Title: Traces of intertwiners for quantum affine sl_2, affine Macdonald conjectures, and Felder-Varchenko functions

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Abstract: <p>This talk concerns a family of special functions common to the study of quantum conformal blocks and hypergeometric solutions to q-KZB type equations. In the first half, I will explain two methods for their construction -- as traces of intertwining operators between representations of quantum affine algebras and as certain theta hypergeometric integrals we term Felder-Varchenko functions. I will then explain our proof by bosonization the first case of Etingof-Varchenko's conjecture that these constructions are related by a simple renormalization.</p>

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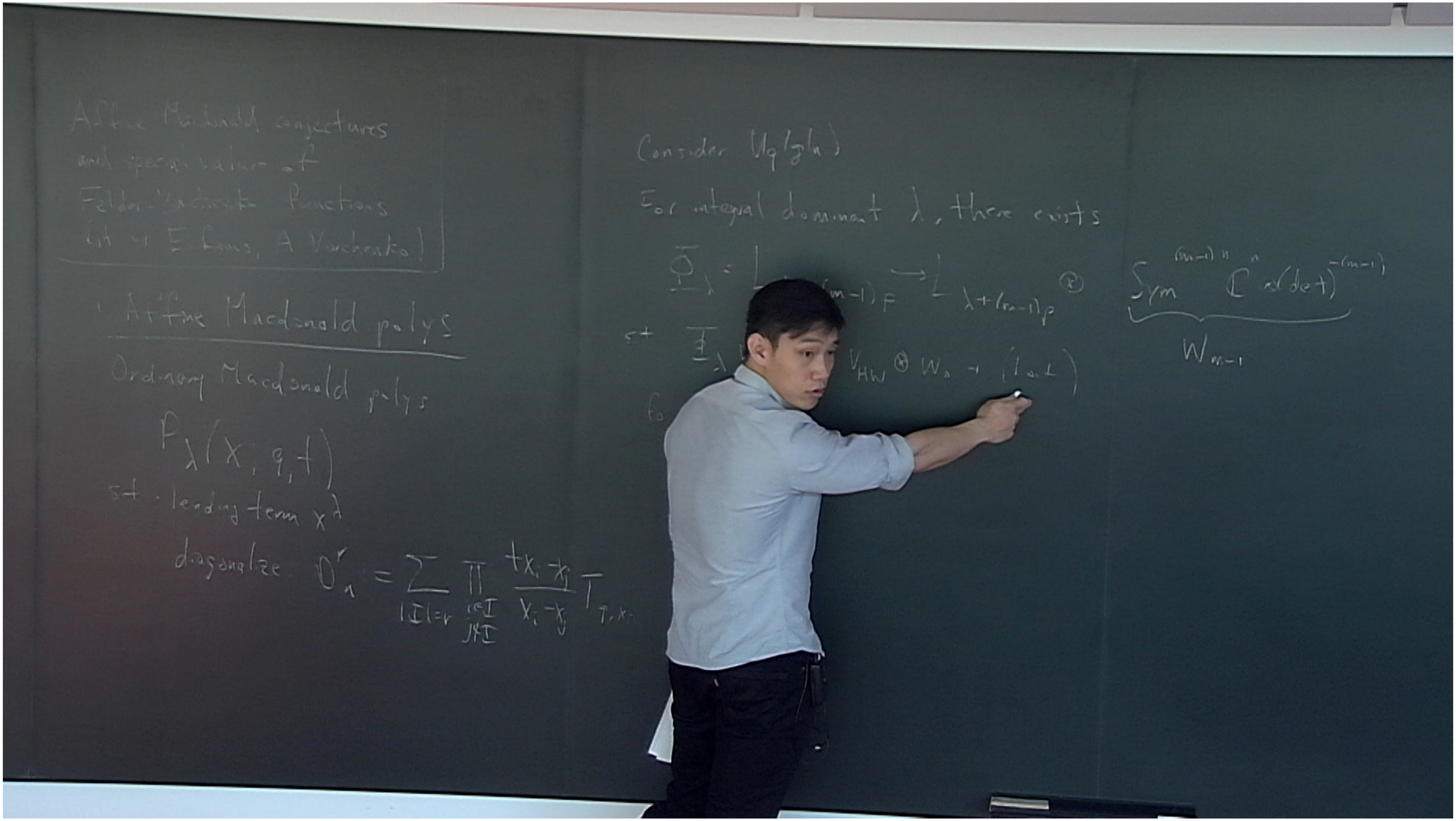
<p>The second half of the talk will concern applications to affine Macdonald theory. I will present refinements of the denominator and evaluation conjectures for affine Macdonald polynomials proposed by Etingof-Kirillov Jr. I will then explain how to prove the first non-trivial cases of these conjectures by combining the methods of the first half and well-chosen applications of the elliptic beta integral.   The second half of this talk is joint work with E. Rains and A. Varchenko.</p>

Affine Macdonald conjectures and special values of Felder-Varchenko functions

(jt w/ E. Rains, A. Varchenko)


Affine Macdonald conjectures
and special values of
Felder–Varchenko functions
(w/ E. Rains, A. Varchenko)
Affine Macdonald polys
Ordinary Macdonald polys
s.t. leading term
diagonalize
Consider
For integral dominant λ, there exists

...ures

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(...henko)

...d polys

... polys

$$= \sum_{|I|=r} \prod_{\substack{i\in I \\ j\notin I}} \frac{tx_i - x_j}{x_i - x_j} T_{q,x_i}$$

Consider $U_q(\mathfrak{gl}_n)$

For integral dominant λ, there exists

$$\Phi_\lambda : L_{\lambda+(m-1)\rho} \longrightarrow L_{\lambda+(m-1)\rho} \otimes \underbrace{\mathrm{Sym}^{(m-1)n}\, \mathbb{C}^n \otimes (\det)^{-(m-1)}}_{W_{m-1}}$$

s.t. $\Phi_\lambda\, v_{HW} \longmapsto v_{HW} \otimes w_0 + (\text{l.o.t})$

for $\mathbb{C}\cdot w_0 \cong W_{m-1}[0]$.

