Title: String theory and integrable lattice models

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Abstract: <p>I discuss a string theoretic approach to integrable lattice models. This approach provides a unified perspective on various important notions in lattice models, and relates these notions to four-dimensional N = 1 supersymmetric field theories and their surface operators. I explain how my construction connects to Costello's work and the Nekrasov-Shatashvili correspondence.</p>

String Theory and Integrable Lattice Models

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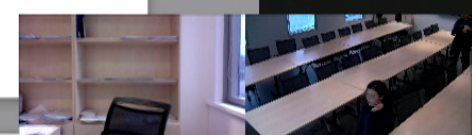
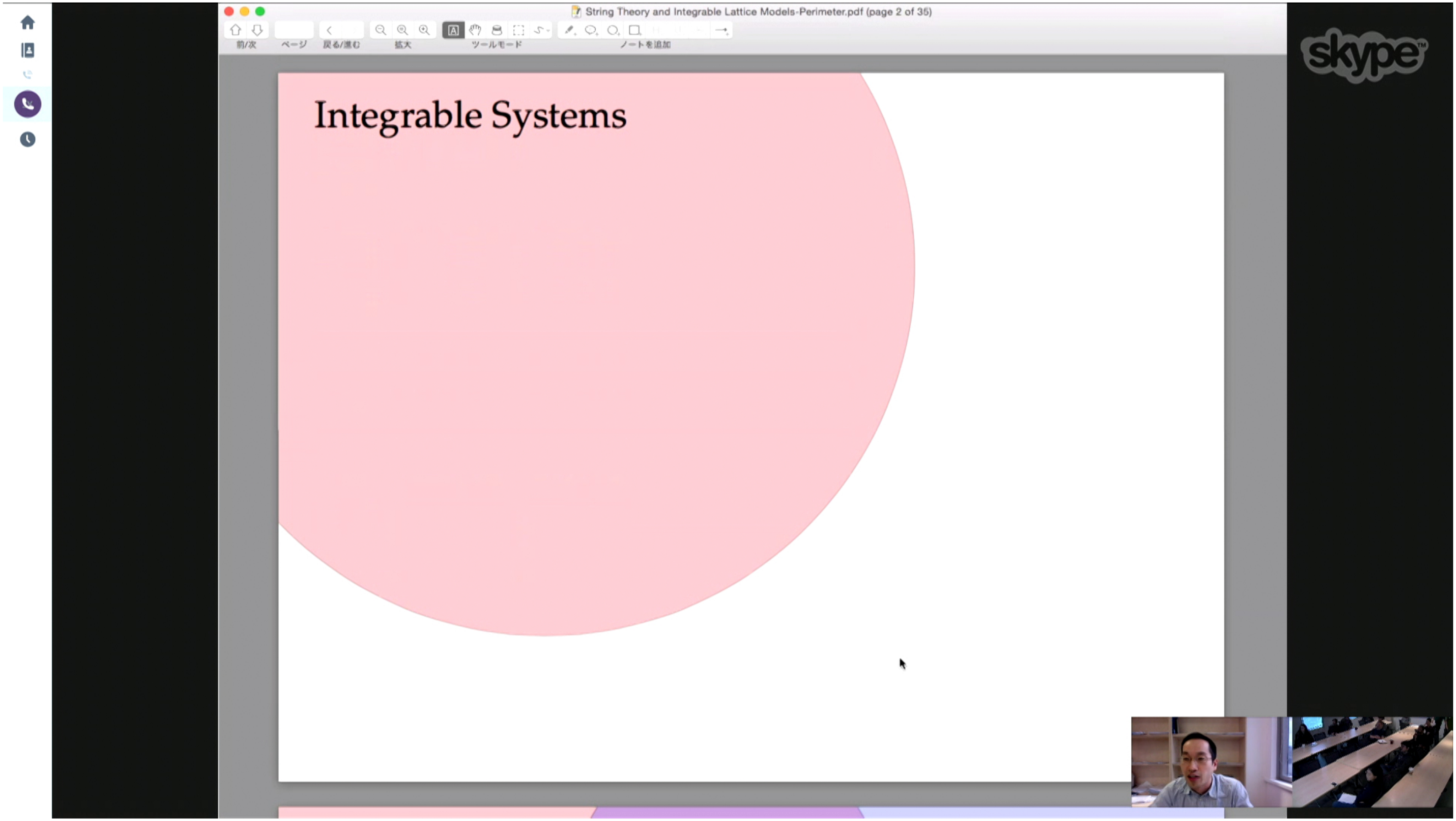
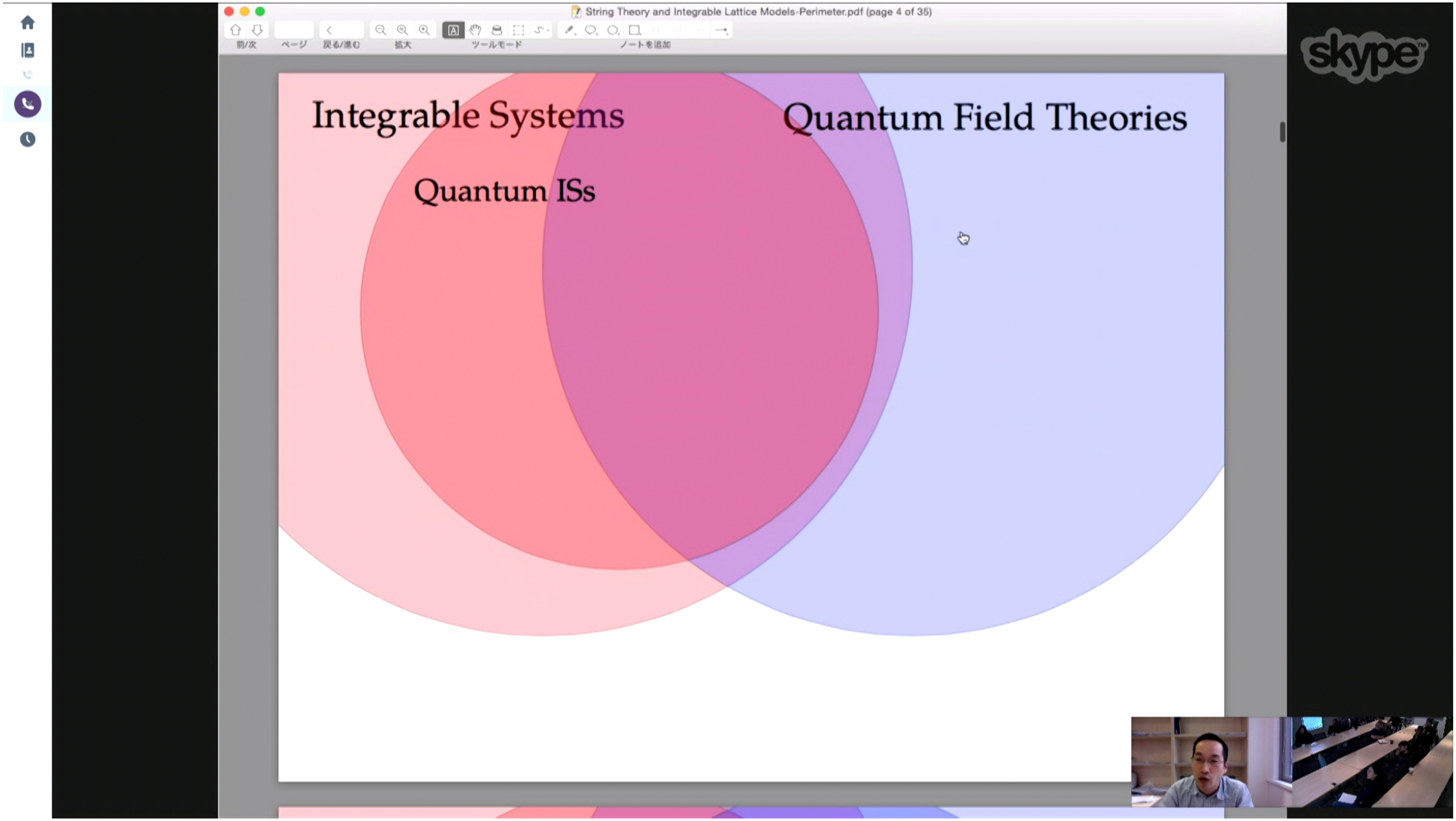
April 26, 2017

