Title: Complexity, Holography & Quantum Field Theory

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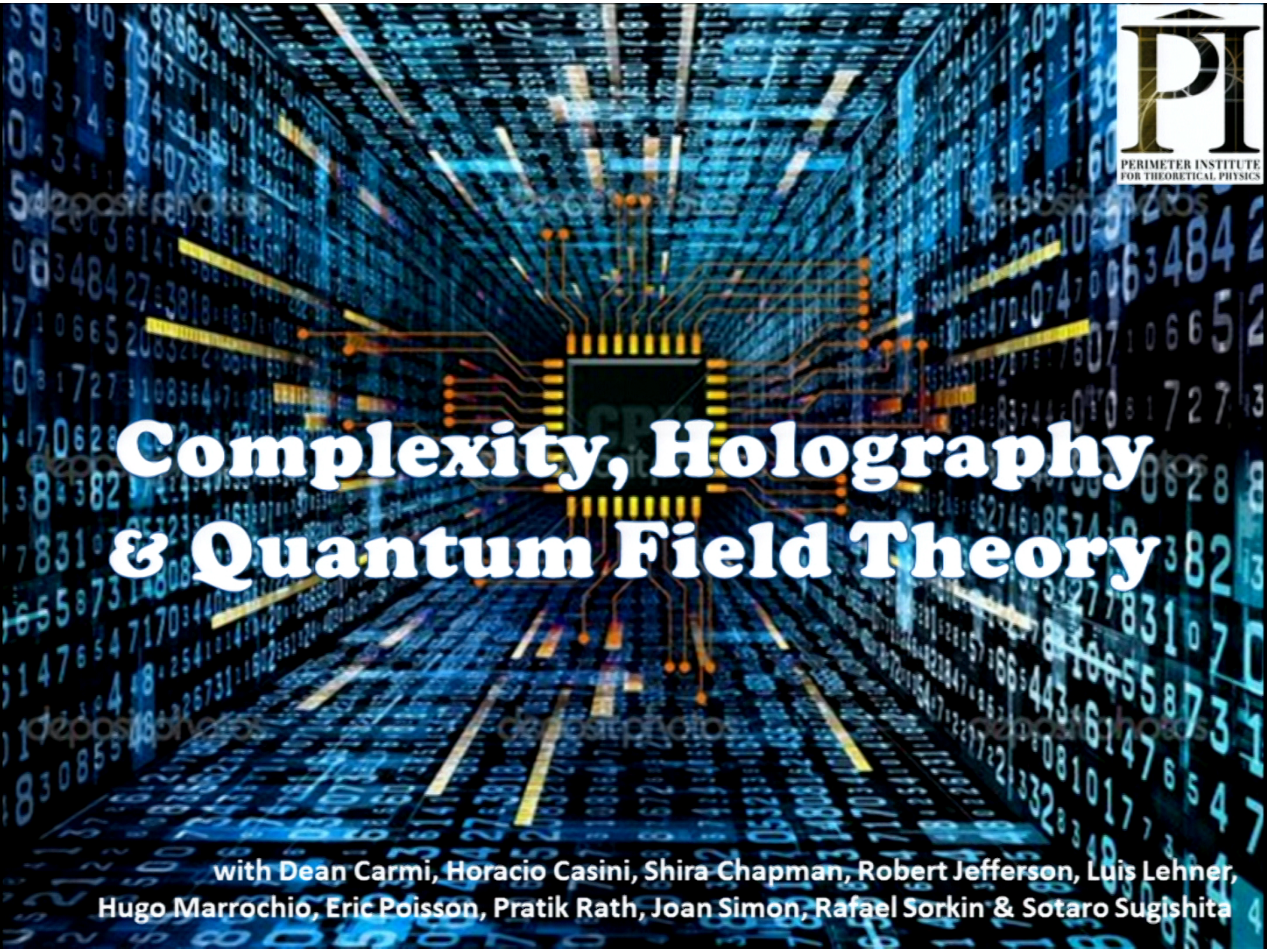
URL: http://pirsa.org/17040050

Abstract: I will describe some recent work studying proposals for computational complexity in holographic theories and in quantum field theories. In particular, I will discuss some interesting properties of the new gravitational observables and of complexity in the boundary theory.

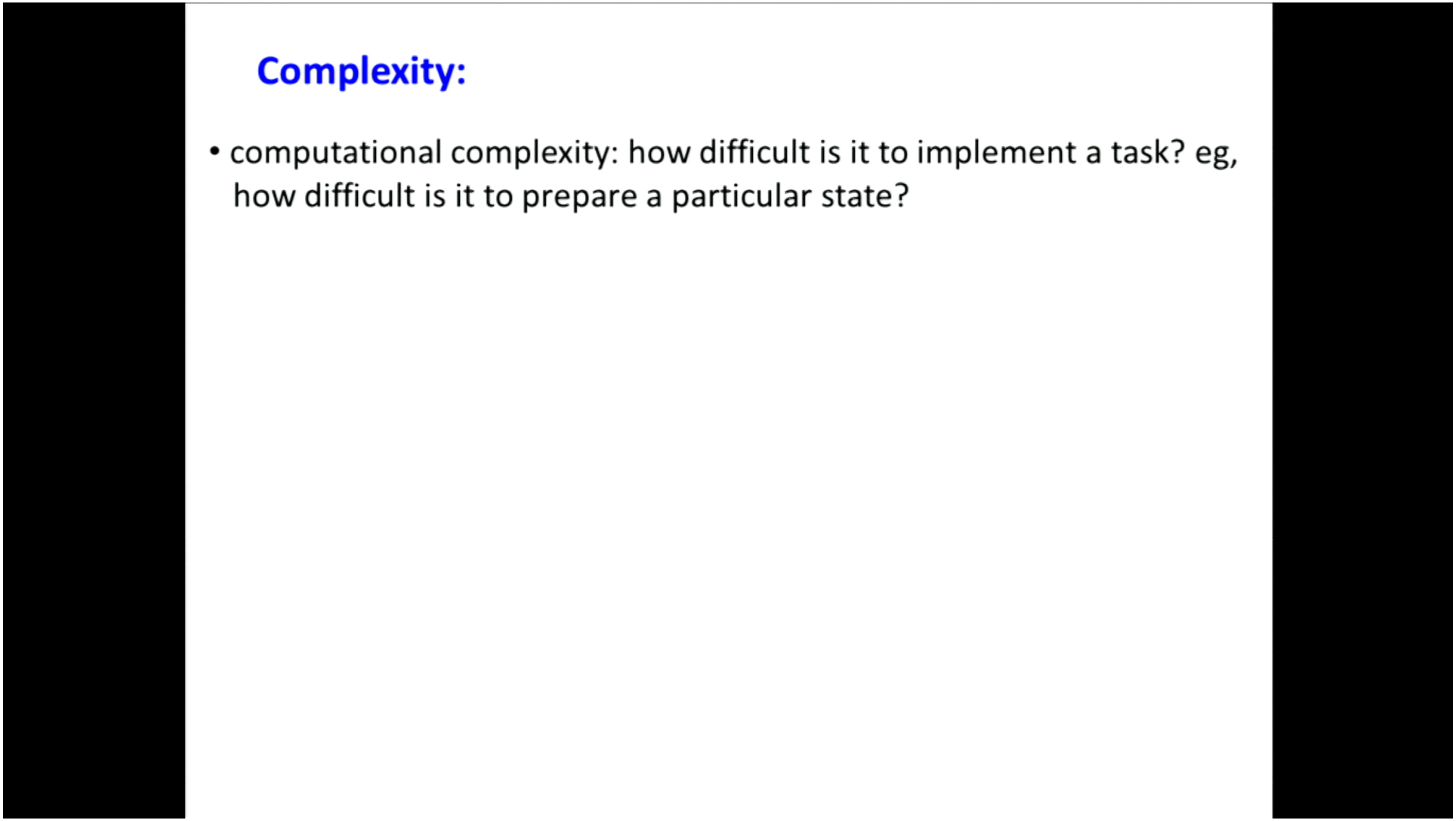


PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

Complexity, Holography & Quantum Field Theory

with Dean Carmi, Horacio Casini, Shira Chapman, Robert Jefferson, Luis Lehner, Hugo Marrochio, Eric Poisson, Pratik Rath, Joan Simon, Rafael Sorkin & Sotaro Sugishita


Disneyland
MICKEY'S TOONTOWN
BIG THUNDER RANCH
CRITTER COUNTRY
FANTASYLAND
PLAZA GARDENS STAGE
FRONTIERLAND
TOMORROWLAND TERRACE
RIVERS OF AMERICA
TOMORROWLAND
NEW ORLEANS SQUARE
PLAZA PAVILION
ADVENTURELAND
MAIN STREET, U.S.A.
CITY HALL
MICKEY & FRIENDS TRAM ROUTE
LION KING TRAM ROUTE
DOWNTOWN DISNEY DISTRICT AND HOTELS OF THE DISNEYLAND RESORT
Disney's CALIFORNIA ADVENTURE



Complexity:

- computational complexity: how difficult is it to implement a task? eg, how difficult is it to prepare a particular state?

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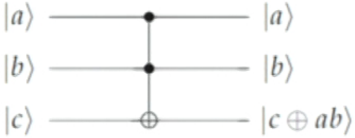
- quantum circuit model:

$$|\psi\rangle = U\,|\psi_0\rangle$$

unitary operator built from set of simple gates

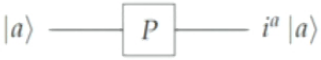
simple reference state eg, $|00000\cdots 0\rangle$

Toffoli gate



Hadamard gate

Phase-shift gate



Ancillary gate

Erasure gate

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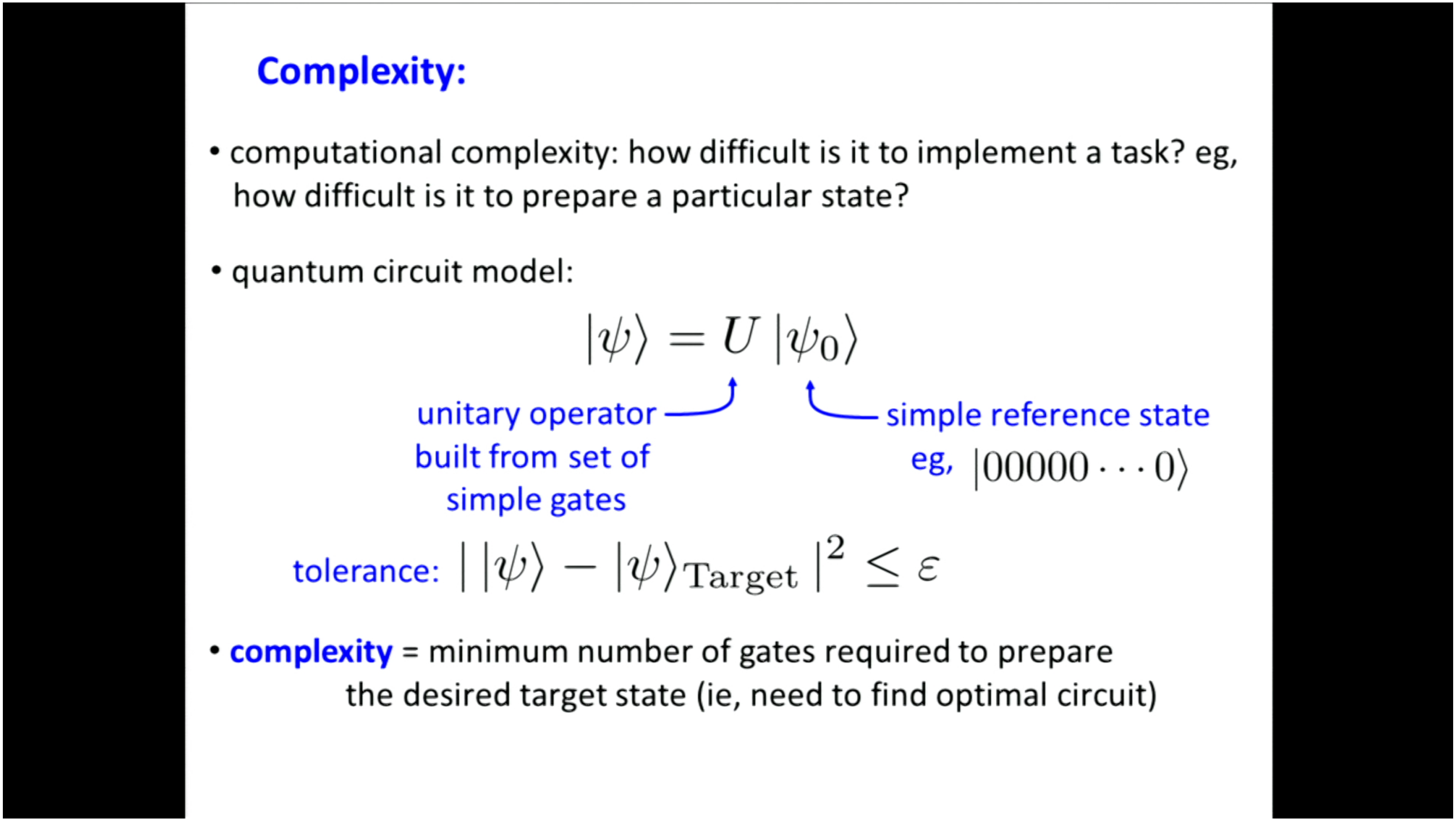