Thm (Etingof–Kirillov Jr, 1994)

$$P_\lambda(x, q^2, q^{2m}) = \frac{\mathrm{Tr}(\Phi_\lambda x^h)}{\mathrm{Tr}(\Phi_0 x^h)}.$$

$$\underbrace{\mathrm{Sym}^{(m-1)n}\, \mathbb{C}^n \otimes (\det)^{-(m-1)}}_{W_{m-1}}$$

Affine setting

$U_q(\mathfrak{gl}_n) \rightsquigarrow U_q(\hat{\mathfrak{sl}}_2)$.

Cartan: $\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$

$\tilde{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$

Verma module: $M_{\mu + k\Lambda_0}$

Irreducible integrable: $L_{\mu + k\Lambda_0}$

Evaluation: $V(z)$ via $U_q(\hat{\mathfrak{sl}}_2) \xrightarrow{ev_z} U_q(\mathfrak{sl}_2)$.

Prop: For $w_0 \in \mathrm{Sym}^{(m-1)n}\, \mathbb{C}^n \otimes (\det)^{-(m-1)}[0]$

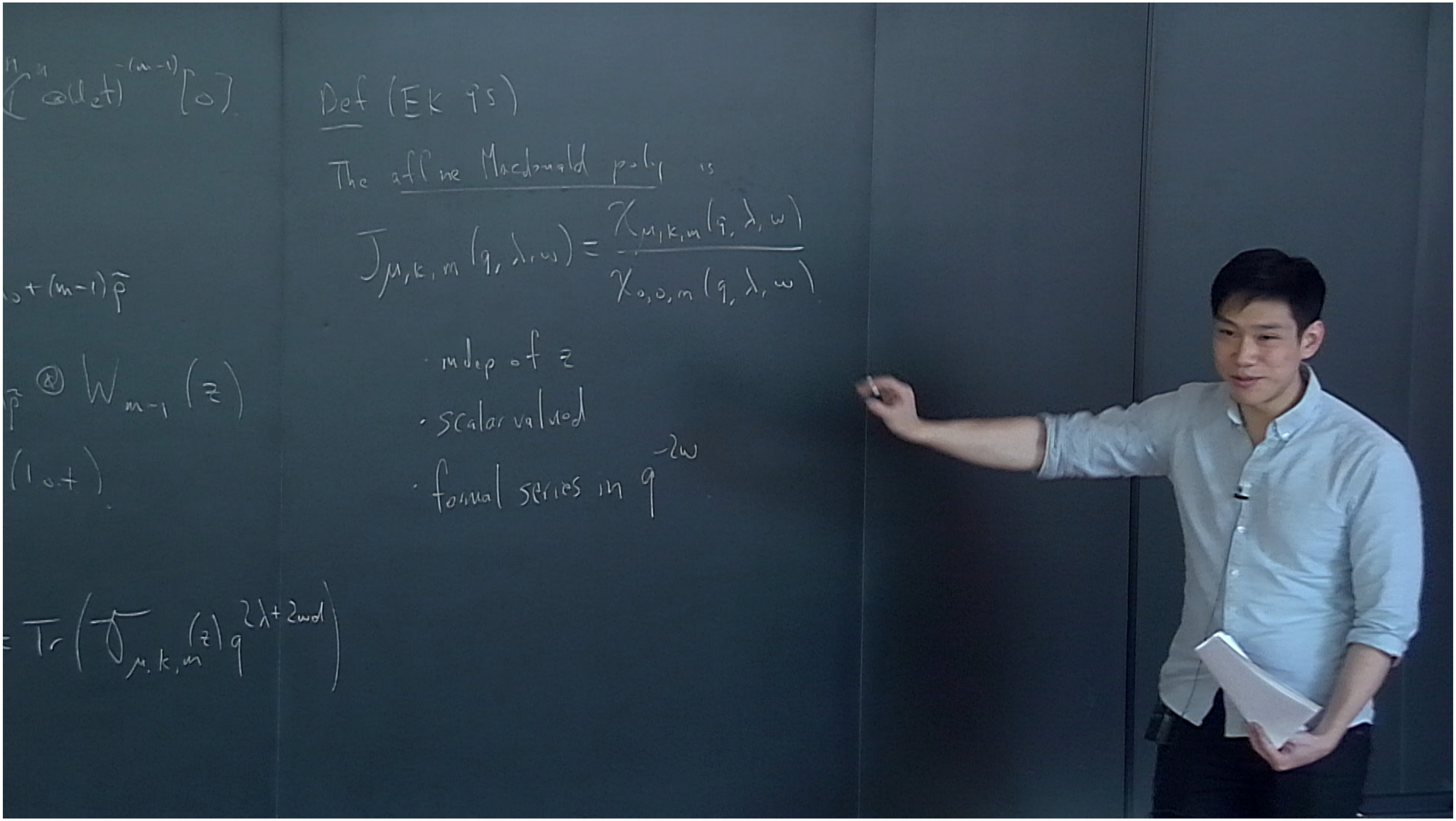
there is a unique

$$\Phi_{\mu,k,m}(z) : L_{\mu + k\Lambda_0 + (m-1)\hat{\rho}} \longrightarrow L_{\mu + k\Lambda_0 + (m-1)\hat{\rho}} \otimes W_{m-1}(z)$$

s.t. $v_{HW} \longmapsto v_{HW} \otimes w_0 + (\text{l.o.t.})$

Def The trace fn is:

$$\chi_{\mu,k,m}(q, \lambda, \omega) = \mathrm{Tr}\left(\Phi_{\mu,k,m}(z)\, q^{2\lambda + 2\omega d}\right)$$


