Title: Cluster duality and mirror symmetry for Grassmannians

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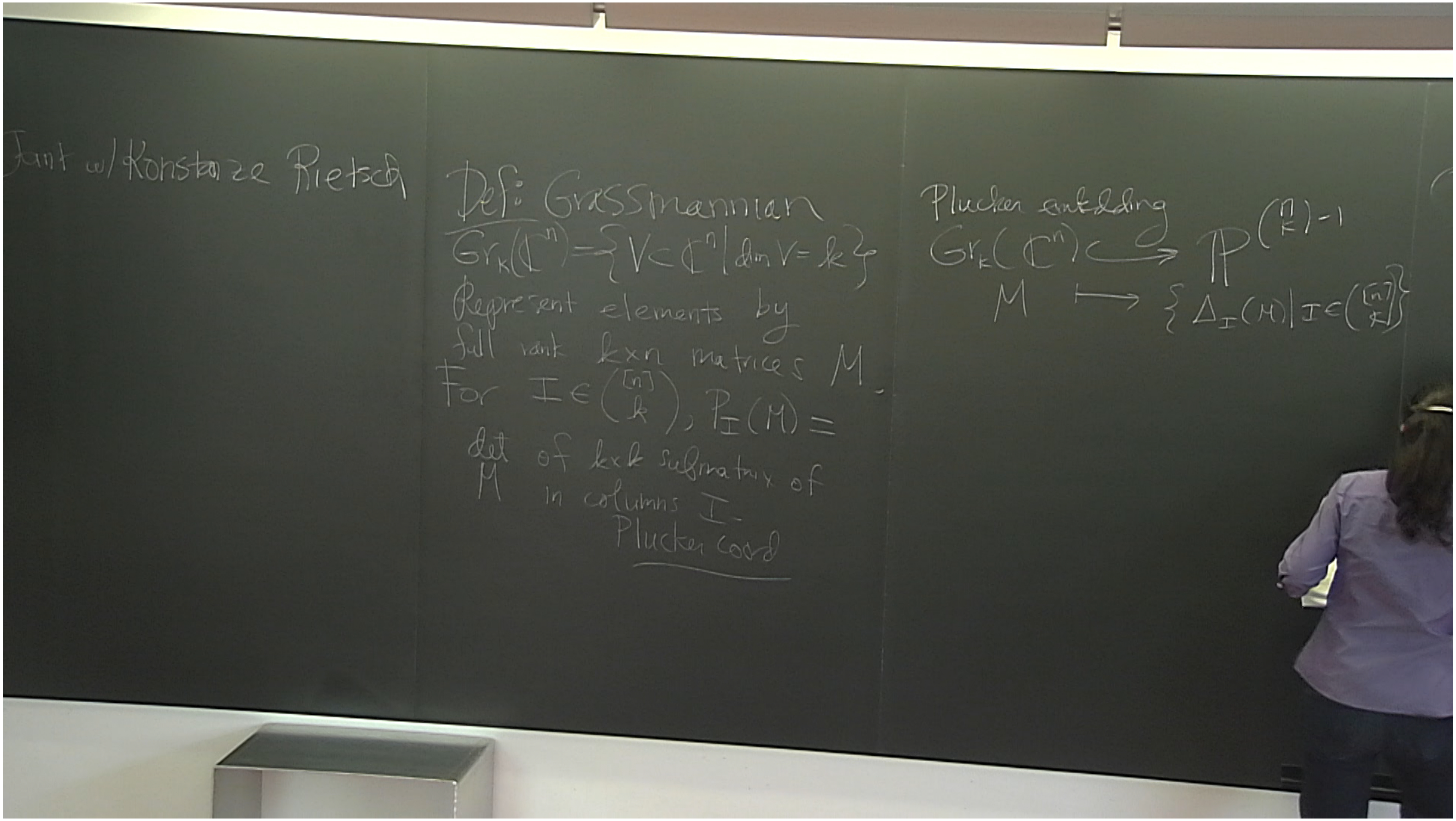
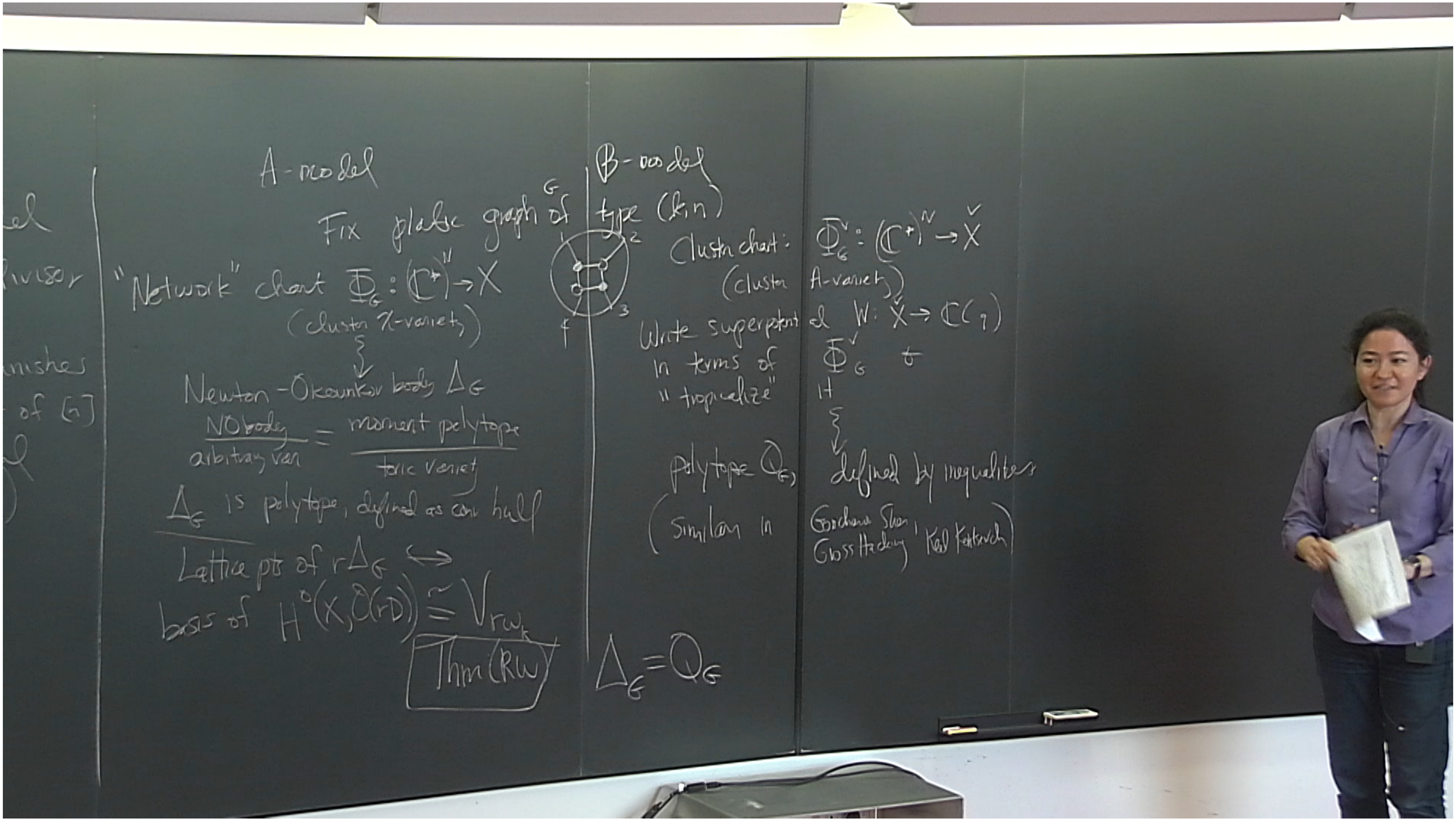
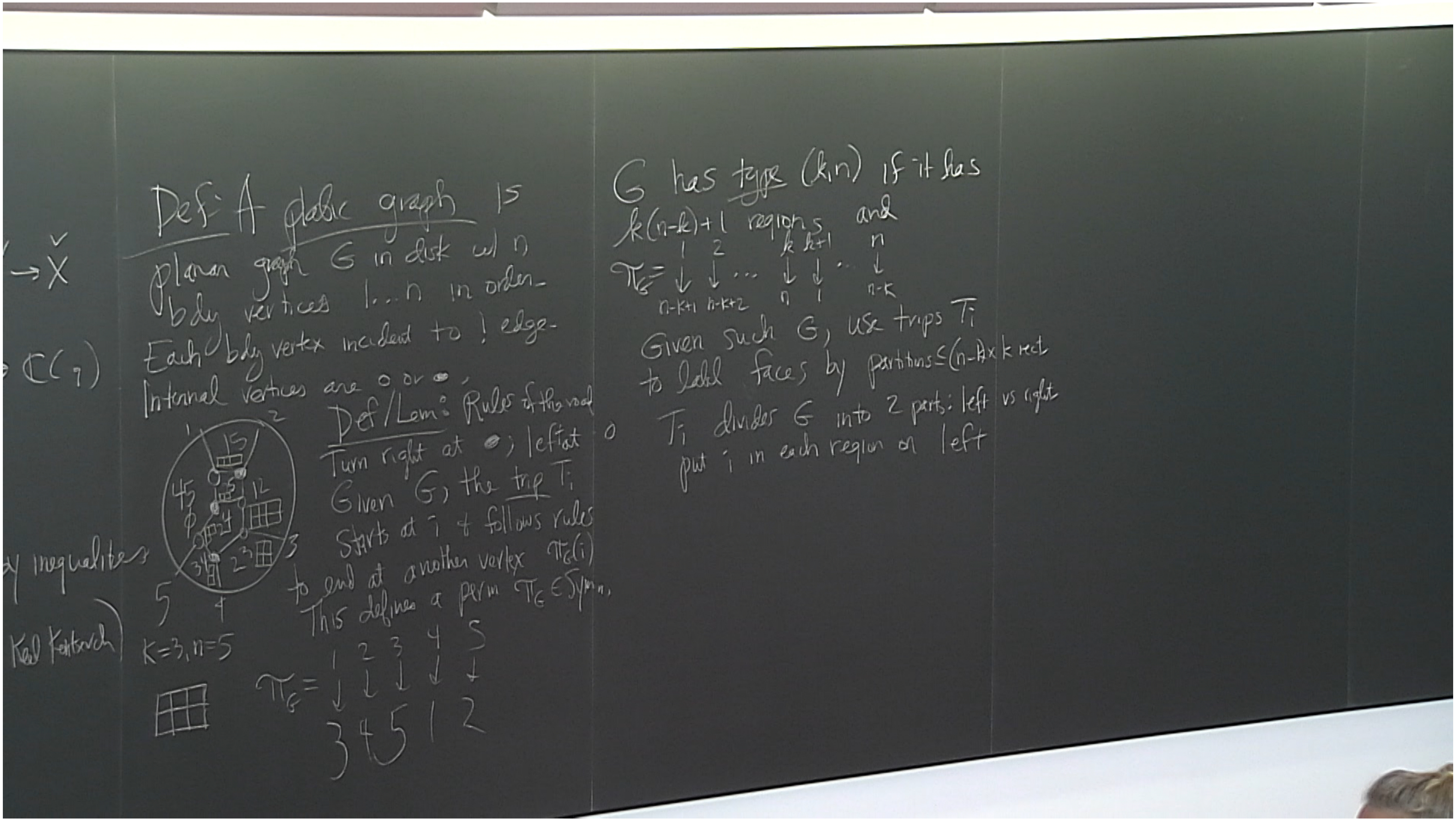
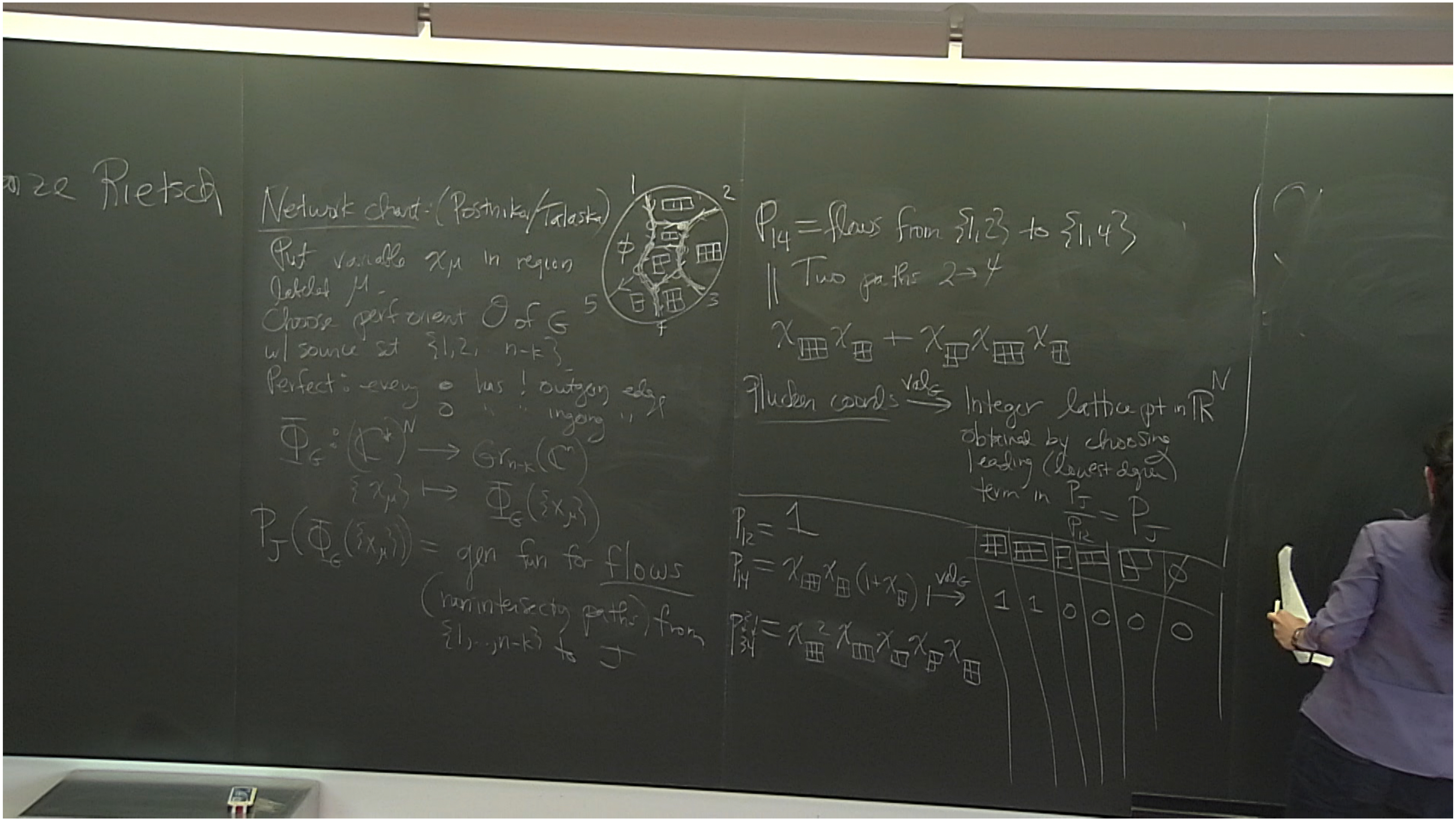
Abstract: <p>We use the cluster structure on the