual lattice

Measured syndrome: end points



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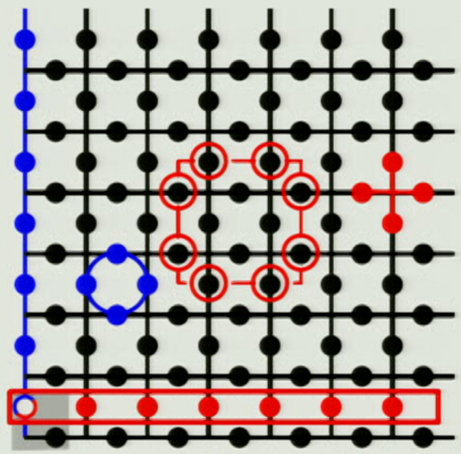


toric code [[98, 2, 7]]

Measured syndrome: end points

Undetectable error: closed loops

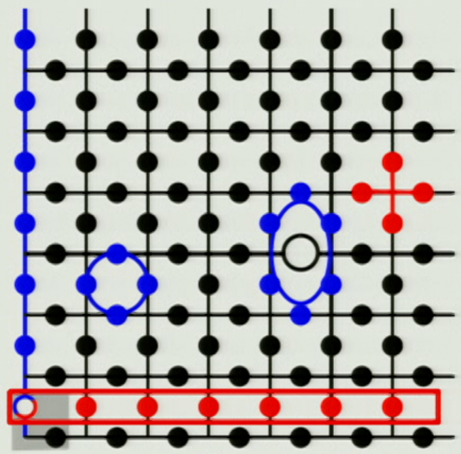
Trivial undetectable error: topologically trivial loops

Bad undetectable error: topologically non-trivial loop $\Rightarrow$ Code distance $d = L \propto \sqrt{n}$.

$[[n = 2L^2, k = 2, d = L]]$

Stability to erasures

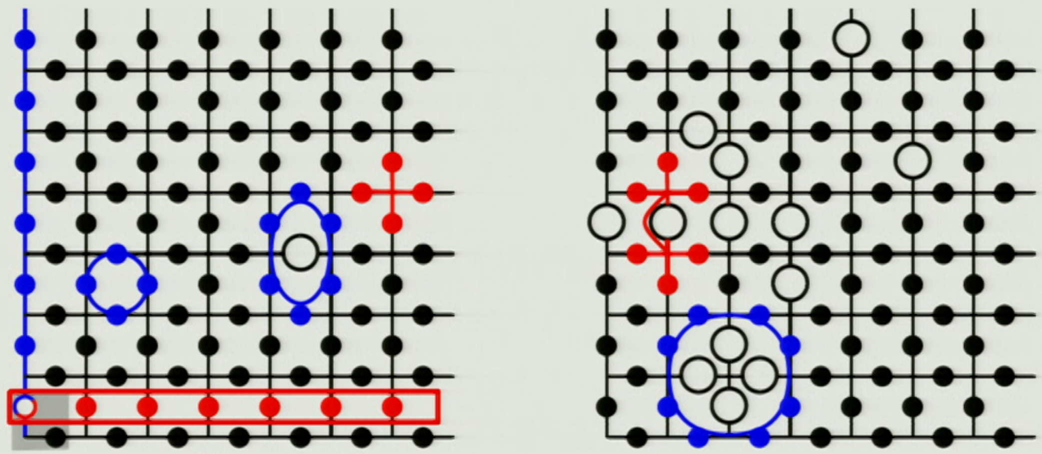
- Erasures: Unknown errors in a known set of qubits
- Modify stabilizer generators and logical operators
- Distance of thus punctured code may decrease



toric code [[98, 2, 7]]

Stability to erasures

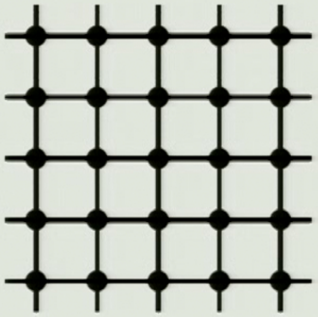
- Erasures: Unknown errors in a known set of qubits
- Modify stabilizer generators and logical operators
- Distance of thus punctured code may decrease
- Encoded qubits preserved as long as no homology is covered
- Clear similarity to percolation



toric code [[98, 2, 7]]