Def (EK q's)
The affine Macdonald poly is
J_{μ,k,m}(q, λ, ω) = χ_{μ,k,m}(q, λ, ω) / χ_{0,0,m}(q, λ, ω)
· indep of z
· scalar valued
· formal series in q^{-2ω}

2. Affine Mac conj

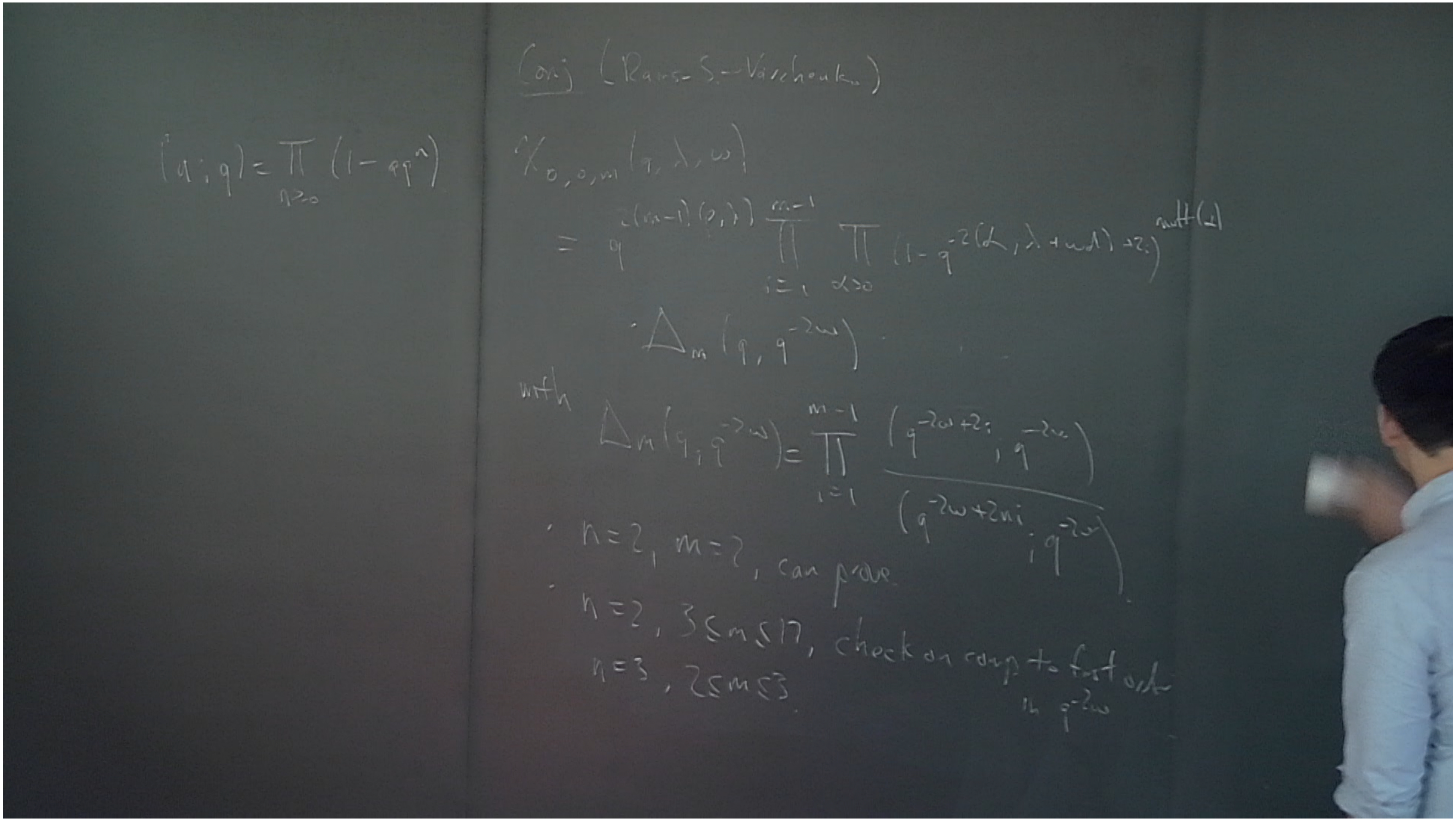
Trig conjectures

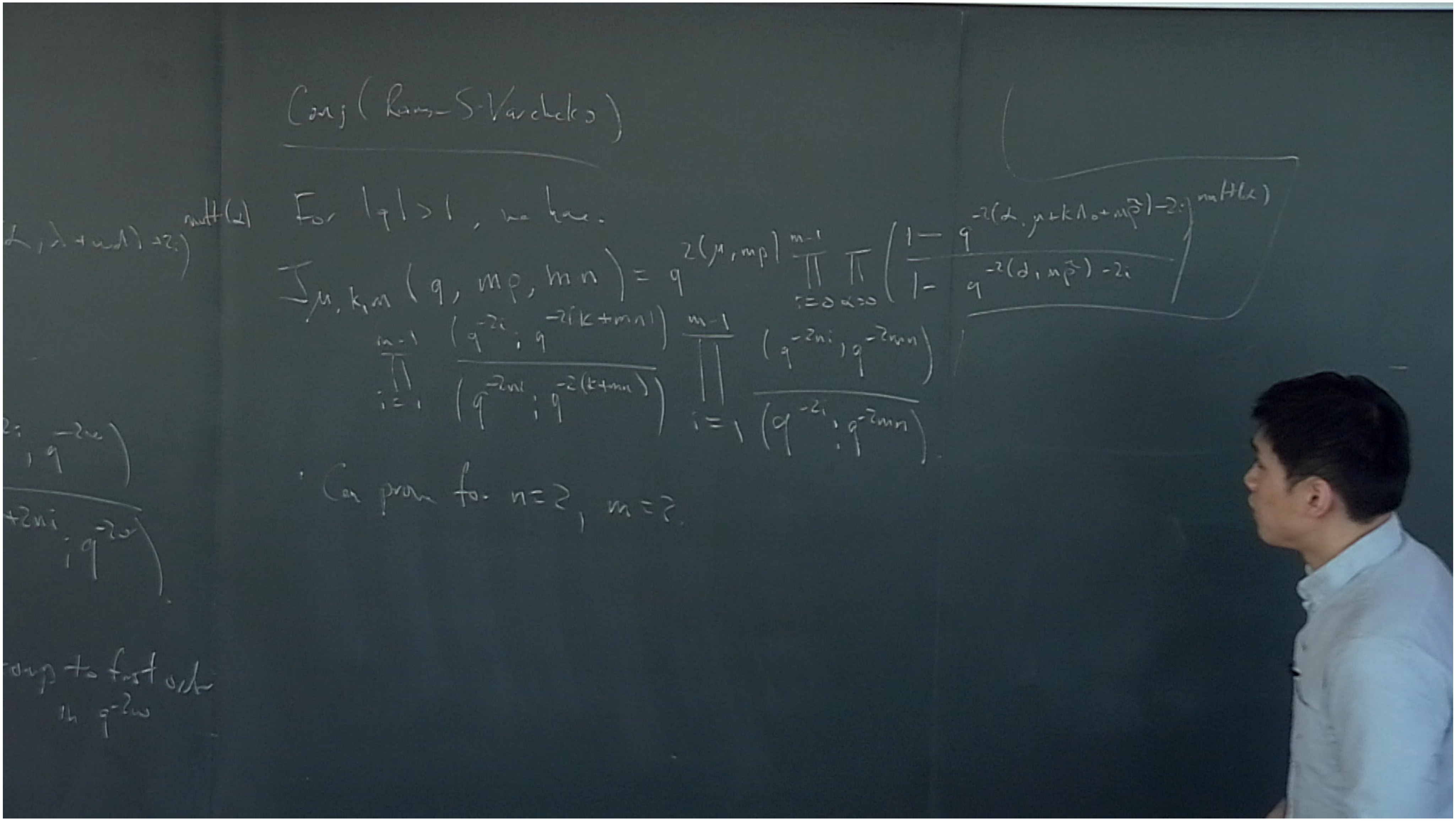
· Denom

$$\mathrm{Tr}\left(\Phi_0 q^{2\lambda}\right) = q^{2(m-1)(\rho,\lambda)} \prod_{i=1}^{m-1} \prod_{\alpha>0} \left(1-q^{-2(\alpha,\lambda)+2i}\right)^{\mathrm{mult}(\alpha)}$$

· Evaluation

$$P_\mu\left(q^{2m\rho}; q^2, q^{2m}\right) = q^{2m(\mu,\rho)} \prod_{i=0}^{m-1} \prod_{\alpha>0} \frac{\left(1-q^{-2(\alpha,\mu+m\rho)-2i}\right)^{\mathrm{mult}(\alpha)}}{\left(1-q^{-2(\alpha,m\rho)2i}\right)^{\mathrm{mult}(\alpha)}}$$


Conj (Rains-S.-Varchenko)
(a; q) = ∏ n≥0 (1 − aq^n)
χ 0,0,m (q, λ, ω)
= q^2(m−1)(ρ,λ) ∏ i=1..m−1 ∏ α>0 (1 − q^−2(α, λ+ωρ)+2i)^mult(α)
· Δm (q, q^−2ω)
with Δm (q, q^−2ω) = ∏ i=1..m−1 (q^−2ω+2i ; q^−2ω) / (q^−2ω+2ni ; q^−2ω)
· n=2, m=2, can prove
· n=2, 3≤m≤17, check on comp. to first order in q^−2ω