Perimeter Insitute for Theoretical Physics

Based on 1504.04055, 1606.01041 (with Kazunobu Maruyoshi), 1610.05584, 1701.05562 & work in progress with Kevin Costello




String Theory and Integrable Lattice Models-Perimeter.pdf (page 2 of 35)
Integrable Systems


String Theory and Integrable Lattice Models-Perimeter.pdf (page 4 of 35)
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Quantum Field Theories
Quantum ISs
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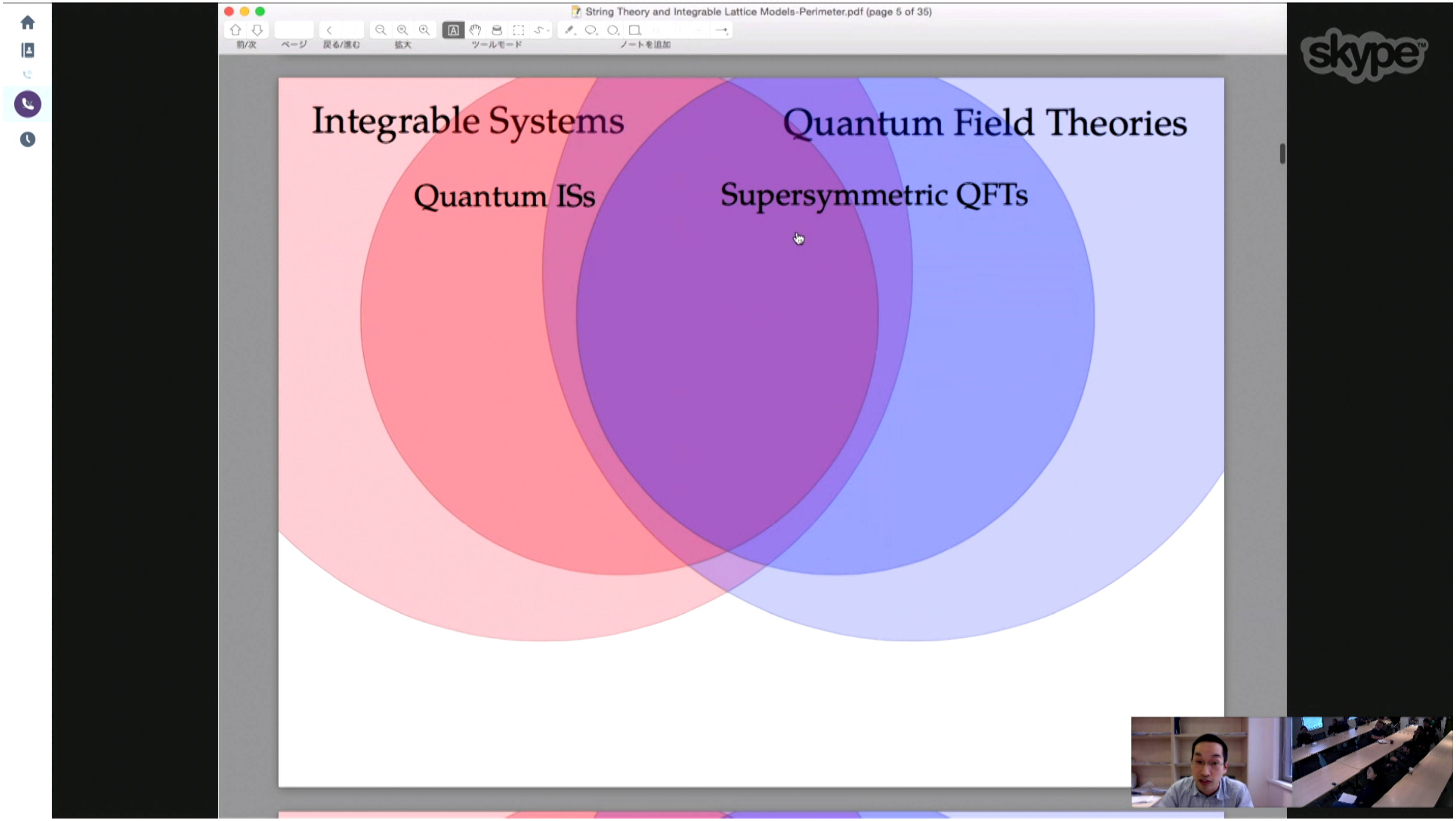
String Theory and Integrable Lattice Models-Perimeter.pdf (page 5 of 35)
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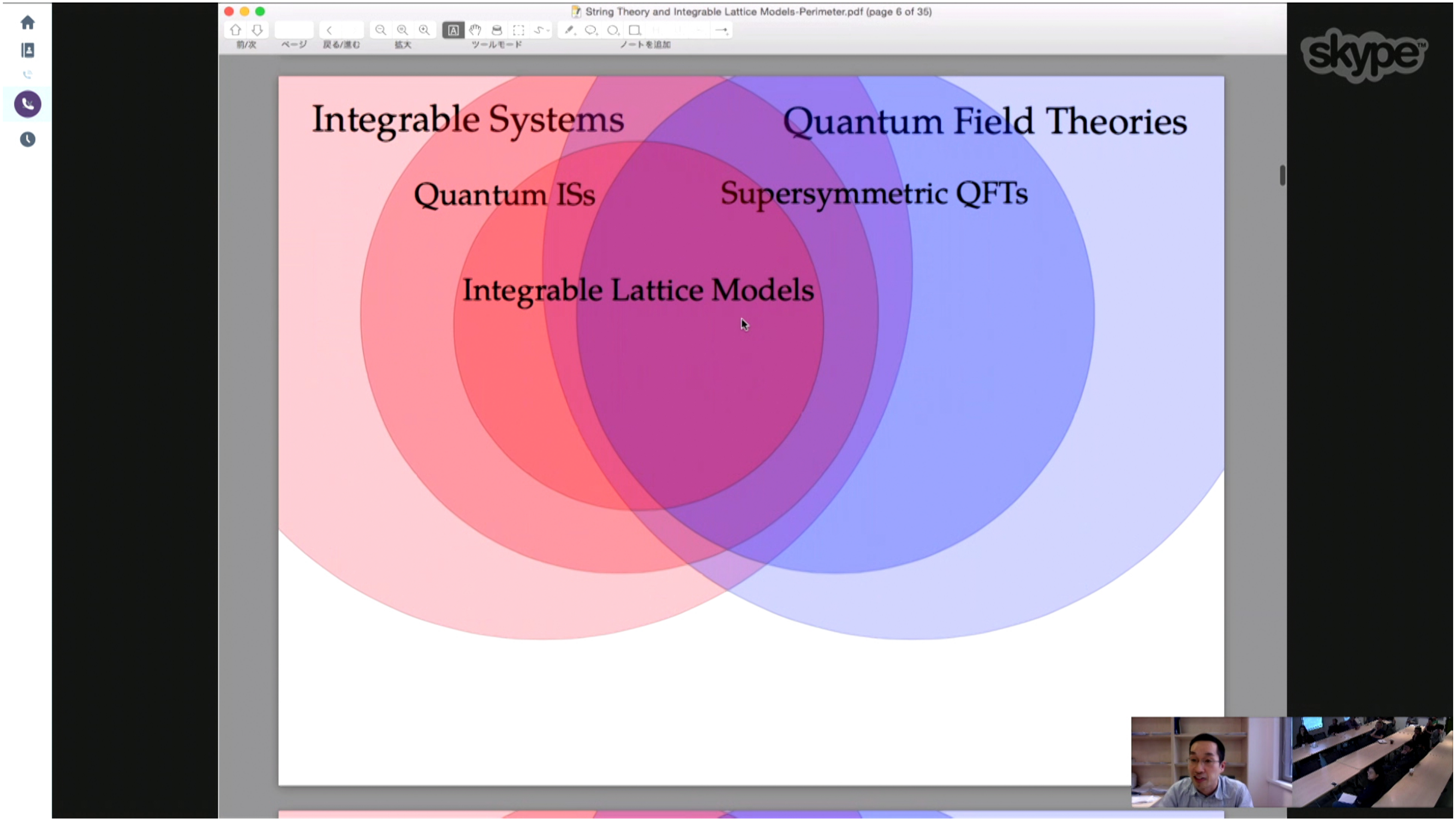
String Theory and Integrable Lattice Models-Perimeter.pdf (page 6 of 35)
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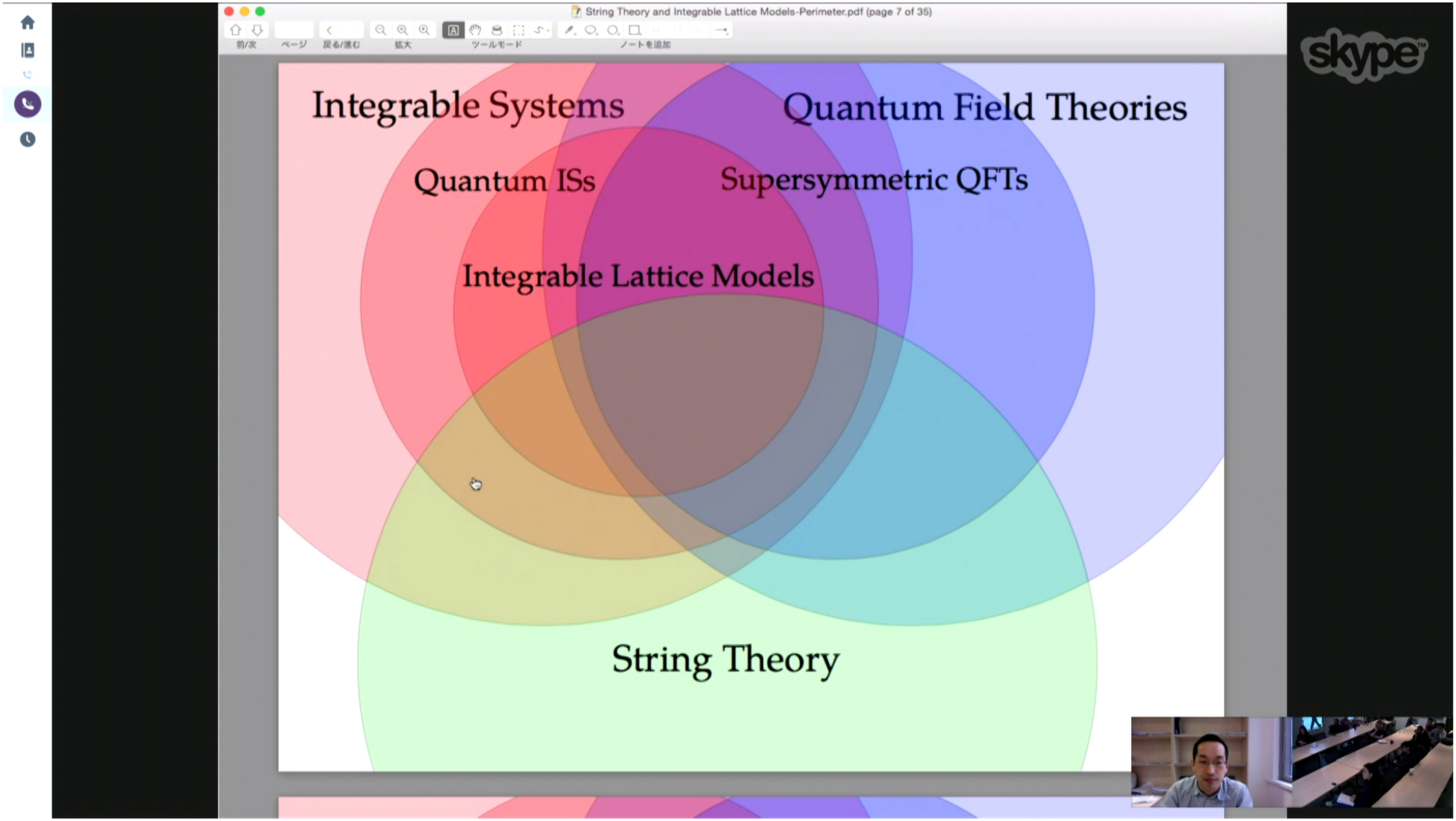
String Theory and Integrable Lattice Models-Perimeter.pdf (page 7 of 35)
Integrable Systems
Quantum Field Theories
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Integrable lattice models can be embedded in string theory.