Edge percolation on a graph

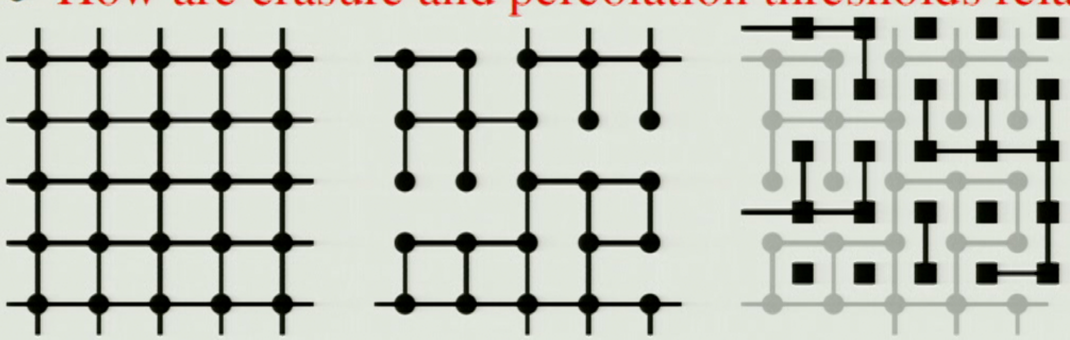
- Each edge is independently labeled "open" with probability p and "closed" with probability $1 - p$.
- Drop closed edges.
- What is the connectivity of the resulting graph?



G

Edge percolation on a graph

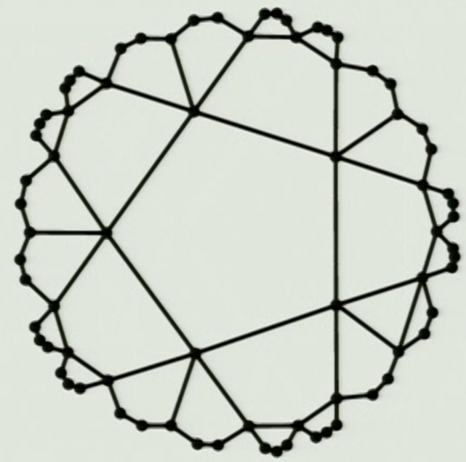
- Each edge is independently labeled "open" with probability p and "closed" with probability $1 - p$.
- Drop closed edges.
- What is the connectivity of the resulting graph?
- Percolation threshold p_c on an infinite graph: smallest p so that a vertex v has probability $\theta_v(p) > 0$ of connecting to infinity
- Not quite the same as covering homologies!
- How are erasure and percolation thresholds related?



G $\quad$ G' $\quad$ Dual to G'

More complicated example

- How about hyperbolic surface codes?
- Codes from $\{5,5\}$ hyperbolic tilings: $[[30,8,3]]$, $[[40,10,4]]$, $[[80,18,5]]$, $[[150,32,6]]$, $[[900,182,8]]$, ...
  - Finite rate family
  - Highly symmetric
  - Logarithmic distance



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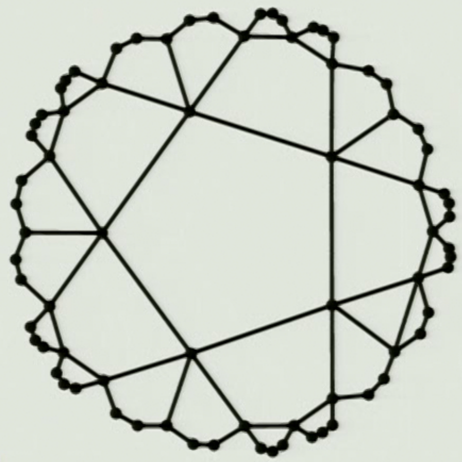


- Expander graphs
  - Perimeter scales as area, $B \geq i_E V$
  - Two percolation transitions
  - p_c: infinite clusters formed for $p > p_c$
  - p_u: unique infinite cluster for $p > p_u$
  - Self-dual graph: $p_c + p_u = 1$

