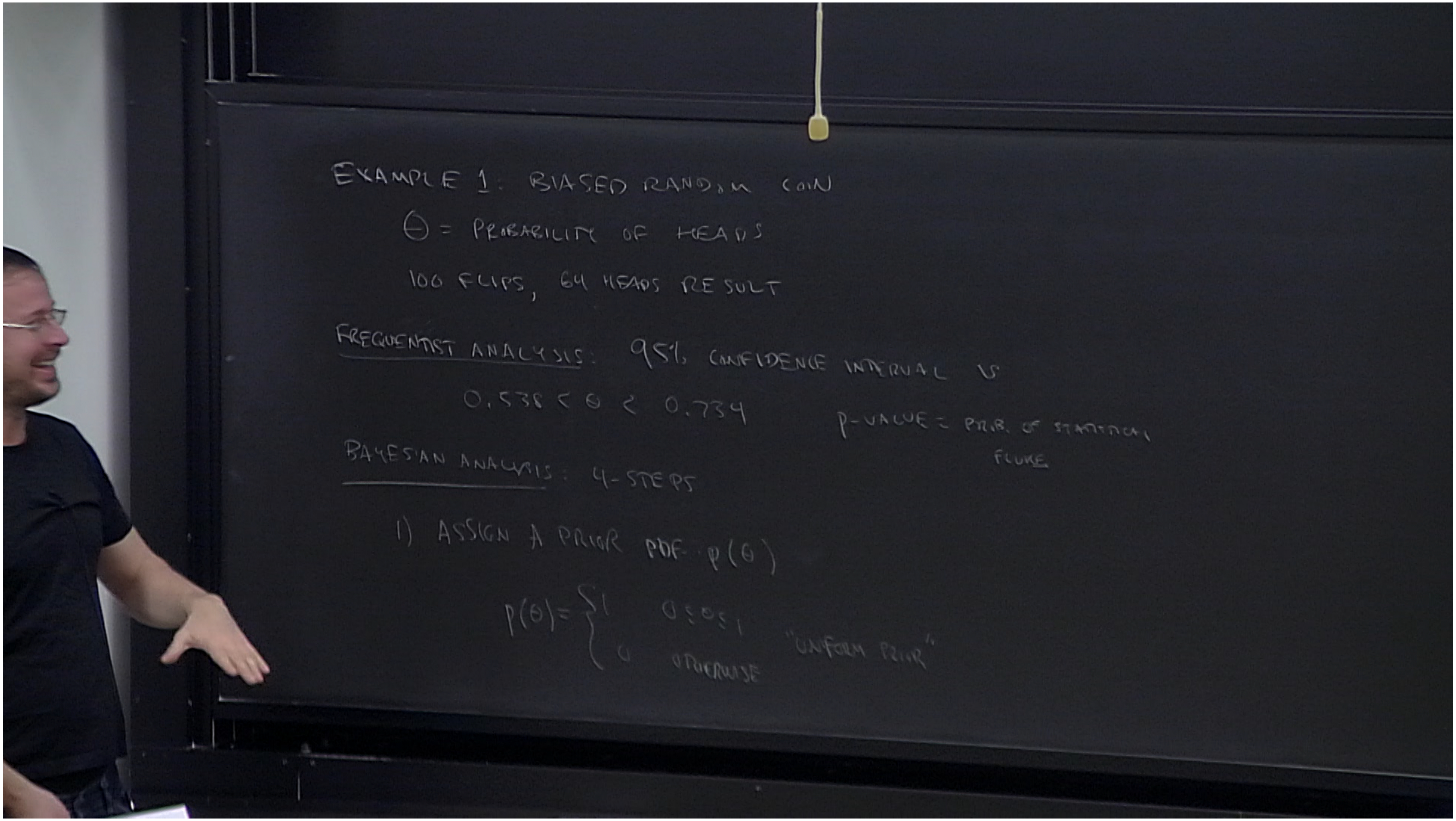
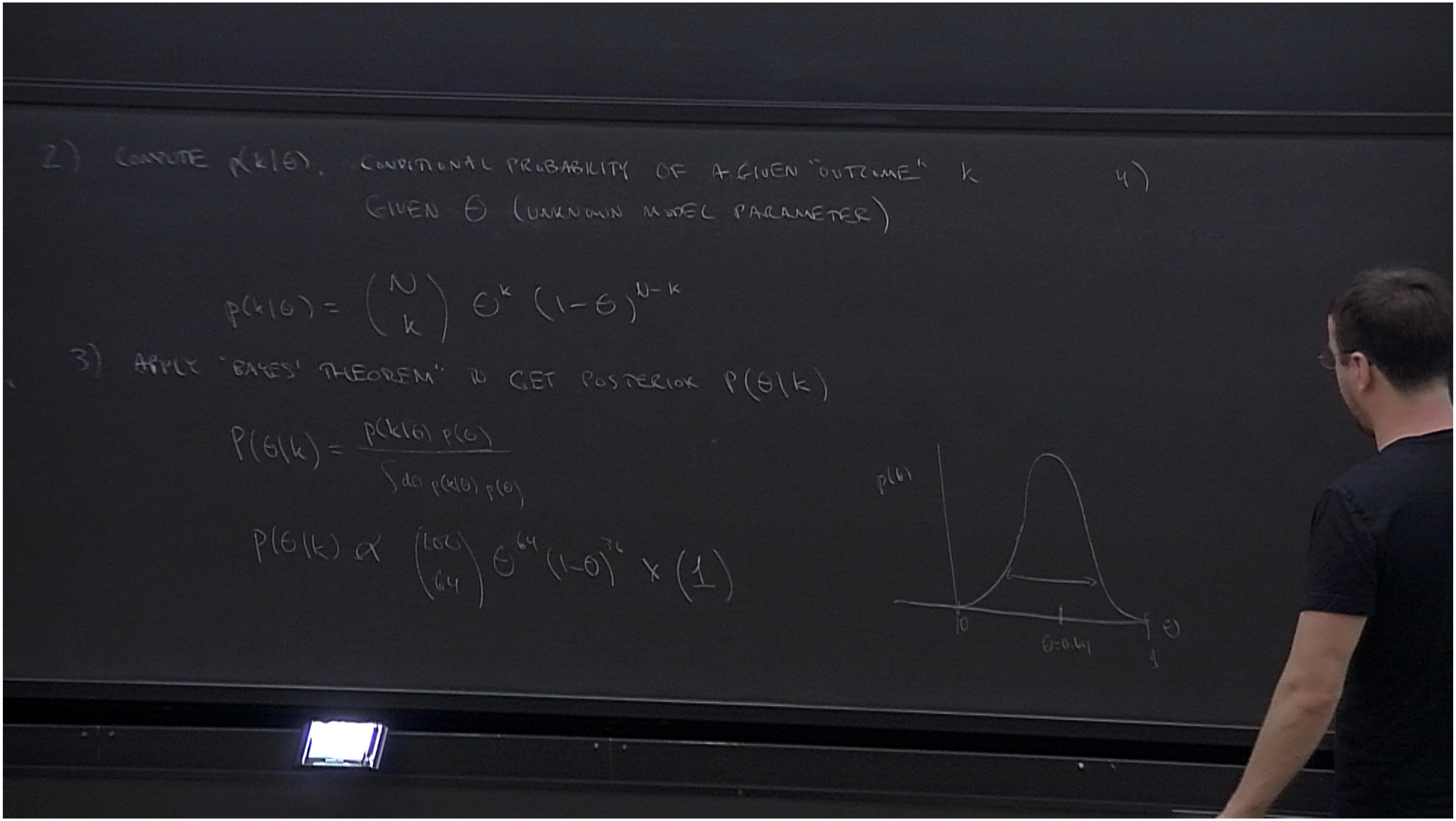
Title: PSI 16/17 Explorations in Cosmology (Kendrick Smith) - Lecture 7

