Title: PSI 16/17 Explorations in Cosmology (Kendrick Smith) - Lecture 8

Date: Apr 19, 2017 10:15 AM

URL: http://pirsa.org/17040060

Abstract:

MODEL SPACE

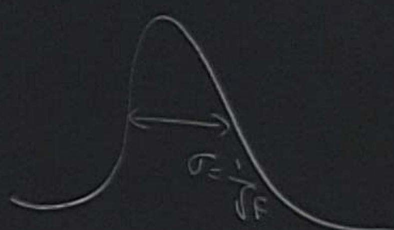
EXAMPLE 3:

$\Theta = \{\theta_i\}$

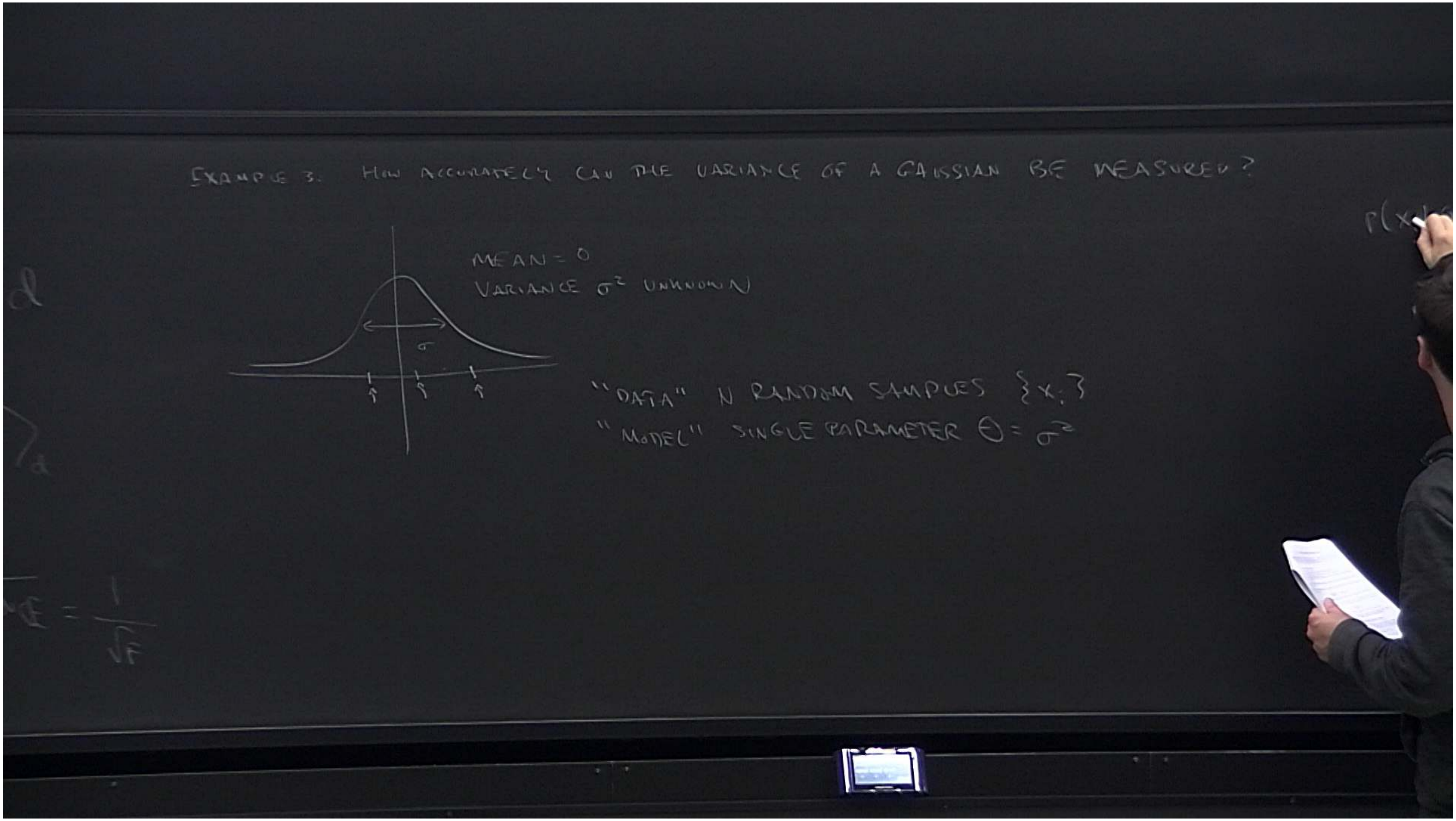
$\longrightarrow d$

$$F_{ij} = \left\langle -\frac{\partial^2 \log[d|\theta]}{\partial\theta_i \, \partial\theta_j} \right\rangle_d$$

$$F = C^{-1}$$



$$\sigma = \sqrt{\text{VARIANCE}} = \frac{1}{\sqrt{F}}$$


EXAMPLE 3: HOW ACCURATELY CAN THE VARIANCE OF A GAUSSIAN BE MEASURED?
MEAN = 0
VARIANCE σ² UNKNOWN
σ
"DATA" N RANDOM SAMPLES {x_i}
"MODEL" SINGLE PARAMETER Θ = σ²

EASURED ?

$$p(x_i|\theta) = \prod_{i=1}^{N} \frac{1}{(2\pi\theta)^{1/2}} \exp\left(-\frac{x_i^2}{2\theta}\right)$$

$$\log p(x_i|\theta) = (\text{CONST.}) - \frac{N}{2}\log\theta - \sum_i \frac{x_i^2}{2\theta}$$

$$\frac{\partial^2 \log p(x_i|\theta)}{\partial\theta^2} = \frac{N}{2}\theta^{-2} - \sum x_i^2\,\theta^{-3}$$

