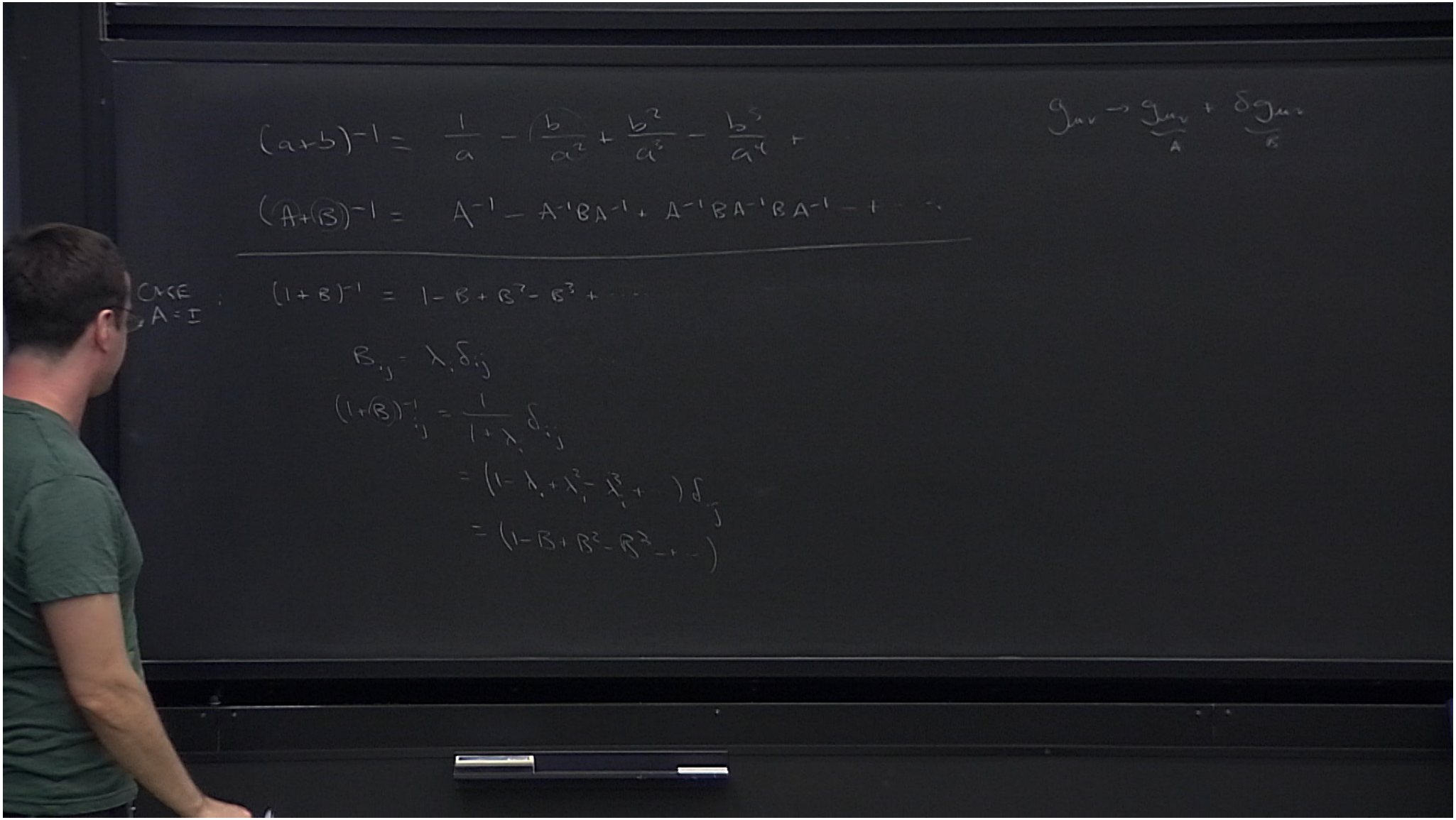
Title: PSI 16/17 Explorations in Cosmology (Kendrick Smith) - Lecture 10

Date: Apr 24, 2017 10:15 AM

URL: http://pirsa.org/17040063

Abstract:


$(a+b)^{-1} = \frac{1}{a} - \frac{b}{a^2} + \frac{b^2}{a^3} - \frac{b^3}{a^4} + \cdots$
$(A+B)^{-1} = A^{-1} - A^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1} - + \cdots$
$g_{\mu\nu} \to \underbrace{g_{\mu\nu}}_{A} + \underbrace{\delta g_{\mu\nu}}_{B}$
CASE $A = I$:
$(1+B)^{-1} = 1 - B + B^2 - B^3 + \cdots$
$B_{ij} = \lambda_i \delta_{ij}$
$(1+B)^{-1}_{ij} = \frac{1}{1+\lambda_i}\delta_{ij}$
$= (1 - \lambda_i + \lambda_i^2 - \lambda_i^3 + \cdots)\delta_{ij}$
$= (1 - B + B^2 - B^3 + \cdots)$

$$(a+b)^{-1} = \frac{1}{a} - \left(\frac{b}{a^2} + \frac{b^2}{a^3} - \frac{b^3}{a^4} + \right.$$

$$\boxed{(A+B)^{-1} = A^{-1} - A^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1} - + \cdots}$$