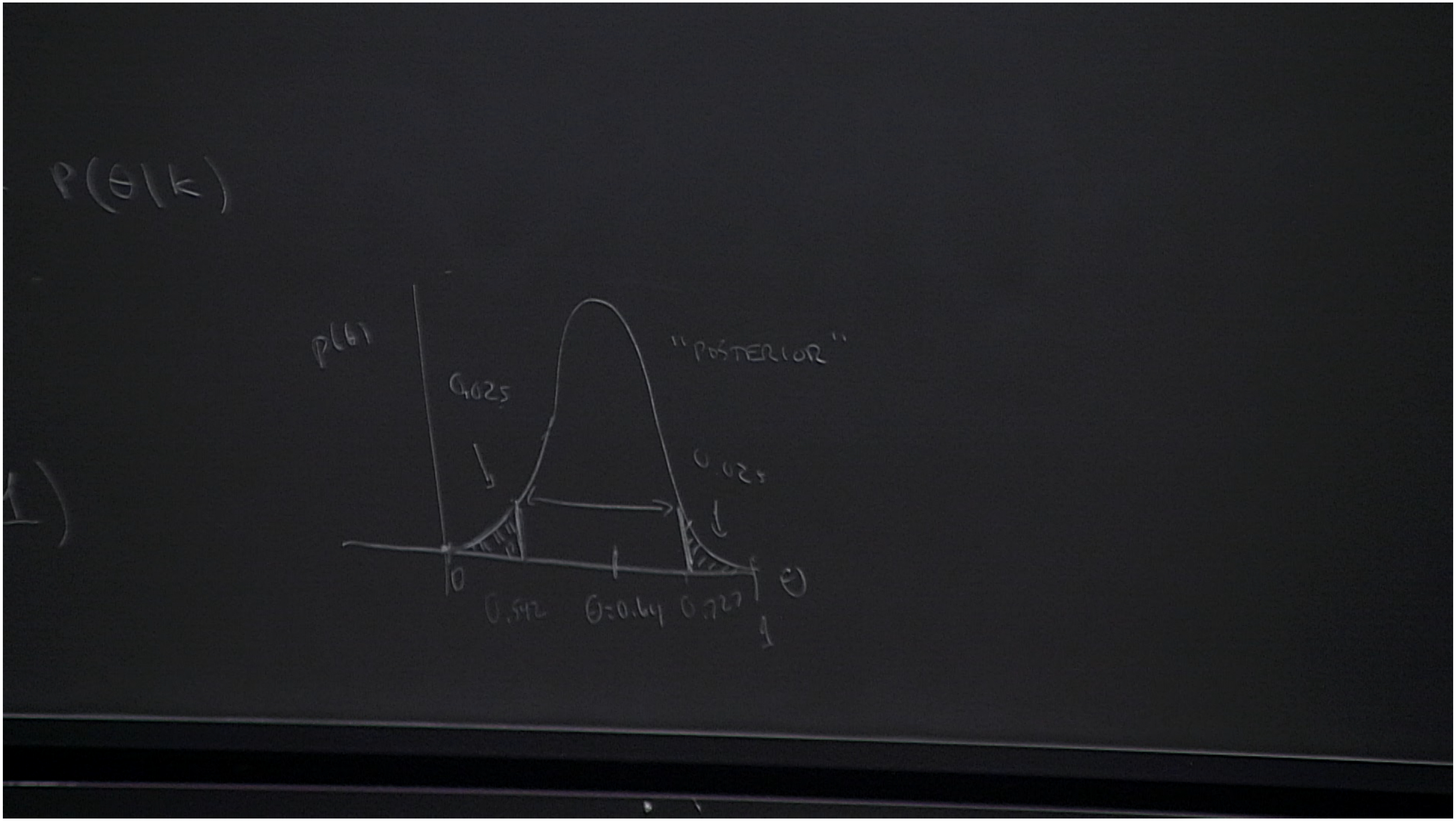
Date: Apr 18, 2017 10:15 AM

URL: http://pirsa.org/17040059

Abstract:


EXAMPLE 1: BIASED RANDOM COIN
Θ = PROBABILITY OF HEADS
100 FLIPS, 64 HEADS RESULT
FREQUENTIST ANALYSIS: 95% CONFIDENCE INTERVAL IS
0.538 < Θ < 0.734
P-VALUE = PROB. OF STATISTICAL FLUKE
BAYESIAN ANALYSIS: 4-STEPS
1) ASSIGN A PRIOR PDF p(Θ)
p(Θ) = 1 0 ≤ Θ ≤ 1
0 OTHERWISE
"UNIFORM PRIOR"