$$F = -\left\langle \frac{\partial^2 \log p(x_i|\theta)}{\partial\theta^2}\right\rangle_{x_i} = -\left(\frac{N}{2}\theta^{-2} - (N\theta)\,\theta^{-3}\right)$$

$$= \frac{N}{2}\theta^{-2}$$

$\sigma(\theta) =$ STATISTICAL ERROR ON Θ

$$= \frac{1}{\sqrt{F}}$$

$$= \sqrt{\frac{2}{N}}\,\Theta$$

HOW ACCURATELY CAN THE CMB POWER SPECTRUM C_ℓ BE MEASURED?

CMB: $a_{\ell m}$

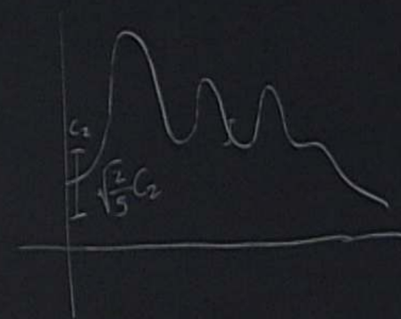
$$a_{20} = 13.09 \ \mu K$$

$$a_{21} = \frac{1}{\sqrt{2}}\left(-2.16 + 3.54\,i\right) \ \mu K$$

$$\vdots$$

$a_{\ell m}$ ARE INDEPENDENT OF EACH OTHER & THERE ARE $(2\ell+1)$ MODES (M-VALUES) FOR EACH ℓ.

$$\boxed{\sigma(C_\ell) = \sqrt{\frac{2}{2\ell+1}}\, C_\ell}$$



HOW ACCURATELY CAN THE CMB POWER SPECTRUM C_ℓ BE MEASURED?

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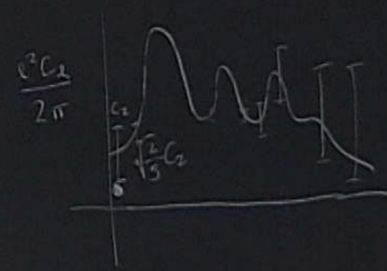


"COSMIC VARIANCE"

$C_\ell^{cmb} \sim \ell^{-2}$

$C_\ell^{noise} \sim 1$

HOW WELL CAN THE COSMOLOGICAL PARAMETERS BE MEASURED FROM THE CMB?