CASE $A = I$: $(1+B)^{-1} = 1 - B + B^2 - B^3 + \cdots$

$$B_{ij} = \lambda_i \delta_{ij}$$

$$(1+B)^{-1}_{ij} = \frac{1}{1+\lambda_i}\delta_{ij}$$

$$= (1 - \lambda_i + \lambda_i^2 - \lambda_i^3 + \cdots)\delta_{ij}$$

$$= (1 - B + B^2 - B^3 - + \cdots)$$

$$(A+B)^{-1} = \left[A^{-1}(1+A^{-1}B)\right]$$

$$= (1+A^{-1}B)^{-1} A^{-1}$$

$$= (1 - A^{-1}B + A^{-1}BA^{-1}B - \cdots)A^{-1}$$

$$= A^{-1} - A^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1} - \cdots$$

$$g_{\mu\nu} \to \underbrace{g_{\mu\nu}}_{A} + \underbrace{\delta g_{\mu\nu}}_{B}$$

$$\sqrt{-g}$$

$$\delta g^{\mu\nu} = -g^{\mu\nu}(\delta g_{\rho\sigma})\, g^{\rho\sigma}$$

$$\delta(A+B)^{-1} = -A^{-1}BA^{-1}$$

$\delta g_{\mu\nu}$

$g^{\mu\nu}\,(\delta g_{\rho\sigma})\,g^{\nu\sigma}$

$(A+B)^{-1} = -A^{-1}BA^{-1}$

$(A+B)^{-1} = A^{-1} - \ldots$

$\mathrm{Det}(A+B) = ?$

$\mathrm{Det}(I+B) = \prod_i (1+\lambda_i)$ IF $B_{ij} = \lambda_i \delta_{ij}$

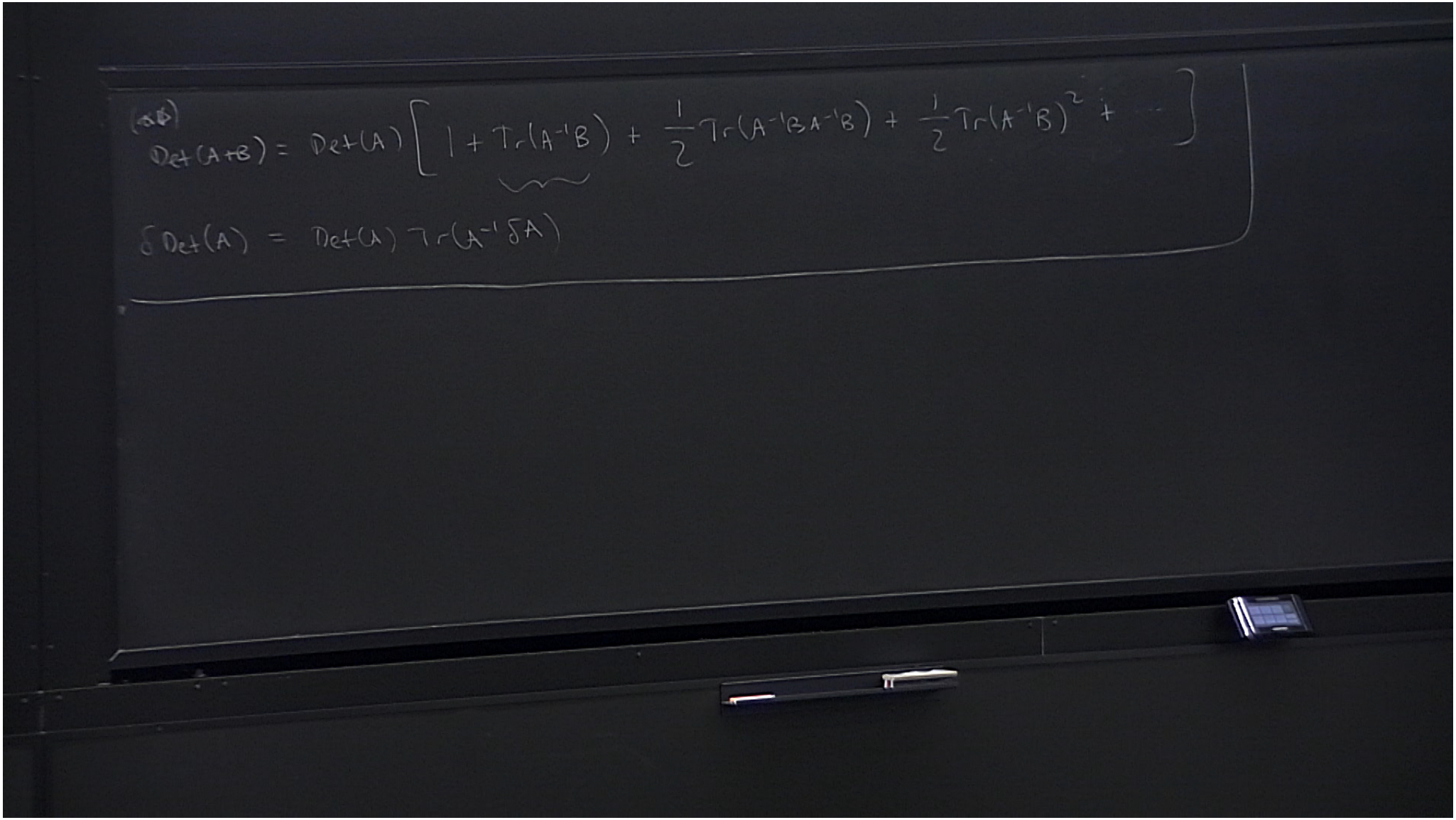
$= \exp\left[\sum_i \log(1+\lambda_i)\right]$