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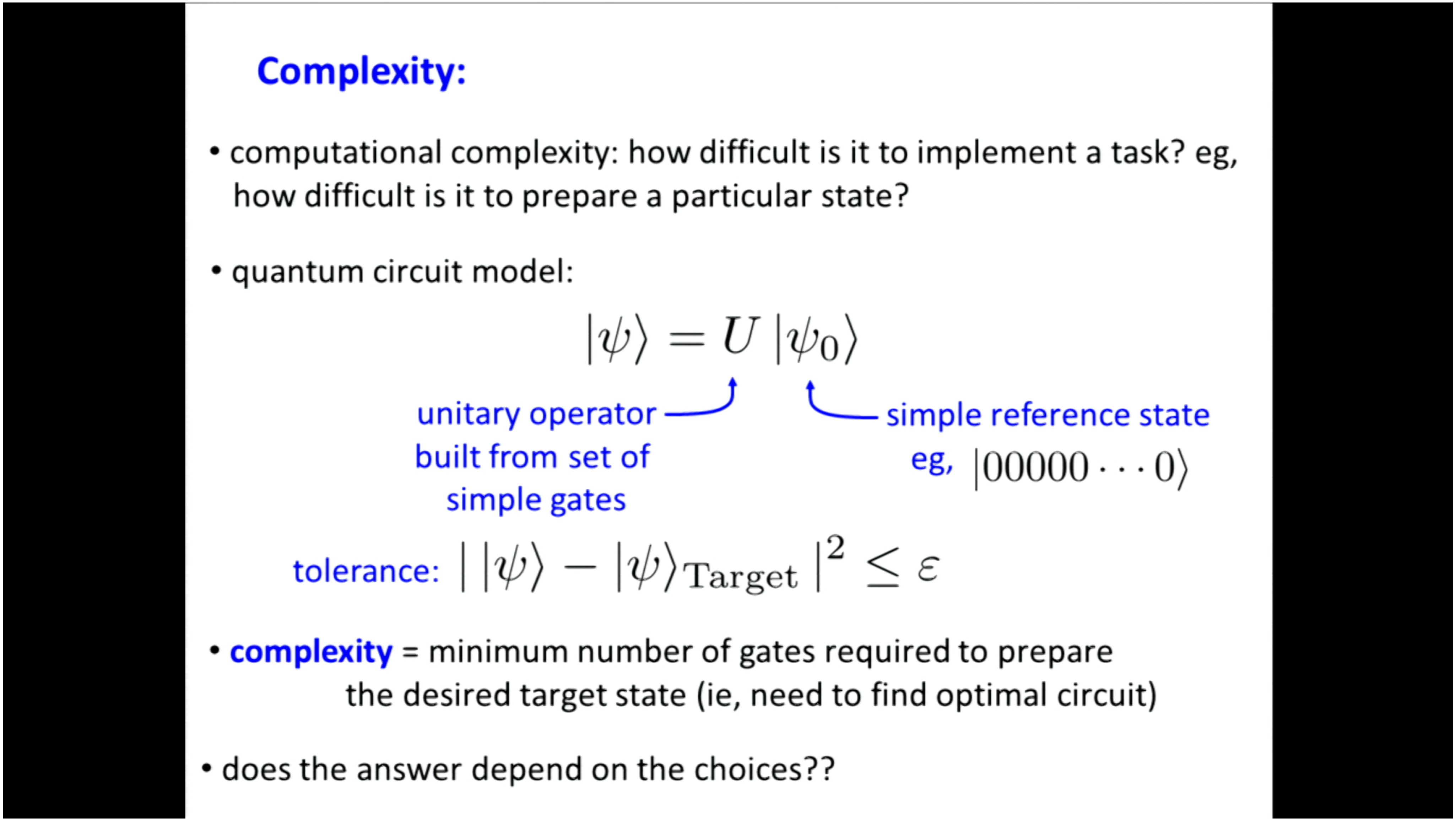

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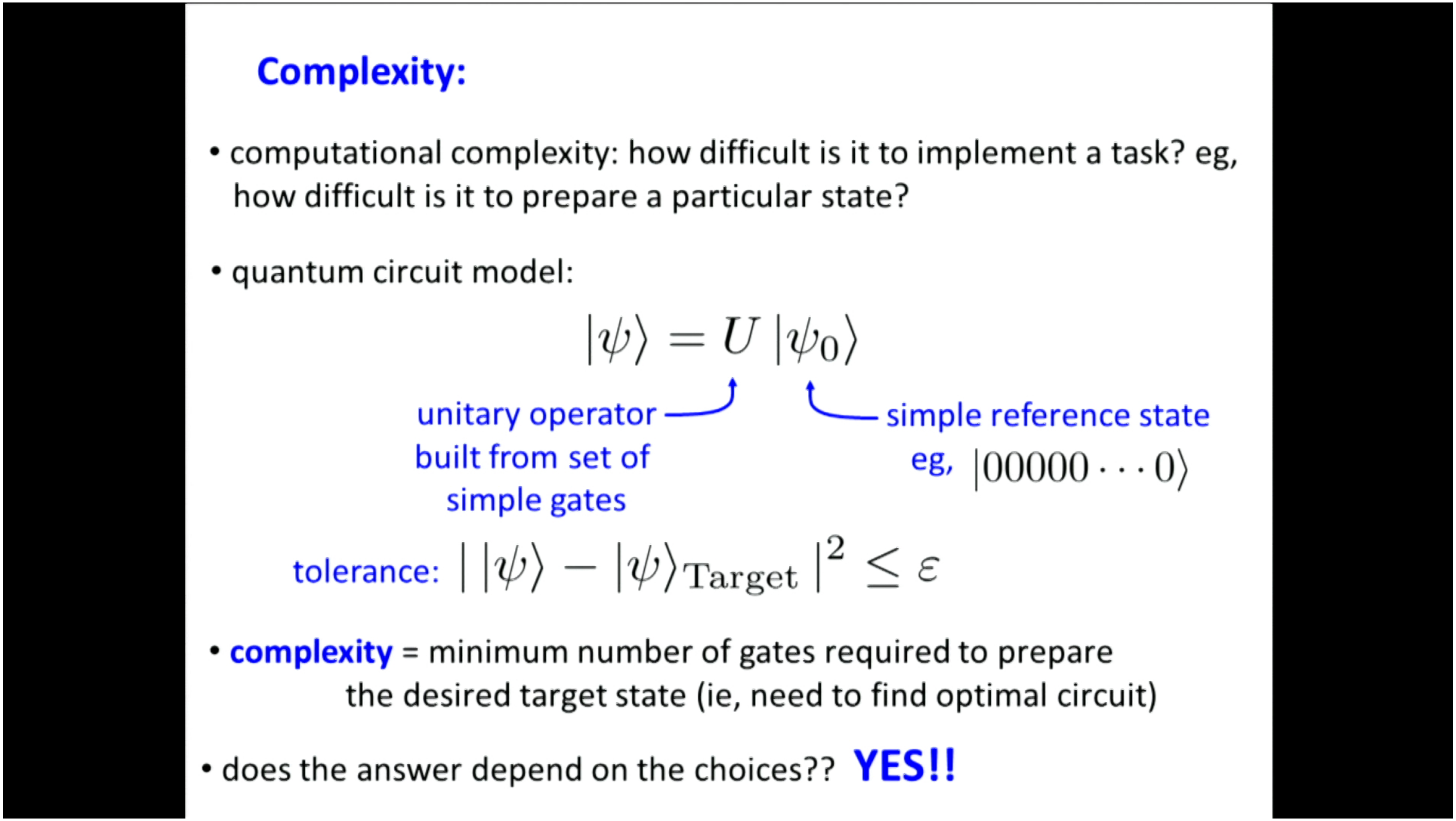

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- does the answer depend on the choices?? **YES!!**

Holography

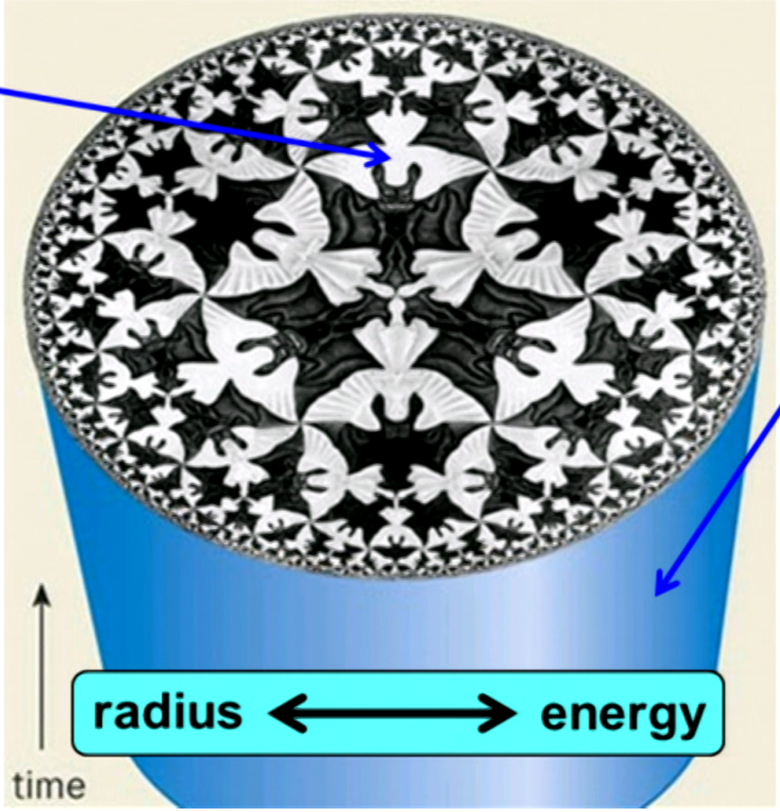
Complexity

Holography: AdS/CFT correspondence

Bulk: gravity with negative Λ
in **d+1** dimensions

↔
"holography"