Grassmannian and the combinatorics of plabic graphs to exhibit a new aspect of mirror symmetry for Grassmannians in terms of polytopes. From a given plabic graph G we have two coordinate systems: we have a network chart for the A-model Grassmannian, and a cluster chart for the B-model (Landau-Ginzburg model) Grassmannian. On the A-model side, we use the network chart from G and an ample divisor D to define an associated Newton-Okounkov polytope NO_G(D). We give explicit formulas for the lattice points in NO_G(D) in terms of the combinatorics of Young diagrams. We then reinterpret NO_G(D) in terms of the superpotential and the cluster chart for the B- model Grassmannian. *This is joint work with Konstanze Rietsch. </p>

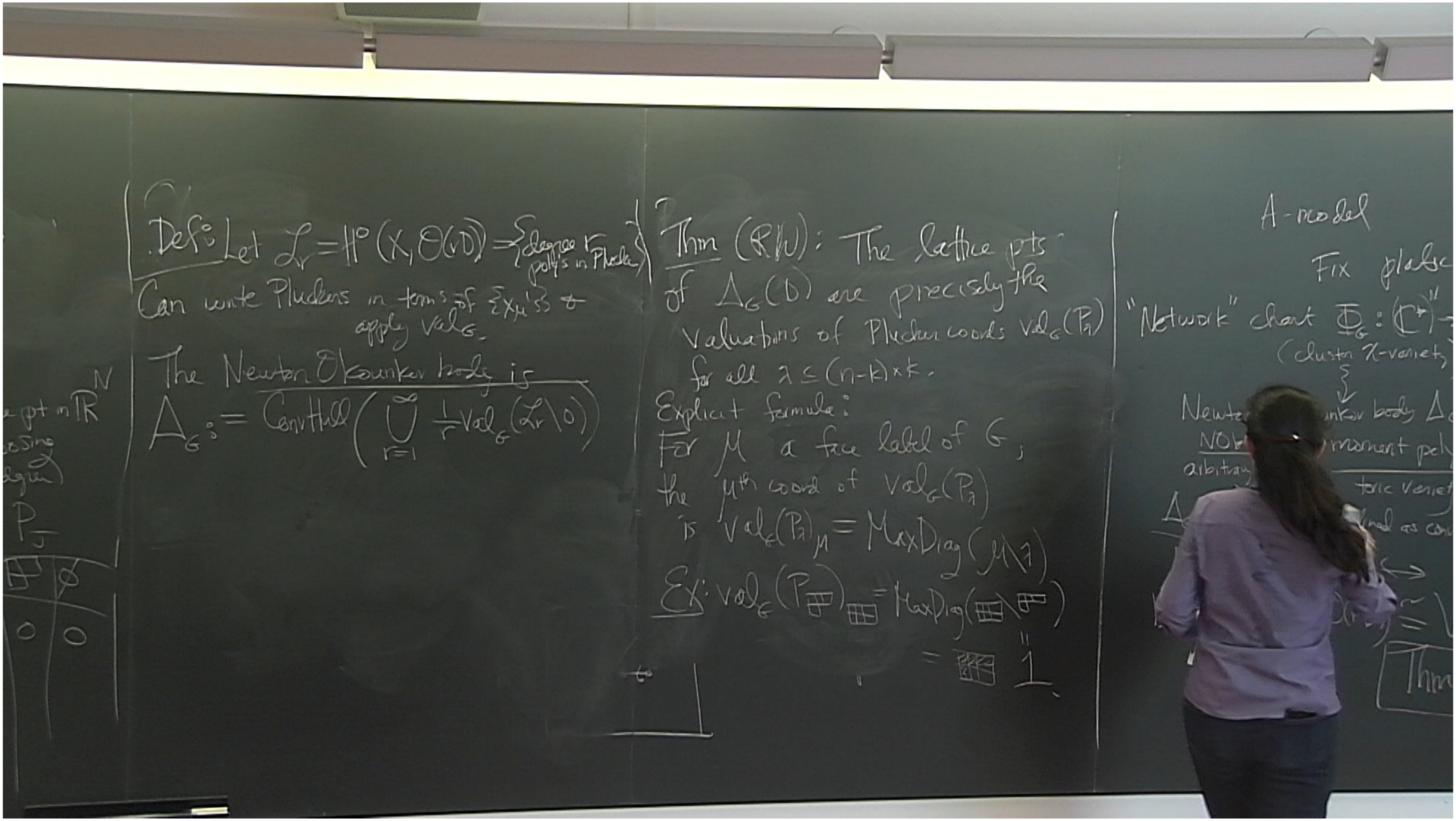

Joint w/ Konstanze Rietsch
Def: Grassmannian
$Gr_k(\mathbb{C}^n) = \{V < \mathbb{C}^n \mid \dim V = k\}$
Represent elements by
full rank k×n matrices M.
For $I \in \binom{[n]}{k}$, $P_I(M) =$
det of k×k submatrix of
M in columns I.