FOR EACH

"MODEL": 6 PARAMETERS $\Theta_i = \{\Omega_b, \Omega_m, \dots\}$

"DATA": CMB MULTIPOLES $a_{\ell m}$. ASSUMED MEASURED WITHOUT NOISE

$$\xrightarrow{\Theta_i} \boxed{\begin{matrix}\text{CAMB}\\ \text{CLASS}\end{matrix}} \longrightarrow C_\ell(\Theta_b, \dots)$$

$$\frac{\partial C_\ell}{\partial \Theta_i} = \frac{C_\ell(\Theta + \Delta\Theta) - C_\ell(\Theta - \Delta\Theta)}{2(\Delta\Theta)}$$

MINOR CHANGE OF NOTATION: $a_{\ell 0} = \tilde{a}_{\ell 0}$ [REAL]

$$a_{\ell m} = \frac{1}{\sqrt{2}}\left(\tilde{a}_{\ell, 2m-1} + i\,\tilde{a}_{\ell, 2m}\right) \quad \text{FOR } 1 \le m \le \ell$$

$\Delta\theta)$

FOR EACH ℓ, $\tilde{a}_{\ell i}$ ARE $(2\ell+1)$ REAL GAUSSIAN NUMBERS

$$\langle \tilde{a}_{\ell i} \tilde{a}_{\ell' i'} \rangle = C_\ell \, \delta_{\ell\ell'} \, \delta_{ii'}$$

$$F_{ij} = -\left\langle \frac{\partial^2 \log p(\tilde{a}_{\ell i} | \theta)}{\partial \theta_i \, \partial \theta_j} \right\rangle_{\tilde{a}_{\ell i}}$$

$$p(\tilde{a}_{\ell i} | \theta) = \prod_{\ell=2}^{\infty} \prod_{i=0}^{2\ell} \frac{1}{(2\pi C_\ell)^{1/2}} \exp\left(-\frac{\tilde{a}_{\ell i}^2}{C_\ell}\right)$$

$$\xrightarrow{\theta_i} \boxed{\begin{matrix}\text{CAMB}\\ \text{CLASS}\end{matrix}} \longrightarrow C_\ell(\theta_i, \ldots)$$

$$\frac{\partial C_\ell}{\partial \theta_i} = \frac{C_\ell(\theta + \Delta\theta) - C_\ell(\theta - \Delta\theta)}{2(\Delta\theta)}$$

$$F_{ij} = \ldots \frac{\ldots}{\partial \theta_i \, \partial \theta_j}$$

$$p(\tilde{a}_\ell \mid \theta) = \prod_{\ell=2}^{\infty} \prod_{i=0}^{2\ell} \frac{1}{(2\pi C_\ell \ldots}$$