Boundary: quantum field theory
without intrinsic scales
in **d** dimensions



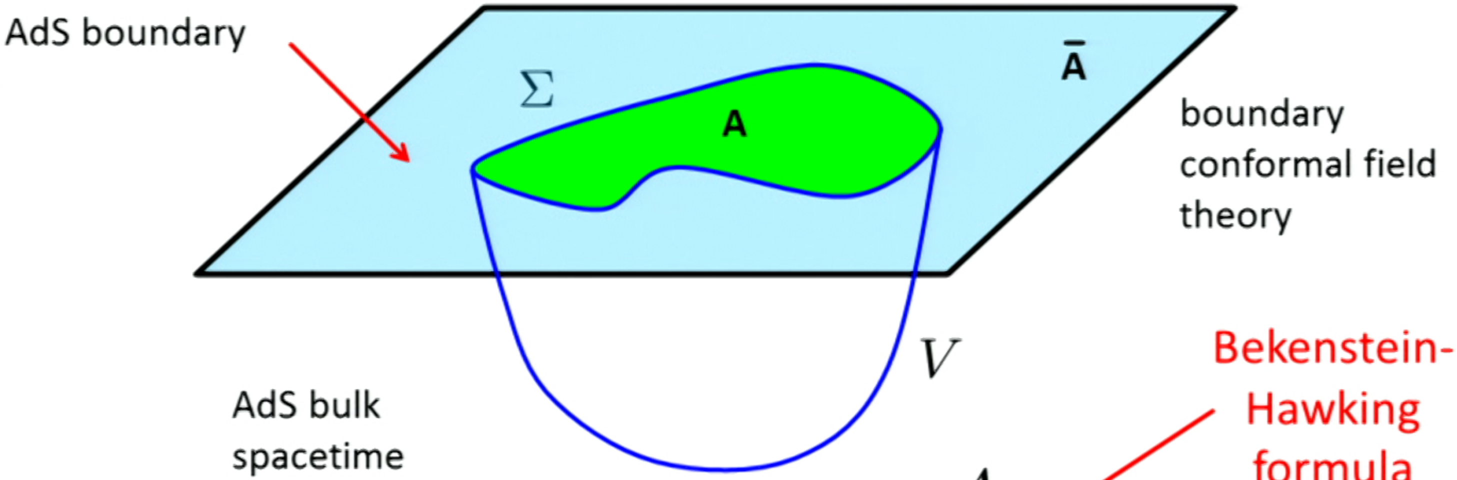
anti-de Sitter space

conformal field theory

(Maldacena '97)

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:

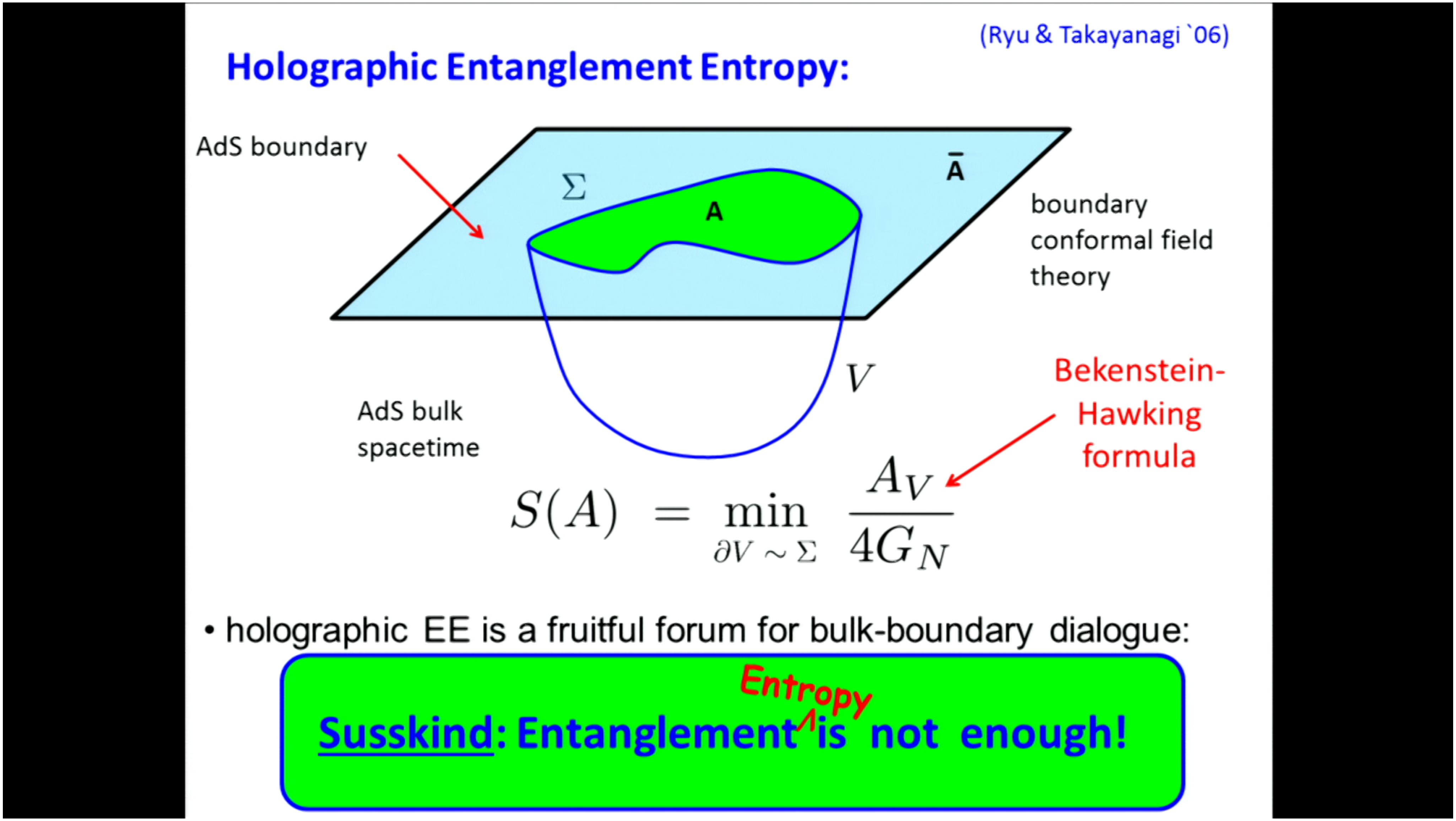


$$S(A) = \min_{\partial V \sim \Sigma} \frac{A_V}{4G_N}$$

Bekenstein-Hawking formula

- holographic EE is a fruitful forum for bulk-boundary dialogue:

Spacetime Geometry = Entanglement



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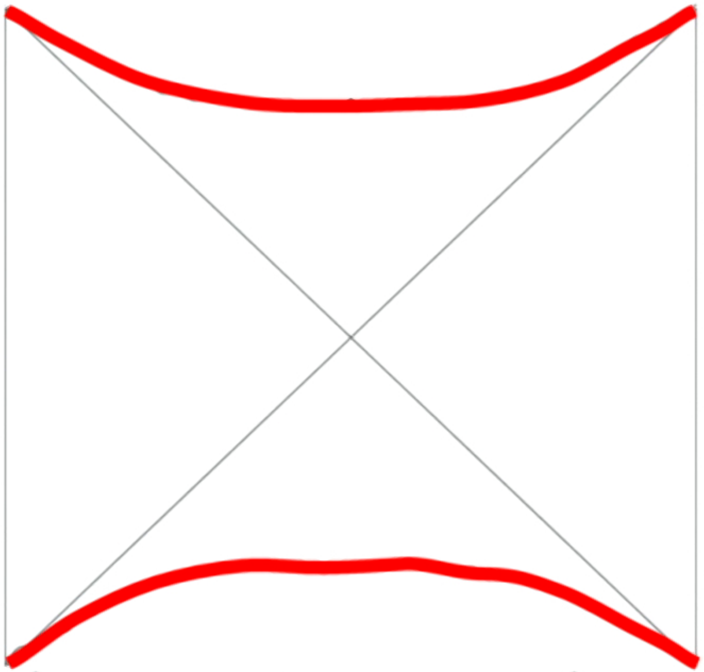

$$ds^2 = -\left(\frac{r^2}{L^2} + 1 - \frac{\mu}{r^{d-2}}\right) dt^2 + \frac{dr^2}{\frac{r^2}{L^2} + 1 - \frac{\mu}{r^{d-2}}} + r^2 d\Omega_{d-1}^2$$

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(Interstellar: black holes go to Hollywood)

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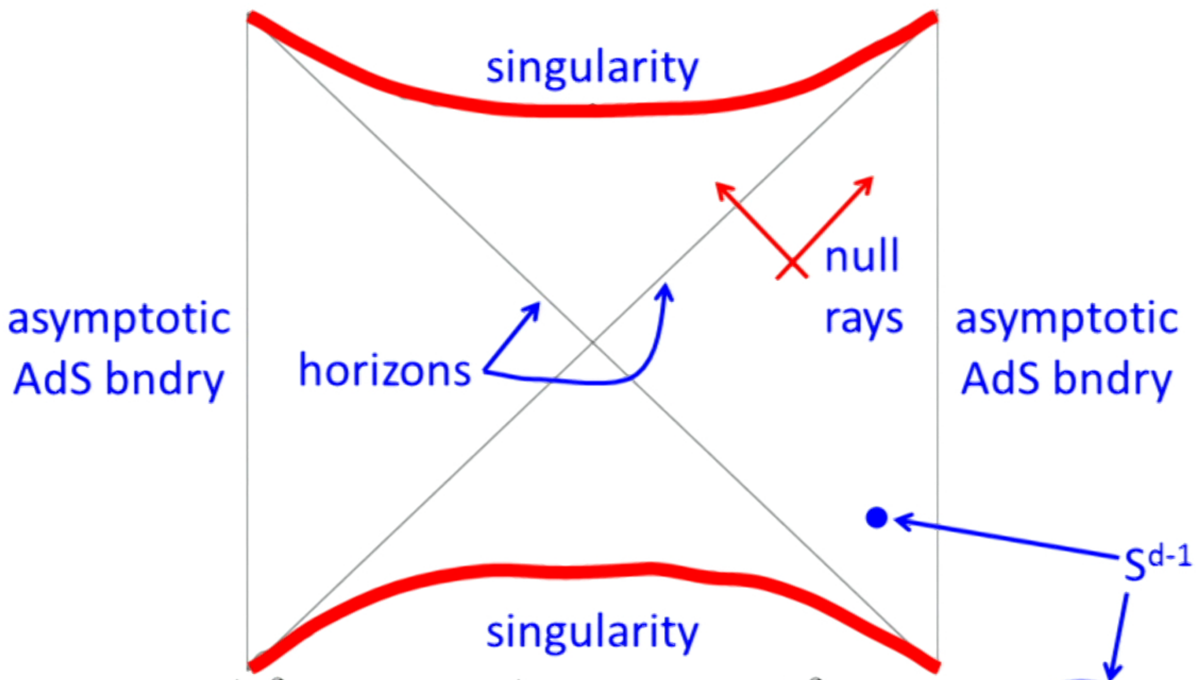

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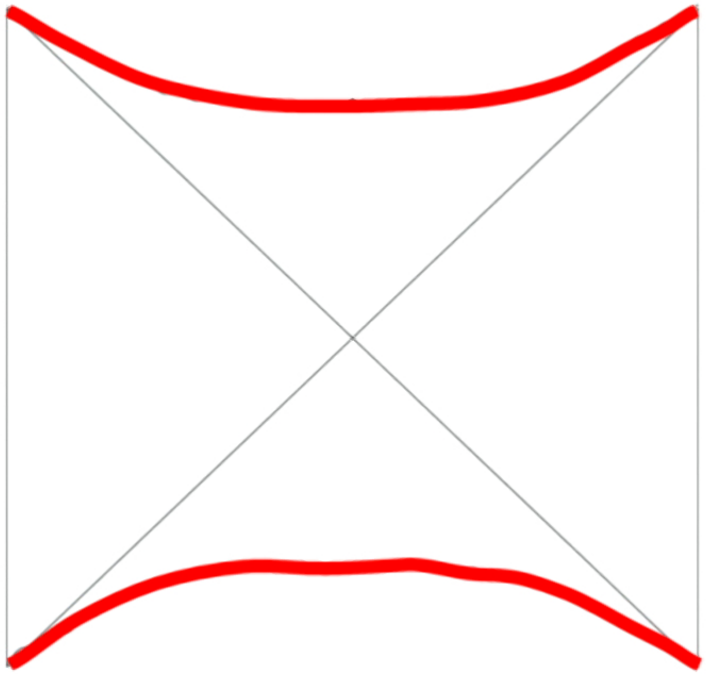

$$|\mathrm{TFD}\rangle \simeq \sum_{\alpha} e^{-E_\alpha/(2T)} |E_\alpha\rangle_L |E_\alpha\rangle_R$$