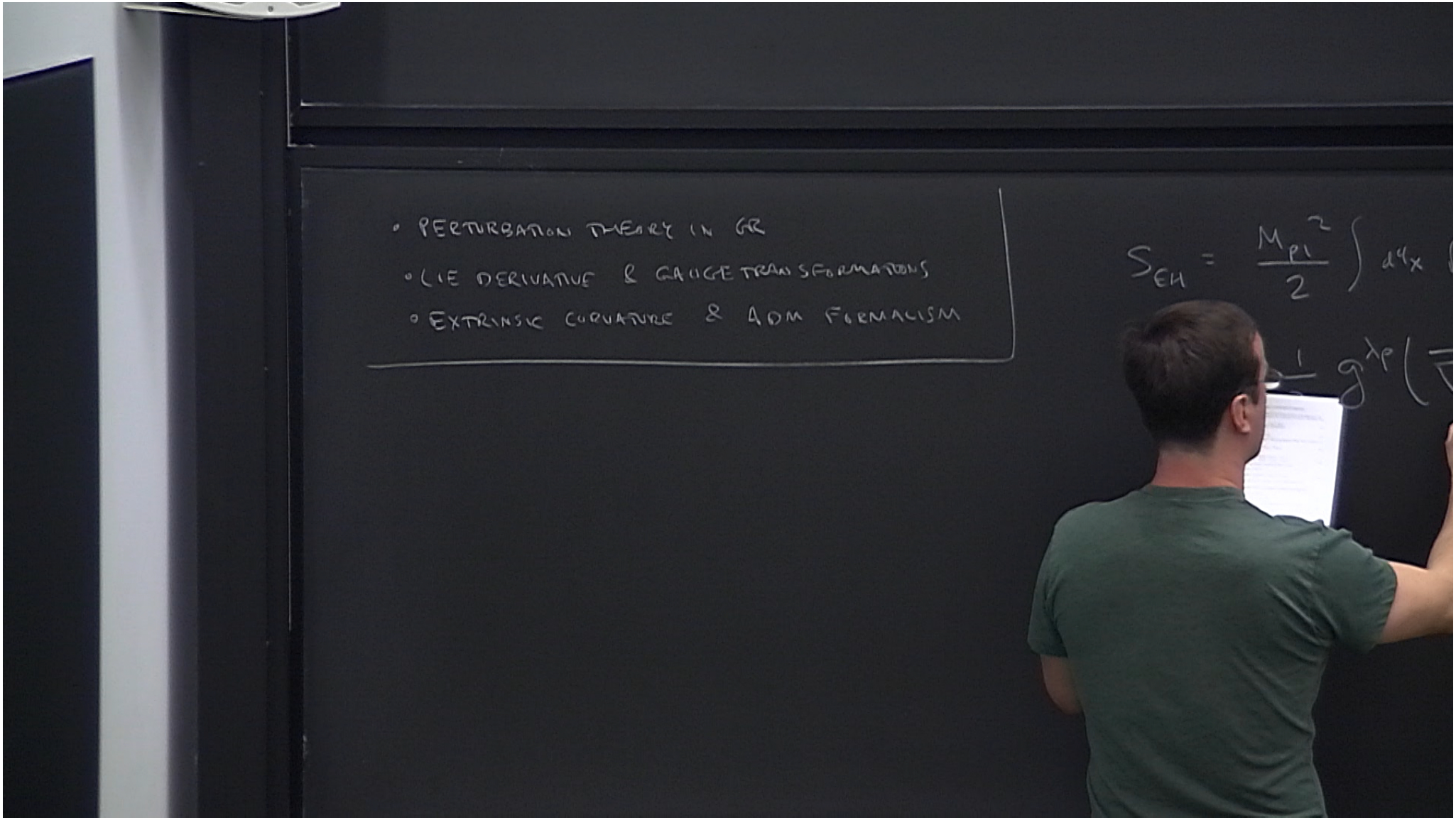
$= \exp\left[\sum_i \lambda_i - \frac{1}{2}\sum_i \lambda_i^2 + \frac{1}{3}\sum_i \lambda_i^3 - \ldots\right]$

$= \exp\left[\mathrm{Tr}(B) - \frac{1}{2}\mathrm{Tr}(B^2) + \frac{1}{3}\mathrm{Tr}(B^3) - \ldots\right]$

$\mathrm{Det}(A+B) = \mathrm{Det}\left[A(1+A^{-1}B)\right]$

$= \mathrm{Det}(A)\exp\left[\mathrm{Tr}(A^{-1}B) - \frac{1}{2}\mathrm{Tr}(A^{-1}BA^{-1}B) + \frac{1}{3}\mathrm{Tr}(A^{-1}BA^{-1}BA^{-1}B)\right]$


(AB)
Det(A+B) = Det(A) [1 + Tr(A^-1 B) + 1/2 Tr(A^-1 B A^-1 B) + 1/2 Tr(A^-1 B)^2 + ...]
δ Det(A) = Det(A) Tr(A^-1 δA)


• PERTURBATION THEORY IN GR
• LIE DERIVATIVE & GAUGE TRANSFORMATIONS
• EXTRINSIC CURVATURE & ADM FORMALISM
$S_{EH} = \frac{M_{Pl}^2}{2} \int d^4x$

$$S_{EH} = \frac{M_{Pl}^2}{2}\int d^4x \sqrt{-g}\, g^{\mu\nu}\left(\bar{R}_{\mu\nu} - C^{\rho}_{\mu\nu} C^{\sigma}_{\rho\sigma} + C^{\sigma}_{\mu\rho} C^{\rho}_{\nu\sigma}\right) \qquad g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

$$C^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left(\bar{\nabla}_{\mu} g_{\nu\rho} + \bar{\nabla}_{\nu} g_{\mu\rho} - \bar{\nabla}_{\rho} g_{\mu\nu}\right) = \mathcal{O}(h)$$