n=3, 2≤m≤3


Conj (Rimanyi-S-Varchenko)
For |q|>1, we have.
J_{μ,k,m}(q, mρ, mn) = q^{2(μ,mρ)} ∏_{i=0}^{m-1} ∏_{α>0} ((1 − q^{-2(α, μ+kΛ_0+mρ̃)-2i}) / (1 − q^{-2(α,mρ̃)-2i}))^{mult(α)}
∏_{i=1}^{m-1} (q^{-2i}; q^{-2i(k+mn)}) / (q^{-2ni}; q^{-2(k+mn)}) ∏_{i=1}^{m-1} (q^{-2ni}; q^{-2mn}) / (q^{-2i}; q^{-2mn}).
Can prove for n=2, m=2.

Conj (Rains–S–Varchenko)

For $|q|>1$, we have.

$$J_{\mu,k,m}(q, m\rho, mn) = q^{2(\mu, m\rho)} \prod_{i=0}^{m-1} \prod_{\alpha>0} \left(\frac{1 - q^{-2(\alpha, \mu+k\Lambda_0+m\hat{\rho}) - 2i}}{1 - q^{-2(\alpha, m\hat{\rho}) - 2i}} \right)^{\mathrm{mult}(\alpha)}$$

$$\prod_{i=1}^{m-1} \frac{(q^{-2i}; q^{-2(k+mn)})}{(q^{-2mi}; q^{-2(k+mn)})} \prod_{i=1}^{m-1} \frac{(q^{-2ni}; q^{-2mn})}{(q^{-2i}; q^{-2mn})}.$$

- Can prove for $n=2$, $m=2$.

Limits of the conj

Trig: $q^{-2w} \to 0$. $\Rightarrow$ indep of t

get trig Mac conj.

Classical: $q = e^{\varepsilon}$, $\lambda = \varepsilon^{-1}\Lambda$, $\omega = \varepsilon^{-1}\Omega$. $\varepsilon \to 0$.