- pure state: $S_{EE} = 0$

$$\rho_R = \mathrm{Tr}_L |\mathrm{TFD}\rangle\langle\mathrm{TFD}|$$
$$\simeq \sum_{\alpha} e^{-E_\alpha/T} |E_\alpha\rangle_R \langle E_\alpha|_R$$

- mixed state: $S_{EE} = {}^{A_H}/_{4G}$

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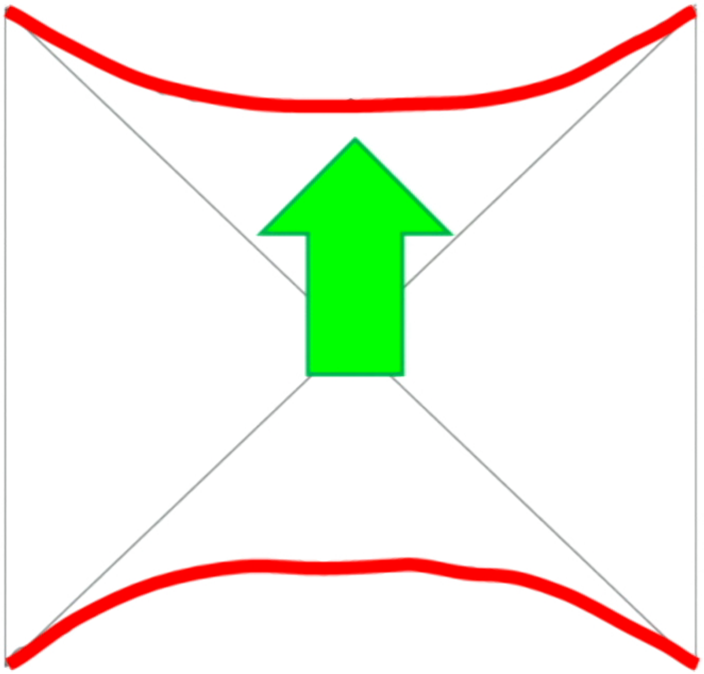

Hawking temperature

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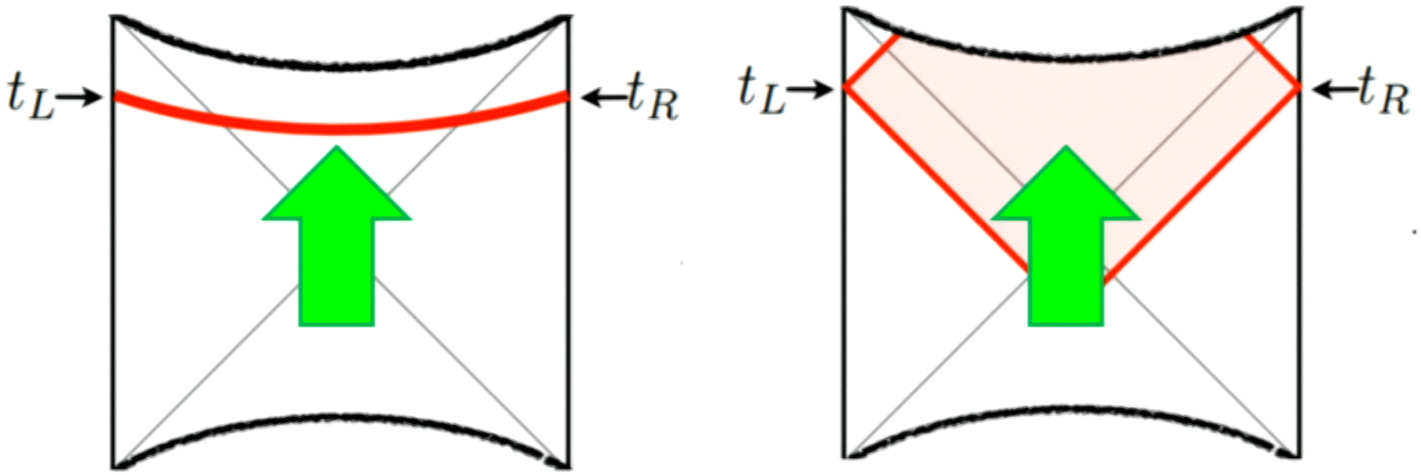
Bekenstein-Hawking entropy

A Tale of Two Dualities: Holographic Complexity

Complexity = Volume

Complexity = Action



$$\mathcal{C}_{\rm V}(\Sigma) = \max_{\Sigma=\partial\mathcal{B}} \left[\frac{\mathcal{V}(\mathcal{B})}{G_N\,\ell}\right]$$

$$\mathcal{C}_{\rm A}(\Sigma) = \frac{I_{\rm WDW}}{\pi\,\hbar}$$

- both of these gravitational "observables" probe the black hole interior (at arbitrarily late times)

(Brown, Roberts, Swingle, Stanford, **Susskind** & Zhao)

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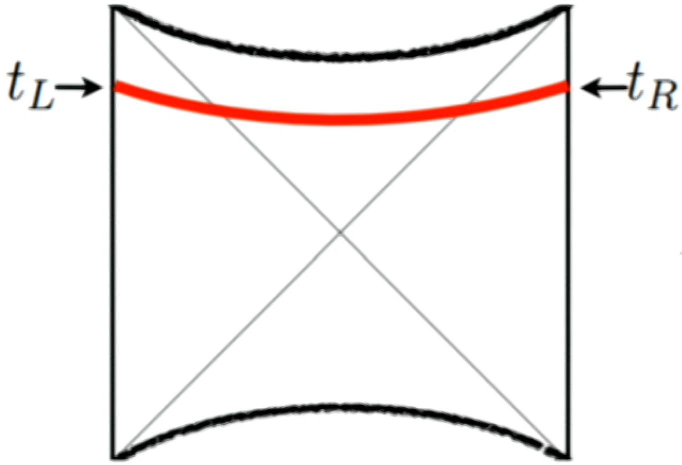

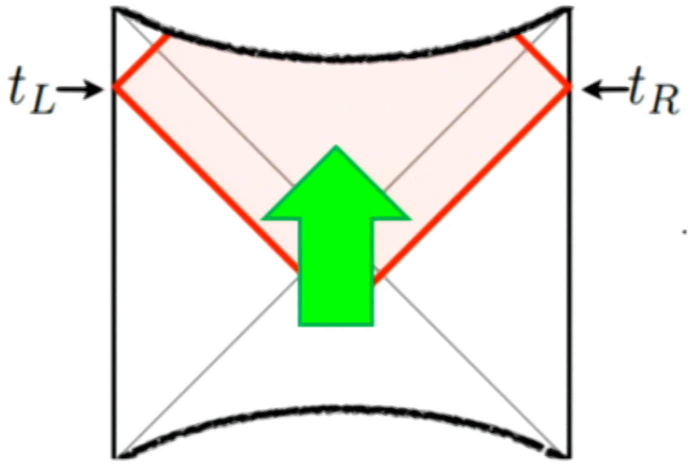

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- evaluate proper volume of extremal codimension-one surface connecting Cauchy surfaces in boundary theory

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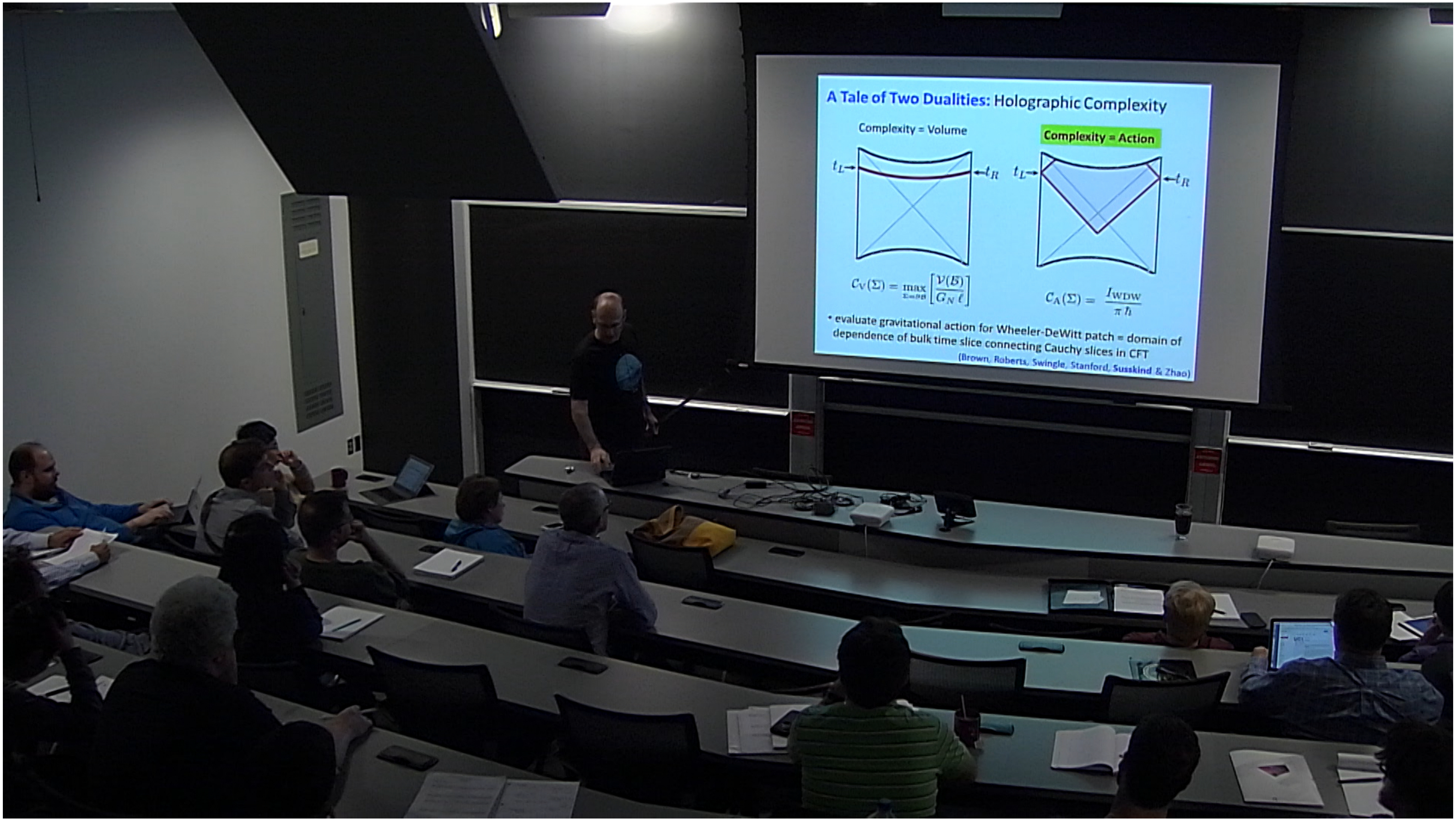

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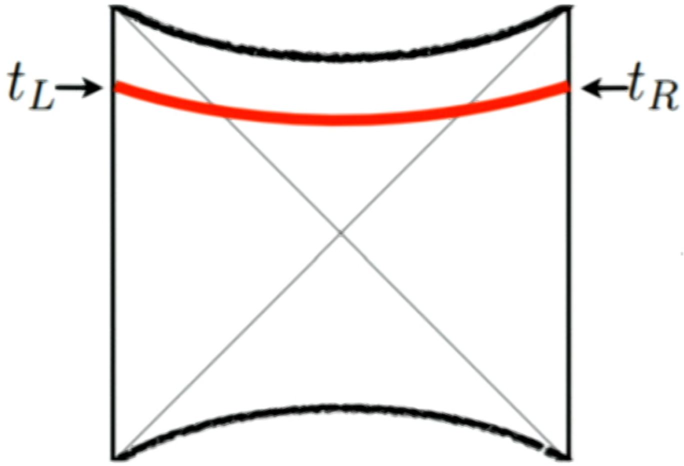

