Title: PSI 16/17 Explorations in Cosmology (Kendrick Smith) - Lecture 12

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Abstract:

$$g^{(4)}_{\mu\nu} = \begin{pmatrix} -N^{-2} + g^{(3)}_{ij} N^i N^j & g^{(3)}_{ij} N^i \\ g^{(3)}_{ij} N^j & g^{(3)}_{ij} \end{pmatrix}$$

$$g^{(4)\mu\nu} = \begin{pmatrix} -N^{-2} & N^{-2} N^i \\ N^{-2} N^i & g^{(3)ij} - N^{-2} N^i N^j \end{pmatrix}$$

$$n_\mu = (-N, 0) \quad \leftarrow \text{SIGN CONVENTION CHANGED!}$$

$$n^\mu = (N^{-1}, -N^{-1} N^i)$$

$g^{(4)\mu\nu} =$

$=$

EXTRINSIC

$$g^{(4)\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & g^{(3)ij} \end{pmatrix} - n^\mu n^\nu$$

$$= \delta^\mu_i \delta^\nu_j g^{(3)ij} - n^\mu n^\nu$$

EXTRINSIC CURVATURE $\quad K_{ij} = \nabla^{(4)}_i n_j$

$$K_{ij} = \partial_i n_j - \Gamma^{(4)\rho}_{ij} n_\rho$$

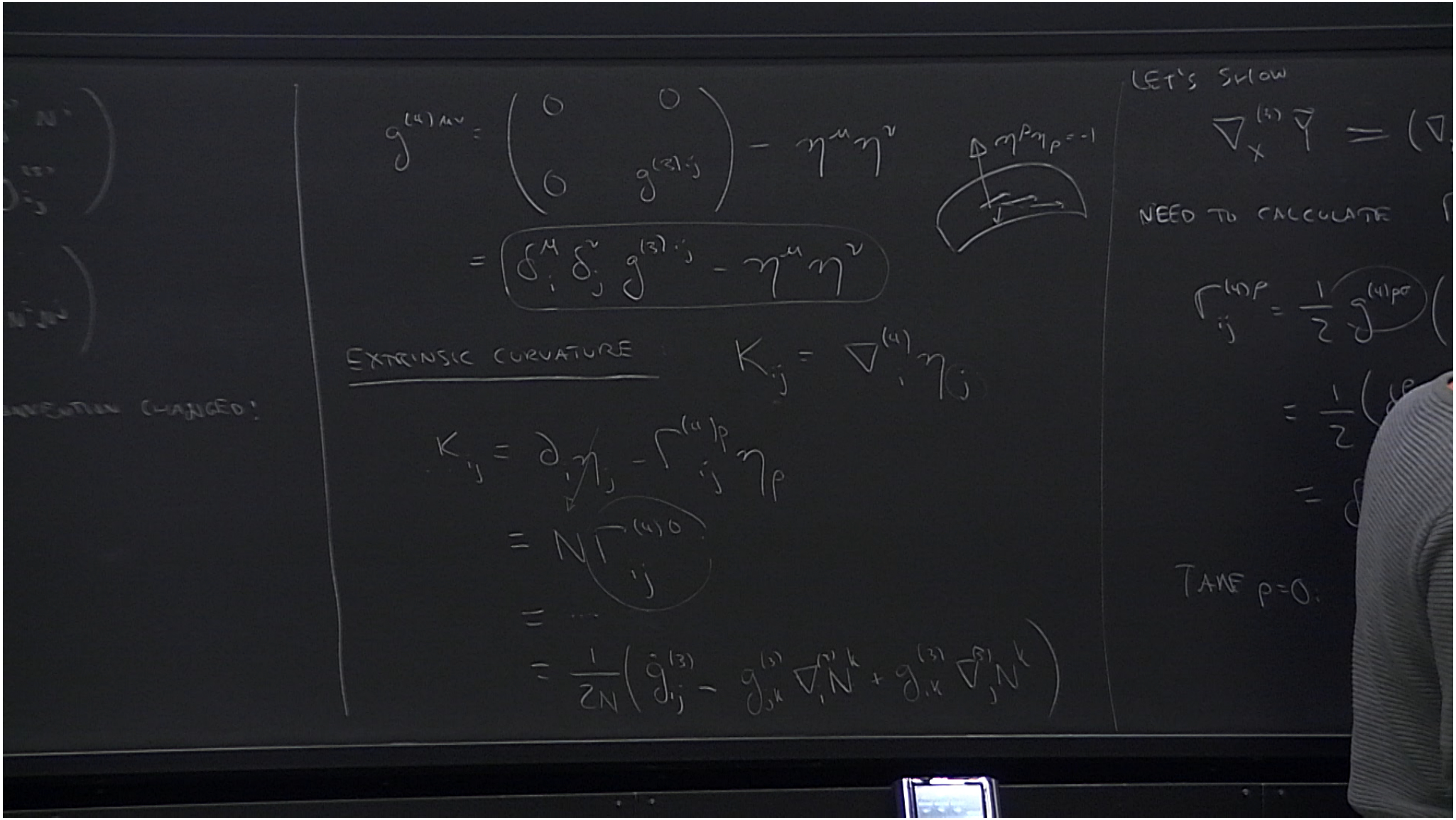
$$= N \Gamma^{(4)0}_{ij}$$

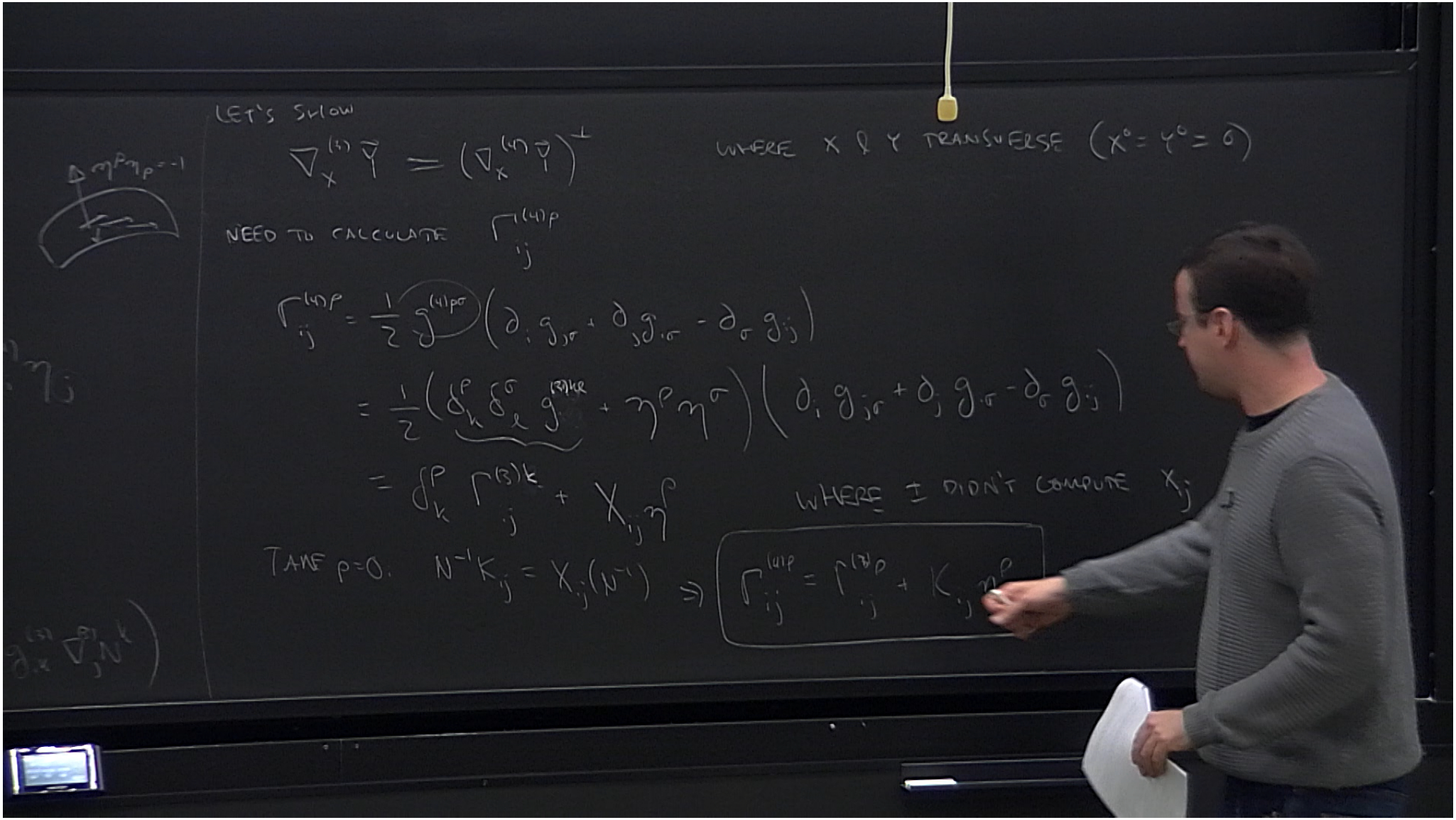
$$= \cdots$$

$$= \frac{1}{2N}\left(\dot{g}^{(3)}_{ij} - g^{(3)}_{jk} \nabla^{(3)}_i N^k + g^{(3)}_{ik} \nabla^{(3)}_j N^k \right)$$

CONVENTION CHANGED!

$\nabla^{(3)}_x$


$g^{(4)\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & g^{(3)ij} \end{pmatrix} - n^\mu n^\nu$
$= \delta^\mu_i \delta^\nu_j g^{(3)ij} - n^\mu n^\nu$
$n^\rho n_\rho = -1$
EXTRINSIC CURVATURE
$K_{ij} = \nabla^{(4)}_i n_j$
$K_{ij} = \partial_i n_j - \Gamma^{(4)\rho}_{ij} n_\rho$
$= N \Gamma^{(4)0}_{ij}$