- aff Mac poly $\to$ aff Jack poly

limit of denom holds

$$(a;q) = \prod_{n\geq 0} (1 - aq^n)$$

$$(a;q,r) = \prod_{n,m\geq 0} (1 - aq^m r^n)$$

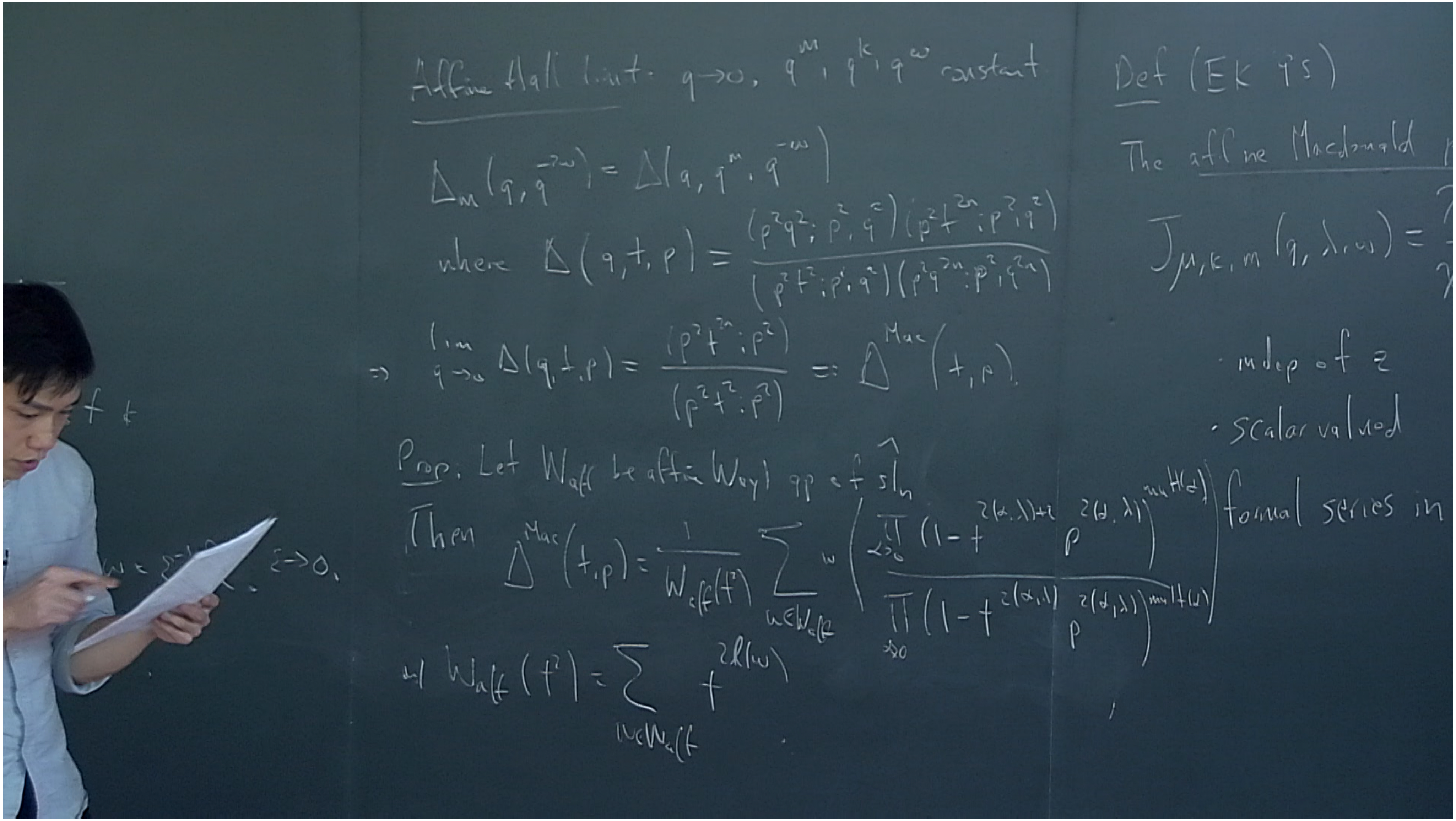
$$\chi_{0,0,m}(q,\lambda,\omega)$$

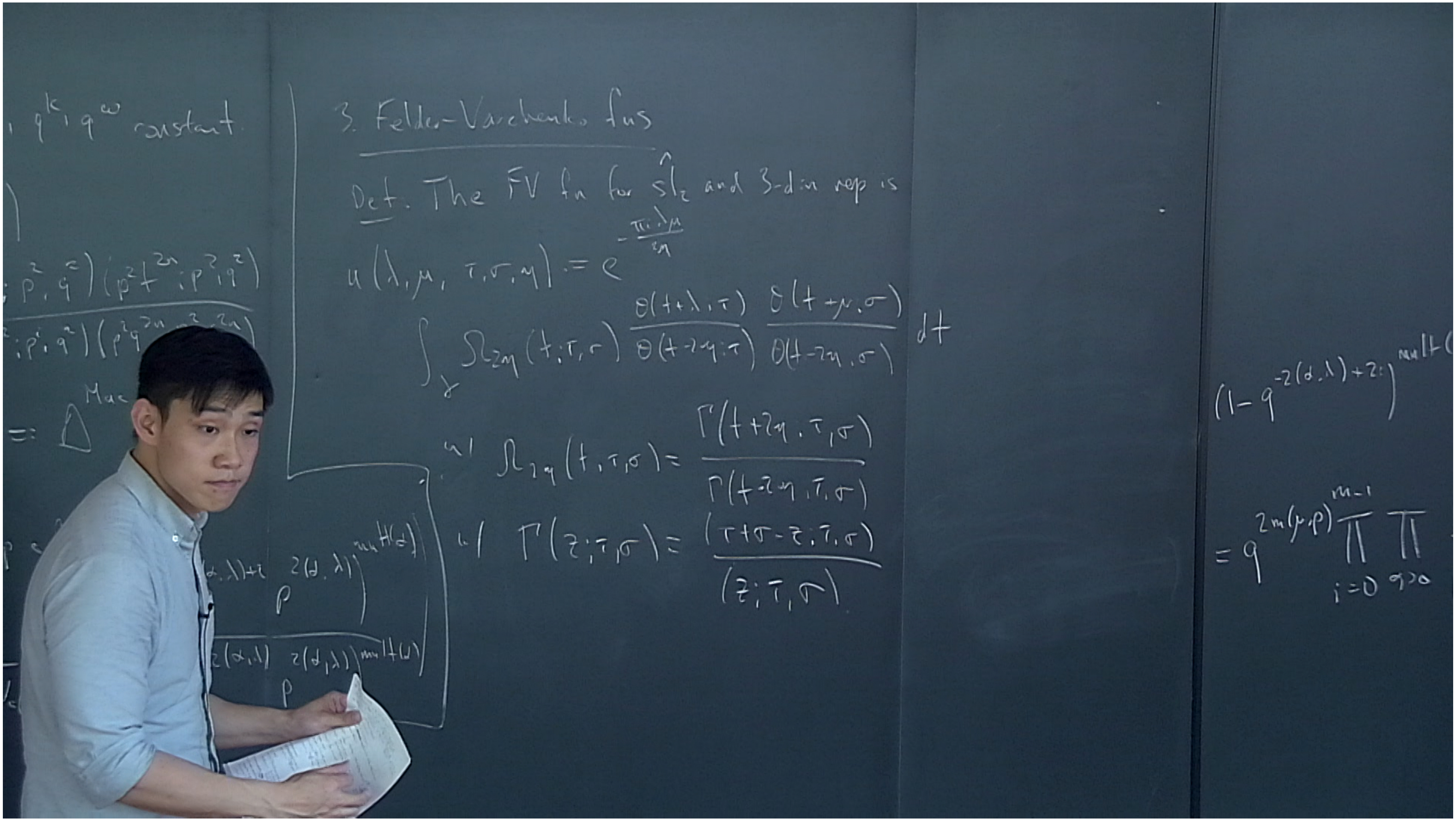
$$= q^{2(m-1)(\ldots)} \ldots$$

$$\cdot \Delta_m(q, \ldots$$

with

$$\Delta_m(q, q^{-2}\omega) \ldots$$


Affine Hall limit: q→∞, q^m, q^k, q^ω constant
Δ_m(q, q^{-2ω}) = Δ(q, q^m, q^{-ω})
where Δ(q,t,p) = (p²q²; p², q²)(p²t^{2n}; p², q²) / (p²t²; p², q²)(p²q^{2n}; p², q^{2n})
⇒ lim_{q→∞} Δ(q,t,p) = (p²t^{2n}; p²) / (p²t²; p²) =: Δ^{Mac}(t,p)
Prop: Let W_aff be affine Weyl gp of ŝl_n
Then Δ^{Mac}(t,p) = 1/W_aff(t²) Σ_{w∈W_aff} w(∏_{α>0} (1 − t^{2(α,λ)+2} p^{2(α,λ)})^{m_α H(α)} / ∏_{α>0} (1 − t^{2(α,λ)} p^{2(α,λ)})^{m_α H(α)})
formal series in
w/ W_aff(t²) = Σ_{w∈W_aff} t^{2ℓ(w)}
Def (EK q's)
The affine Macdonald
J_{μ,k,m}(q, λ, ω) =
· indep of z
· scalar valued
ε→0.


3. Felder-Varchenko fns
Def. The FV fn for $\hat{sl}_2$ and 3-dim rep is