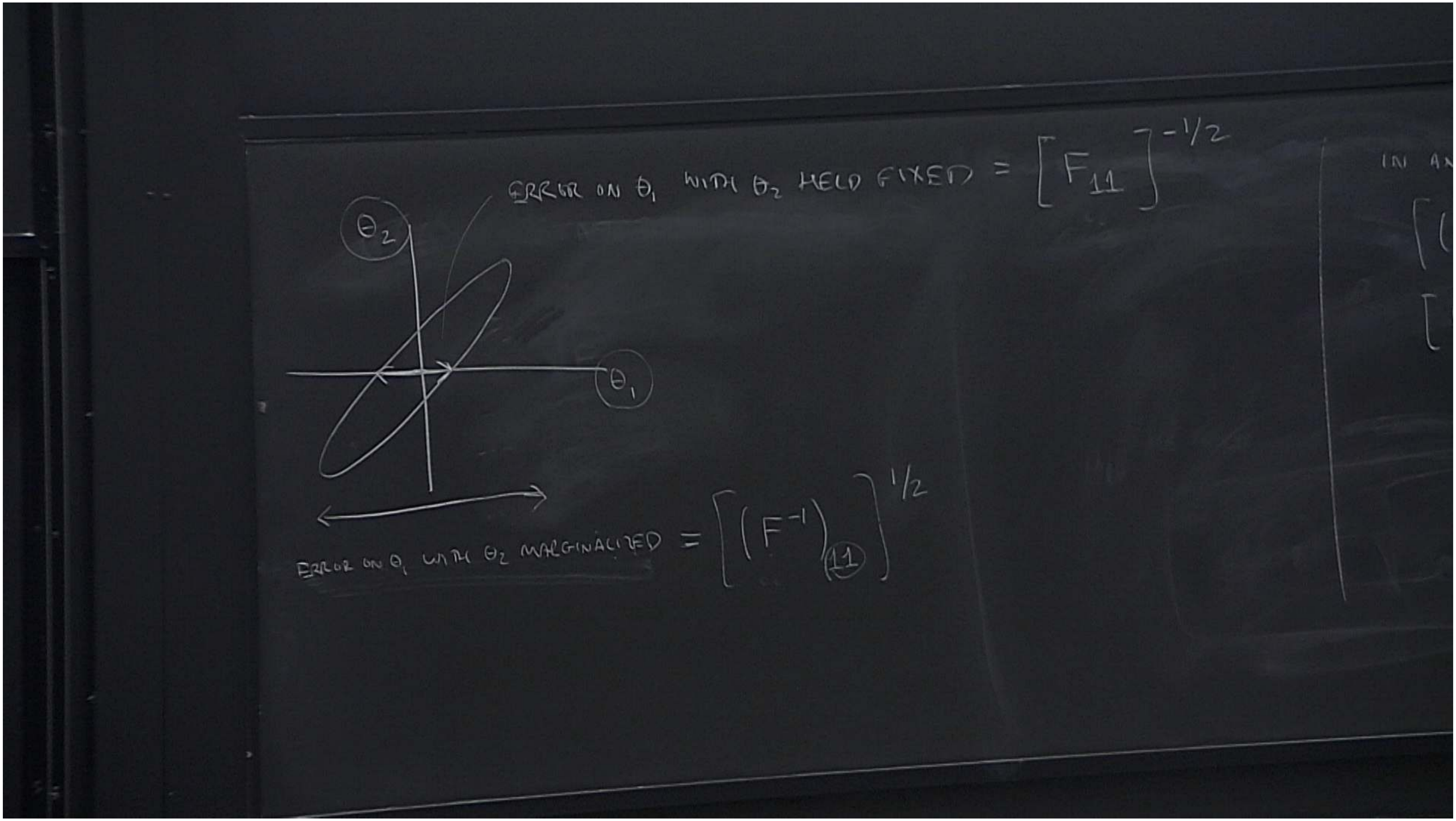
MINOR CHANGE OF NOTATION: $a_{\ell 0} = \tilde{a}_{\ell 0}$ [REAL]

$a_{\ell m} = \frac{1}{\sqrt{2}}\left(\tilde{a}_{\ell, 2m-1} + i\,\tilde{a}_{\ell, 2m}\right)$ FOR $1 \le m \le \ell$

$$\frac{\partial^2 \log p(\tilde{a} \mid C_\ell)}{\partial \theta_a \, \partial \theta_b} = \sum_\ell \left(\frac{2\ell+1}{2} \frac{1}{C_\ell^2} - \frac{\sum_i \tilde{a}_{\ell i}^2}{C_\ell^3} \right) \frac{\partial C_\ell}{\partial \theta_a} \frac{\partial C_\ell}{\partial \theta_b} + \sum_\ell \left(-\frac{2\ell+1}{2} \frac{1}{C_\ell} + \frac{\sum_i \tilde{a}_{\ell i}^2}{2 C_\ell^2} \right) \frac{\partial^2 C_\ell}{\partial \theta_a \, \partial \theta_b}$$