$= \frac{1}{2N}\left(\dot{g}^{(3)}_{ij} - g^{(3)}_{jk} \nabla_i N^k + g^{(3)}_{ik} \nabla_j N^k \right)$
LET'S SHOW
$\nabla^{(4)}_X Y =$
NEED TO CALCULATE
$\Gamma^{(4)\rho}_{ij} = \frac{1}{2} g^{(4)\rho\sigma}$
$= \frac{1}{2}$
$=$
TAKE $\rho = 0$.


LET'S SHOW
$\nabla_X^{(4)} Y = (\nabla_X^{(4)} Y)^\perp$
WHERE X & Y TRANSVERSE $(X^0 = Y^0 = 0)$
NEED TO CALCULATE $\Gamma_{ij}^{(4)\rho}$
$\Gamma_{ij}^{(4)\rho} = \frac{1}{2} g^{(4)\rho\sigma} (\partial_i g_{j\sigma} + \partial_j g_{i\sigma} - \partial_\sigma g_{ij})$
$= \frac{1}{2} (\delta_k^\rho \delta_\ell^\sigma g^{(3)k\ell} + \eta^\rho \eta^\sigma)(\partial_i g_{j\sigma} + \partial_j g_{i\sigma} - \partial_\sigma g_{ij})$
$= \delta_k^\rho \Gamma_{ij}^{(3)k} + X_{ij} \eta^\rho$
WHERE I DIDN'T COMPUTE X_{ij}
TAKE $\rho = 0$. $N^{-1} K_{ij} = X_{ij} (N^{-1})$ $\Rightarrow$ $\Gamma_{ij}^{(4)\rho} = \Gamma_{ij}^{(3)\rho} + K_{ij} n^\rho$

LET'S SHOW

$$\nabla^{(3)}_X Y = (\nabla^{(4)}_X Y)^{\perp}$$

WHERE X & Y TRANSVERSE $(X^0 = Y^0 = 0)$

NEED TO CALCULATE $\Gamma^{(4)\rho}_{ij}$

$$\Gamma^{(4)\rho}_{ij} = \frac{1}{2} g^{(4)\rho\sigma}\left(\partial_i g_{j\sigma} + \partial_j g_{i\sigma} - \partial_\sigma g_{ij}\right)$$

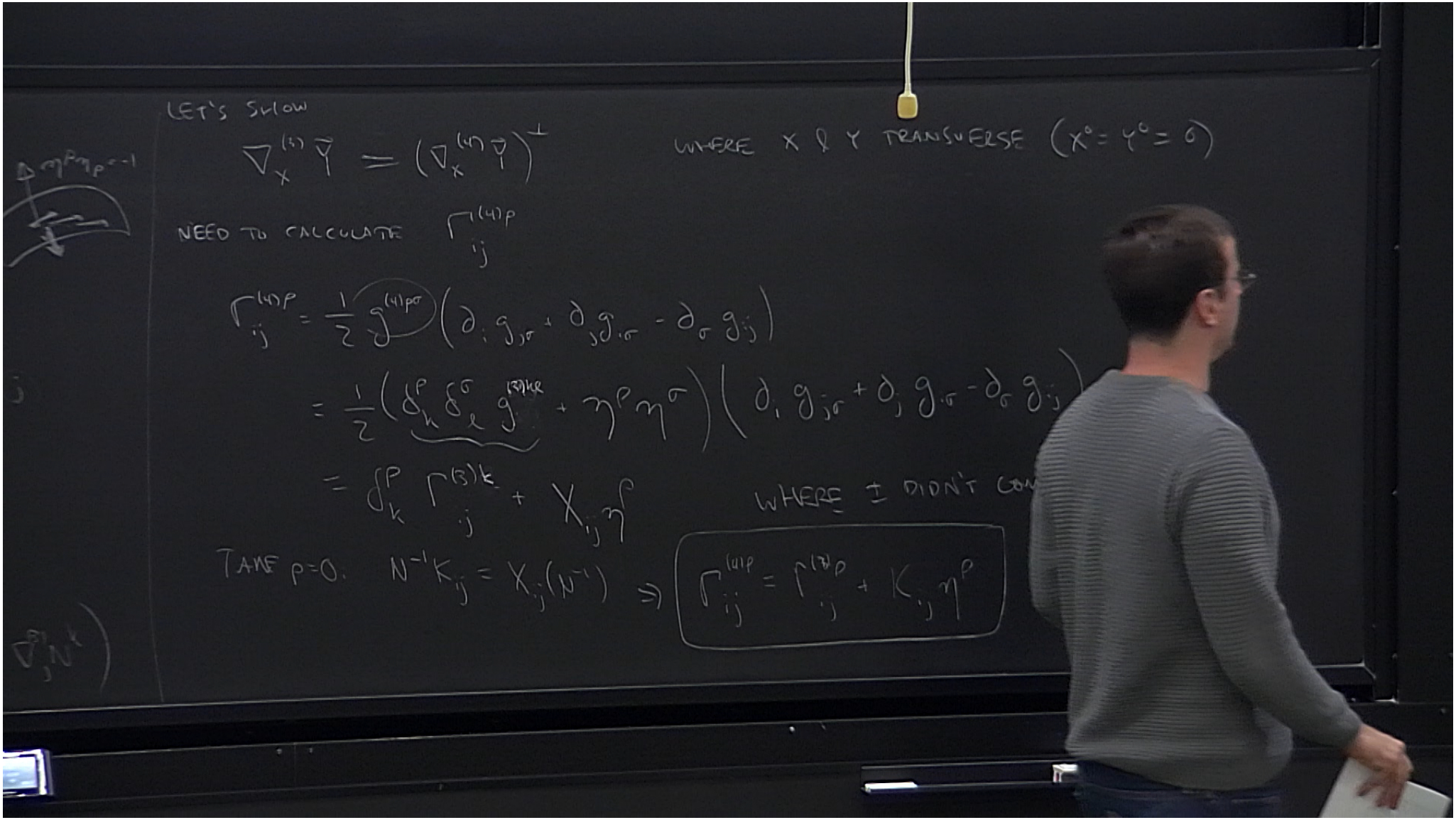
$$= \frac{1}{2}\left(\delta^\rho_k \delta^\sigma_\ell g^{(3)k\ell} + \eta^\rho \eta^\sigma\right)\left(\partial_i g_{j\sigma} + \partial_j g_{i\sigma} - \partial_\sigma g_{ij}\right)$$