$$= \Gamma^{\lambda}_{\mu\nu} - \bar{\Gamma}^{\lambda}_{\mu\nu}$$

ALL INDICES HAVE BEEN RAISED/LOWERED WITH $\bar{g}$

$$S_{EH} = \frac{M_{Pl}^2}{2}\int d^4x \sqrt{-\bar{g}}\left(\bar{R} - h^{\mu\nu}\left(\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R}\right) + \left(h^{\mu\rho}h^{\nu}_{\rho} - \frac{1}{2}h^{\rho}_{\rho}h^{\mu\nu}\right)\left(\bar{R}_{\mu\nu} - \frac{1}{4}\bar{g}_{\mu\nu}\bar{R}\right)\right.$$
$$\left. - \frac{1}{2}\left(\bar{\nabla}_{\mu}h^{\mu\rho}\right)\left(\bar{\nabla}_{\rho}h^{\sigma}_{\sigma}\right) + \frac{1}{4}\left(\bar{\nabla}^{\rho}h^{\mu}_{\mu}\right)\left(\bar{\nabla}_{\rho}h^{\sigma}_{\sigma}\right) + \frac{1}{2}\left(\bar{\nabla}_{\mu}h_{\rho\sigma}\right)\left(\bar{\nabla}^{\sigma}h^{\mu\rho}\right) - \frac{1}{4}\left(\bar{\nabla}_{\sigma}h_{\mu\rho}\right)\left(\bar{\nabla}^{\sigma}h^{\mu\rho}\right)\right] + \mathcal{O}(h^3)$$

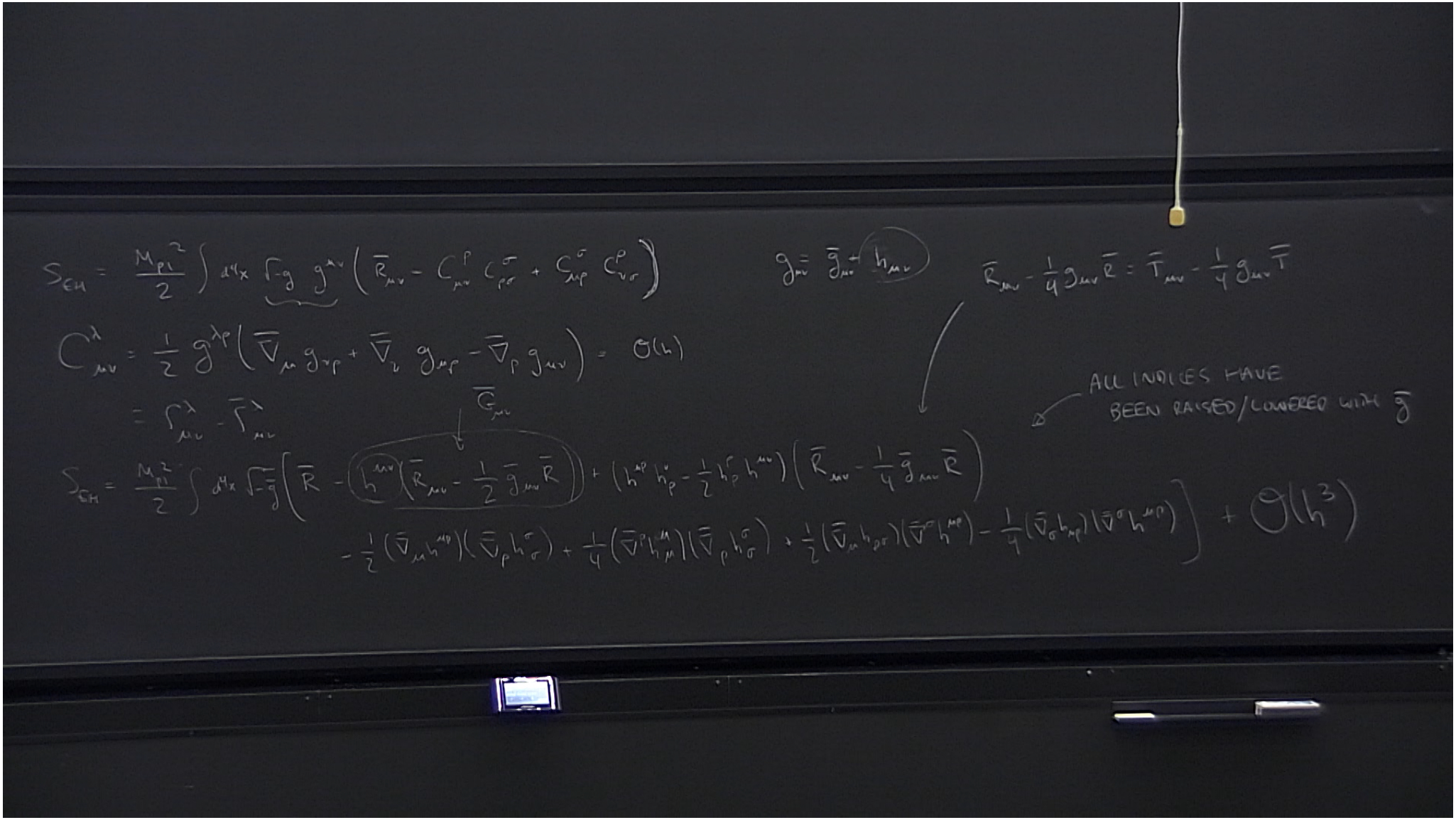
$$S_{EH} = \frac{M_{Pl}^2}{2} \int d^4x \sqrt{-g}\, g^{\mu\nu} \left(\bar{R}_{\mu\nu} - C^{\rho}_{\mu\nu} C^{\sigma}_{\rho\sigma} + C^{\sigma}_{\mu\rho} C^{\rho}_{\nu\sigma} \right) \qquad g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