Average cluster size χ_v diverges at p_c

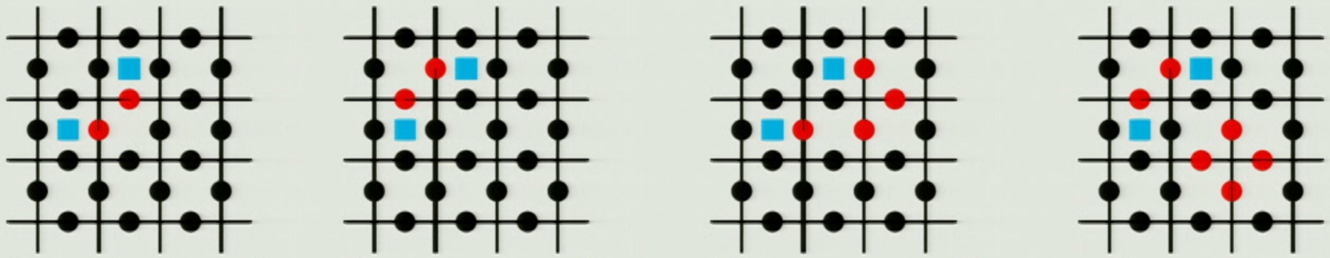
Connectivity τ_{uv} decays exponentially below p_u

$\chi_v = \sum_u \tau_{vu}$

Maximum likelihood decoding

Syndrome-based decoding: find likely error, given the syndrome

Can only be done up to degeneracy class



$p^2(1-p)^{n-2}$ $\quad$ $p^2(1-p)^{n-2}$ $\quad$ $p^4(1-p)^{n-4}$ $\quad$ $p^6(1-p)^{n-6}$

Trivial sector probabilities: P_0

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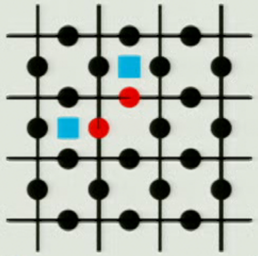
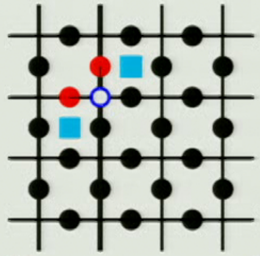
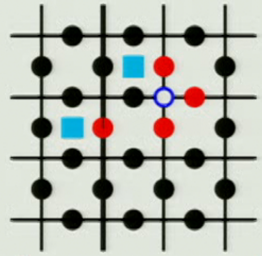
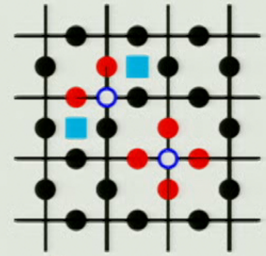
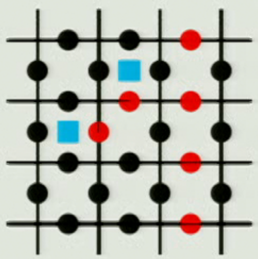
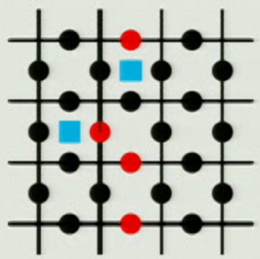
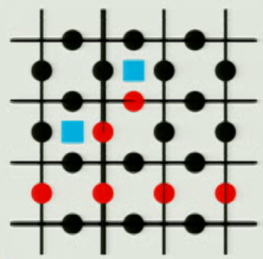
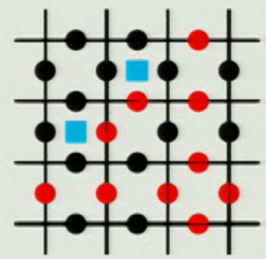





$p^2(1-p)^{n-2}$ $p^2(1-p)^{n-2}$ $p^4(1-p)^{n-4}$ $p^6(1-p)^{n-6}$

Trivial sector probabilities: P_0 Random Bond Ising model map

Homologically non-trivial sectors (domain walls):