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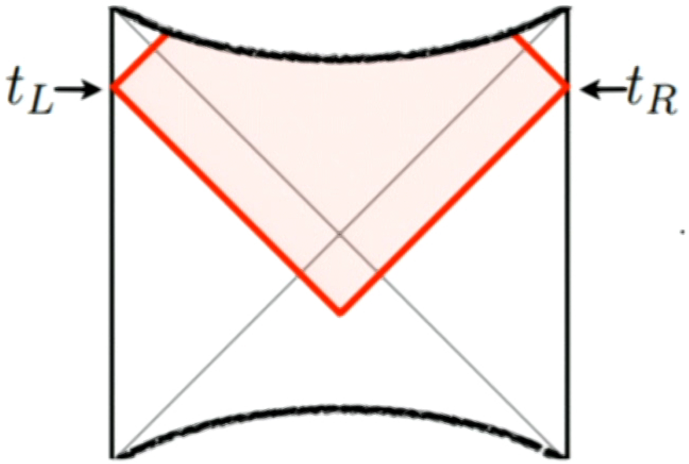

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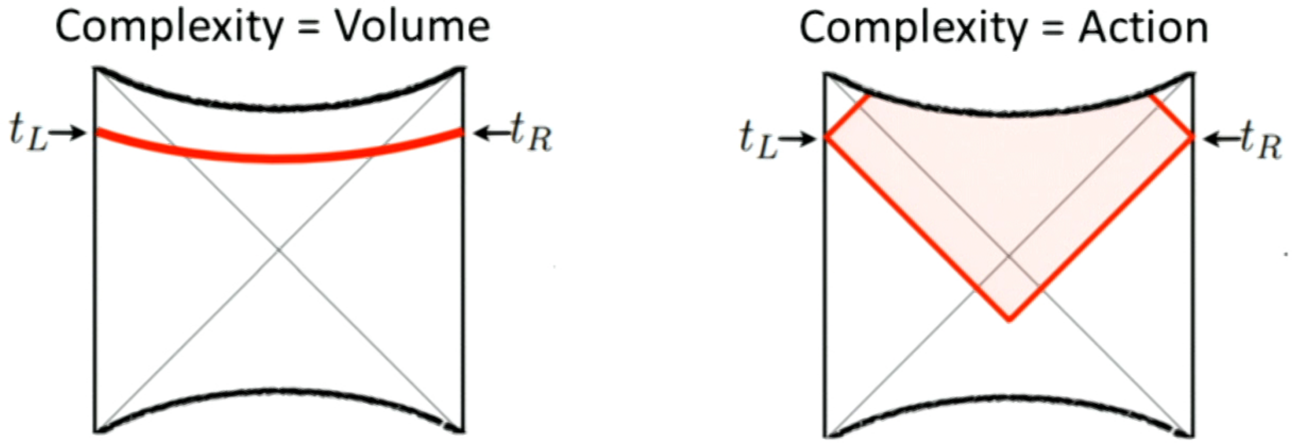
Complexity = Volume

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WHY COMPLEXITY??

- connection of complexity=volume to AdS/MERA
- linear growth (at late times)

$(d = \text{boundary dimension})$

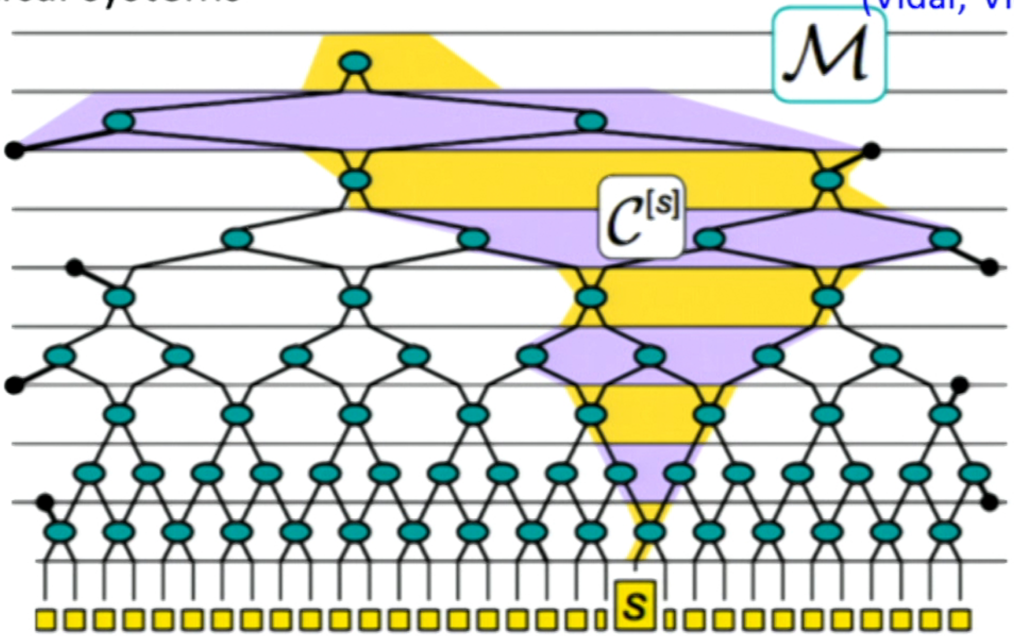
$$\left.\frac{d\mathcal{C}_{\rm V}}{dt}\right|_{t\to\infty} = \frac{8\pi}{d-1}\,M \quad \text{(planar)} \qquad \left.\frac{d\mathcal{C}_{\rm A}}{dt}\right|_{t\to\infty} = \frac{2M}{\pi}$$

- "switchback" effect (out of time-order correlators)

AdS/MERA:

- MERA (Multi-scale Entanglement Renormalization Ansatz) provides efficient tensor network representation of ground-state wave-function in d=2 critical systems

(Vidal; Vidal & Evenbly)



this is a quantum circuit!!

(Swingle)

- has been argued that MERA gives a discrete representation of a time slice in AdS space!

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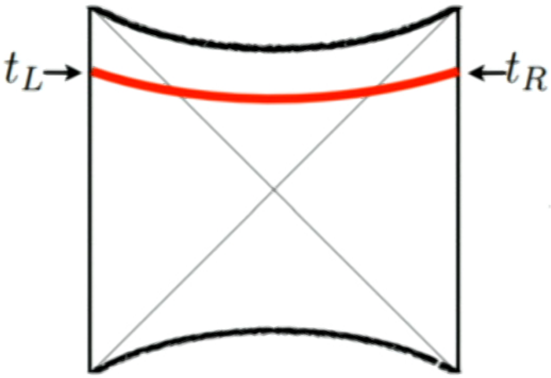

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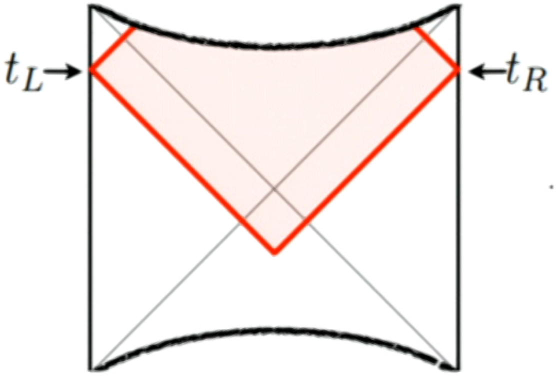

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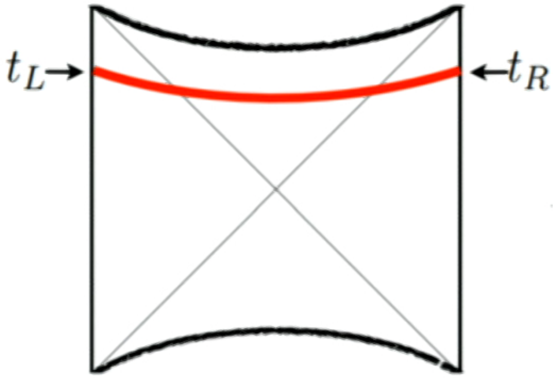

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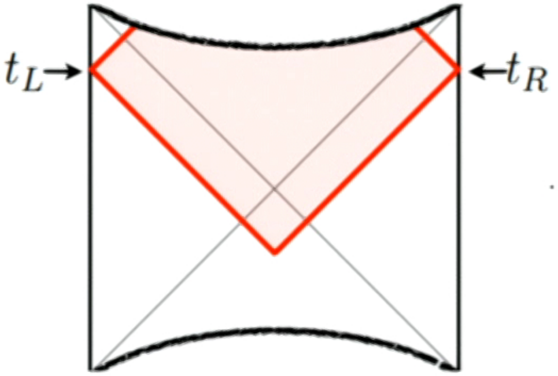

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- new gravitational observables: what are properties/features?
- linear growth (at late times)

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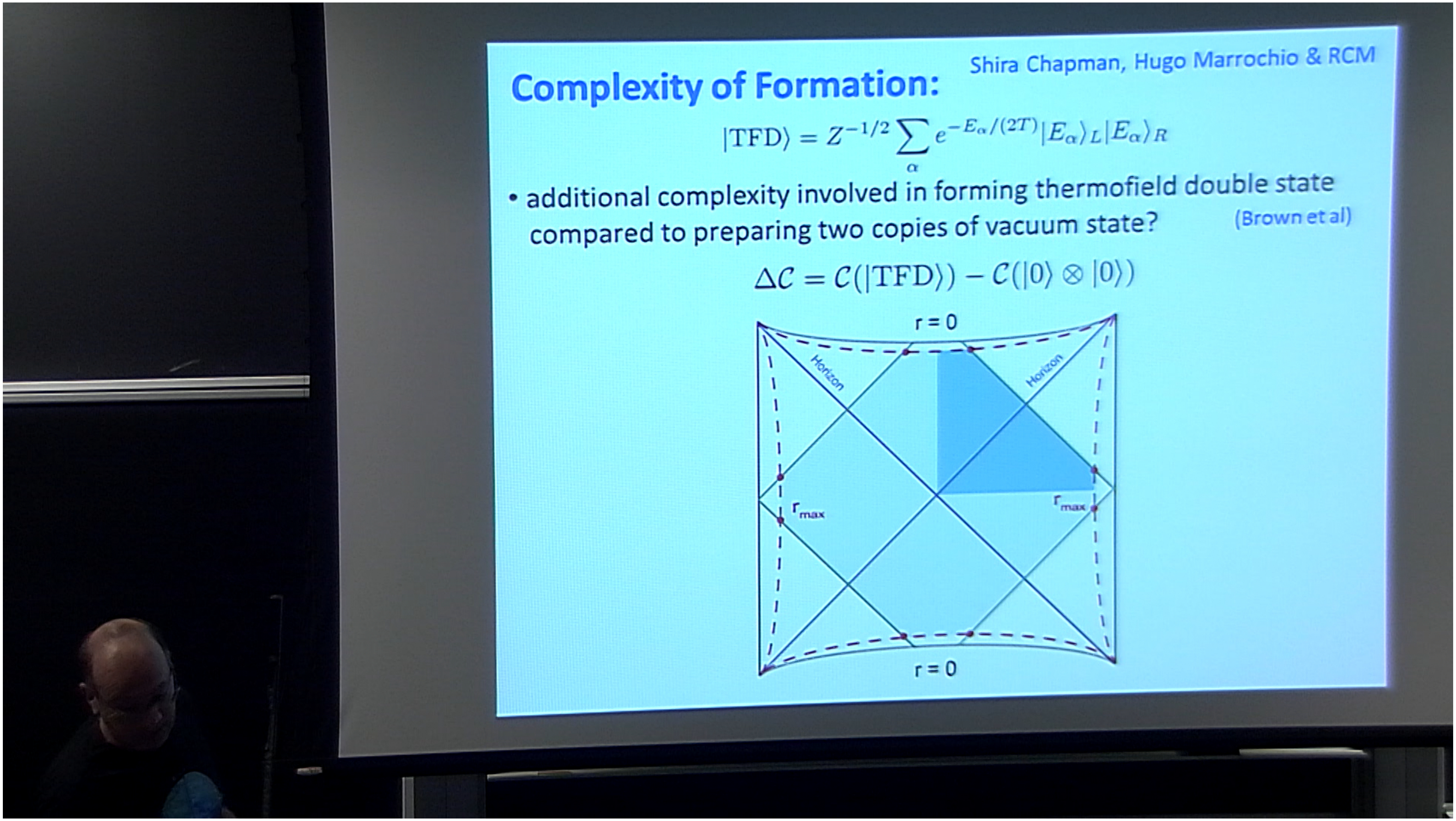