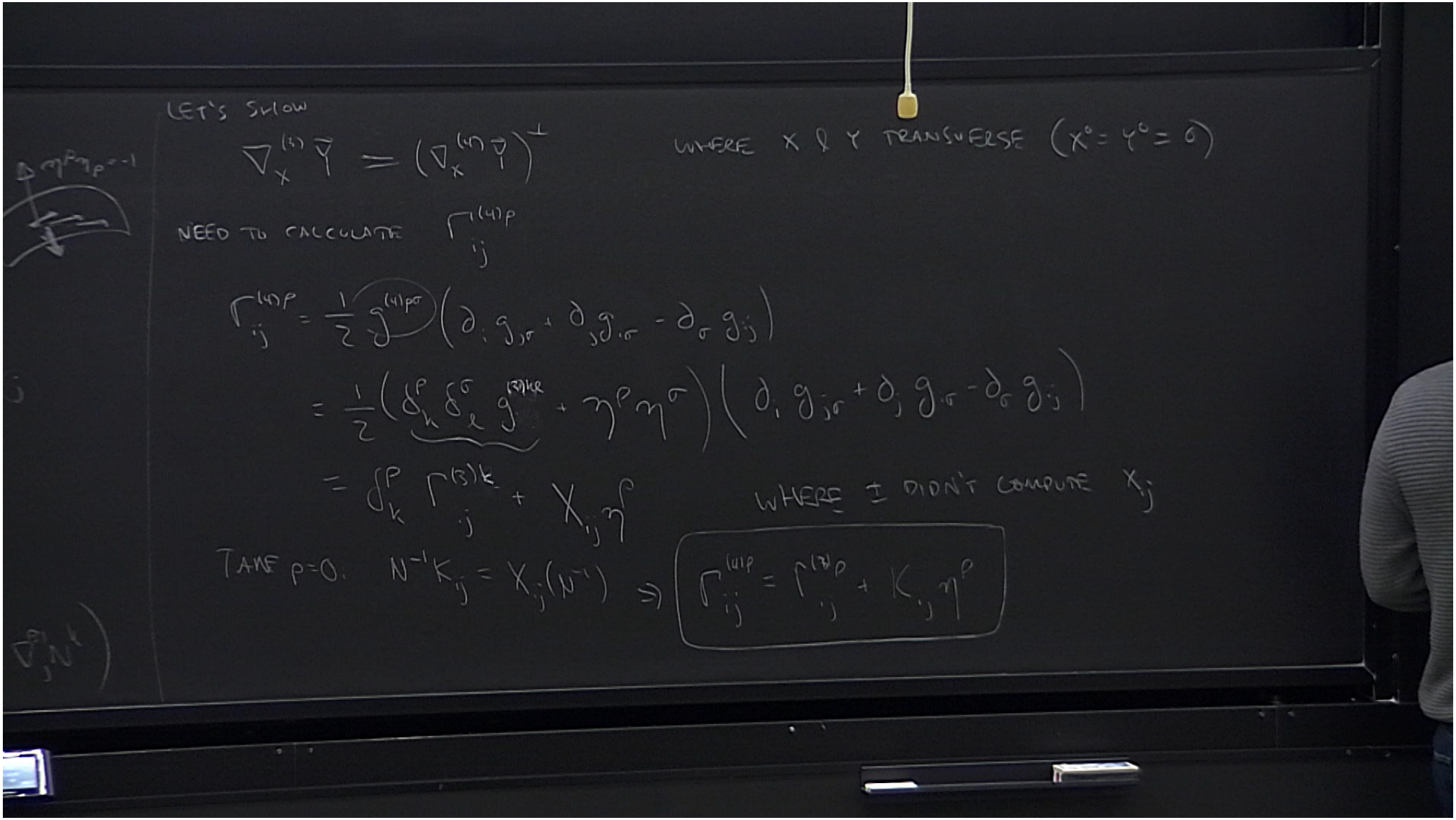
$$= \delta^\rho_k \Gamma^{(3)k}_{ij} + X_{ij}\eta^\rho$$