Plucker coord
Plucker embedding
$Gr_k(\mathbb{C}^n) \hookrightarrow \mathbb{P}^{\binom{n}{k}-1}$
$M \longmapsto \{\Delta_I(M) \mid I \in \binom{[n]}{k}\}$

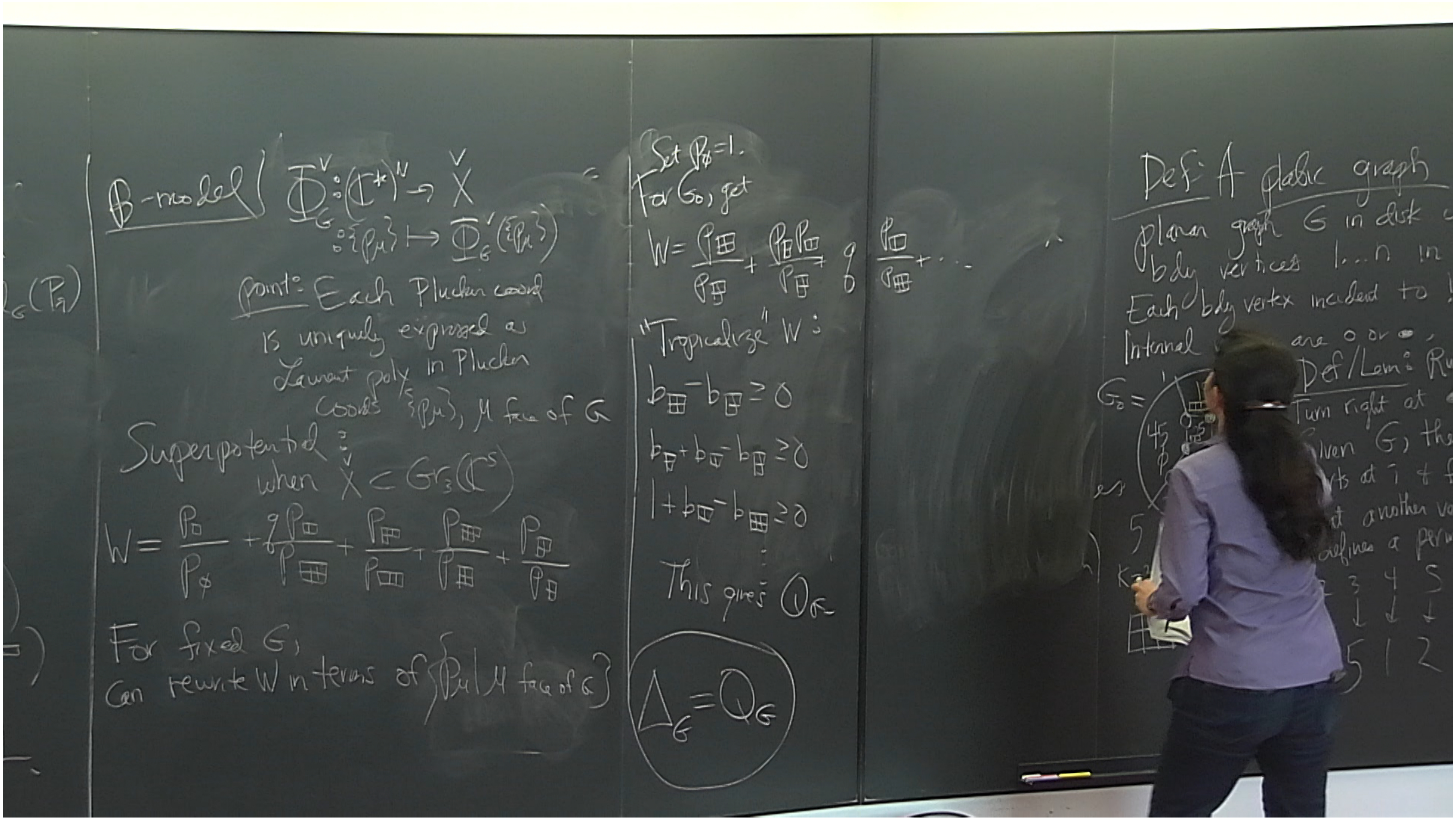

A-model
Fix plabic graph G of type (k,n)
"Network" chart Φ_G : (C*)^N → X
(cluster X-variety)
Newton-Okounkov body Δ_G
NO body / arbitrary var = moment polytope / toric variety
Δ_G is polytope, defined as conv hull
Lattice pts of rΔ_G ↔
basis of H^0(X, O(rD)) ≅ V_rω_k
Thm (RW)
B-model
Cluster chart: Φ_G^∨ : (C*)^N → X^∨
(cluster A-variety)
Write superpotential W: X^∨ → C
in terms of Φ_G^∨ &
"tropicalize"
polytope Q_G, defined by inequalities
(similar in Goncharov Shen, Gross Hacking Keel Kontsevich)
Δ_G = Q_G
divisor
vanishes
of [n]


Def: A plabic graph is
planar graph G in disk w/ n
bdy vertices 1...n in order.
Each bdy vertex incident to ! edge.
Internal vertices are ○ or ●.