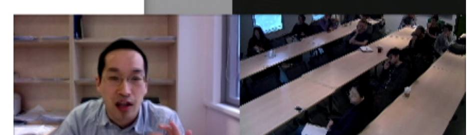
What does this buy us? A lot!

Today, I will

- construct an integrable lattice model that unifies three lattice models,
- relate it to 4d $\mathcal{N} = 1$ SUSY QFTs and
- elucidate surface operators in these theories.

I will also explain relations to

- Costello's construction of 8-vertex model from gauge theory [Costello '13, Costello–Witten–Yamazaki] and
- Nekrasov–Shatashvili correspondence.



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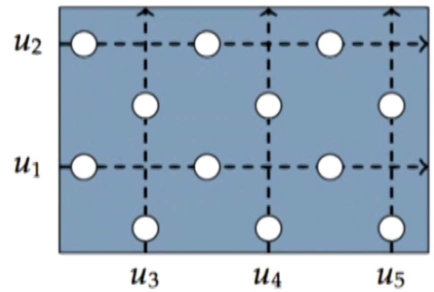
Belavin model [Belavin '81]

Belavin model is a lattice model in statistical mechanics:

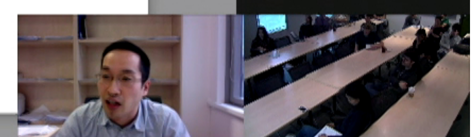


Spins ○ live on edges, interact at vertices.