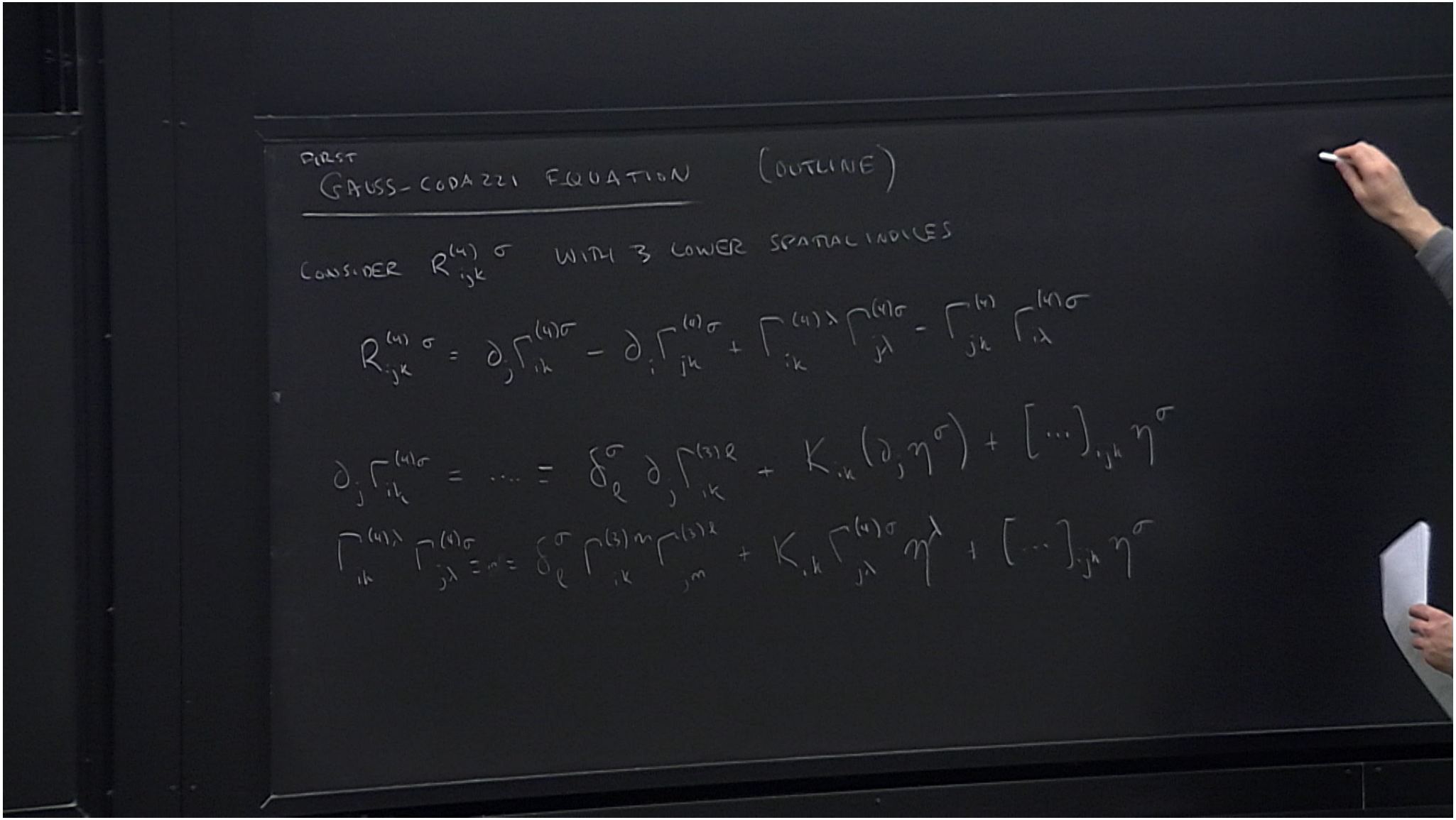
WHERE I DIDN'T COMPUTE X_{ij}

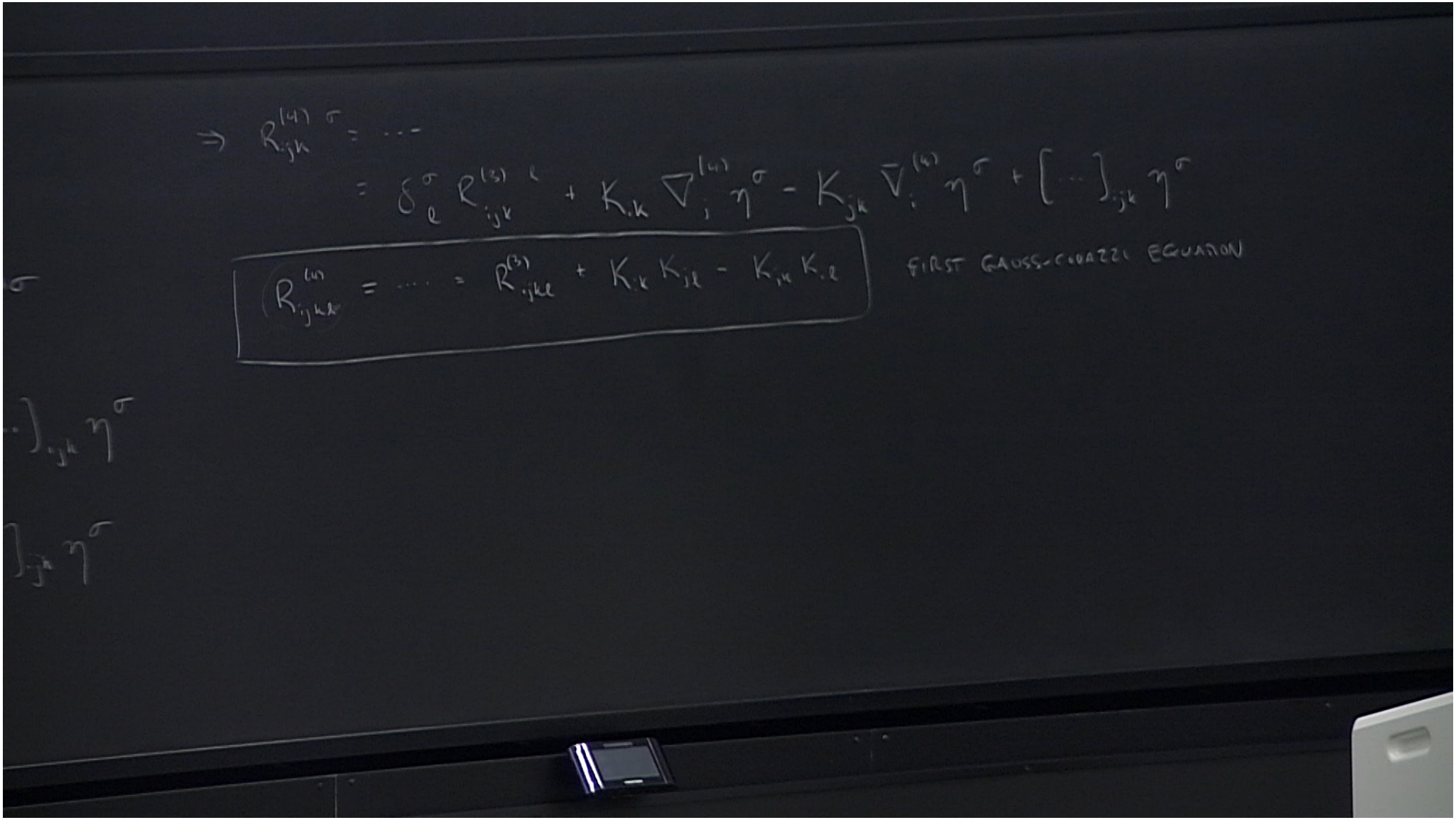
TAKE $\rho = 0$: $N^{-1}K_j = X_{ij}(N^{-1}) \Rightarrow$

$$\boxed{\Gamma^{(4)\rho}_{ij} = \Gamma^{(3)\rho}_{ij} + K_{ij}\eta^\rho}$$


LET'S SHOW
$\nabla_X^{(3)} \bar{Y} = (\nabla_X^{(4)} \bar{Y})^{\perp}$
WHERE X & Y TRANSVERSE $(X^0 = Y^0 = 0)$
NEED TO CALCULATE $\Gamma_{ij}^{(4)\rho}$
$\Gamma_{ij}^{(4)\rho} = \frac{1}{2} g^{(4)\rho\sigma} (\partial_i g_{j\sigma} + \partial_j g_{i\sigma} - \partial_\sigma g_{ij})$
$= \frac{1}{2} (\delta_k^\rho \delta_\ell^\sigma g^{(3)k\ell} + \eta^\rho \eta^\sigma)(\partial_i g_{j\sigma} + \partial_j g_{i\sigma} - \partial_\sigma g_{ij})$
$= \delta_k^\rho \Gamma_{ij}^{(3)k} + X_{ij} \eta^\rho$
WHERE I DIDN'T CON
TAKE $\rho = 0$. $N^{-1} K_{ij} = X_{ij} (N^{-1})$ $\Rightarrow$ $\Gamma_{ij}^{(4)\rho} = \Gamma_{ij}^{(3)\rho} + K_{ij} \eta^\rho$


LET'S SHOW
$\nabla^{(3)}_X \bar{Y} = (\nabla^{(4)}_X \bar{Y})^\perp$
WHERE X & Y TRANSVERSE $(X^0 = Y^0 = 0)$
NEED TO CALCULATE $\Gamma^{(4)p}_{ij}$
$\Gamma^{(4)p}_{ij} = \frac{1}{2} g^{(4)p\sigma}(\partial_i g_{j\sigma} + \partial_j g_{i\sigma} - \partial_\sigma g_{ij})$
$= \frac{1}{2}(\delta^p_k \delta^\sigma_\ell g^{(3)k\ell} + \eta^p \eta^\sigma)(\partial_i g_{j\sigma} + \partial_j g_{i\sigma} - \partial_\sigma g_{ij})$
$= \delta^p_k \Gamma^{(3)k}_{ij} + X_{ij}\eta^p$
WHERE I DIDN'T COMPUTE X_{ij}
TAKE $p=0$. $N^{-1}K_{ij} = X_{ij}(N^{-1})$ $\Rightarrow$ $\Gamma^{(4)p}_{ij} = \Gamma^{(3)p}_{ij} + K_{ij}\eta^p$