$u(\lambda,\mu,\tau,\sigma,\eta) := e^{-\frac{\pi i \lambda\mu}{2\eta}}$
$\int_\gamma \Omega_{2\eta}(t;\tau,\sigma) \frac{\theta(t+\lambda,\tau)}{\theta(t-2\eta;\tau)} \frac{\theta(t+\mu,\sigma)}{\theta(t-2\eta,\sigma)} dt$
w/ $\Omega_{2\eta}(t,\tau,\sigma) = \frac{\Gamma(t+2\eta,\tau,\sigma)}{\Gamma(t-2\eta,\tau,\sigma)}$
w/ $\Gamma(z;\tau,\sigma) = \frac{(\tau+\sigma-z;\tau,\sigma)}{(z;\tau,\sigma)}$
constant

$\cdot\ \hat{\Delta}_{\mu,k}(\lambda,\tau,\eta)$

$$= (\ ) \int_\gamma \Omega_{2\eta}(t;\tau,-2\eta k)\,\frac{\theta(t+\lambda,\tau)}{\theta(t-2\eta,\tau)}\,\frac{\theta(t+2\eta\mu;-2\eta k)}{\theta(t-2\eta,-2\eta k)}$$

$$\theta_0\left(\tfrac{1}{2}+\mu\tau+k\tau-k\lambda+2t;\,2k\tau\right)dt.$$

$$\Delta_{\mu,k}(\lambda,\tau,\eta) = \hat{\Delta}_{\mu,k}(\lambda,\tau,\eta) - \hat{\Delta}_{\mu,k}(-\lambda,\tau,\eta)$$

Def (FV '04)

Elliptic Mac poly at $t=q^2$

$$P_{\mu,k}(\lambda,\tau,\eta) = (\ )\,\frac{\Delta_{\mu+\rho,k}(\lambda,\tau,\eta)}{\theta(\lambda-2\eta,\tau)\,\theta(\lambda,\tau)\,\theta(\lambda+2\eta,\tau)}$$