Spin variables $i, j, k, l, \ldots \in \{1, \ldots, N\}$, $N \geq 2$

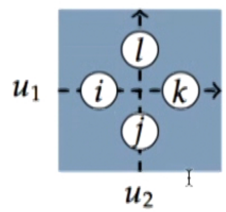
A spectral parameter $u \in \mathbb{C}$ assigned to each line

Lattice can be drawn on any surface Σ; in this talk $\Sigma = T^2$.



Interaction is governed by the R-matrix $R^{\mathrm{B}}(u) \in \mathrm{End}(\mathbb{C}^N \otimes \mathbb{C}^N)$.

The local Boltzmann weight is given by its matrix elements:

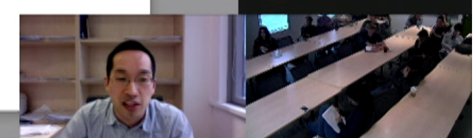


$$= R^{\mathrm{B}}(u_{12})^{kl}_{ij}\,, \quad u_{12} = u_1 - u_2\,.$$

R^{B} depends on three parameters, u and $\tau, \gamma \in \mathbb{C}$, $\mathrm{Im}\,\tau > 0$.

Partition function

$$Z = \sum_{\text{spin configs}} \prod_{\text{vertices}} \text{local Boltzmann weight}\,.$$



String Theory and Integrable Lattice Models-Perimeter.pdf (page 12 of 35)

Explicitly,

$$R^{\mathrm{B}}(u)_{ij}^{kl} = \delta_{i+j,k+l}\frac{\theta_1(\gamma)}{\theta_1(u+\gamma)}\frac{\theta^{(k-l)}(u+\gamma)}{\theta^{(k-i)}(\gamma)\theta^{(i-l)}(u)}\frac{\prod_{m=0}^{N-1}\theta^{(m)}(u)}{\prod_{n=1}^{N-1}\theta^{(n)}(0)}\,.$$

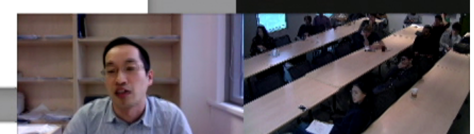
Here i, j, k, l are treated mod N and

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(u|\tau) = \sum_{n=-\infty}^{\infty} e^{\pi \mathrm{i}(n+a)^2\tau + 2\pi\mathrm{i}(n+a)(u+b)}\,,$$

$$\theta^{(j)}(u) = \theta\begin{bmatrix} 1/2 - j/N \\ 1/2 \end{bmatrix}(u|N\tau)\,,$$

$$\theta_1(u) = -\theta\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}(u|\tau)\,.$$

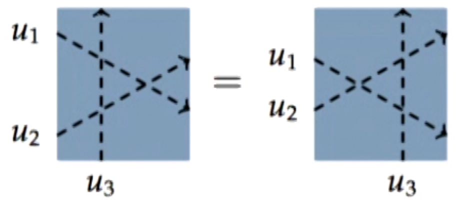
For $N = 2$, we get Baxter's 8-vertex model [Baxter '71].



R^{B} satisfies the Yang–Baxter equation

$$R^{\mathrm{B}}_{12}(u_{12})R^{\mathrm{B}}_{13}(u_{13})R^{\mathrm{B}}_{23}(u_{23}) = R^{\mathrm{B}}_{23}(u_{23})R^{\mathrm{B}}_{13}(u_{13})R^{\mathrm{B}}_{12}(u_{12})\,.$$

Graphically,



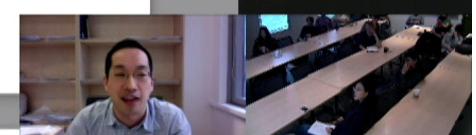
Highly overconstrained: N^6 equations for N^4 unknowns.

It follows the model is integrable.

This is quantum integrability:

$$\text{2d lattice model} \leftrightarrow \text{1d quantum spin chain}\,.$$

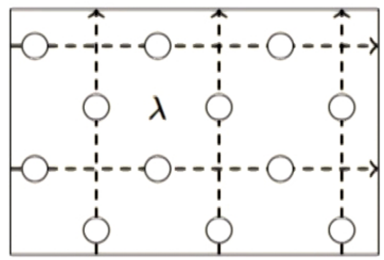
Belavin model $\leftrightarrow$ $\mathfrak{sl}_N$ XYZ spin chain.



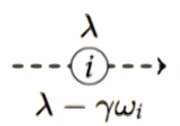
Jimbo–Miwa–Okado model [JMO '87]

JMO model has an additional parameter $\lambda \in \mathfrak{h} \subset \mathfrak{sl}_N$.

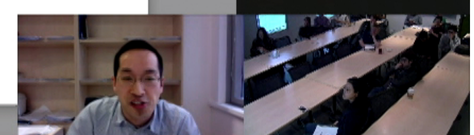
The dynamical variable λ is specified on one face:



Across a line λ jumps: if $\lambda = (\lambda_1, \dots, \lambda_N)$, $\sum_{i=1}^N \lambda_i = 0$, then



where $\omega_i = e_i - \frac{1}{N}\sum_{j=1}^N e_j$ is the weight of $e_i \in \mathbb{C}^N$, $(e_i)_j = \delta_{ij}$.



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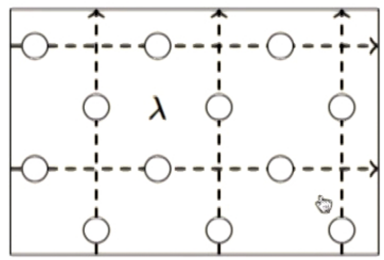

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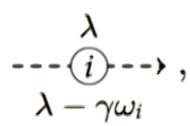

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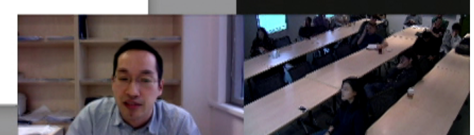
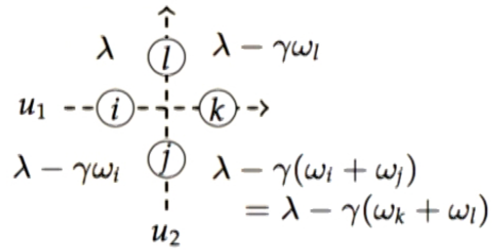


String Theory and Integrable Lattice Models-Perimeter.pdf (page 15 of 35)

The Boltzmann weight is given by Felder's R-matrix [Felder '94]:

$$R^{\mathrm{F}}(u_{12}, \lambda)_{ij}^{kl} =$$

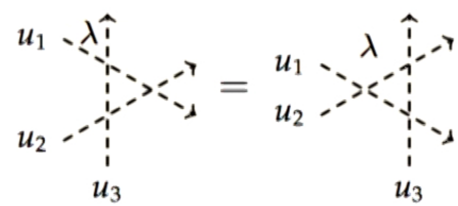


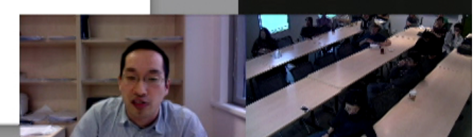
$(R^{\mathrm{F}})_{ij}^{kl} = 0$ unless $\{i, j\} = \{k, l\}$ for consistency, and

$$(R^{\mathrm{F}})_{ii}^{ii} = 1\,, \quad (R^{\mathrm{F}})_{ij}^{ij} = \frac{\theta_1(u)\theta_1(\lambda_{ij}+\gamma)}{\theta_1(u+\gamma)\theta_1(\lambda_{ij})}\,, \quad (R^{\mathrm{F}})_{ij}^{ji} = \frac{\theta_1(\gamma)\theta_1(u+\lambda_{ij})}{\theta_1(u+\gamma)\theta_1(\lambda_{ij})}\,.$$

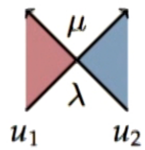
For $N = 2$, we get the 8VSOS model [Baxter '73].