FIRST
GAUSS-CODAZZI EQUATION (OUTLINE)
CONSIDER $R^{(4)\sigma}_{ijk}$ WITH 3 LOWER SPATIAL INDICES
$R^{(4)\sigma}_{ijk} = \partial_j \Gamma^{(4)\sigma}_{ik} - \partial_i \Gamma^{(4)\sigma}_{jk} + \Gamma^{(4)\lambda}_{ik} \Gamma^{(4)\sigma}_{j\lambda} - \Gamma^{(4)}_{jk} \Gamma^{(4)\sigma}_{i\lambda}$
$\partial_j \Gamma^{(4)\sigma}_{ik} = \cdots = \delta^\sigma_\ell \, \partial_j \Gamma^{(3)\ell}_{ik} + K_{ik} (\partial_j n^\sigma) + [\cdots]_{ijk} n^\sigma$
$\Gamma^{(4)\lambda}_{ik} \Gamma^{(4)\sigma}_{j\lambda} = \cdots = \delta^\sigma_\ell \Gamma^{(3)m}_{ik} \Gamma^{(3)\ell}_{jm} + K_{ik} \Gamma^{(4)\sigma}_{j\lambda} n^\lambda + [\cdots]_{ijk} n^\sigma$


$\Rightarrow R^{(4)\,\sigma}_{\cdot jk\cdot} = \cdots$
$= \delta^{\sigma}_{\ell} R^{(3)\,\ell}_{\cdot jk\cdot} + K_{\cdot k} \nabla^{(4)}_{j} \eta^{\sigma} - K_{jk} \bar{\nabla}^{(4)}_{\cdot} \eta^{\sigma} + [\cdots]_{\cdot jk} \eta^{\sigma}$
$R^{(4)}_{\cdot jk\ell\cdot} = \cdots = R^{(3)}_{\cdot jk\ell\cdot} + K_{\cdot k} K_{j\ell} - K_{\cdot \ell} K_{jk}$
FIRST GAUSS-CODAZZI EQUATION

$$S_{EH} = \frac{M_{Pl}^2}{2} \int d^4x \sqrt{-g^{(4)}}\; g^{(4)\mu\rho}\, g^{(4)\nu\sigma}\, R^{(4)}_{\mu\nu\rho\sigma}$$

1) $$\int d^4x \sqrt{-g^{(4)}}\; \eta^\mu \eta^\rho \eta^\nu \eta^\sigma R^{(4)}_{\mu\nu\rho\sigma} = 0$$

2) $$\int d^4x \sqrt{-g^{(4)}}\; \eta^\mu \eta^\rho\, g^{(4)\nu\sigma} R^{(4)}_{\mu\nu\rho\sigma} = \int d^4x \sqrt{-g^{(4)}}\; \eta^\mu \delta^\nu_\sigma R^{(4)}_{\mu\nu\rho}{}^\sigma \eta^\rho$$

$$= \int d^4x \sqrt{-g^{(4)}}\; \eta^\mu \delta^\nu_\sigma \left(-\nabla^{(4)}_\mu \nabla^{(4)}_\nu + \nabla^{(4)}_\nu \nabla^{(4)}_\mu \right) \eta^\sigma$$

$$= \int d^4x \sqrt{-g^{(4)}}\; \eta^\mu \left(-\nabla^{(4)}_\mu \nabla^{(4)}_\nu + \nabla^{(4)}_\nu \nabla^{(4)}_\mu \right) \eta^\nu$$

$$= \int d^4x \sqrt{-g^{(4)}} \left(\nabla^{(4)}_\mu \eta^\mu \right)\left(\nabla^{(4)}_\nu \eta^\nu \right) - \left(\nabla^{(4)}_\nu \eta^\mu \right)\left(\nabla^{(4)}_\mu \right.$$

$$S_{EH} = \frac{M_{Pl}^2}{2} \int d^4x \sqrt{-g^{(4)}}\, g^{(4)\mu\rho}\, g^{(4)\nu\sigma}\, R^{(4)}_{\mu\nu\rho\sigma}$$

1) $$\int d^4x \sqrt{-g^{(4)}}\, \eta^\mu \eta^\rho \eta^\nu \eta^\sigma R^{(4)}_{\mu\nu\rho\sigma} = 0$$

2) $$\int d^4x \sqrt{-g^{(4)}}\, \eta^\mu \eta^\rho\, g^{(4)\nu\sigma}\, R^{(4)}_{\mu\nu\rho\sigma} = \int d^4x \sqrt{-g^{(4)}}\, \eta^\mu\, \delta^\nu_\sigma\, R^{(4)}_{\mu\nu\rho}{}^{\sigma}\, \eta^\rho$$

$$= \int d^4x \sqrt{-g^{(4)}}\, \eta^\mu \delta^\nu_\sigma \left(-\nabla^{(4)}_\mu \nabla^{(4)}_\nu + \nabla^{(4)}_\nu \nabla^{(4)}_\mu \right) \eta^\sigma$$

$$= \int d^4x \sqrt{-g^{(4)}}\, \eta^\mu \left(-\nabla^{(4)}_\mu \nabla^{(4)}_\nu + \nabla^{(4)}_\nu \nabla^{(4)}_\mu \right) \eta^\nu$$