P_1 P_1 P_2 P_3

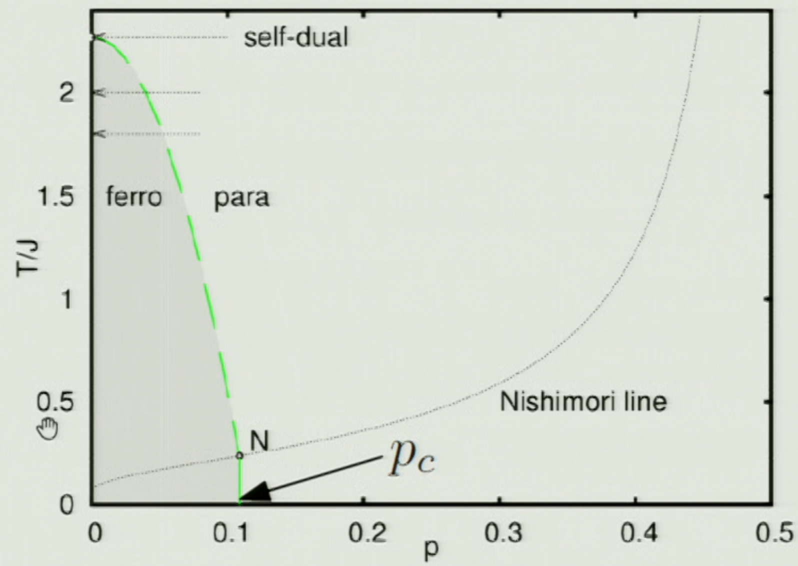





ML dec success: $\mathbb{P} = \dfrac{P_0}{P_0 + P_1 + P_2 + P_3}$ [Dennis et al, 2002]

Spin glass physics and decoding threshold

Toric code: decoding X or Z errors $\leftrightarrow$ random bond Ising model on the Nishimori line, $e^{-2K} \equiv e^{-2J/T} = p/(1-p)$.

Points away from the Nishimori line: suboptimal decoder (assumed p is wrong)

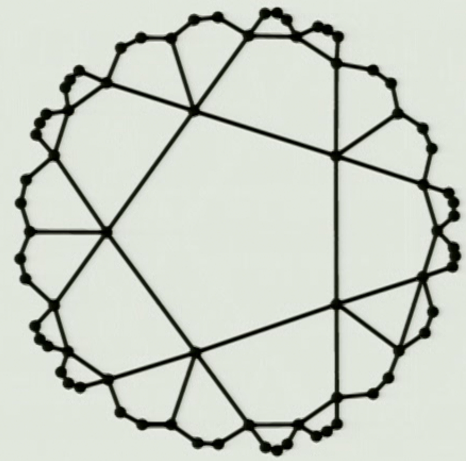


Multicritical point at $p_c \approx 0.109$ is the highest point where FM phase is possible

$$\mathbb{P}_{\text{succ}} = \frac{P_0}{P_0 + P_1 + P_2 + P_3}$$

Here, decodable phase is the FM phase with non-zero domain wall tension: homologically non-trivial sectors are suppressed in thermodynamical limit

Example: hyperbolic surface codes



- Shown: (5,5) hyperbolic tiling. Planar expander graph. Not embeddable in 2D without large deformations
- Surface code on related finite graphs: weight=5 generators; rate $R = 1/5$.
- Ground state degeneracy 2^k
- ML decoding → random bond Ising

$$\mathbb{P}_{\rm succ} = \frac{P_0}{P_0 + P_1 + \ldots + P_{2^k-1}}$$

- Generally different from those defined using
  - Domain wall line tension
  - Correlation function $\tau_{ab} \equiv \langle s_a s_b \rangle$
  - Susceptibility at a: $\chi_a = \sum_b \tau_{ab}$.
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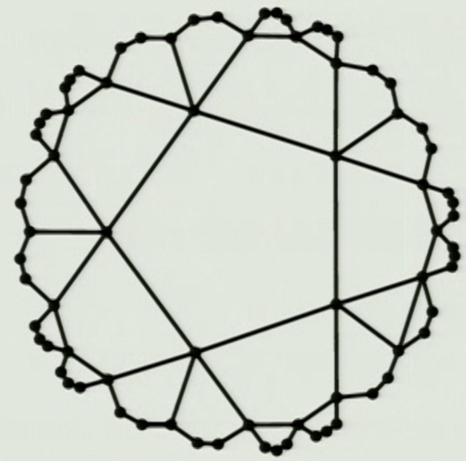

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Motivating questions

- Can we convert these into maps for threshold parameters?
  - Erasure and percolation thresholds
  - ML decoding threshold vs. the transition in RB Ising model
  - Same for qu$q$it code and random bond q-state Potts
  - Same for GKP code and charge-q compact QED...
- Is there associated critical scaling?
- What determines the critical exponents?
- Is there a meaningful notion of convergence for sequences of quantum codes?
- Is there a meaningful definition of a family of quantum codes, comparable to ensembles of random classical codes?