This time, we have the dynamical YBE:





We assign Boltzmann weights



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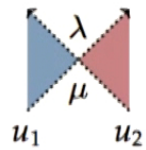
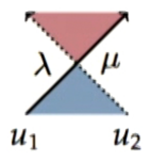


$$= \prod_{i,j} \Gamma\left(e^{2\pi \mathrm{i}(u_2-u_1)/N+\mu_j-\lambda_i}\right) / \Gamma\left(e^{2\pi \mathrm{i}(u_2-u_1)}\right),$$



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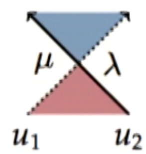


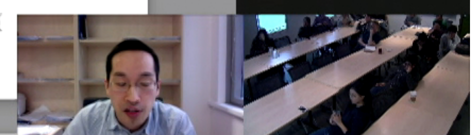
$$= \prod_{i,j} \Gamma\left(\sqrt{pq}\, e^{2\pi \mathrm{i}(u_1-u_2)/N+\mu_j-\lambda_i}\right),$$

where $p = e^{2\pi \mathrm{i}\tau}$, $q = e^{2\pi \mathrm{i}\gamma}$ and Γ is the elliptic gamma function

$$\Gamma(z;p,q) = \prod_{m,n=0}^{\infty} \frac{1 - p^{m+1}q^{n+1}/z}{1 - p^m q^n z}.$$

Each spin λ is integrated over with measure

$$\frac{\left(\prod_{n=1}^{\infty}(1-p^n)(1-q^n)\right)^{N-1}}{N!} \int_{[0,1]^{N-1}} \frac{\prod_{k=1}^{N-1} \mathrm{d}\lambda_k}{\prod_{i\neq j}\Gamma\left(e^{2\pi \mathrm{i}(\lambda_i-\lambda_j)}\right)}.$$

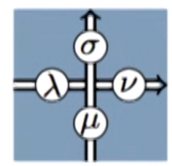


We can reformulate the BS model in terms of an R-matrix.

Introduce double line notation $(u,v) \Longrightarrow \; = \; \begin{matrix} v \cdots\cdots\rightarrow \\ u \longrightarrow \end{matrix}$.

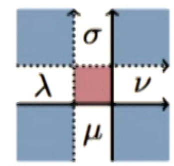
BS–Derkachov–Spiridonov R-operator R^{BSDS} is given by

$$R^{\mathrm{BSDS}}((u_1,v_1),(u_2,v_2))^{\nu\sigma}_{\lambda\mu} = (u_1,v_1)$$



(u_2,v_2)

$=$ v_1 u_1



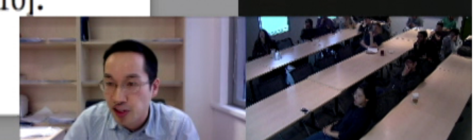
$v_2 \; u_2$.

[BS '11, Derkachov–Spiridonov '12, Maruyoshi–Y '16].

Lattice made from R^{BSDS} gives the tricolor checkerboard.

R^{BSDS} is an ∞-dim R-matrix; it's an integral operator.

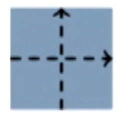
YBE follows from an integral identity for Γ [Spiridonov '03 , Rains '10].



Summary of the three models

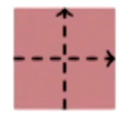
Belavin:

$R^{\mathrm{B}}(u_{12}) = u_1$



u_2

$, \quad \widetilde{R}^{\mathrm{B}}(u_{12}) = u_1$



u_2

$= \left(R^{\mathrm{B}}(u_{12})\right)^T$

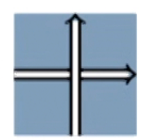
Jimbo–Miwa–Okado/Felder:

$R^{\mathrm{F}}(u_{12}, \lambda) = u_1$

u_2

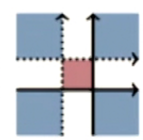
Bazhanov–Sergeev:

$R^{\mathrm{BSDS}}((u_1, v_1), (u_2, v_2)) = (u_1, v_1)$

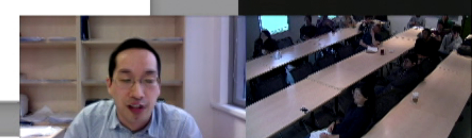


(u_2, v_2)

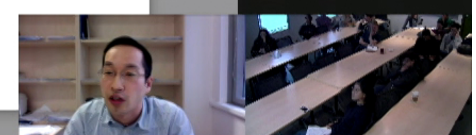
$= \begin{matrix} v_1 \\ u_1 \end{matrix}$



$v_2 u_2$



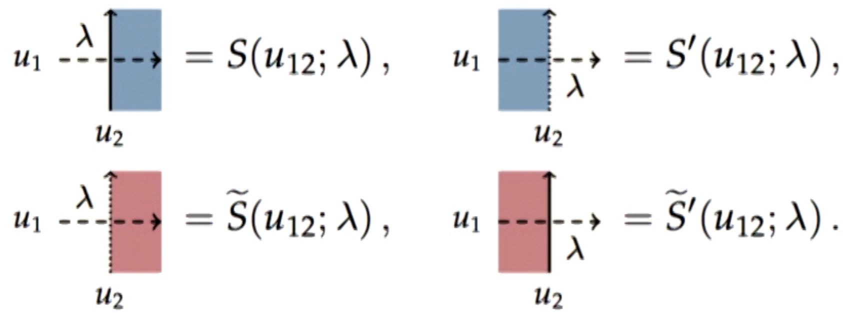
String theory tells me the three models can be unified [Y '17].



String Theory and Integrable Lattice Models-Perimeter.pdf (page 21 of 35)

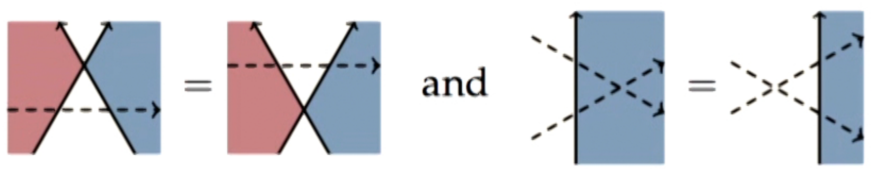
Intertwining operators

If different kinds of lines can coexist, we have more crossings:

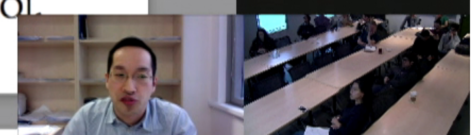


These intertwining operators are $N \times N$ matrices of functions.

They must solve many Yang–Baxter equations such as



The second one is the vertex–face correspondence [Baxter '73, JMO], relating R^{B} and R^{F}.