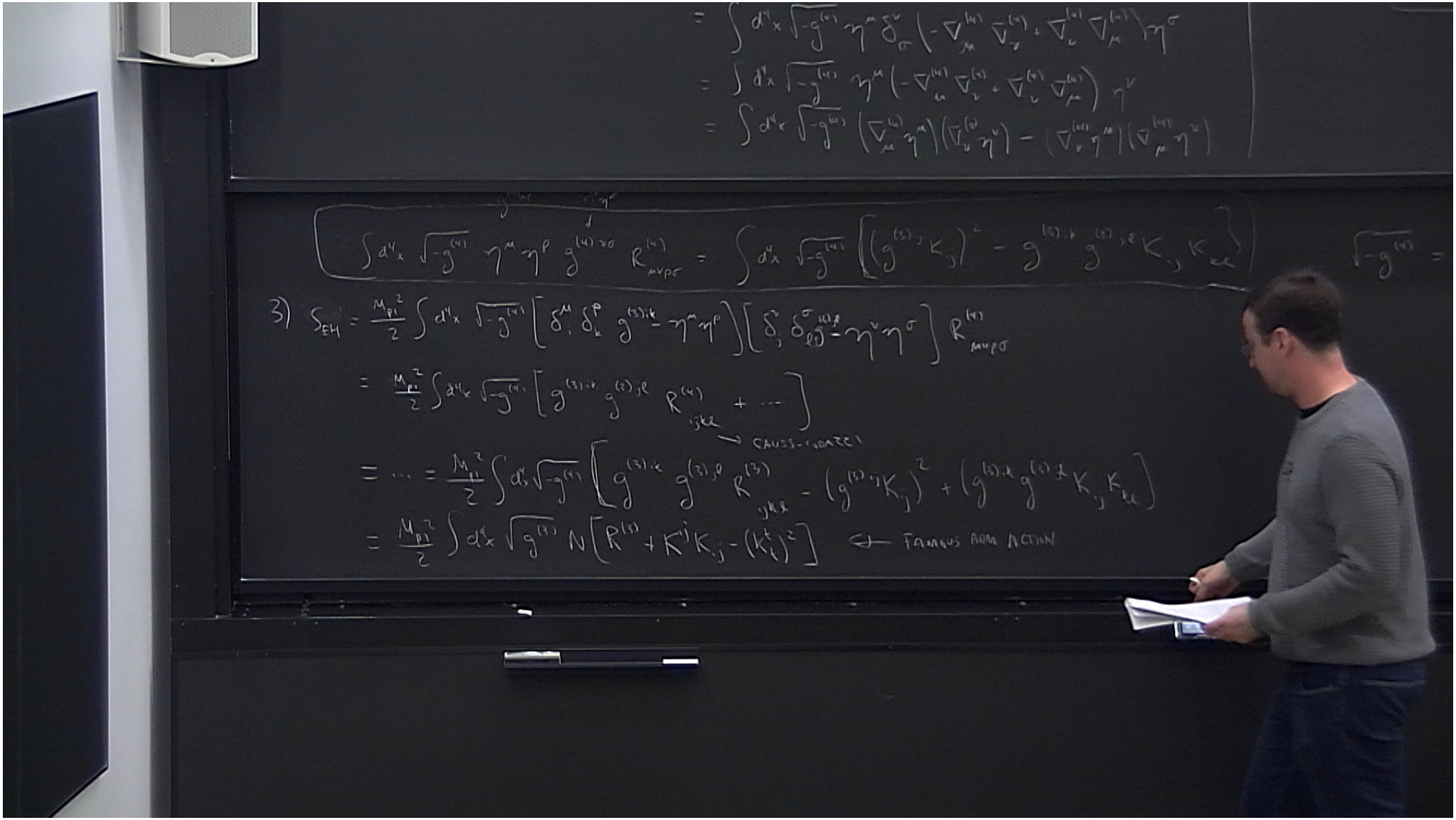
$$= \int d^4x \sqrt{-g^{(4)}} \left(\nabla^{(4)}_\mu \eta^\mu \right)\left(\nabla^{(4)}_\nu \eta^\nu \right) - \left(\nabla^{(4)}_\nu \eta^\mu \right)\left(\nabla^{(4)}_\mu \eta^\nu \right)$$

$$\left(\nabla^{(4)}_{\mu}\eta^{\mu}\right) = g^{(4)\mu\nu}\left(\nabla^{(4)}_{\mu}\eta_{\nu}\right)$$

$$= \left(\delta^{\mu}_{i}\delta^{\nu}_{j}\,g^{(3)ij} - \eta^{\mu}\eta^{\nu}\right)\left(\nabla^{(4)}_{\mu}\eta_{\nu}\right)$$

$$= \boxed{g^{(3)ij}K_{ij} + 0} \qquad \text{BECAUSE} \quad \left(\eta^{\nu}\nabla^{(4)}_{\mu}\eta_{\nu}\right) = \frac{1}{2}\nabla^{(4)}_{\mu}\underbrace{\left(\eta^{\nu}\eta_{\nu}\right)}_{-1} = 0$$

SIMILARLY, $\boxed{\left(\nabla^{(4)}_{\nu}\eta^{\mu}\right)\left(\nabla^{(4)}_{\mu}\eta^{\nu}\right) = g^{(3)ik}g^{(3)jl}K_{ij}K_{kl}}$


$= \int d^4x \sqrt{-g^{(4)}}\, \eta^\mu \delta^\nu_\sigma \left(-\nabla^{(4)}_\mu \nabla^{(4)}_\nu + \nabla^{(4)}_\nu \nabla^{(4)}_\mu \right) \eta^\sigma$
$= \int d^4x \sqrt{-g^{(4)}}\, \eta^\mu \left(-\nabla^{(4)}_\mu \nabla^{(4)}_\nu + \nabla^{(4)}_\nu \nabla^{(4)}_\mu \right) \eta^\nu$
$= \int d^4x \sqrt{-g^{(4)}} \left(\nabla^{(4)}_\mu \eta^\mu \right) \left(\nabla^{(4)}_\nu \eta^\nu \right) - \left(\nabla^{(4)}_\nu \eta^\mu \right) \left(\nabla^{(4)}_\mu \eta^\nu \right)$
$\int d^4x \sqrt{-g^{(4)}}\, \eta^\mu \eta^\rho\, g^{(4)\,\sigma\delta}\, R^{(4)}_{\mu\nu\rho\sigma} = \int d^4x \sqrt{-g^{(4)}} \left[\left(g^{(3)\,ij} K_{ij} \right)^2 - g^{(3)\,ik} g^{(3)\,jl} K_{ij} K_{kl} \right]$
$\sqrt{-g^{(4)}} =$
3) $S_{EH} = \frac{M_{pl}^2}{2} \int d^4x \sqrt{-g^{(4)}} \left[\delta^\mu_i \delta^\rho_k\, g^{(3)\,ik} - \eta^\mu \eta^\rho \right] \left[\delta^\sigma_j \delta^\delta_l\, g^{(3)\,jl} - \eta^\nu \eta^\sigma \right] R^{(4)}_{\mu\nu\rho\sigma}$
$= \frac{M_{pl}^2}{2} \int d^4x \sqrt{-g^{(4)}} \left[g^{(3)\,ik}\, g^{(3)\,jl}\, R^{(4)}_{ijkl} + \cdots \right]$
→ GAUSS-CODAZZI
$= \cdots = \frac{M_{pl}^2}{2} \int d^4x \sqrt{-g^{(4)}} \left[g^{(3)\,ik}\, g^{(3)\,jl}\, R^{(3)}_{ijkl} - \left(g^{(3)\,ij} K_{ij} \right)^2 + \left(g^{(3)\,ik}\, g^{(3)\,jl}\, K_{ij} K_{kl} \right) \right]$
$= \frac{M_{pl}^2}{2} \int d^4x \sqrt{g^{(3)}}\, N \left[R^{(3)} + K^{ij} K_{ij} - (K^i_i)^2 \right]$ ← FAMOUS ADM ACTION

$$= \int d^4x \sqrt{-g^{(4)}} \left(\nabla^{(4)}_\mu n^\mu\right)\left(\nabla^{(4)}_\nu n^\nu\right)$$

1) WRITE ACTION IN ADM VARIABLES: $S = S_{EH} + S_m$

$$S_{EH} = \frac{M_{Pl}^2}{2} \int d^4x \sqrt{g^{(3)}} \left(N R^{(3)} + N^{-1} E_{ij} E^{ij} - N^{-1} E^2 \right)$$

$$E_{ij} = \frac{1}{2}\left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\right)$$