$$C^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\bar{\nabla}_{\mu} g_{\nu\rho} + \bar{\nabla}_{\nu} g_{\mu\rho} - \bar{\nabla}_{\rho} g_{\mu\nu} \right) = \mathcal{O}(h)$$

$$= \Gamma^{\lambda}_{\mu\nu} - \bar{\Gamma}^{\lambda}_{\mu\nu}$$

ALL INDICES HAVE BEEN RAISED/LOWERED WITH $\bar{g}$

$$S_{EH} = \frac{M_{Pl}^2}{2} \int d^4x \sqrt{-\bar{g}} \left(\bar{R} - h^{\mu\nu} \left(\bar{R}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{R} \right) + \left(h^{\mu\rho} h^{\nu}_{\rho} - \frac{1}{2} h^{\rho}_{\rho} h^{\mu\nu} \right) \left(\bar{R}_{\mu\nu} - \frac{1}{4} \bar{g}_{\mu\nu} \bar{R} \right) \right.$$
$$\left. - \frac{1}{2} \left(\bar{\nabla}_{\mu} h^{\mu\rho} \right) \left(\bar{\nabla}_{\rho} h^{\sigma}_{\sigma} \right) + \frac{1}{4} \left(\bar{\nabla}^{\rho} h^{\mu}_{\mu} \right) \left(\bar{\nabla}_{\rho} h^{\sigma}_{\sigma} \right) + \frac{1}{2} \left(\bar{\nabla}_{\mu} h_{\rho\sigma} \right) \left(\bar{\nabla}^{\sigma} h^{\mu\rho} \right) - \frac{1}{4} \left(\bar{\nabla}_{\sigma} h_{\mu\rho} \right) \left(\bar{\nabla}^{\sigma} h^{\mu\rho} \right) \right] + \mathcal{O}(h^3)$$


$S_{EH} = \frac{M_{Pl}^2}{2}\int d^4x \sqrt{-g}\, g^{\mu\nu}\left(\bar{R}_{\mu\nu} - C^{\rho}_{\mu\nu} C^{\sigma}_{\rho\sigma} + C^{\sigma}_{\mu\rho} C^{\rho}_{\nu\sigma}\right)$
$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$
$\bar{R}_{\mu\nu} - \frac{1}{4} g_{\mu\nu} \bar{R} = \bar{T}_{\mu\nu} - \frac{1}{4} g_{\mu\nu} \bar{T}$
$C^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left(\bar{\nabla}_{\mu} g_{\nu\rho} + \bar{\nabla}_{\nu} g_{\mu\rho} - \bar{\nabla}_{\rho} g_{\mu\nu}\right) = \mathcal{O}(h)$
$= \Gamma^{\lambda}_{\mu\nu} - \bar{\Gamma}^{\lambda}_{\mu\nu}$
ALL INDICES HAVE BEEN RAISED/LOWERED WITH $\bar{g}$
$\bar{G}_{\mu\nu}$
$S_{EH} = \frac{M_{Pl}^2}{2}\int d^4x \sqrt{-\bar{g}}\left(\bar{R} - h^{\mu\nu}\left(\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R}\right) + \left(h^{\mu\rho}h^{\nu}_{\rho} - \frac{1}{2}h^{\rho}_{\rho}h^{\mu\nu}\right)\left(\bar{R}_{\mu\nu} - \frac{1}{4}\bar{g}_{\mu\nu}\bar{R}\right)\right.$
$\left. - \frac{1}{2}\left(\bar{\nabla}_{\mu}h^{\mu\rho}\right)\left(\bar{\nabla}_{\rho}h^{\sigma}_{\sigma}\right) + \frac{1}{4}\left(\bar{\nabla}^{\rho}h^{\mu}_{\mu}\right)\left(\bar{\nabla}_{\rho}h^{\sigma}_{\sigma}\right) + \frac{1}{2}\left(\bar{\nabla}_{\mu}h_{\rho\sigma}\right)\left(\bar{\nabla}^{\sigma}h^{\mu\rho}\right) - \frac{1}{4}\left(\bar{\nabla}_{\sigma}h_{\mu\rho}\right)\left(\bar{\nabla}^{\sigma}h^{\mu\rho}\right)\right] + \mathcal{O}(h^3)$

$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$ $\qquad$ $g_{\mu\nu} = g_{\mu\nu}(t)$

$\phi = \bar{\phi} + \delta\phi$ $\qquad$ $\phi = \phi(t)$





$\phi = \phi(x,t)$

$= \phi(t) + \delta\phi$

SOME MODES $\begin{pmatrix} \delta g_{\mu\nu} \\ \delta\phi \end{pmatrix}$ ARE "GAUGE MODES"

WHAT ARE THE GAUGE MODES?

$\Rightarrow$ LIE DERIVATIVE

SUPPOSE S^{μ}_{ν} IS A TENSOR

$$S'^{\mu}_{\nu}(x') = \left(\frac{\partial x'^{\mu}}{\partial x^{\rho}}\right)\left(\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\right) S^{\rho}_{\sigma}(x)$$ TRANSFORMATION LAW