String Theory and Integrable Lattice Models-Perimeter.pdf (page 22 of 35)

Solutions exist [Y '17]:

$$S(u;\lambda) = e^{-\pi i u}\Psi(u,\lambda)\,, \quad \widetilde{S}'(u;\lambda) = e^{-\pi i u} Z^{N/2}\Phi(u,\lambda)^T\,,$$

$$S'(u;\lambda) = S(u';\lambda)^{-1}\,, \quad \widetilde{S}(u;\lambda) = (-1)^{N-1}\widetilde{S}'(u';\lambda)^{-1} Z^N\,,$$

where $u' = u + N(\tau+\gamma)/2 - \gamma$ and

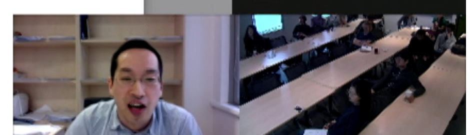
$$Z = \mathrm{diag}(e^{2\pi i \lambda_1}, \dots, e^{2\pi i \lambda_N})\,,$$

$$\Phi(u,\lambda)^j_i = \theta^{(j)}\left(u - N\lambda_i + \frac{N-1}{2}\right)\,,$$

$$\Psi(u,\lambda)^j_i = \Phi(u,-\lambda)^j_i \Big/ \prod_{k(\neq i)} \theta_1(\lambda_{ki})\,.$$

Partial results in [Sergeev '92, Quano–Fujii '93].

Similar analysis for $N=2$ in [Derkachov–Spiridonov].

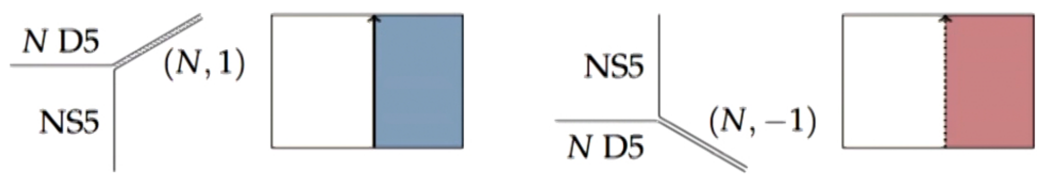


Brane construction [Yamazaki '13, Maruyoshi–Y '16, Y '17]

Our model can be constructed from branes in string theory.

Wrap a stack of N D5-branes on $S^3 \times S^1 \times \Sigma$.

Let an NS5-brane hit the D5s. They form bound states:



In the lattice model, the junction is a solid/dotted line and the bound state region is colored blue/red.

A TQFT argument shows $Z_{\text{brane system}} = Z_{\text{integrable lattice model}}$ [Costello '13, Y '15, '16].

YBE becomes brane movements!

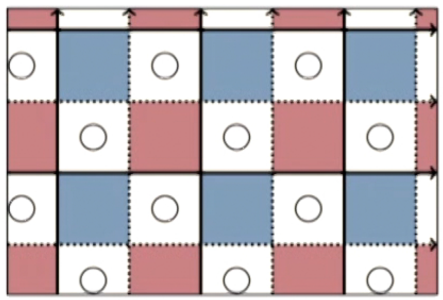
String Theory and Integrable Lattice Models-Perimeter.pdf (page 24 of 35)

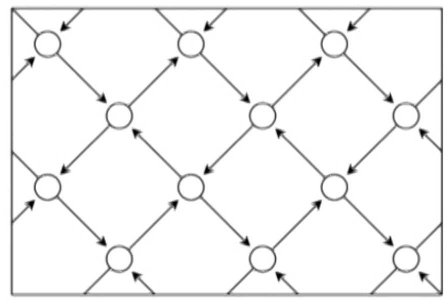
Correspondence with 4d SUSY QFTs

The D5s support 6d SYM on $S^3 \times S^1 \times \Sigma$. NS5s wrap $S^3 \times S^1$.

Making Σ small, we get an $\mathcal{N} = 1$ SUSY theory on $S^3 \times S^1$.

It is a gauge theory described by a planar quiver:

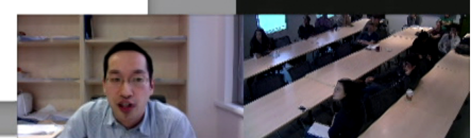




Now ○ is an $SU(N)$ gauge group. $\longrightarrow$ is a chiral multiplet.

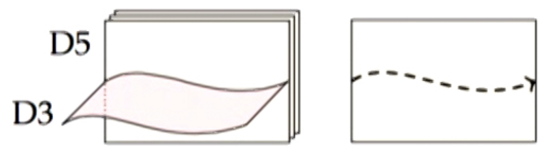
SUSY index of this theory on $S^3 = Z_{BS}$ [Spiridonov '10, Yamazaki '13].

YBE follows from Seiberg duality.

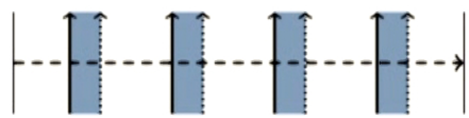


Surface defects as transfer matrices [Maruyoshi–Y, Y '17]

A D3 hitting the D5s creates a dashed line:



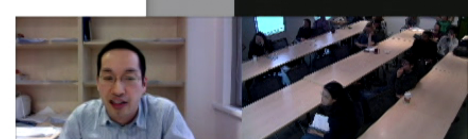
It acts on the BS model by a transfer matrix, e.g.

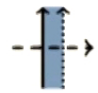
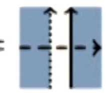


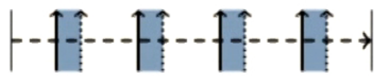
This is a difference operator shifting dynamical variables.

The D3 wraps $S^1 \times S^1 \subset S^3 \times S^1$, so it creates a surface operator in the 4d $\mathcal{N} = 1$ theory.

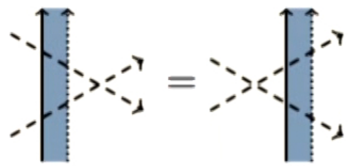
A new way of understanding surface operators



In terms of L-operators $L^{\mathrm{F}} =$  and $L^{\mathrm{B}} =$  ,

 $= \mathrm{Tr}(L^{\mathrm{F}}L^{\mathrm{F}}L^{\mathrm{F}}L^{\mathrm{F}}) = \mathrm{Tr}(L^{\mathrm{B}}L^{\mathrm{B}}L^{\mathrm{B}}L^{\mathrm{B}})$.

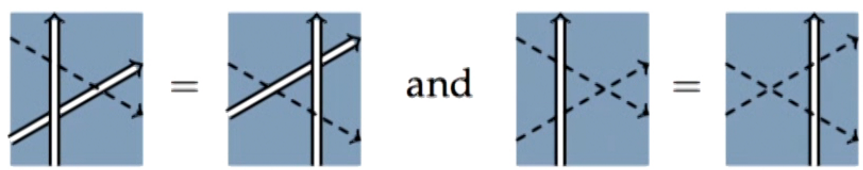
L^{F} satisfies an RLL relation with Felder's R-matrix,



$$\Longleftrightarrow \quad R^{\mathrm{F}}L^{\mathrm{F}}_1L^{\mathrm{F}}_2 = L^{\mathrm{F}}_2L^{\mathrm{F}}_1R^{\mathrm{F}} ,$$

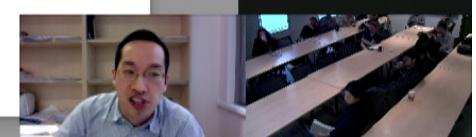
giving an ∞-dim rep of the elliptic quantum group $E_{\tau,\gamma/2}(\mathfrak{sl}_N)$.

L^{B} satisfies two RLL relations, with R^{B} and R^{BSDS}:



This is the elliptic lift of the chiral Potts/six-vertex relation

[Bazhanov–Stroganov '90, Bazhanov–Kashaev–Mangazeev–Stroganov '91].