Def/Lem: Rules of the road
Turn right at ●; left at ○
Given G, the trip T_i
Starts at i & follows rules
to end at another vertex π_G(i)
This defines a perm π_G ∈ Sym_n.
k=3, n=5
π_G =
1 2 3 4 5
3 4 5 1 2
G has type (k,n) if it has
k(n-k)+1 regions and
Given such G, use trips T_i
to label faces by partitions ⊆ (n-k) x k rect
T_i divides G into 2 parts: left vs right
put i in each region on left
inequalities



Network chart: (Postnikov/Talaska)
Put variable x_μ in region labeled μ.
Choose perf orient $\mathcal{O}$ of G
w/ source set $\{1,2,\ldots,n-k\}$
Perfect: every • has ! outgoing edge
○ " " ingoing "
$\Phi_G : (\mathbb{C}^*)^N \to Gr_{n-k}(\mathbb{C}^n)$
$\{x_\mu\} \mapsto \Phi_G(\{x_\mu\})$
$P_J(\Phi_G(\{x_\mu\}))$ = gen fun for flows (nonintersecting paths) from $\{1,\ldots,n-k\}$ to J
P_{14} = flows from $\{1,2\}$ to $\{1,4\}$
Two paths $2 \to 4$
Plucker coords $\xrightarrow{val_G}$ Integer lattice pt in $\mathbb{R}^N$