$$E = g^{ij} E_{ij}$$

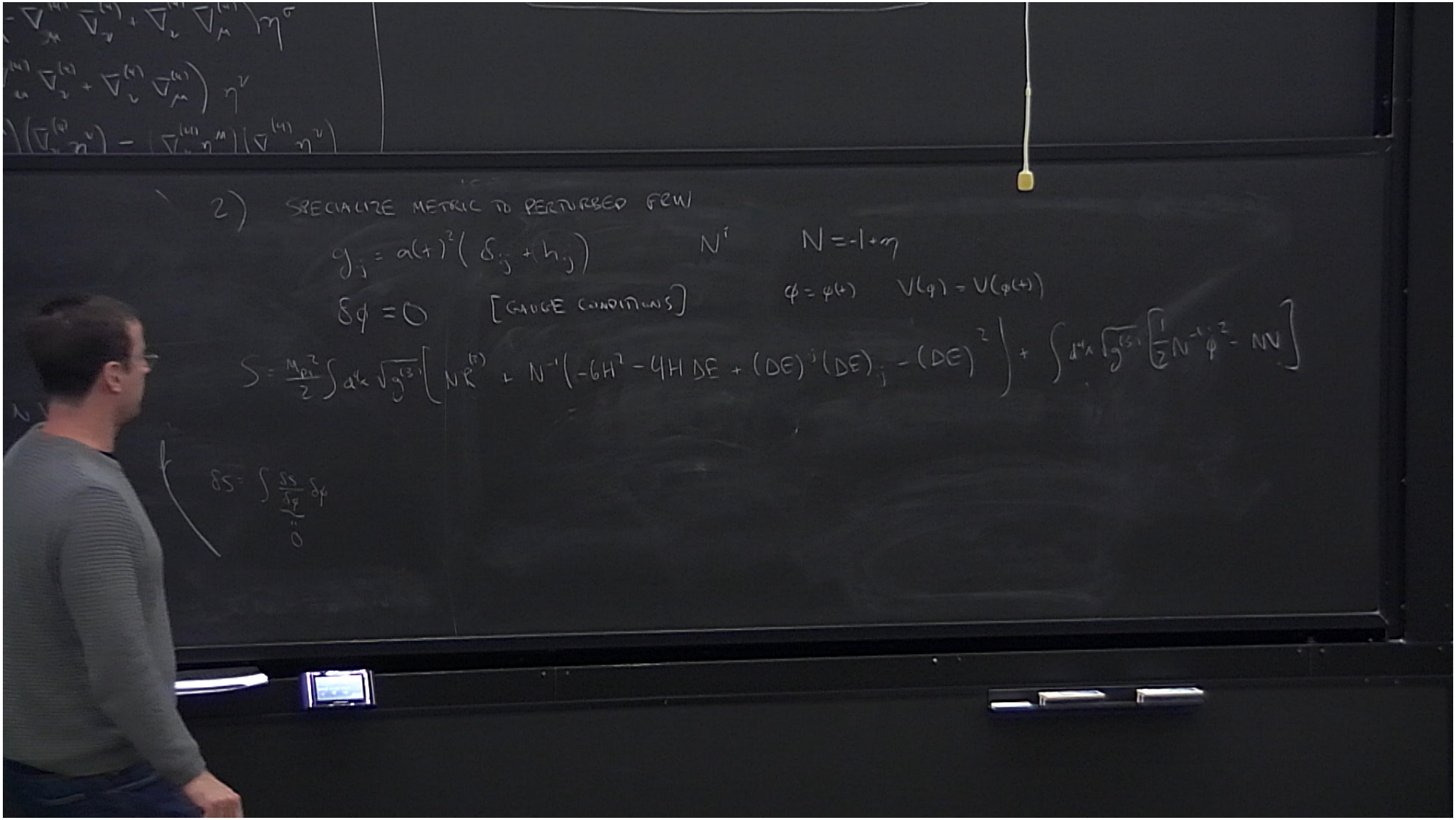
$$S_m = \int d^4x \sqrt{g^{(3)}} \left(\frac{1}{2} N^{-1} \left(\dot{\phi} - N^i \partial_i \phi\right)^2 - \frac{1}{2} g^{ij} (\partial_i \phi)(\partial_j \phi) - N V(\phi) \right)$$