$$F_{ab} = -\left\langle \frac{\partial^2 \log p(\tilde{a} \mid C_\ell)}{\partial \theta_a \, \partial \theta_b} \right\rangle_{\tilde{a}} = -\sum_\ell \left[\left(\frac{2\ell+1}{2 C_\ell^2} - \frac{(2\ell+1) C_\ell}{C_\ell^3} \right) \frac{\partial C_\ell}{\partial \theta_a} \frac{\partial C_\ell}{\partial \theta_b} + \left(-\frac{2\ell+1}{2} \frac{1}{C_\ell} + \frac{(2\ell+1) C_\ell}{2 C_\ell^2} \right) \frac{\partial^2 C_\ell}{\partial \theta_a \, \partial \theta_b} \right]$$

$$\boxed{F_{ab}(\theta_i) = \sum_\ell \frac{2\ell+1}{2} \frac{1}{C_\ell^2} \frac{\partial C_\ell}{\partial \theta_a} \frac{\partial C_\ell}{\partial \theta_b}}$$


ERROR ON θ_1 WITH θ_2 HELD FIXED $= \left[F_{11}\right]^{-1/2}$
θ_2
θ_1
ERROR ON θ_1 WITH θ_2 MARGINALIZED $= \left[(F^{-1})_{11}\right]^{1/2}$

IN AN N-DIM. SPACE

$[(F^{-1})_{ii}]^{1/2}$ = ERROR ON PARAMETER i, WITH $j \neq i$ MARGINALIZED

$[F_{ii}]^{-1/2}$ = ______ " ______ HELD FIXED TO THEIR FIDUCIAL VALUES

$d = (d', d'')$ $\quad p[d|\theta] = p[d'|\theta]\, p[d''|\theta]$

$$F_{ij} = -\left\langle \frac{\partial^2 \log p[d|\theta]}{\partial\theta_i \partial\theta_j} \right\rangle$$

$$= F'_{ij} + F''_{ij}$$

FOR INDEPENDENT EXPERIMENTS ON $\{\theta\}$

$$F = F_1^{planck} + F_2^{SDSS} + \dots + F^{new}$$

(f)

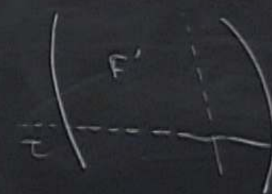
$$a_{\ell m} = \frac{1}{\sqrt{2}}\left(\tilde{a}_{\ell,2m-1} + i\,\tilde{a}_{\ell,2m}\right) \quad \text{FOR } 1 \le m \le \ell$$

BEHAVIOR OF THE FISHER MATRIX UNDER REPARAMETERIZATION $\Theta' \longrightarrow \Theta$ $\qquad \Theta_i = f_i(\Theta'_j)$

$$F'_{ij} = \left(\frac{\partial \Theta_k}{\partial \Theta'_i}\right)\left(\frac{\partial \Theta_\ell}{\partial \Theta'_j}\right) F_{k\ell}$$

[PROOF OMITTED]

$$g'_{\mu\nu} = \left(\frac{\partial x_\rho}{\partial x'_\mu}\right)\left(\frac{\partial x_\sigma}{\partial x'_\nu}\right) g_{\mu\nu}$$



E.G. CMB FISHER MATRIX $\underbrace{\Theta \rightarrow C_\ell}_{\text{SUBMODEL}} \longrightarrow \tilde{a}_\ell$.

$$F_{\ell\ell'} = \frac{2\ell+1}{2}\, C_\ell^{-2}\, \delta_{\ell\ell'}$$

$$F_{ab} = \sum_{\ell\ell'} \left(\frac{\partial C_\ell}{\partial \theta_a}\right)\left(\frac{\partial C_{\ell'}}{\partial \theta_b}\right) F_{\ell\ell'} = \sum_\ell \frac{2\ell+1}{2}\, C_\ell^{-2} \left(\frac{\partial C_\ell}{\partial \theta_a}\right)\left(\frac{\partial C_\ell}{\partial \theta_b}\right)$$