2) COMPUTE $p(k|\theta)$. CONDITIONAL PROBABILITY OF A GIVEN "OUTCOME" k
GIVEN θ (UNKNOWN MODEL PARAMETER)
4)
$p(k|\theta) = \binom{N}{k} \theta^{k} (1-\theta)^{N-k}$
3) APPLY "BAYES' THEOREM" TO GET POSTERIOR $P(\theta|k)$
$P(\theta|k) = \frac{p(k|\theta)\, p(\theta)}{\int d\theta\, p(k|\theta)\, p(\theta)}$
$P(\theta|k) \propto \binom{100}{64} \theta^{64} (1-\theta)^{36} \times (1)$
$p(\theta)$
0
$\theta = 0.64$
1
θ


P(θ|k)
p(θ)
0.025
"POSTERIOR"
0.025
0
0.542
θ=0.64
0.727
1
θ

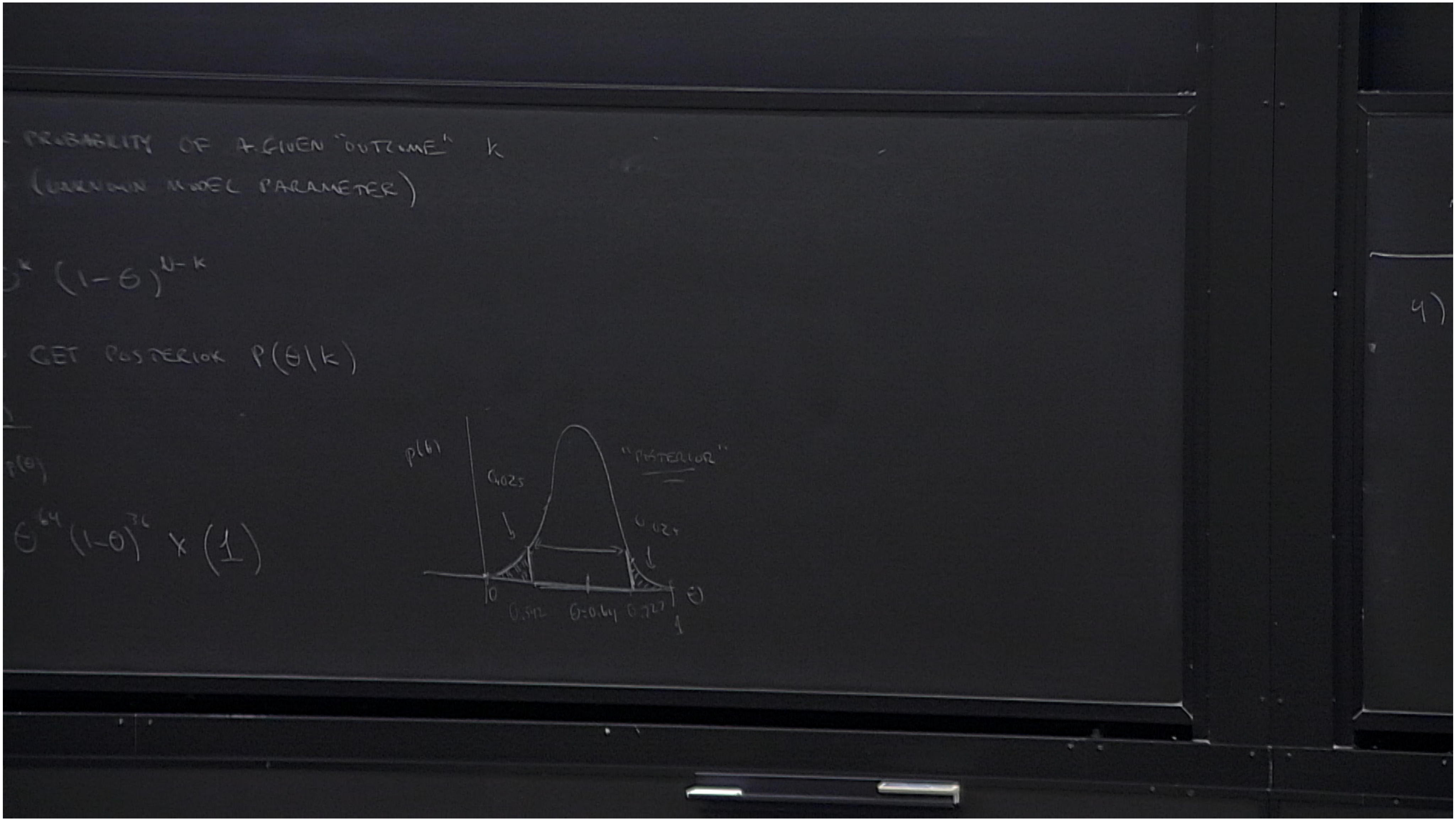
$p(X|Y) =$ "PROBABILITY OF X, GIVEN Y"