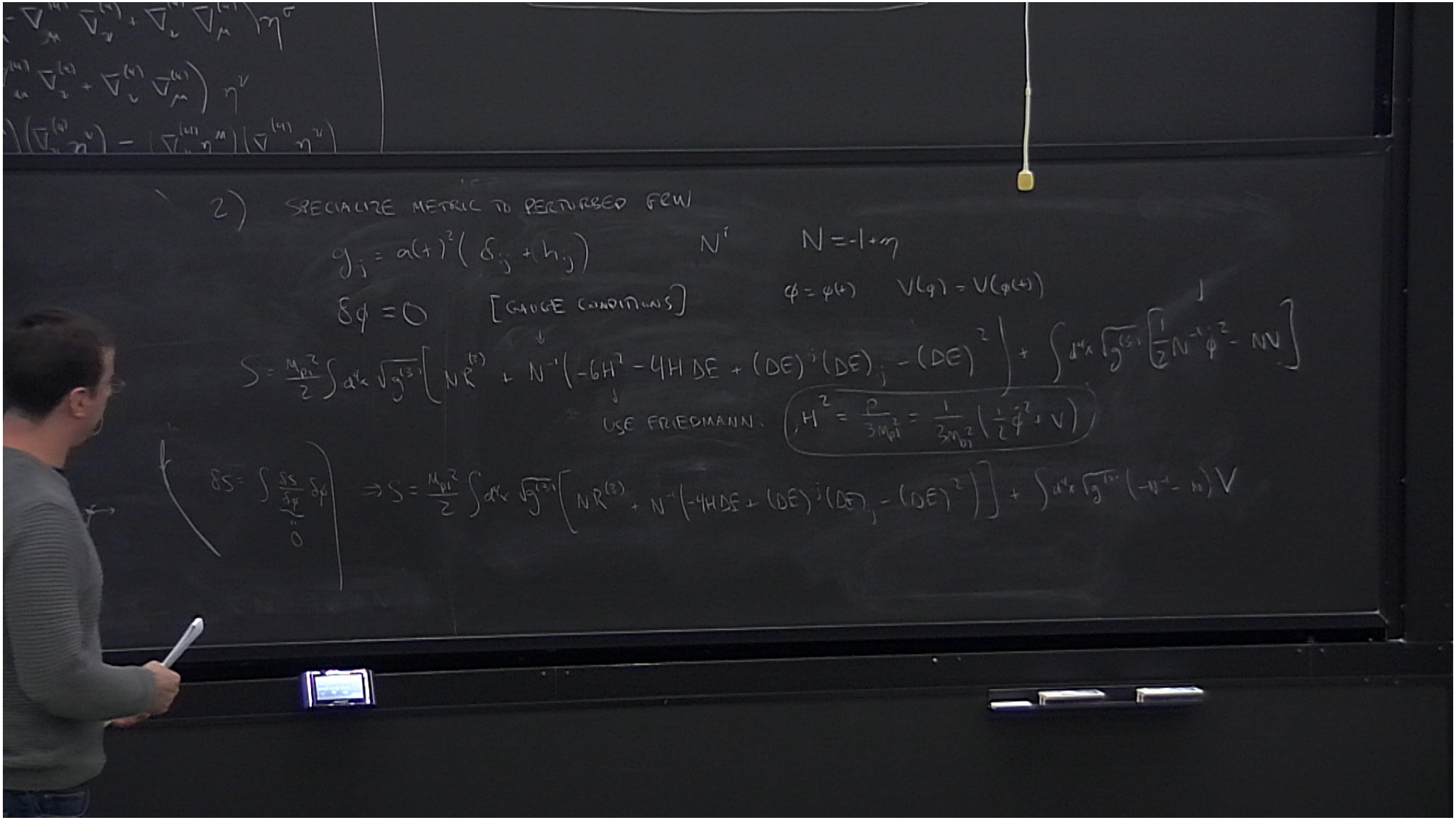
2) SPECIALIZE METRIC TO PERTURBED FRW

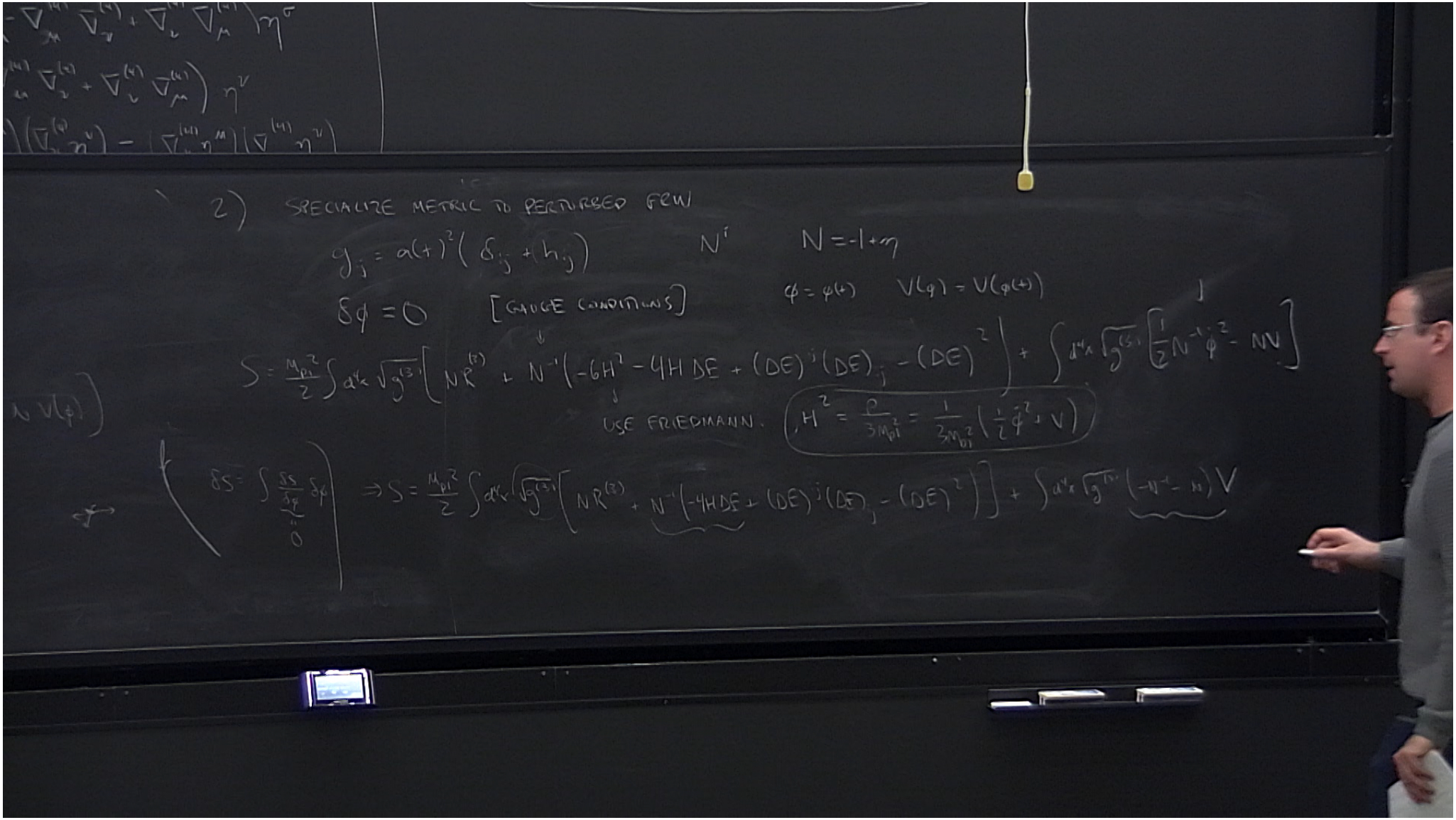
$g_{ij} = a(t)^2(\delta_{ij} + h_{ij})$ $\quad N^i$ $\quad N = -1 + \eta$

$\delta\phi = 0$ [GAUGE CONDITIONS] $\quad \phi = \phi(t)$ $\quad V(\phi) = V(\phi(t))$

$$S = \frac{M_{Pl}^2}{2}\int d^4x \sqrt{g^{(3)}}\left(N R^{(3)} + N^{-1}\left(-6H^2 - 4H\,\delta E + (\delta E)^{ij}(\delta E)_{ij} - (\delta E)^2\right)\right) + \int d^4x \sqrt{g^{(4)}}\left[\frac{1}{2}N^{-1}\dot{\phi}^2 - NV\right]$$


2) SPECIALIZE METRIC TO PERTURBED FRW
$g_{ij} = a(t)^2(\delta_{ij} + h_{ij})$ N^i $N = -1 + \eta$
$\delta\phi = 0$ [GAUGE CONDITIONS] $\phi = \phi(t)$ $V(\phi) = V(\phi(t))$
$S = \frac{M_{Pl}^2}{2}\int d^4x \sqrt{g^{(3)}}\left(N R^{(3)} + N^{-1}(-6H^2 - 4H\Delta E + (\Delta E)^{ij}(\Delta E)_{ij} - (\Delta E)^2)\right) + \int d^4x \sqrt{g^{(3)}}\left[\frac{1}{2}N^{-1}\dot{\phi}^2 - NV\right]$
$\delta S = \int \frac{\delta S}{\delta \phi}\delta\phi$
$= 0$


2) SPECIALIZE METRIC TO PERTURBED FRW
$g_{ij} = a(t)^2(\delta_{ij} + h_{ij})$ N^i $N = -1 + \eta$
$\delta\phi = 0$ [GAUGE CONDITIONS] $\phi = \phi(t)$ $V(\phi) = V(\phi(t))$
USE FRIEDMANN.
$H^2 = \frac{\rho}{3M_{Pl}^2} = \frac{1}{3M_{Pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + V\right)$


2) SPECIALIZE METRIC TO PERTURBED FRW
$g_{ij} = a(t)^2(\delta_{ij} + h_{ij})$ N^i $N = -1 + \eta$
$\delta\phi = 0$ [GAUGE CONDITIONS] $\phi = \phi(t)$ $V(\phi) = V(\phi(t))$
USE FRIEDMANN.
$\Rightarrow S = \frac{M_{Pl}^2}{2}\int d^4x \sqrt{g^{(3)}}\left[N R^{(3)} + N^{-1}\left(-4H\Delta E + (\Delta E)^{ij}(\Delta E)_{ij} - (\Delta E)^2\right)\right] + \int d^4x \sqrt{g^{(4)}}\left(-N^{-1} - N\right)V$

$$= \int d^4x \sqrt{-g^{(4)}} \left(\nabla^{(4)}_\mu n^\mu \right) \left(\nabla^{(4)}_\nu n^\nu \right) - \left(\nabla^{(4)}_\mu n^\mu \right) \left(\nabla^{(4)} n^\nu \right)$$

1) WRITE ACTION IN ADM VARIABLES: $S = S_{EH} + S_m$

2)

$$S_{EH} = \frac{M_{Pl}^2}{2} \int d^4x \sqrt{g^{(3)}} \left(N R^{(3)} + N^{-1} E_{ij} E^{ij} - N^{-1} E^2 \right)$$

$$E_{ij} = \frac{1}{2} \left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i \right)$$

$$E = g^{ij} E_{ij}$$

$$S_m = \int d^4x \sqrt{g^{(3)}} \left(\frac{1}{2} N^{-1} \left(\dot{\varphi} - N^i \partial_i \varphi \right)^2 - \frac{1}{2} g^{ij} (\partial_i \varphi)(\partial_j \varphi) - N V(\varphi) \right)$$

$$\Delta E_{ij} = \frac{1}{2} a^2 \dot{h}_{ij} - \frac{1}{2} \left(\nabla_i N_j + \nabla_j N_i \right)$$

$$\Delta E = g^{ij} \Delta E_{ij}$$

$$S = \frac{M_{Pl}^2}{2}$$

$$\delta S = \int \frac{\delta S}{\delta g_{ij}} \delta g_{ij} = 0$$