String Theory and Integrable Lattice Models-Perimeter.pdf (page 27 of 35)

$\mathrm{Tr}(\underbrace{L \cdots L}_{k})$ is a surface op for class-$\mathcal{S}_k$ theories [Gaiotto–Razamat '15].

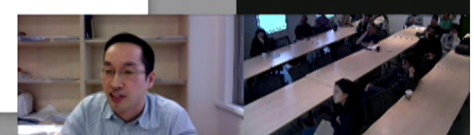
For $k = 1$, we get surface ops in class-$\mathcal{S}$ theories, which play important roles in the AGT correspondence.

They come from codim-4 defects in 6d $(2, 0)$ theory, classified by irreps of $\mathfrak{sl}_N$.

Correspondingly, dashed lines in any irrep R,

$$R \dashrightarrow ,$$

can be constructed by fusion.



String Theory and Integrable Lattice Models-Perimeter.pdf (page 28 of 35)

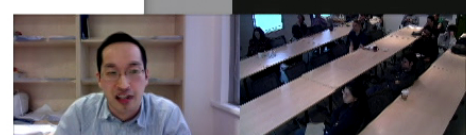
The transfer matrices match gauge theory results:

- $R = \mathbb{C}^N, k = 1$ [Gaiotto–Rastelli–Razamat '13, Gadde–Gukov '13]
- $R = \mathbb{C}^N, k > 1$ [Gaiotto–Razamat, Maruyoshi–Y, Ito–Yoshida]
- $R = \bigwedge^n \mathbb{C}^N, n > 1, k = 1$: Ruijsenaars' ops [Bullimore–Fluder–Hollands–Richmond '14]

Comparison in progress [Vaško–Y]:

- $R = S^n\mathbb{C}^N, n > 1, k = 1$ [Gaiotto–Rastelli–Razamat, Gadde–Gukov]
- $R = S^n\mathbb{C}^N, n > 1, k > 1$: partial results [Ito–Yoshida, Vaško–Y]

Other cases: no QFT results yet.



String Theory and Integrable Lattice Models-Perimeter.pdf (page 29 of 35)

Relation to Costello's theory [Costello–Y, work in progress]

Replace $S^3 \times_{\tau,\gamma} S^1$ with $\mathbb{R}^2 \times_\epsilon T^2_\tau$ in our brane setup:

spacetime	$\mathbb{R}^2$	$\times_\epsilon$	T^2	$\times_{-\epsilon}$	$\mathbb{R}^2$	$\times$	$T^*\Sigma$
N D5s	$\mathbb{R}^2$	$\times_\epsilon$	T^2	$\times$	0	$\times$	Σ
NS5	$\mathbb{R}^2$	$\times_\epsilon$	T^2	$\times$	0	$\times$	Σ'
D3	0	$\times$	T^2	$\times$	0	$\times$	Σ''

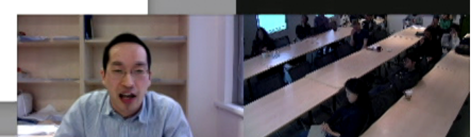
$\Sigma \cap \Sigma' =$ solid/dotted line, $\Sigma \cap \Sigma'' =$ dashed line.

Apply S-duality $\to$ T-duality on T^2 $\to$ S-duality:

spacetime	$\mathbb{R}^2$	$\times$	$\widetilde{T}^2$	$\times$	$\mathbb{R}^2$	$\times$	$T^*\Sigma$
N D5s	$\mathbb{R}^2$	$\times$	$\widetilde{T}^2$	$\times$	0	$\times$	Σ
D3	$\mathbb{R}^2$	$\times$	pt	$\times$	0	$\times$	Σ'
F1	0	$\times$	pt	$\times$	0	$\times$	Σ''

in a fluxtrap background ($C_2 \neq 0$) [Hellerman–Orland–Reffert '12].

The coordinates on $\widetilde{T}^2$ are the spectral parameters.



String Theory and Integrable Lattice Models-Perimeter.pdf (page 29 of 35)

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$$\begin{array}{rccccccc}
\text{spacetime} & \mathbb{R}^2 & \times_\epsilon & T^2 & \times_{-\epsilon} & \mathbb{R}^2 & \times & T^*\Sigma \\
N\ \text{D5s} & \mathbb{R}^2 & \times_\epsilon & T^2 & \times & 0 & \times & \Sigma \\
\text{NS5} & \mathbb{R}^2 & \times_\epsilon & T^2 & \times & 0 & \times & \Sigma' \\
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\text{spacetime} & \mathbb{R}^2 & \times & \widetilde{T}^2 & \times & \mathbb{R}^2 & \times & T^*\Sigma \\
N\ \text{D5s} & \mathbb{R}^2 & \times & \widetilde{T}^2 & \times & 0 & \times & \Sigma \\
\text{D3} & \mathbb{R}^2 & \times & \text{pt} & \times & 0 & \times & \Sigma' \\
\text{F1} & 0 & \times & \text{pt} & \times & 0 & \times & \Sigma''
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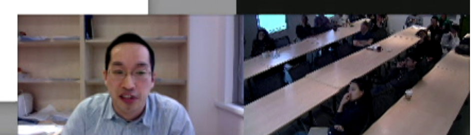

String Theory and Integrable Lattice Models-Perimeter.pdf (page 30 of 35)

$$
\begin{array}{rccccccc}
\text{spacetime} & \mathbb{R}^2_\epsilon & \times & \widetilde{T}^2 & \times & \mathbb{R}^2_{-\epsilon} & \times & T^*\Sigma \\
N\ \text{D5s} & \mathbb{R}^2_\epsilon & \times & \widetilde{T}^2 & \times & 0 & \times & \Sigma \\
\text{D3} & \mathbb{R}^2_\epsilon & \times & \text{pt} & \times & 0 & \times & \Sigma' \\
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\end{array}
$$

D5s in the fluxtrap gives Ω-deformed 6d SYM on $\mathbb{R}^2_\epsilon \times \widetilde{T}^2 \times \Sigma$.

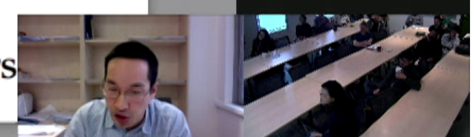
D3s create codim-3 defects. F1s create Wilson lines.

Localization leads to Costello's theory with action

$$\frac{1}{\epsilon}\int_{T^2\times\Sigma} d\bar{z} \wedge CS(A)\,.$$

Wilson lines support Belavin/JMO model [Costello '13, Costello–Witten–Yamazaki].

Codim-3 defects can be used to construct Baxter's Q-operators



spacetime	$\mathbb{R}^2_\epsilon$	$\times$	$\widetilde{T}^2$	$\times$	$\mathbb{R}^2_{-\epsilon}$	$\times$	$T^*\Sigma$
N D5s	$\mathbb{R}^2_\epsilon$	$\times$	$\widetilde{T}^2$	$\times$	0	$\times$	Σ
D3	$\mathbb{R}^2_\epsilon$	$\times$	pt	$\times$	0	$\times$	Σ'
F1	0	$\times$	pt	$\times$	0	$\times$	Σ''

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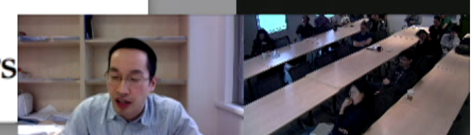


String Theory and Integrable Lattice Models-Perimeter.pdf (page 31 of 35)

Relation to Nekrasov–Shatashvili [Costello–Y, work in progress]

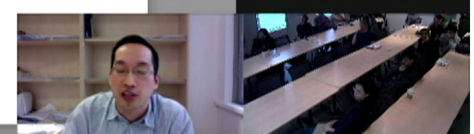
Put a 4d $\mathcal{N} = 2$ gauge theory on $\mathbb{R}^3 \times S^1$.