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- Is there a meaningful notion of convergence for sequences of quantum codes?
- Is there a meaningful definition of a family of quantum codes, comparable to ensembles of random classical codes?
- Percolation is the simplest example
- Also the transition most easy to understand
- Of current independent interest (network theory)
- Some of these questions may have already been answered

Erasure threshold for hyperbolic surface codes

- Consider a sequence of hyperbolic surface codes $[[n_t, k_t, d_t]]$ from transitive graphs H_t, $t \in \mathbb{N} \longrightarrow \{r, r\}$ regular tiling H

- Assume $d_t \geq 2t + 1$, then H_t converge to H locally, meaning that balls of radius t look the same on H and H_t.

Theorem: For surface codes defined by a sequence of homologically non-trivial graphs locally convergent to a planar infinite graph, $p_E \leq p_c$.

- Proof based on the cluster expansion: Denote $N^{(v)}_{V,E,B}$ number of edge-connected clusters with V vertices, E edges, and B "boundary" edges, which include the chosen vertex v, then

$$1 - \theta_p(v) = \sum_{V,E,B} N^{(v)}_{V,E,B} p^E (1-p)^B$$

Erasure threshold (cont'd)

- For any $p > p_E$, probability to cover a homology: $\mathbb{P} \geq \epsilon > 0$

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- Proof: Assume $p_E > p_c$, and choose p in between. For such a p, only homologically trivial clusters are covered at large t. These can be mapped $1 : 1$ to those at H. Any finite cluster on H will be encountered at large enough t, and these are all likely to appear, thus $\theta_p = 0$. Contradiction.

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- Note: p_E is not necessarily the same as p_c: $t \times L_t$ toric codes
  - Infinite graph limit = square lattice
  - Percolation threshold $p_c = 1/2$
  - For $L_t = 2^{t \ln t}$, $p_E = 0$.
  - For $L_t = m^t$, $p_E \leq 1/m$.

$$\mathbb{P}_{\text{vert}} \geq 1-(1-p^t)^{L_t} \geq 1 - e^{-p^t L_t} \to 1, \quad \forall p > 0.$$

- Same *may be* true for hyperbolic codes: $p_E < p_c$ (log distance)

Homological threshold

- Consider a finite-rate family of surface codes associated with a sequence (G_t) of finite locally planar transitive graphs weakly convergent to G (rate $R = \lim_t k_t/n_t$).
- Necessarily, these are some hyperbolic tilings.

Let p_H^f be the threshold to cover fraction f of homologies: for $p > p_H^f$, fraction $\eta_p > f$ is covered, while for $p < p_H^f$, $\eta_p < f$.

Define $p_H^0 = \lim_{f\downarrow 0} p_H^f, \quad p_H^1 = \lim_{f\uparrow 1} p_H^f$

Theorem: $p_c \leq p_H^0 \leq p_H^1 \leq p_u$ [Version of Delfosse, Zemor 2010]

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Theorem: $p_c \le p_H^0 \le p_H^1 \le p_u$ [Version of Delfosse, Zemor 2010]

- Notice: p_c and p_u are mutually dual, and so are p_H^0 and p_H^1. On self-dual graphs, $p_c + p_u = 1$, $p_H^0 + p_H^1 = 1$.

Proof: For graphs with maximum vertex degree Δ, $\theta_p \ge fR\dfrac{2}{\Delta}$.