Thm (S '16)

In certain numerical range:

$$J_{\mu,k,2}(q,\lambda,\omega) = (\ )\,P_{\mu,k}(2\eta\lambda,-2\eta\omega,\eta)$$

1. Reduce to trace over Verma

$$= q^{2(\mu, m\rho)} \prod_{i=0}^{m-1} \prod_{\alpha>0} \left(\frac{1 - q^{-2(\alpha, \mu+k\Lambda_0+m\tilde{\rho}) - 2i}}{1 - q^{-2(\alpha, m\tilde{\rho}) - 2i}} \right)^{\mathrm{mult}(\alpha)}$$

$$\prod_{i=1}^{m-1} \frac{(q^{-2ni}; q^{-2mn})}{(q^{-2i}; q^{-2mn})}$$

$= ?$

<u>Step 1</u>: $\Psi_{\mu,k}(z) : M_{\mu+k\Lambda_0} \to M_{\mu+k\Lambda_0} \otimes W_{m-1}(z)$

<u>Def.</u> $T(q, \lambda, \omega, \mu, k) = \mathrm{Tr}\left(\Psi_{\mu,k}(z)\, q^{2\lambda + 2\omega d} \right)$.

<u>"Thm"</u>: T is a renormalization of φ.

<u>Thm</u> (Matsuo '94)

$$M_{2\mu+k\Lambda_0} =$$

$$F_{\mu,s,s} = F_{\mu,\beta} \otimes F_{s,\alpha} \otimes I_{s,I}$$

$$\prod_{\alpha>0}\left(\frac{1-q^{-2(\alpha,\mu+k\Lambda_0+m\beta)-2\ldots}}{1-q^{-2(\alpha,m\beta)-2\ldots}}\right)^{m_{\ldots}}$$

Step 1: $\Psi_{\mu,k}(z): M_{\mu+k\Lambda_0} \to M_{\mu+k\Lambda_0}\otimes W_{n-1}(z)$

Def. $T(q,\lambda,\omega,\mu,k) = \mathrm{Tr}\left(\Psi_{\mu,k}(z)\, q^{2\lambda+2\omega d}\right)$

"**Thm**": T is a renormalization of ψ.

Thm (Matsuo '94)

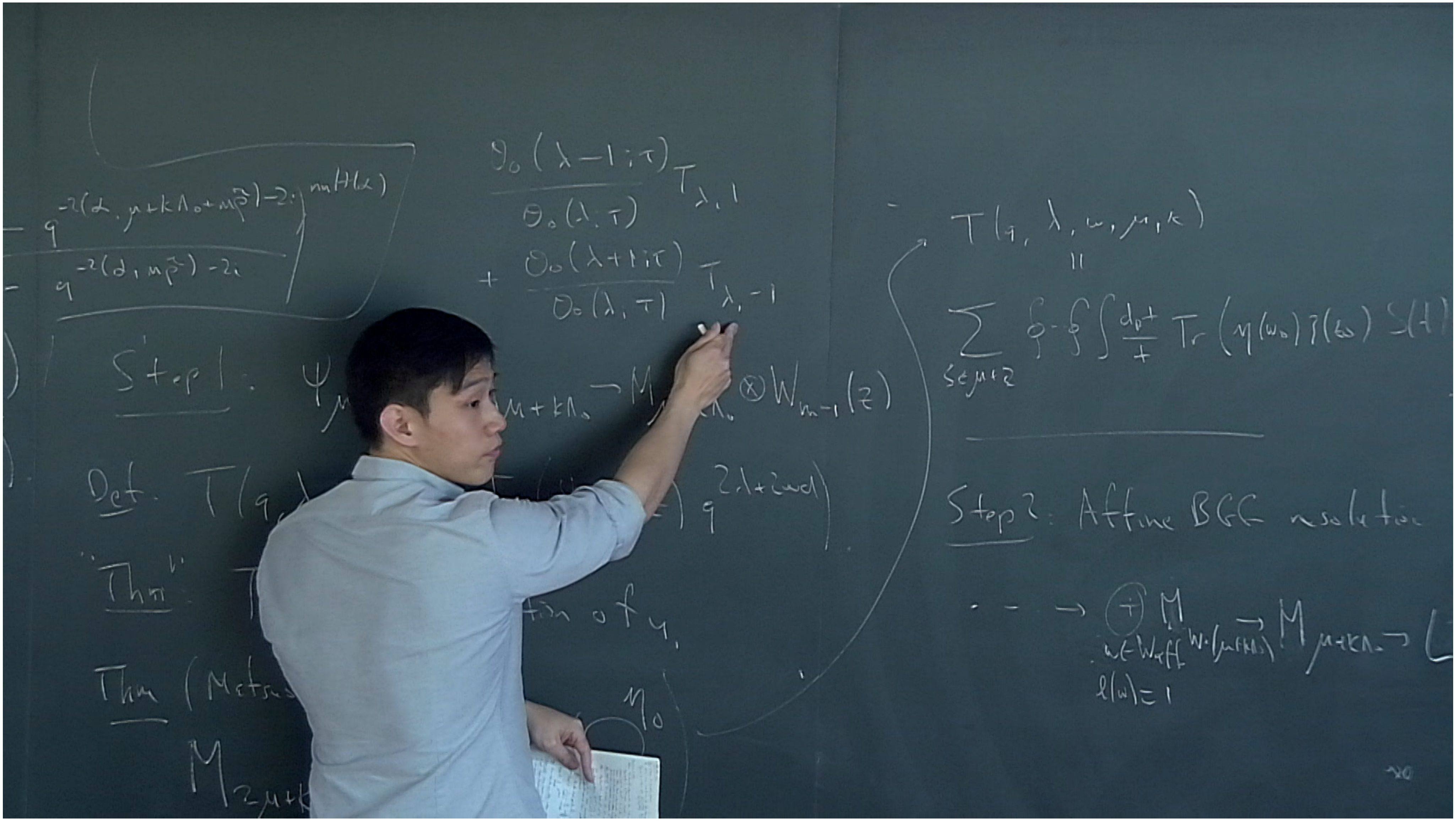
$$M_{2\mu+k\Lambda_0} = \ker\left(\bigoplus_{s\in\mu+Z} F_{\mu,s,s}\right)^{\eta_0}$$