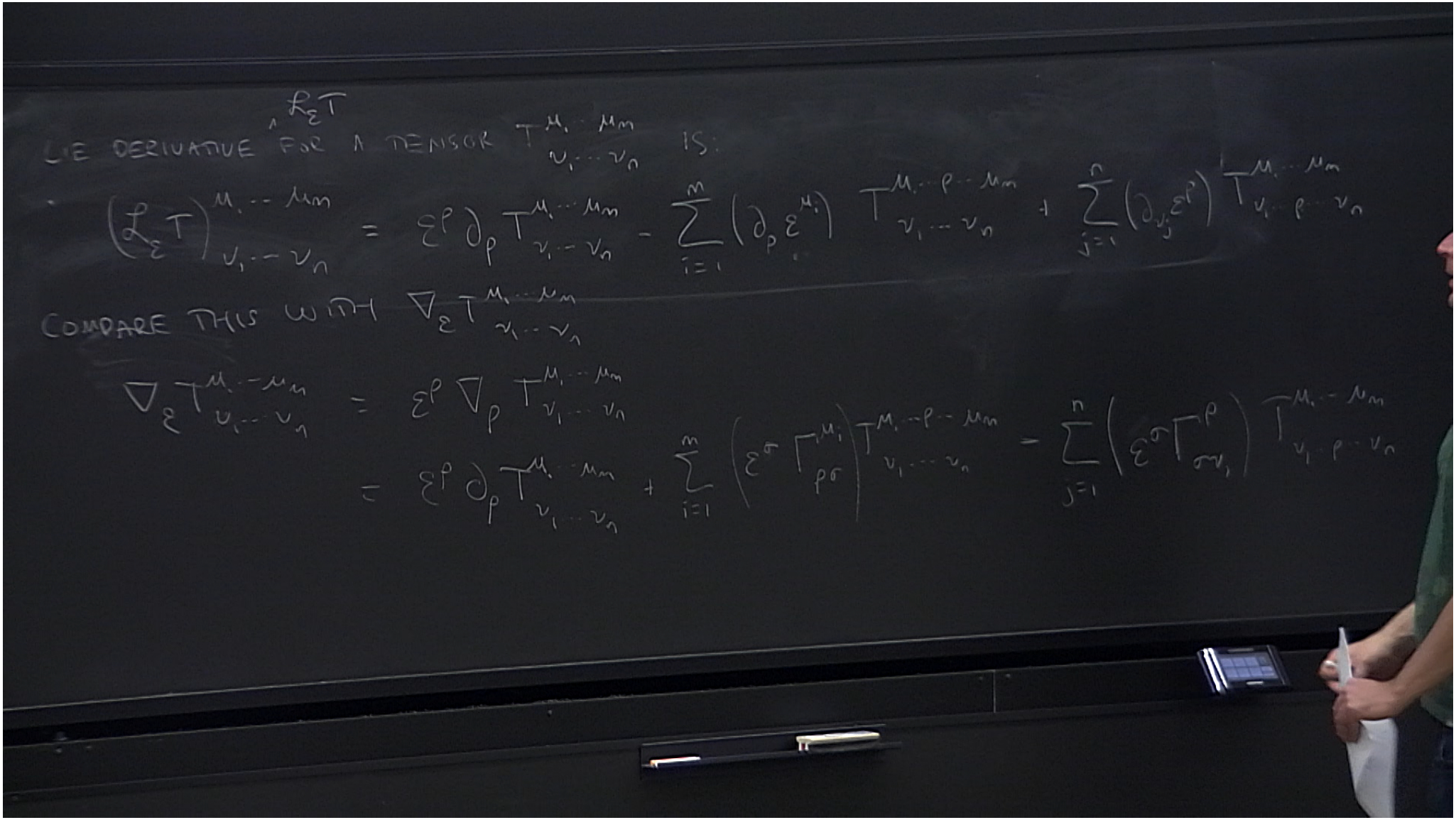
TAKE LIMIT $x'^{\mu} = x^{\mu} - \varepsilon^{\mu}(x)$ WHERE ε^{μ} IS A "FIRST ORDER" VECTOR FIELD

$$x^{\mu} = x'^{\mu} + \varepsilon^{\mu}(x)$$

EXPAND RHS TO $\mathcal{O}(\varepsilon)$.

$$S'^{\mu}_{\nu}(x') = \left(\delta^{\mu}_{\rho} - \partial_{\rho}\varepsilon^{\mu}\right)\left(\delta^{\sigma}_{\nu} + \partial_{\nu}\varepsilon^{\sigma}\right)\left(S^{\rho}_{\sigma}(x') + \varepsilon^{\lambda}\partial_{\lambda}S^{\rho}_{\sigma}(x')\right)$$

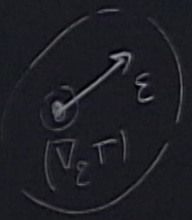
$$= S^{\mu}_{\nu}(x') + \underbrace{\varepsilon^{\lambda}\partial_{\lambda}S^{\mu}_{\nu}(x') - (\partial_{\rho}\varepsilon^{\mu})S^{\rho}_{\nu}(x') + (\partial_{\nu}\varepsilon^{\rho})S^{\sigma}_{\mu}(x')}_{\stackrel{\text{def}}{=} \mathcal{L}_{\varepsilon}(S)^{\mu}_{\nu}}$$