At low energies on the Coulomb branch, we get a 3d $\mathcal{N} = 4$ sigma model with hyperkähler target $\mathcal{M}$.

In one complex structure, $\mathcal{M}$ is the phase space of a complex integrable system:

$$\begin{array}{ccc} \text{complex torus} & \longrightarrow & \mathcal{M} \\ & & \downarrow \\ & & \text{Coulmb branch on } \mathbb{R}^4 \end{array}$$

Ω-deformation $\mathbb{R}^3 \times S^1 \to \mathbb{R}^2_\epsilon \times \mathbb{R} \times S^1$ quantizes $\mathcal{M}$.



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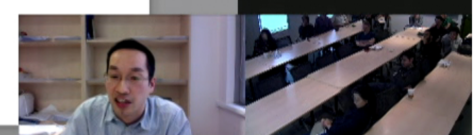


String Theory and Integrable Lattice Models-Perimeter.pdf (page 32 of 35)

Consder $\mathcal{N} = 2$ SQCD with $G = \mathrm{SU}(N)$ and $N_f = 2N$.

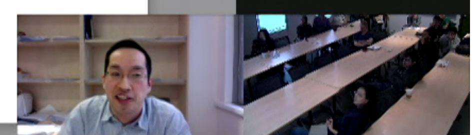
In this case, we get the quantum trigonometric Gaudin model.

This is spectral dual to $\mathfrak{sl}_2$ XXX spin chain of length N.

Indeed [Chen–Dorey–Hollowood–Lee '11],

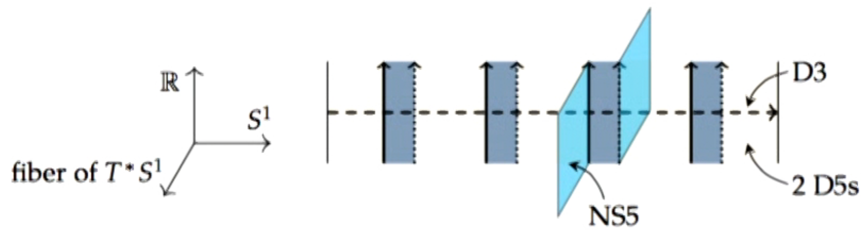
$$\text{vacua of the } \Omega\text{-def. theory} \leftrightarrow \text{states in the spin chain}\,,$$
$$\text{Seiberg–Witten curve} \leftrightarrow \text{spectral curve } \det(x - T(u)) = 0\,.$$

Where is the spin chain hiding?



String Theory and Integrable Lattice Models-Perimeter.pdf (page 33 of 35)

In our construction, T of the XYZ spin chain is given by

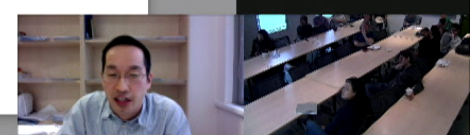


Take $\Sigma = \mathbb{R} \times S^1$ in our brane setup:

spacetime	$\mathbb{R}^2$	$\times_\epsilon$	T^2	$\times_{-\epsilon}$	$\mathbb{R}^2$	$\times$	$T^*\mathbb{R}$	$\times$	T^*S^1
2 D5s	$\mathbb{R}^2$	$\times_\epsilon$	T^2	$\times$	0	$\times$	$\mathbb{R}$	$\times$	S^1
N NS5s	$\mathbb{R}^2$	$\times_\epsilon$	T^2	$\times$	0	$\times$	$\mathbb{R}$	$\times$	fiber
D3	0	$\times$	T^2	$\times$	0	$\times$	fiber	$\times$	S^1

Apply S-duality $\to$ T-duality on S^1 $\to$ T-duality on T^2:

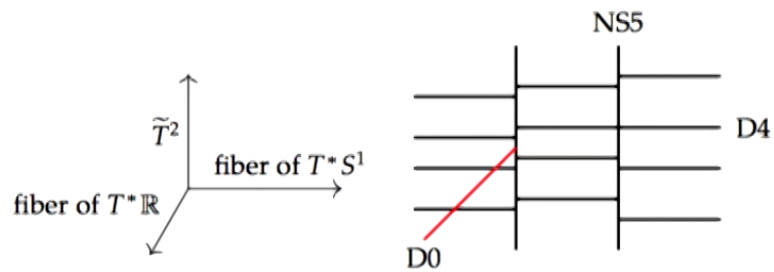
spacetime	$\mathbb{R}^2_\epsilon$	$\times$	$\widetilde{T}^2$	$\times$	$\mathbb{R}^2_{-\epsilon}$	$\times$	$T^*\mathbb{R}$	$\times$	T^*S^1
2 NS5s	$\mathbb{R}^2_\epsilon$	$\times$	$\widetilde{T}^2$	$\times$	0	$\times$	$\mathbb{R}$	$\times$	S^1
N D4s	$\mathbb{R}^2_\epsilon$	$\times$	pt	$\times$	0	$\times$	$\mathbb{R}$	$\times$	T^*S^1
D0	0	$\times$	pt	$\times$	0	$\times$	fiber	$\times$	pt



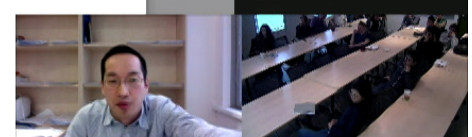
String Theory and Integrable Lattice Models-Perimeter.pdf (page 34 of 35)

spacetime	$\mathbb{R}^2_\epsilon$	$\times$	$\widetilde{T}^2$	$\times$	$\mathbb{R}^2_{-\epsilon}$	$\times$	$T^*\mathbb{R}$	$\times$	T^*S^1
2 NS5s	$\mathbb{R}^2_\epsilon$	$\times$	$\widetilde{T}^2$	$\times$	0	$\times$	$\mathbb{R}$	$\times$	S^1
N D4s	$\mathbb{R}^2_\epsilon$	$\times$	pt	$\times$	0	$\times$	$\mathbb{R}$	$\times$	T^*S^1
D0	0	$\times$	pt	$\times$	0	$\times$	fiber	$\times$	pt

In the limit $\widetilde{T}^2 \to \mathbb{R}^2$, this realizes the 4d $\mathcal{N} = 2$ theory on $\mathbb{R}^2_\epsilon \times \mathbb{R} \times S^1$ + instanton:



In this limit, XYZ $\to$ XXX.



String Theory and Integrable Lattice Models-Perimeter.pdf (page 35 of 35)

Conclusion

String theory allows us to construct integrable lattice models and relate them to 4d SUSY QFTs.

Further directions:

- Change $S^3 \times S^1$ to $M_3 \times S^1$ to get new R-matrices: $M_3 = S^3/\mathbb{Z}_r$ [Yamazaki, Kels], $S^2 \times S^1, \Sigma \times S^1, \ldots$
- Zamolodchikov's tetrahedron equation [Y '15]
- Chiral Potts model and monopoles [Atiyah '91]
- Geometric Langlands and AGT correspondences
- Categorification of lattice models
- Little String Theory, AdS/CFT correspondence, ...