4) ASSIGN CONFIDENCE REGIONS BASED ON POSTERIOR

$$\int_0^{0.542} p(\theta|k)\, d\theta = 0.025$$

$$\int_{0.727}^{1} p(\theta|k)\, d\theta = 0.025$$

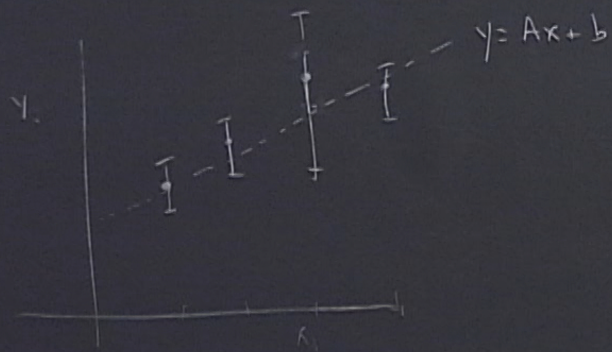
$\Rightarrow$ CONFIDENCE INTERVAL IS $0.542 < \theta < 0.727$


PROBABILITY OF A GIVEN "OUTCOME" k
(UNKNOWN MODEL PARAMETER)
(1-θ)^N-k
GET POSTERIOR P(θ|k)
p(θ)
θ^64 (1-θ)^36 x (1)
p(θ)
"POSTERIOR"
0.025
0.025
0
1
θ
θ=0.64

"LIKELIHOOD" = "PDF"

BAYESIAN FROM NOW ON!

EXAMPLE 2: FITTING A STRAIGHT LINE



SETUP:

(x_i, y_i) $i = 1, \dots, N$

y_i HAS A GAUSSIAN ERROR WITH VARIANCE σ_i^2

2-BY-2 MATRIX NOTATION

$\vec{\theta}_a = (\theta_1, \theta_2) = (A, b)$ MODEL PARAMETERS

$f_a(x) = (x, 1)$

$$y = \sum_a \theta_a f_a(x)$$

$$\int_{0.727} p(\theta|y)\, d\theta = 0.02\ldots$$

$\Rightarrow$ CONFIDENCE INTERVAL IS $0.542 < \theta < 0.727$

1) PRIOR $p(\vec{\theta}) = 1$ UNIFORM PRIOR

2) CONDITIONAL PROBABILITY OF DATA, GIVEN MODEL

$$P(\vec{y}|\vec{\theta}) = \prod_i \frac{1}{(2\pi\sigma_i^2)^{1/2}} \exp\left[-\frac{1}{2\sigma_i^2}\left(y_i - \sum_a \theta_a f_a(x_i)\right)^2 \right]$$

GAUSSIAN PDF FOR y_i:
MEAN $= \sum_a \theta_a f_a(x_i)$
VARIANCE $= \sigma_i^2$

3) GET POSTERIOR & SIMPLIFY

$$p(\vec{\theta}|y) \propto \prod_i \exp\left(-\frac{(y_i - \theta_a f_a(x_i))^2}{\sigma_i^2} \right)$$

$$= \exp\left(-\sum_i \frac{y_i^2}{2\sigma_i^2} + \sum_{ia} \frac{\theta_a f_a(x_i)}{\sigma_i^2} - \sum_{iab} \frac{\theta_a \theta_b f_a(x_i) f_b(x_i)}{2\sigma_i^2} \right)$$