obtained by choosing leading (lowest degree) term in $\frac{P_J}{P_{12}} = P_J$
$P_{12} = 1$
$P_{14} = x\,x\,(1+x) \overset{val_G}{\longmapsto}$
1 1 0 0 0 0


Def: Let $\mathcal{L}_r = H^0(X, \mathcal{O}(rD))$ = {degree r polys in Plücker}
Can write Plückers in terms of {x_μ}'s & apply val_G.
The Newton Okounkov body is
$\Delta_G := \mathrm{ConvHull}\left(\bigcup_{r=1}^{\infty} \frac{1}{r} val_G(\mathcal{L}_r \setminus 0)\right)$
Thm (RW): The lattice pts of $\Delta_G(D)$ are precisely the valuations of Plücker coords $val_G(P_\lambda)$ for all $\lambda \subseteq (n-k)\times k$.
Explicit formula:
For μ a face label of G, the μ^{th} coord of $val_G(P_\lambda)$ is $val_G(P_\lambda)_\mu = \mathrm{MaxDiag}(\mu \setminus \lambda)$
Ex:
A-model
"Network" chart
(cluster X-variety)
NOBody
moment poly
arbitrary
toric variety
Thm


B-model
point: Each Plucker coord is uniquely expressed as Laurent poly in Plucker coords {p_μ}, μ face of G