Homological threshold (cont'd)

- Notice that p_H^0 can be viewed as a logical bit erasure threshold: for $p > p_H^0$, there is a finite probability that a randomly chosen homology is covered
- Calculate the fraction of covered homologies η_p: [Delfosse & Zemor]
  - Finite graph with V_t vertices, E_t edges, k_t homologies.
  - Take a subgraph G_t' induced by $E \leq E_t$ open edges
  - G_t' has $K = V_t - E + C$ connected components, $C = F + H$ cycles, F faces, and H homologies ← [Euler]
  - For the dual graph, $\widetilde{K} = 1 + F$.
  - This gives $H = K - \widetilde{K} + E - V_t + 1$
  - Rescale and take the expectation

$$\frac{k_t}{V_t}\eta_p = \kappa_{p,0}(G_t) - \kappa_{1-p,0}(\widetilde{G}_t) + \frac{pE_t - 1}{V_t} - 1$$

$$\kappa_{p,h}(G) \equiv \sum \frac{1}{V} N_{V,E,B}\, e^{hV} p^E (1-p)^B$$ (vertex-average

Homological threshold (cont'd)

Take the infinite graph limit:

$$\frac{2R}{d}\eta_p = \kappa_{p,0}(G) - \kappa_{1-p,0}(\widetilde{G}) + \frac{2}{d}p - 1$$

- $\kappa_{p,h}(G)$ is an analog of the free energy density:

$$\kappa_{p,h}(G) \equiv \sum\nolimits_V e^{hV}\frac{1}{V}\underbrace{\sum\nolimits_{E,B} N_{V,E,B}\, p^E(1-p)^B}$$

$$1-\theta_p = \partial_h \kappa_{p,h}\Big|_{h=0}$$

$$\chi_p = \partial_h^2 \kappa_{p,h}\Big|_{h=0}$$

$\mathbb{P}_V(p)$: Probability of a cluster of size V

- Transitive graph: $\mathbb{P}_V(p)$ decay exponentially for any $p < p_c$ $\Rightarrow \kappa_{p,0}$ analytic
- Scaling form: $\mathbb{P}_s(p) = s^{-\tau} f\left((p-p_c)s^{1/\sigma}\right)$ [conjectured]
- $\kappa_{p,0}$ has non-analyticity at p_c [$\partial_p^4 \kappa_{p,0}$ diverges]

Analyticity bound

Any cluster on a graph with Cheeger constant i_E has $B \geq i_E V$.

With handshaking lemma, this also gives: $E \leq \frac{\Delta - i_E}{2} V$.

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With handshaking lemma, this also gives: $E \leq \frac{\Delta - i_E}{2} V$.

For some p_0, let $p > p_0$, then $1 - p < 1 - p_0$, and we have

$$\begin{aligned}
\mathbb{P}_v(p) &= \sum_{B,E} N_{V,E,B}\, p^E (1-p)^B \\
&\leq \sum_{B,E} N_{V,E,B}\, p_0^E (1-p_0)^B \left(\frac{p}{p_0}\right)^{(\Delta - i_E)V/2} \left(\frac{1-p}{1-p_0}\right)^{i_E V} \\
&= \mathbb{P}_v(p_0) \left(\frac{p^{(\Delta - i_E)/2}(1-p)^{i_E}}{p_0^{(\Delta - i_E)/2}(1-p_0)^{i_E}}\right)^V \leq \left(\frac{p^{(\Delta - i_E)/2}(1-p)^{i_E}}{p_0^{(\Delta - i_E)/2}(1-p_0)^{i_E}}\right)^V
\end{aligned}$$

Take $p_0 = \frac{(\Delta - i_E)/2}{(\Delta - i_E)/2 + i_E} = \frac{\Delta - i_E}{\Delta + i_E}$, the maximum of $p^{(\Delta - i_E)/2}(1-p)^{i_E}$,

$\Rightarrow$ Exponential decay of $\mathbb{P}_v(p)$ for any $p > p_0$ $\Rightarrow$ Analyticity of $\kappa_p(G)$

$\Rightarrow \kappa_{1-p}$ analytic for $p < 1 - p_0 = \frac{2 i_E}{\Delta + i_E}$. Also known: $\frac{1}{\Delta - 1} \leq p_c \leq \frac{1}{i_E + 1}$

Summary for erasure and percolation

- For self-dual hyperbolic tilings
  - $p_E \leq p_c = p_H^0 < p_H^1 = p_u$
  - Expect $p_E < p_c$
  - Effectively infinite dimension — MF critical exponents
  - Proof relied on analyticity of $\kappa_{p,0}(G)$ (cf. Lee-Yang zeros!)
- For codes with power-law distance based on amenable graphs ($i_E = 0$) — e.g., $\mathbb{Z}^D$:
  - $p_E = p_c = p_u$, p_H is not meaningful
  - MF exponents for $D \geq 6$ (upper critical dimension)
- These results require local convergence of $G_t \xrightarrow{t} G$.