$$f_a(x) = (x, 1)$$

$$y = \sum_a \Theta_a f_a(x)$$

...USSIAN PDF FOR y_i:
MEAN = $\sum_a \Theta_a f_a(x_i)$
VARIANCE = σ_i^2

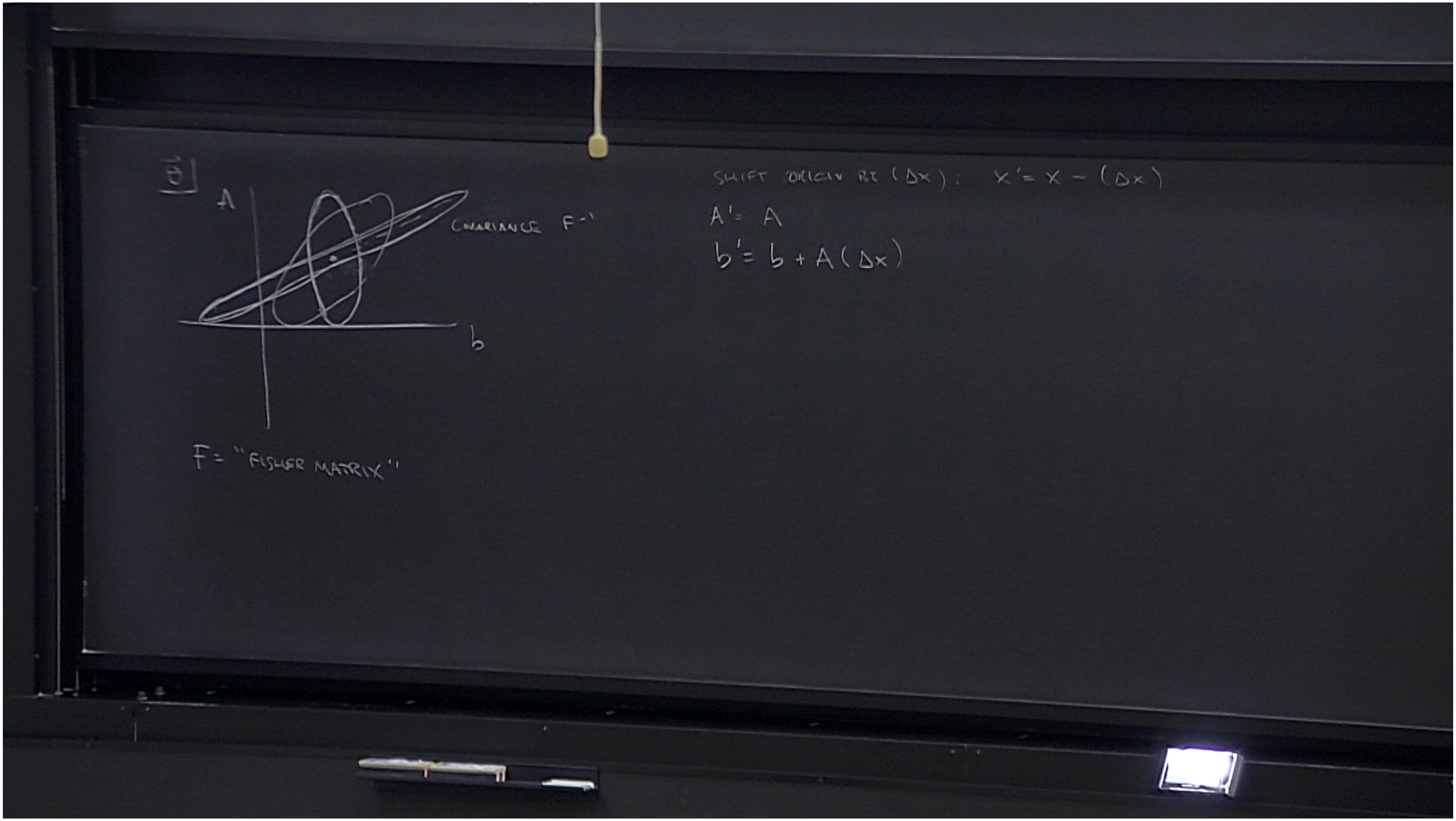
$$p(\theta|y) \propto \exp\left[-\frac{1}{2}\theta^T F \theta + \theta^T v\right] \leftarrow \text{GAUSSIAN}$$

$$F_{ab} = \sum_i \frac{f_a(x_i) f_b(x_i)}{\sigma_i^2}$$

$$v_a = \frac{f_a(x_i)\, y_i}{\sigma_i^2}$$

$$p(\theta|y) \propto \exp\left[-\frac{1}{2}(\theta - F^{-1}v)^T F (\theta - F^{-1}v)\right] \exp\left[-\frac{1}{2} v^T F^{-1} v\right]$$