Superpotential: when X ⊂ Gr_3(C^5)
For fixed G, can rewrite W in terms of {p_μ | μ face of G}
Set p_∅ = 1.
For G_0, get
"Tropicalize" W:
This gives Q_G
Δ_G = Q_G
Def: A plabic graph
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bdy vertices 1...n
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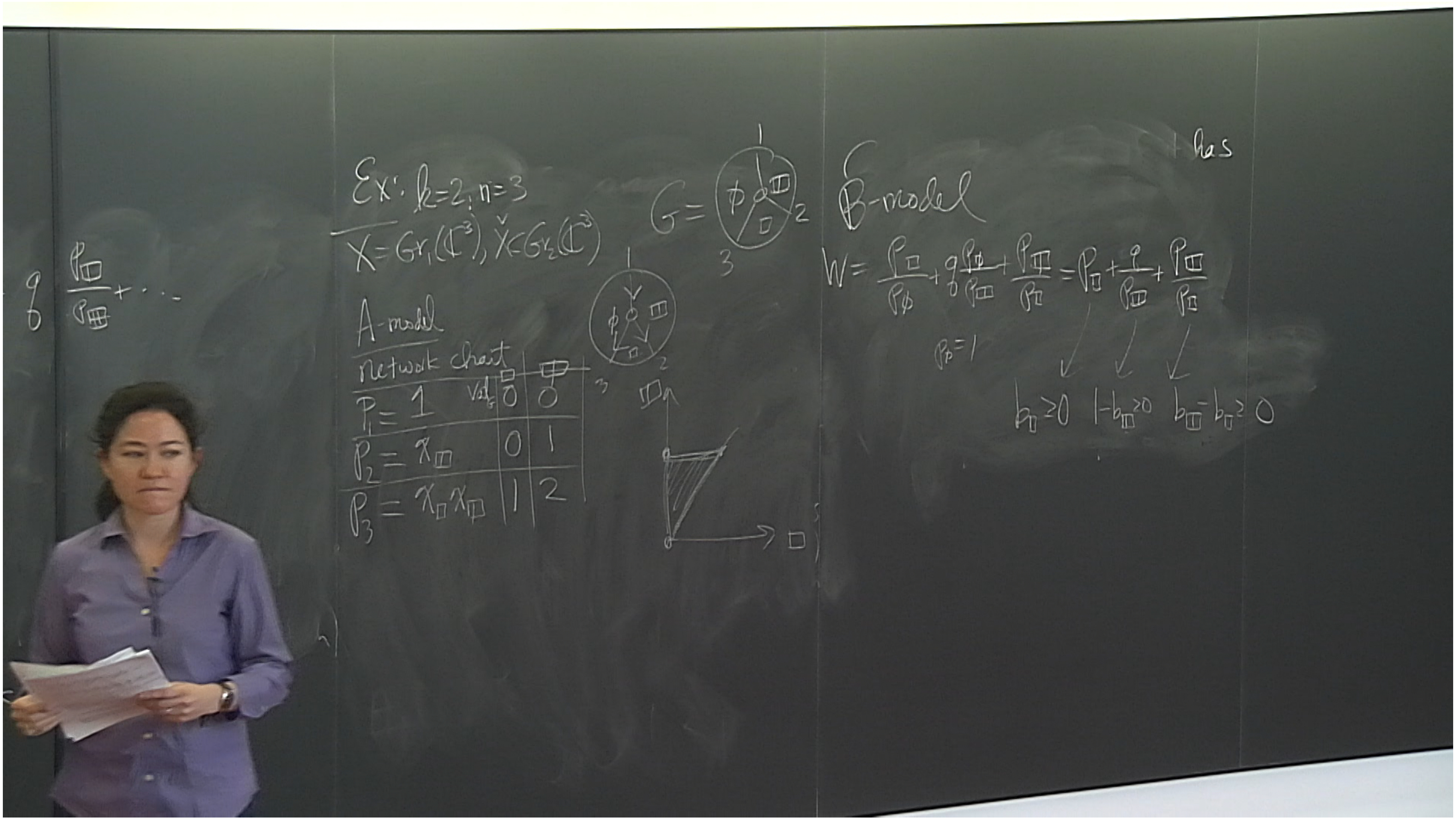
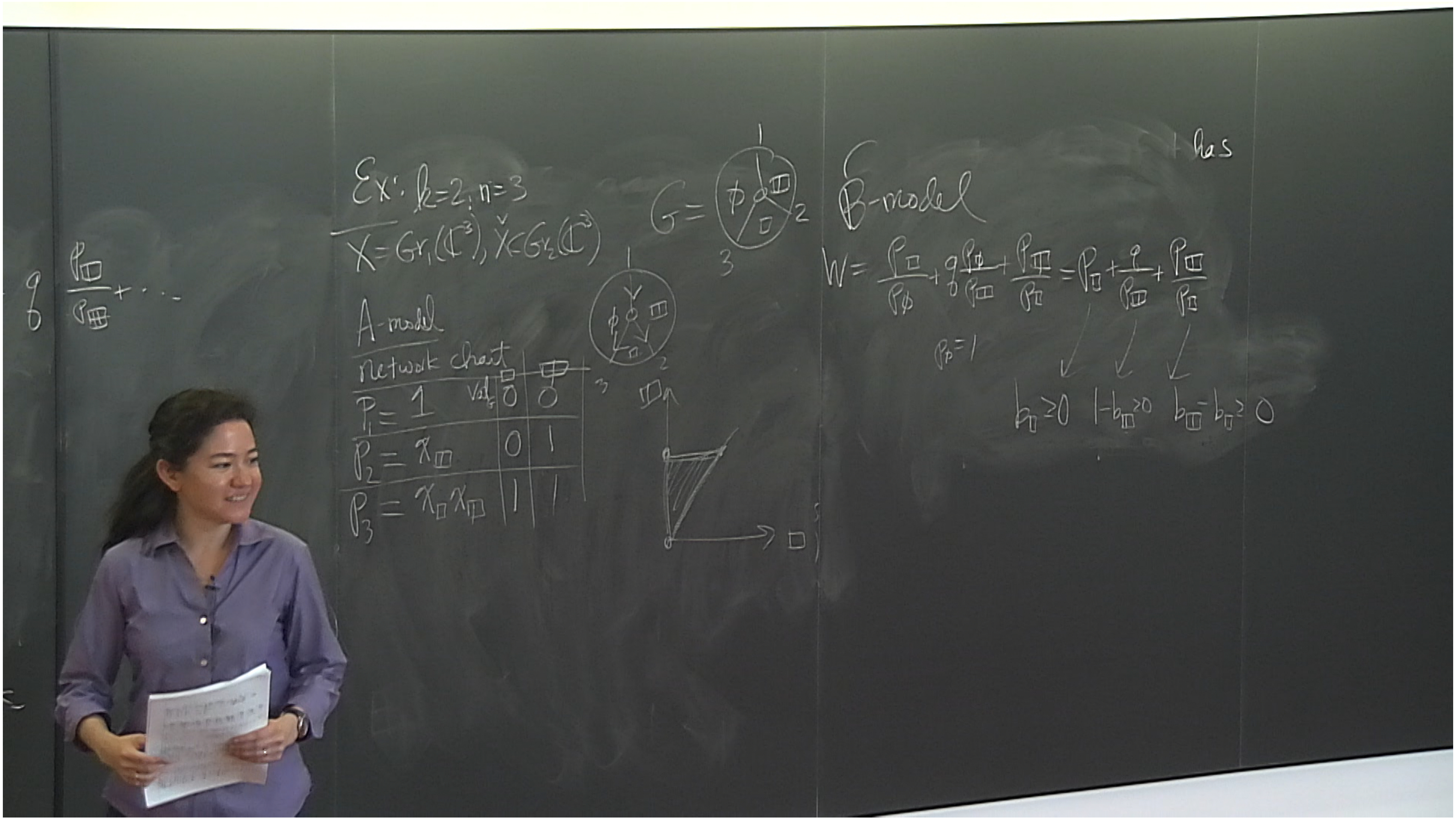
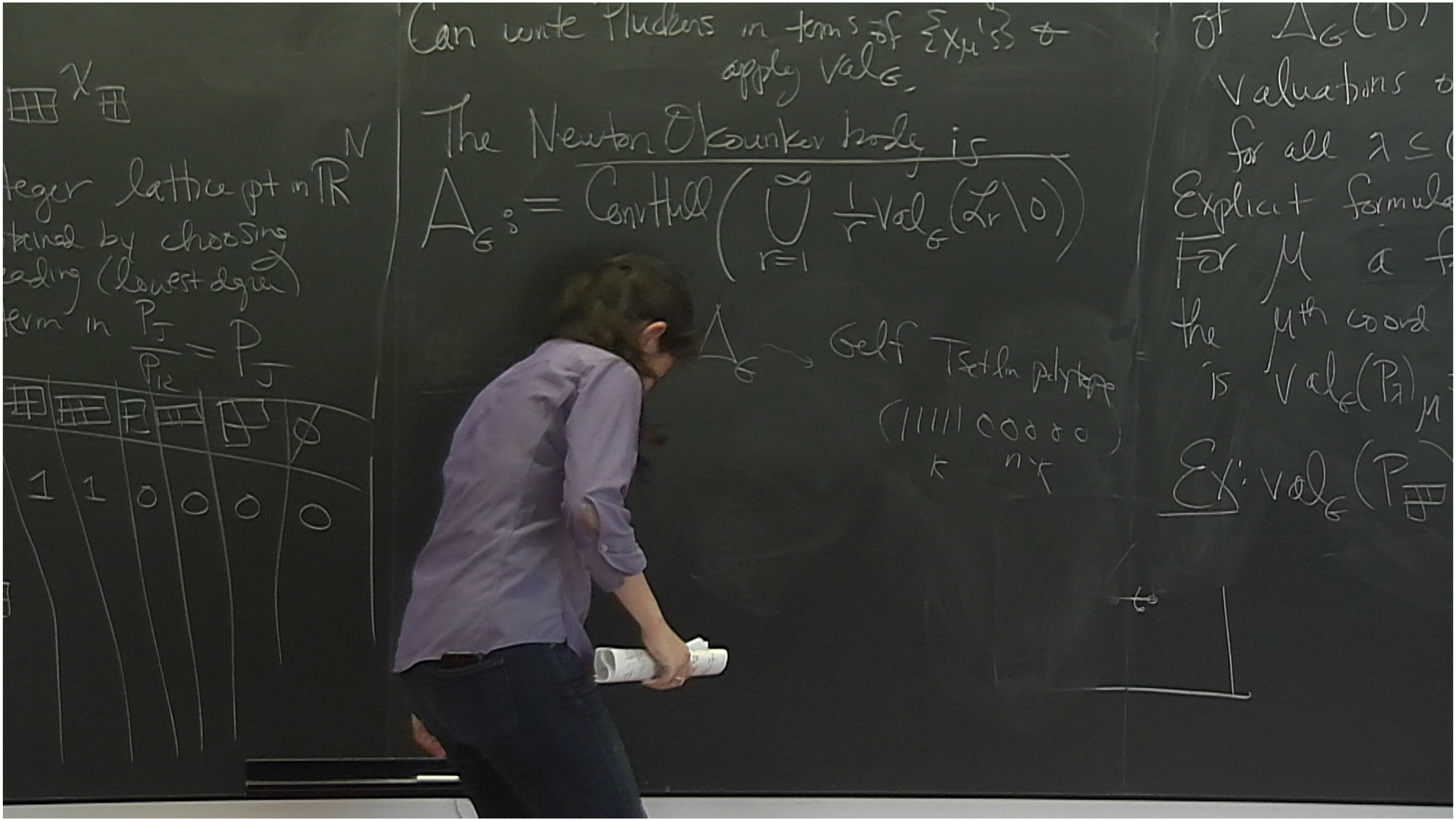

Ex: k=2, n=3
A-model
network chart
B-model
has


Ex: k=2, n=3
X = Gr1(C^3), X̌ = Gr2(C^3)
G =
A-model
network chart
B-model
W =
has


Can write Pluckers in terms of {x_μ's} & apply val_G.
The Newton Okounkov body is
Δ_G := ConvHull(⋃_{r=1}^∞ 1/r val_G(L_r \ 0))
Δ_G ⇝ Gelf Tsetlin polytope
Valuations
for all λ ≤
Explicit formula
For μ a
the μth coord
is val_G(P_λ)_μ
Ex: val_G(P
teger lattice pt in R^N
tained by choosing
ading (lowest degree)
term in P_J / P_K = P_J
1 1 0 0 0 0