Benjamini-Schramm convergence

Definition: consider a sequence of finite graphs (G_t), $t \in \mathbb{N}$, with vertex degrees uniformly bounded by Δ. For each $L = 1, 2, \ldots$, let $B_t(v, L)$ be the ball of radius L in G_t, centered at its vertex v. Let us choose v randomly, and denote $\pi_t(L; H)$ the probability that $B_t(v, L)$ is isomorphic to H.

The sequence is said to be Benjamini-Schramm convergent if the sequence $\pi_t(L; H)$, $t \in \mathbb{N}$ is convergent for every L and H.

- This convergence guarantees only local properties
- Yet it is enough to construct an expansion of the limiting $\kappa_{p,h}$ to an arbitrary order.
- This is enough to fix the location of percolation transition, p_c
- In some cases, also the uniqueness, p_u, and the homological transitions, p_H^0.
- If distance scaling is also known, a bound on p_E can be made.

Extension to other iid error models

- BS convergence of Tanner graphs $\Rightarrow$ analyticity of free energy per edge, $f(T)$, at high enough temperatures

  [generalization of Borgs et al, 2012]
- This also defines a code family
- $\mathbb{P}_{\text{dec}} = \dfrac{Z_0}{Z_{\text{tot}}}$; $\mathbb{P}_{\text{dec}} = 1$ for small enough T and p
- $\dfrac{1}{n} \ln \mathbb{P}_{\text{dec}} = f_0(T) - f_{\text{tot}}(T) \stackrel{T\to\infty}{\to} -R \ln 2$

  $\tanh \widetilde{\beta} = e^{-2\beta}$
  $\beta \equiv 1/T$
- At $p = 0$ use duality: $\dfrac{1}{n} \ln \mathbb{P}_{\text{dec}} = f_0(T) - f_0(\widetilde{T})$

  Only bounded-weight terms are involved
- In general, $T_{\text{dec}} \le \widetilde{T}_c$ (dual of upper critical temperature).
- With known scaling of the distance, this allows to make a good lower bound for the decoding threshold, T_{dec}.

Conclusions

- Erasures on graph-based LDPC codes: $p_E \le p_c \le p_H^0 \le p_H^1 \le p_u$
  - Self-dual hyperbolic codes: $p_c = p_H^0$
  - Codes on quasi-transitive graphs with $d \propto n^\alpha$: $p_E = p_c$.
- Here p_H^0 is the "logical bit" erasure threshold
- Families of quantum LDPC codes can be introduced in terms of the distributional limit of the corresponding Tanner graphs
- This gives asymptotic distribution of linked clusters to any size — analog of the asymptotic weight distribution for ensembles of random classical codes, or a free energy density analytic in a finite range of parameters
- Thus defined thermodynamic properties are insensitive to local properties affecting sublinear fractions of the qubits
- For a well defined error correction threshold, additional information about the global properties of the codes is needed: e.g., distance scaling with n

Conclusions

- Erasures on graph-based LDPC codes: $p_E \le p_c \le p_H^0 \le p_H^1 \le p_u$
  - Self-dual hyperbolic codes: $p_c = p_H^0$
  - Codes on quasi-transitive graphs with $d \propto n^\alpha$: $p_E = p_c$.
- Here p_H^0 is the "logical bit" erasure threshold
- Families of quantum LDPC codes can be introduced in terms of the distributional limit of the corresponding Tanner graphs
- This gives asymptotic distribution of linked clusters to any size — analog of the asymptotic weight distribution for ensembles of random classical codes, or a free energy density analytic in a finite range of parameters
- Thus defined thermodynamic properties are insensitive to local properties affecting sublinear fractions of the qubits
- For a well defined error correction threshold, additional information about the global properties of the codes is needed: e.g., distance scaling with n