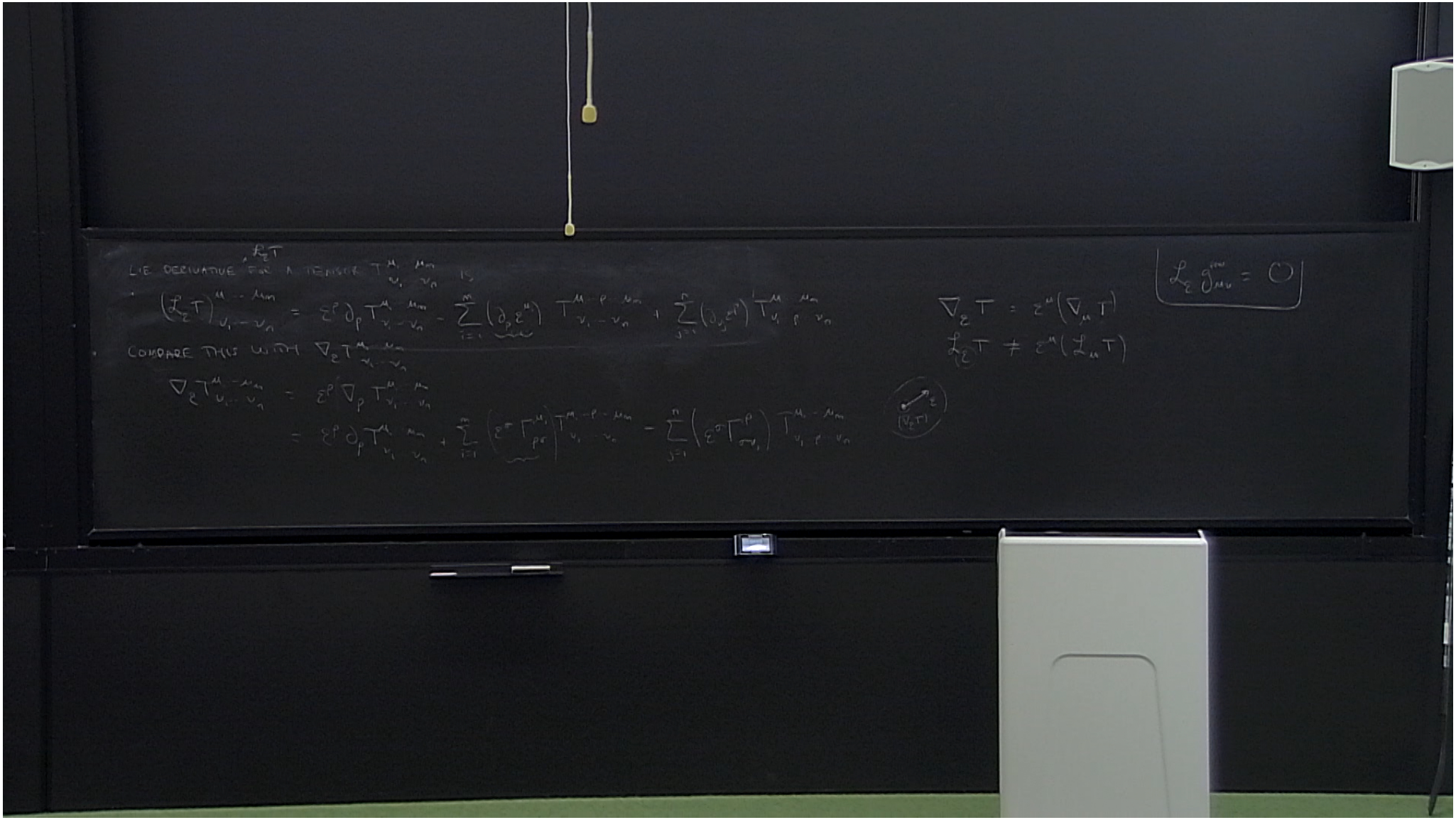

LIE DERIVATIVE FOR A TENSOR IS:
COMPARE THIS WITH

$$\mathcal{L}_\xi g^{FRW}_{\mu\nu} = 0$$

$$\nabla_\xi T = \xi^\mu (\nabla_\mu T)$$

$$\mathcal{L}_\xi T \neq \xi^\mu (\mathcal{L}_\mu T)$$



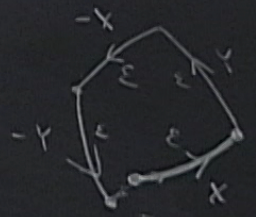

LIE DERIVATIVE $\mathcal{L}_\xi T$ FOR A TENSOR $T^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n}$ IS
$(\mathcal{L}_\xi T)^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n} = \xi^\rho \partial_\rho T^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n} - \sum_{i=1}^{m} (\partial_\rho \xi^{\mu_i}) T^{\mu_1 \cdots \rho \cdots \mu_m}_{\nu_1 \cdots \nu_n} + \sum_{j=1}^{n} (\partial_{\nu_j} \xi^\rho) T^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \rho \cdots \nu_n}$
COMPARE THIS WITH $\nabla_\xi T^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n}$
$\nabla_\xi T^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n} = \xi^\rho \nabla_\rho T^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n}$
$= \xi^\rho \partial_\rho T^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n} + \sum_{i=1}^{m} (\xi^\sigma \Gamma^{\mu_i}_{\rho\sigma}) T^{\mu_1 \cdots \rho \cdots \mu_m}_{\nu_1 \cdots \nu_n} - \sum_{j=1}^{n} (\xi^\sigma \Gamma^{\rho}_{\sigma\nu_j}) T^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \rho \cdots \nu_n}$
$\nabla_\xi T = \xi^\mu (\nabla_\mu T)$
$\mathcal{L}_\xi T \neq \xi^\mu (\mathcal{L}_\mu T)$
$\mathcal{L}_\xi g_{\mu\nu} = 0$
ξ
$(\mathcal{L}_\xi T)$

$(\mathcal{L}_X Y)^\mu = X^\nu(\partial_\nu Y^\mu) - (\partial_\nu X^\mu) Y^\nu$ ANTISYMMETRIC IN X, Y !

$[X, Y] \overset{\text{def}}{=} \mathcal{L}_X Y$

$$[X, Y] = \mathcal{L}_X Y = \nabla_X Y - \nabla_Y X$$

$\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$



$[X,Y]\varepsilon^2 + \mathcal{O}(\varepsilon^3)$

GAUGE MODES IN SINGLE-FIELD INFLATION?