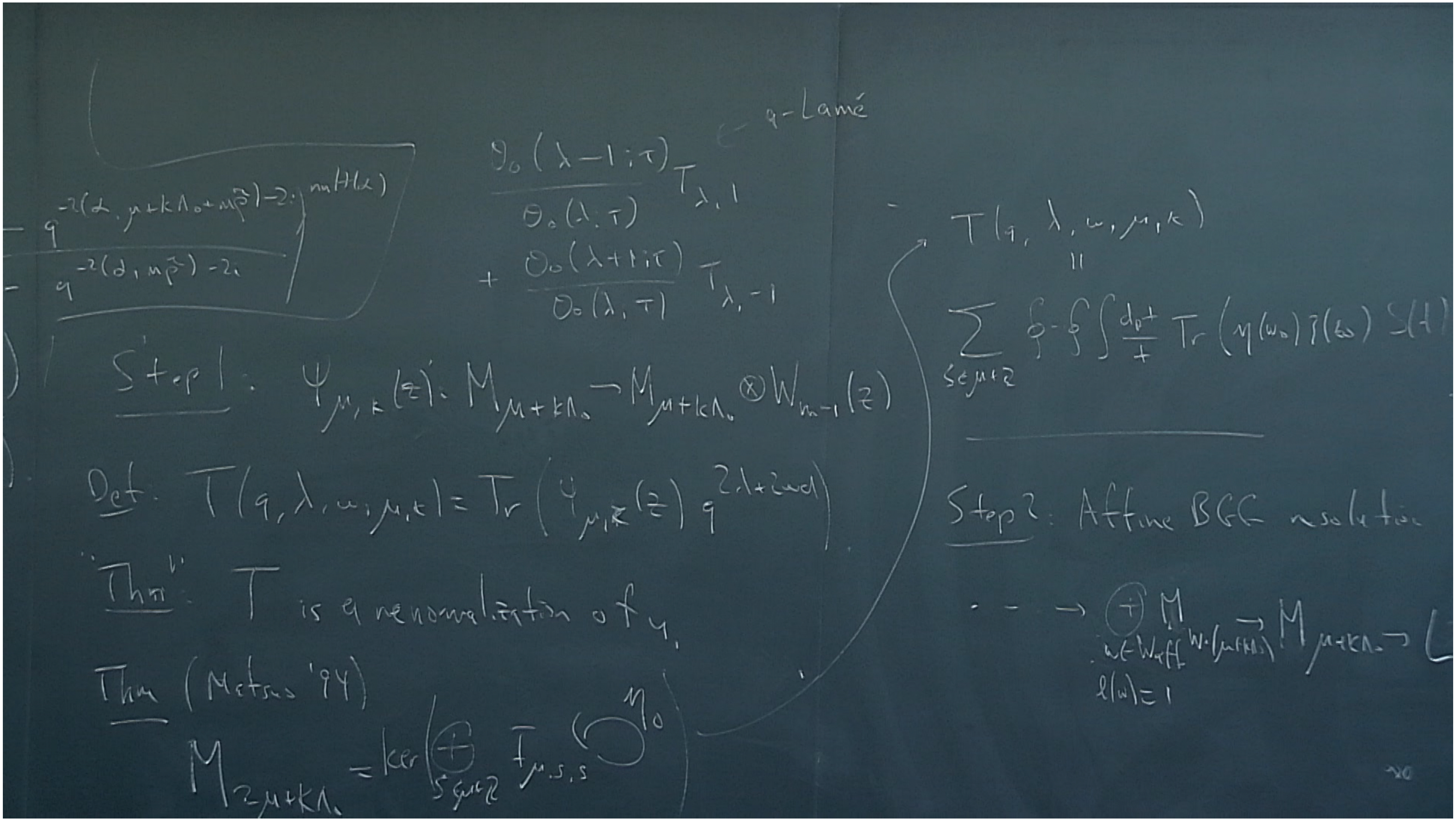
$$F_{\mu,s,s} = F_{\mu,\beta}\otimes F_{s,\alpha}\otimes I_{s,I}$$

$$T(q,\lambda,\omega,\mu,k) = \sum_{s\in\mu+Z} \oint\cdots\oint \frac{dt}{t}\,\mathrm{Tr}\left(\eta(\omega_0)\,\xi(\omega)\,S(t)\left[A(z),X^{\ldots}\right]_q \cdot q^{2\lambda\ldots+2\omega d}\right)$$

Step 2: Affine BGG resolution

$$\cdots \to \bigoplus_{\substack{w\in W_{aff}\\ \ell(w)=1}} M_{w\cdot(\mu+k\Lambda_0)} \to M_{\mu+k\Lambda_0} \to L_{\mu+k\Lambda_0}$$


Step 1:
Def.
Thm
Step 2: Affine BGG


q-Lamé
Step 1: Ψμ,k(z): Mμ+kΛ0 → Mμ+kΛ0 ⊗ Wn-1(z)
Def: T(q, λ, ω, μ, k) = Tr(Ψμ,k(z) q^(2λ+2ωd))
"Thm": T is a renormalization of fq.
Thm (Matsuo '94)
T(q, λ, ω, μ, k)
Step 2: Affine BGG resolution
ℓ(w) = 1