Shira Chapman, Hugo Marrochio & RCM

Complexity of Formation:

$$|\mathrm{TFD}\rangle = Z^{-1/2}\sum_{\alpha} e^{-E_\alpha/(2T)}|E_\alpha\rangle_L|E_\alpha\rangle_R$$

- additional complexity involved in forming thermofield double state compared to preparing two copies of vacuum state? (Brown et al)

$$\Delta\mathcal{C} = \mathcal{C}(|\mathrm{TFD}\rangle) - \mathcal{C}(|0\rangle\otimes|0\rangle)$$

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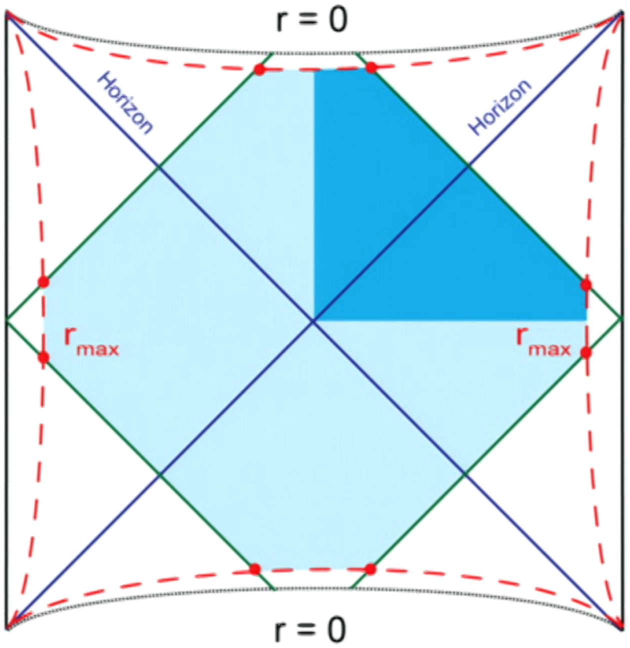
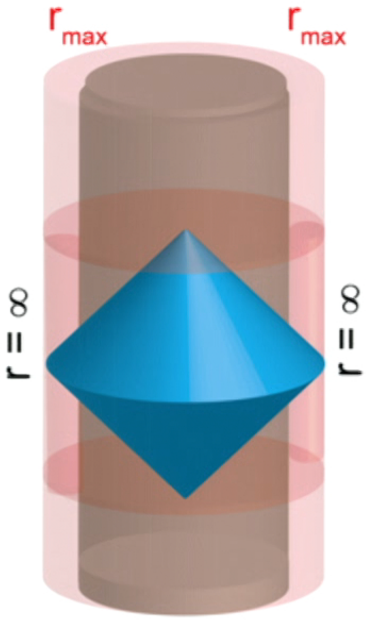


$-\ 2\times$



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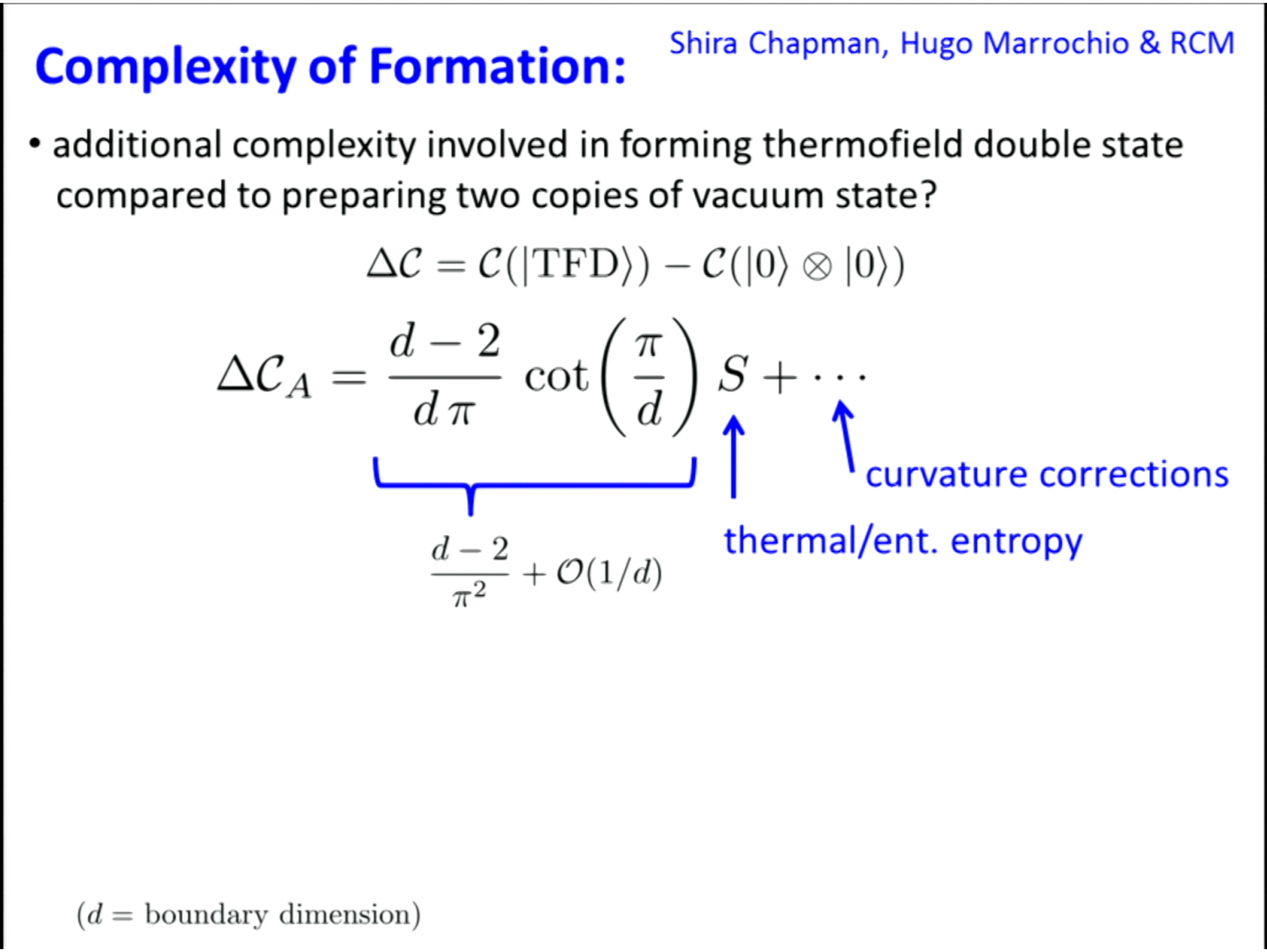
curvature corrections

thermal/ent. entropy

$(d = \text{boundary dimension})$



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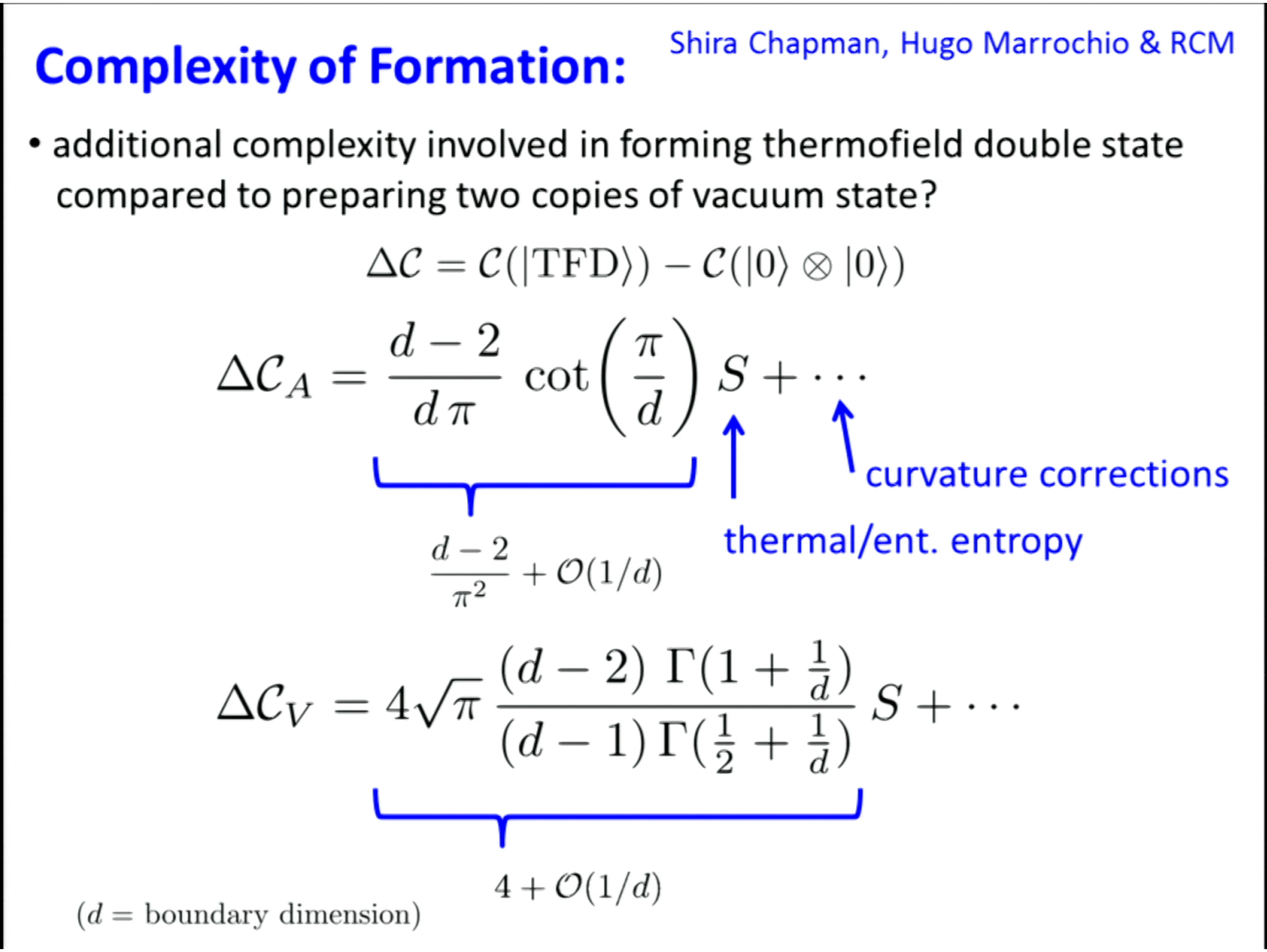

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$$\Delta\mathcal{C}_V = \underbrace{4\sqrt{\pi}\,\frac{(d-2)\;\Gamma(1+\frac{1}{d})}{(d-1)\,\Gamma(\frac{1}{2}+\frac{1}{d})}}_{4+\mathcal{O}(1/d)}\, S + \cdots$$