GAUSSIAN W/ MEAN
$\bar{\theta}_{mean} = F^{-1} v$
COVARIANCE
F^{-1}


$\vec{\theta}$
A
COVARIANCE F^{-1}
b
F = "FISHER MATRIX"
SHIFT ORIGIN BY (Δx): $x' = x - (\Delta x)$
$A' = A$
$b' = b + A(\Delta x)$

ABSTRACT DEFINITION OF FISHER MATRIX

SETUP: MODEL SPACE (θ_i) + DATA d

EXAMPLE 1: (RANDOM COIN) $\theta =$ P(HEADS) $d = k =$ # OF HEADS

EXAMPLE 2: (CURVE FITTING) $\theta_\alpha = (A, b)$ $d = \{y_i\}$

- SIMULATION: GIVEN A MODEL θ_i, HOW DO I SIMULATE d?
- ANALYSIS: GIVEN d, WHAT ARE THE CONSTRAINTS ON θ_i?
- FORECASTING: GIVEN A FIDUCIAL MODEL θ_{fid}, WHAT CONSTRAINTS ON θ_i WILL I GET, FOR A "TYPICAL" REALIZATION OF d?

$\leftarrow$ FISHER MATRIX IS A TOOL FOR FORECASTING

• FOR

FORECASTING: HAVE θ_{FID}, NO d

ASSUMPTIONS:

• THE POSTERIOR $p(\theta|d)$ WILL BE GAUSSIAN

$$\exp\left[-A_{ij}\theta_i\theta_j - b\theta_i\right]$$

$$\Leftrightarrow \quad \frac{\partial^2 \log p(\theta|d)}{\partial\theta_i \, \partial\theta_j} \quad \text{IS CONSTANT IN } \theta.$$

[NOT ASSUMED CONSTANT IN d]

• $$F_{ij} \overset{\text{DEF}}{=} \left\langle -\frac{\partial^2 \log p(\theta|d)}{\partial\theta_i \, \partial\theta_j} \right\rangle_d$$

$\langle \cdot \rangle_d$ = EXPECTATION VALUE OVER REALIZATIONS OF DATA d, ASSUMING MODEL PARAMETERS θ_{fid}.

FORECASTING: HAVE θ_{FID}, NO d

ASSUMPTIONS:

- THE POSTERIOR $p(\theta|d)$ WILL BE GAUSSIAN

$\exp\left[-A_{ij}\theta_i\theta_j - b\theta_i\right]$

$\Leftrightarrow \quad \dfrac{\partial^2 \log p(\theta|d)}{\partial\theta_i \, \partial\theta_j}$ IS CONSTANT IN θ.

[NOT ASSUMED CONSTANT IN d]

- $F_{ij} \overset{DEF}{=} \left\langle -\dfrac{\partial^2 \log p(\theta_{fid}|d)}{\partial\theta_i \, \partial\theta_j} \right\rangle_d$

$\langle \cdot \rangle_d$ = EXPECTATION VALUE OVER REALIZATIONS OF DATA d, ASSUMING MODEL PARAMETERS θ_{fid}.

• FORECASTING: GIVEN A FIDUCIAL MODEL θ_{fid}, WHAT CONSTRAINTS ON θ WILL I GET, FOR A "TYPICAL" REALIZATION OF d?

← FISHER MATRIX IS A TOOL FOR FORECASTING

...ATIONS

...AMETERS θ_{fid}.

$$F = \frac{N}{\theta(1-\theta)} \leftarrow$$

$\theta_{true} \sim 0$ $\qquad \theta_{true} < 0.02$

$\theta_{true} \sim 0.5$ $\qquad$ 55 $\qquad$ 0.4 $\quad$ 0.6