"LIE DERIVATIVE EVERYTHING IN SIGHT"

$g_{\mu\nu}$ = FLRW METRIC

$\phi = \phi(t)$

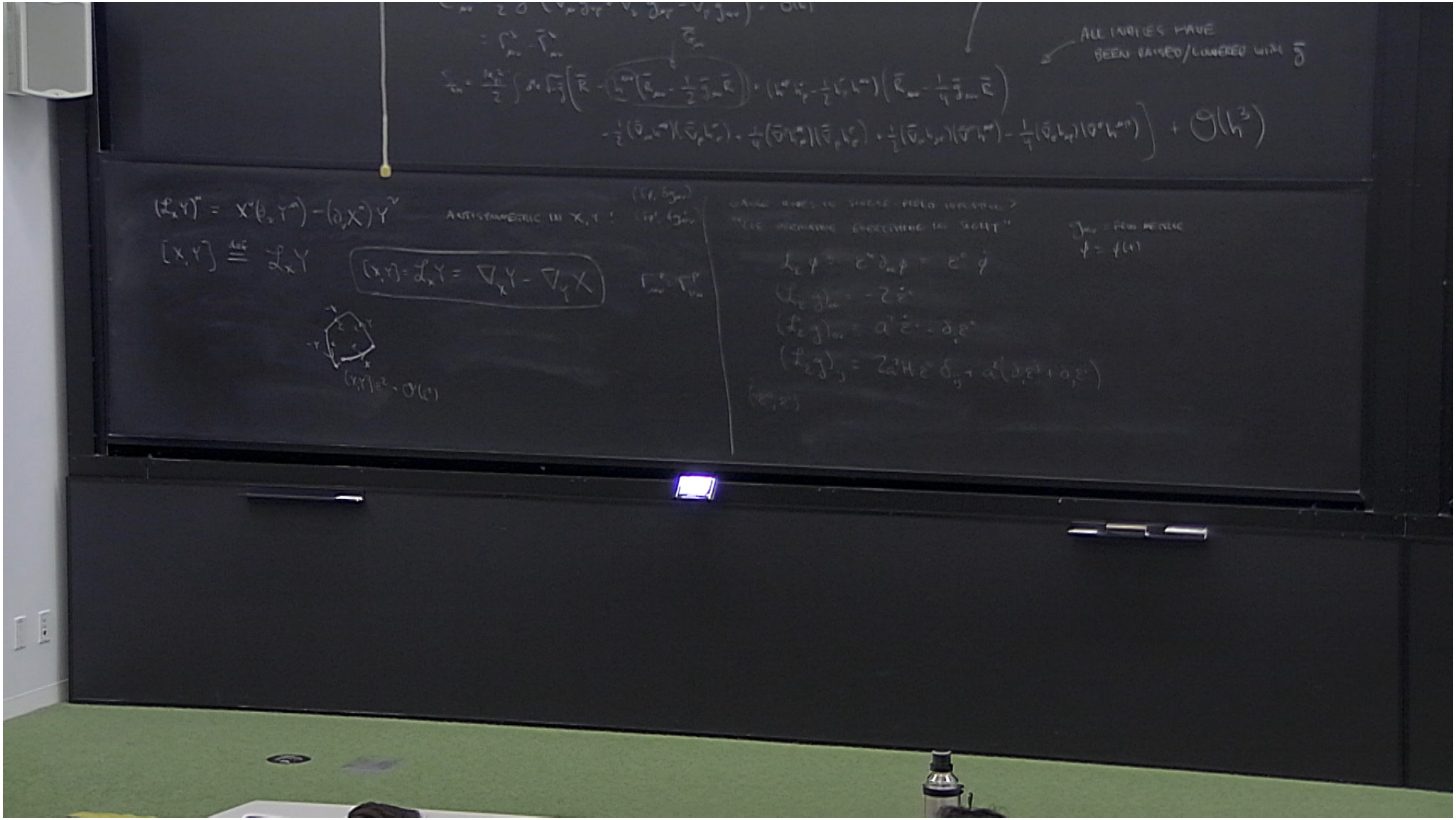
$$\mathcal{L}_\xi \phi = \xi^\mu \partial_\mu \phi = \xi^0 \dot{\phi}$$

$$(\mathcal{L}_\xi g)_{00} = -2\dot{\xi}^0$$

$$(\mathcal{L}_\xi g)_{0i} = a^2 \dot{\xi}^i - \partial_i \xi^0$$

$$(\mathcal{L}_\xi g)_{ij} = 2a^2 H \xi^0 \delta_{ij} + a^2(\partial_i \xi_j + \partial_j \xi_i)$$

(ξ^0, ξ^i)


ALL INDICES HAVE
BEEN RAISED/LOWERED WITH $\bar{g}$
$(\mathcal{L}_X Y)^\mu = X^\nu(\partial_\nu Y^\mu) - (\partial_\nu X^\mu) Y^\nu$
ANTISYMMETRIC IN X, Y !
$[X,Y] \overset{\text{def}}{=} \mathcal{L}_X Y$
$[X,Y] = \mathcal{L}_X Y = \nabla_X Y - \nabla_Y X$
$\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$
$[X,Y]^2 = O(\epsilon)$
$\mathcal{L}_\xi \phi = \xi^\mu \partial_\mu \phi = \xi^\mu \dot{\phi}$
$(\mathcal{L}_\xi g)_{00} = -2\ddot{\xi}^0$
$(\mathcal{L}_\xi g)_{0i} = a^2 \dot{\xi}^i - \partial_i \xi^0$
$(\mathcal{L}_\xi g)_{ij} = 2a^2 H \xi^0 \delta_{ij} + a^2(\partial_i \xi^j + \partial_j \xi^i)$