$(d = \text{boundary dimension})$

Compare?

$$R_{\text{form}} = \frac{\Delta \mathcal{C}_A}{\Delta \mathcal{C}_V} = \frac{d-1}{4\pi^{3/2}} \frac{\Gamma\left(1-\frac{1}{d}\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{d}\right)} \simeq \frac{d}{4\pi^2}$$

$$R_{\text{rate}} = \frac{d\mathcal{C}_A/dt}{d\mathcal{C}_V/dt} = \frac{d-1}{4\pi^2} \simeq \frac{d}{4\pi^2}$$

($d =$ boundary dimension)

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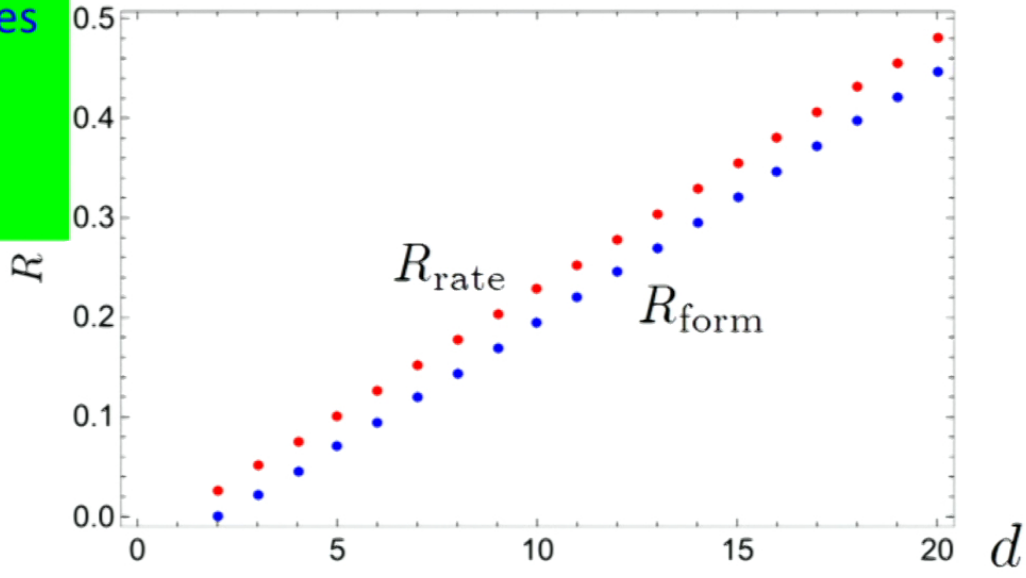
$$R_{\text{rate}} - R_{\text{form}} = \frac{\log 2}{2\pi^2} + \mathcal{O}(1/d)$$

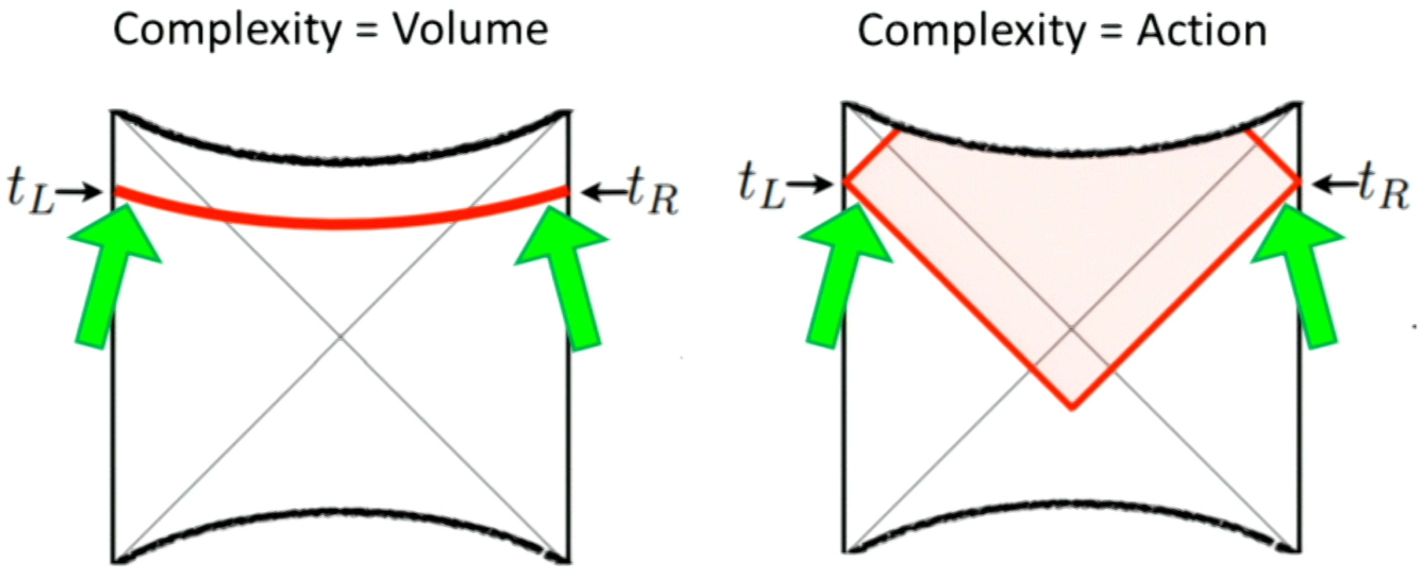
reinforce consistency
of C=V and C=A dualities
up to differences in
microscopic rules,
eg, gate set



(d = boundary dimension)

Holographic Complexity:

Dean Carmi, RCM & Pratik Rath



Complexity = Volume

Complexity = Action

- UV divergences naturally associated with establishing correlations or entanglement down to arbitrarily small length scales
- regulate volume/action with the introduction of UV regulator surface at large radius ($r_{max} = L^2/\delta$), as usual

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$$\mathcal{C}_V(\Sigma) = \frac{8\pi^{\frac{d+2}{2}}\,\Gamma(d/2)}{\Gamma(d+2)}\,C_T \int_\Sigma d^{d-1}\sigma\sqrt{h}\left[\frac{1}{\delta^{d-1}} - \frac{(d-1)}{2(d-2)(d-3)\delta^{d-3}}\left(\mathcal{R}^a_a - \frac{1}{2}\mathcal{R} - \frac{(d-2)^2}{(d-1)^2}K^2\right) + \cdots\right]$$

- UV divergences appear as local integrals of geometric invariants (as with holographic entanglement entropy)

Holographic Complexity:

- UV divergences naturally associated with establishing correlations down to arbitrarily small length scales
- regulate volume/action with the introduction of UV regulator surface at large radius ($r_{max} = L^2/\delta$), as usual

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$$\left.-\frac{(d-1)}{2(d-2)(d-3)\delta^{d-3}}\left(\mathcal{R}^a_a-\frac{1}{2}\mathcal{R}-\frac{(d-2)^2}{(d-1)^2}K^2\right)+\cdots\right]$$

$$= \#\;C_T\,\frac{V_\Sigma}{\delta^{d-1}} \;+\; \text{``curvature corrections''}$$

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Holographic Complexity:

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- regulate volume/action with the introduction of UV regulator surface at large radius ($r_{max} = L^2/\delta$), as usual

$$\mathcal{C}_V(\Sigma) = \frac{1}{\delta^{d-1}} \int_\Sigma d^{d-1}\sigma \sqrt{h}\; v_0(\mathcal{R}, K)$$

$$\mathcal{C}_A(\Sigma) = \frac{1}{\delta^{d-1}} \int_\Sigma d^{d-1}\sigma \sqrt{h} \left[v_1(\mathcal{R}, K) + \underline{\log\left(\frac{L}{\alpha\,\delta}\right)}\; v_2(\mathcal{R}, K) \right]$$

from boundary terms in action

with

$$v_k(\mathcal{R}, K) = \sum_{n=0}^{\lfloor \frac{d-1}{2} \rfloor} \sum_i c_{i,n}^{[k]}(d)\; \delta^{2n}\; [\mathcal{R}, K]_i^{2n}$$

- UV divergences appear as local integrals of geometric invariants (as with holographic entanglement entropy)

Holographic Complexity:

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AdS scale

normalization

$\boldsymbol{k} \cdot \hat{\boldsymbol{t}} = \pm\alpha$

$$\mathcal{C}_A(\Sigma) = \frac{1}{\delta^{d-1}} \int_\Sigma d^{d-1}\sigma \sqrt{h} \left[v_1(\mathcal{R}, K) + \underline{\log\left(\frac{L}{\alpha\,\delta}\right)}\; v_2(\mathcal{R}, K) \right]$$

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- UV divergences appear as local integrals of geometric invariants (as with holographic entanglement entropy)

Holographic Complexity:

- U with establishing correlations
  c s
- r duction of UV regulator surface
  a al

$$\alpha = L/\ell$$

- eliminates AdS scale ✓
- $\ell \sim V^{1/(d-1)}$: complexity is superextensive
- $\ell \sim \delta$: IR contributions depend on UV cutoff, eg, $d\mathcal{C}_A/dt$
- $\ell = \ell_0$: complexity depends on new (unphysical) scale

$$\mathcal{C}_V \quad K)$$

AdS scale

normalization

$\boldsymbol{k}\cdot\hat{\boldsymbol{t}} = \pm\alpha$

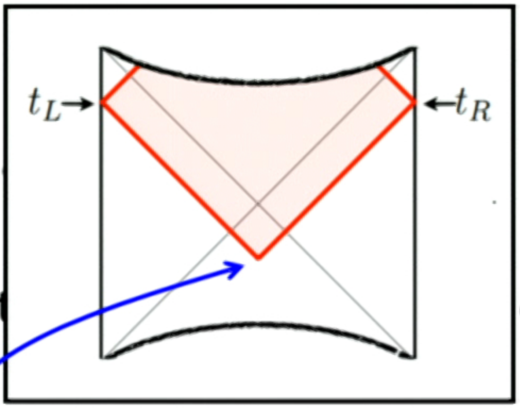
$$\mathcal{C}_A \quad , K) + \underline{\log\left(\frac{L}{\alpha\,\delta}\right)}\; v_2(\mathcal{R},K)\Big]$$

from boundary terms in action

$$\sum_{n=0} \sum_i c_{i,n}^{[k]}(d)\,\delta^{2n}\,[\mathcal{R},K]_i^{2n}$$

- UV divergences appear as local integrals of geometric invariants (as with holographic entanglement entropy)

Holographic Complexity:



- U… vith … c… s
- n… duct… ce a… al

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$\mathcal{C}_V$ … $K)$

normalization

$\boldsymbol{k}\cdot\hat{\boldsymbol{t}} = \pm\alpha$

AdS scale

$$\mathcal{C}_A \; \ldots \; , K) + \underbrace{\log\left(\frac{L}{\alpha\,\delta}\right)}\; v_2(\mathcal{R}, K)\Big]$$

from boundary terms in action

$$\sum_{n=0} \sum_i \; c^{[k]}_{i,n}(d)\;\delta^{2n}\;[\mathcal{R}, K]^{2n}_i$$

- UV divergences appear as local integrals of geometric invariants (as with holographic entanglement entropy)

Questions?

- What is "holographic complexity"?
  - ➢ QFT/path integral description of "complexity" in boundary CFT?
  - ➢ what is boundary dual of these gravitational observables?

- is there a privileged role for (states on) null Cauchy surfaces?
  - ➢ provide distinguished reference states?

- is there a "renormalized holographic complexity"?
  - ➢ what's it good for?; (EE vs mutual information versions of F)

- ambiguities? ambiguities? ambiguities?
  - ➢ connections between ambiguities in gravity and boundary?

- more boundary terms: higher codim. intersections; "complex" joint contributions; boundary "counterterms"
- why is complexity of formation positive?
- C_A contribution of spacetime singularity?
- subregion complexity?

Questions?

- What is "holographic complexity"?
  - ➢ QFT/path integral description of "complexity" in boundary CFT?
  - ➢ what is boundary dual of these gravitational observables?
- is there a … s?
  - ➢ provi
- is there a
  - ➢ what … s of F)

What does "complexity" mean in a quantum field theory?

- ambiguities? ambiguities? ambiguities?
  - ➢ connections between ambiguities in gravity and boundary?
- more boundary terms: higher codim. intersections; "complex" joint contributions; boundary "counterterms"
- why is complexity of formation positive?
- $\mathcal{C}_A$ contribution of spacetime singularity?
- subregion complexity?


Disneyland
Holography
CRITTER COUNTRY
FANTASYLAND
BIG THUNDER RANCH
PLAZA GARDENS STAGE
FRONTIERLAND
TOMORROWLAND TERRACE
TOMORROWLAND
RIVERS OF AMERICA
Complexity
PLAZA PAVILION
ADVENTURELAND
MAIN STREET, U.S.A.
CITY HALL
MICKEY & FRIENDS TRAM ROUTE
LION KING TRAM ROUTE
DOWNTOWN DISNEY, DISTRICT AND HOTELS OF THE DISNEYLAND, RESORT
Disney's CALIFORNIA ADVENTURE

Holography

Quantum Field Theory

Complexity

Quantum Field Theory:

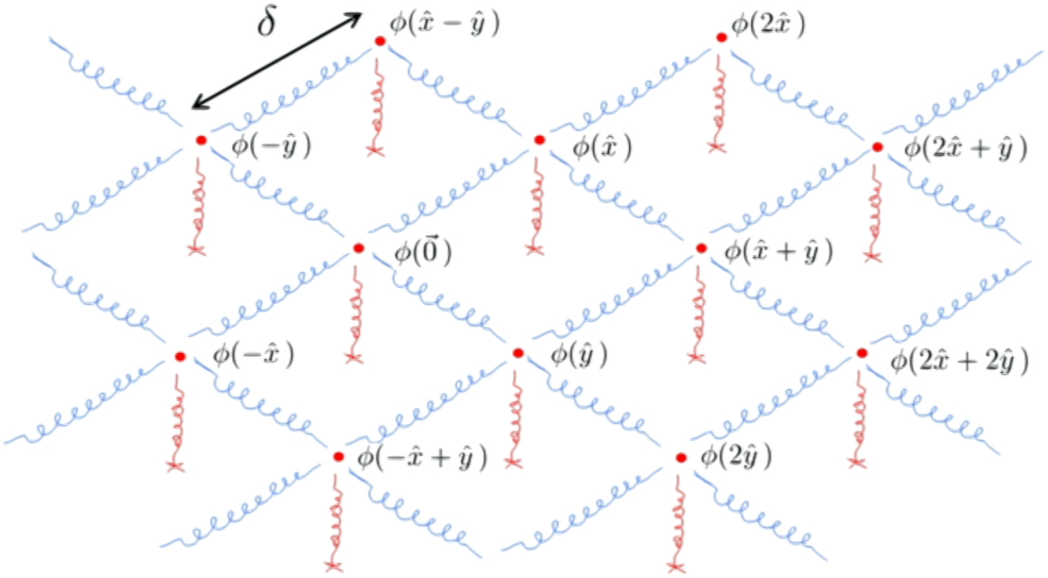
- scalar field theory (in d spacetime dimensions)

$$H = \frac{1}{2}\int d^{d-1}x \left[\pi(x)^2 + \vec{\nabla}\phi(x)^2 + U(\phi(x)) \right]$$

Quantum Field Theory:

- **free** scalar field theory (in d spacetime dimensions)

$$
\begin{aligned}
H &= \frac{1}{2}\int d^{d-1}x\left[\,\pi(x)^2+\vec{\nabla}\phi(x)^2+m^2\phi(x)^2\,\right] \\
&= \frac{1}{2}\sum_{\vec{n}}\left[\frac{p(\vec{n})^2}{\delta^{d-1}}+\delta^{d-1}\left\{\frac{1}{\delta^2}\sum_i[\phi(\vec{n}\,)-\phi(\vec{n}-\hat{x}_i\,)]^2+m^2\phi(\vec{n}\,)^2\right\}\right]
\end{aligned}
$$



Quantum Field Theory:

- an infinite family of coupled harmonic oscillators

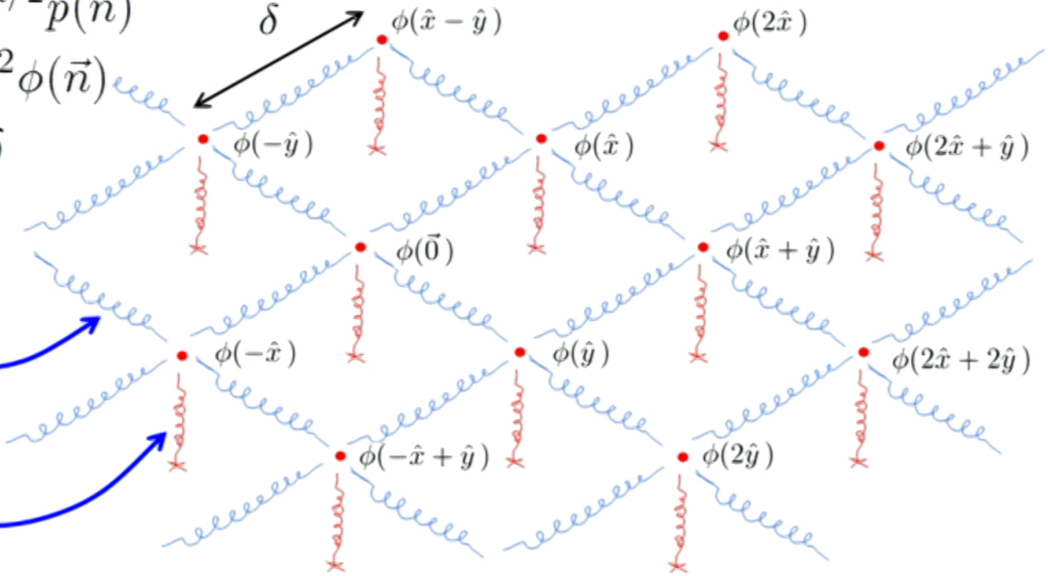
$$H = \frac{1}{2}\int d^{d-1}x \left[\pi(x)^2 + \vec{\nabla}\phi(x)^2 + m^2\phi(x)^2\right]$$

$$= \frac{1}{2}\sum_{\vec{n}} \left[\frac{P(\vec{n})^2}{M} + M\left\{\Omega^2 \sum_i [X(\vec{n}) - X(\vec{n} - \hat{x}_i)]^2 + \omega^2 X(\vec{n})^2\right\}\right]$$

$$P(\vec{n}) = \delta^{-d/2} p(\vec{n})$$

$$X(\vec{n}) = \delta^{d/2}\phi(\vec{n})$$

$$M = 1/\delta$$



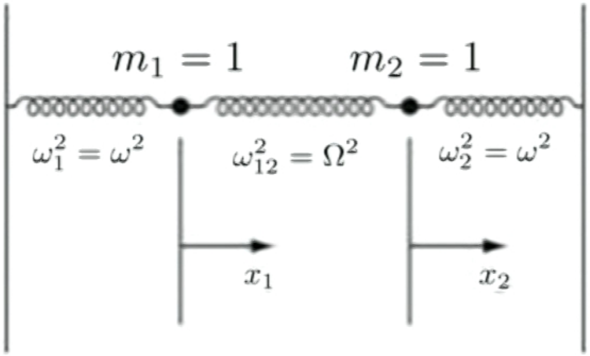
$$\Omega^2 = 1/\delta^2$$

$$\omega^2 = m^2$$

- **QFT** $\longrightarrow$ **two coupled harmonic oscillators**

$$H = \frac{1}{2}\left[p_1^2 + p_2^2 + \omega^2\left(x_1^2 + x_2^2\right) + \Omega^2\left(x_1 - x_2\right)^2\right] \qquad (M = 1)$$

start with "Mad Hatter's Tea Party"



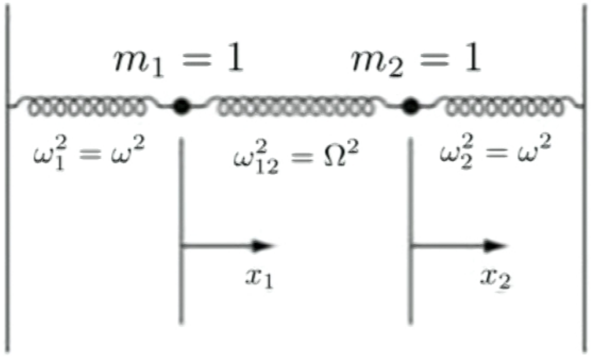
- **QFT** ⟶ **two coupled harmonic oscillators**

$$H = \frac{1}{2}\left[p_1^2 + p_2^2 + \omega^2\left(x_1^2 + x_2^2\right) + \Omega^2\left(x_1 - x_2\right)^2\right] \qquad (M = 1)$$

$$= \frac{1}{2}\left[p_+^2 + \omega_+^2 x_+^2 + p_-^2 + \omega_-^2 x_-^2\right] \quad \text{with} \quad x_\pm = \frac{1}{\sqrt{2}}\left(x_1 \pm x_2\right),$$

$$\omega_+^2 = \omega^2, \quad \omega_- = \omega^2 + 2\Omega^2$$

find normal modes; problem reduces to two independent SHO's



- **QFT ⟶ two coupled harmonic oscillators**

$$\begin{aligned} H &= \frac{1}{2}\left[p_1^2 + p_2^2 + \omega^2\left(x_1^2 + x_2^2\right) + \Omega^2\left(x_1 - x_2\right)^2\right] \qquad (M = 1) \\ &= \frac{1}{2}\left[p_+^2 + \omega_+^2 x_+^2 + p_-^2 + \omega_-^2 x_-^2\right] \ \text{ with } \ x_\pm = \frac{1}{\sqrt{2}}\left(x_1 \pm x_2\right) , \\ & \qquad\qquad \omega_+^2 = \omega^2 , \quad \omega_-^2 = \omega^2 + 2\Omega^2 \end{aligned}$$

- ground-state wave-function:

$$\Psi_0(x_+, x_-) = \Psi_0(x_+)\,\Psi_0(x_-) = \frac{(\omega_+\omega_-)^{1/4}}{\sqrt{\pi}} \exp\left[-\frac{1}{2}\left(\omega_+ x_+^2 + \omega_- x_-^2\right)\right]$$

$$\begin{aligned} \Psi_0(x_1, x_2) &= \frac{(\omega_+\omega_-)^{1/4}}{\sqrt{\pi}} \exp\left[-\frac{1}{4}\left(\omega_+\left(x_1 + x_2\right)^2 + \omega_-(x_1 - x_2)^2\right)\right] \\ &= \frac{(\omega_+\omega_-)^{1/4}}{\sqrt{\pi}} \exp\left[-\frac{1}{2}\omega_1\, x_1^2 - \frac{1}{2}\omega_2\, x_2^2 - \beta\, x_1 x_2\right] \end{aligned}$$

$$\text{with} \quad \omega_1 = \omega_2 = \frac{\omega_+ + \omega_-}{2} \quad \text{and} \quad \beta = \frac{\omega_+ - \omega_-}{2} < 0$$

- **QFT** ⟶ **two coupled harmonic oscillators**

$$
\begin{aligned}
H &= \frac{1}{2}\left[p_1^2 + p_2^2 + \omega^2\left(x_1^2 + x_2^2\right) + \Omega^2\left(x_1 - x_2\right)^2\right] \qquad\qquad (M = 1)\\
&= \frac{1}{2}\left[p_+^2 + \omega_+^2 x_+^2 + p_-^2 + \omega_-^2 x_-^2\right] \text{ with } x_\pm = \frac{1}{\sqrt{2}}\left(x_1 \pm x_2\right),\\
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\end{aligned}
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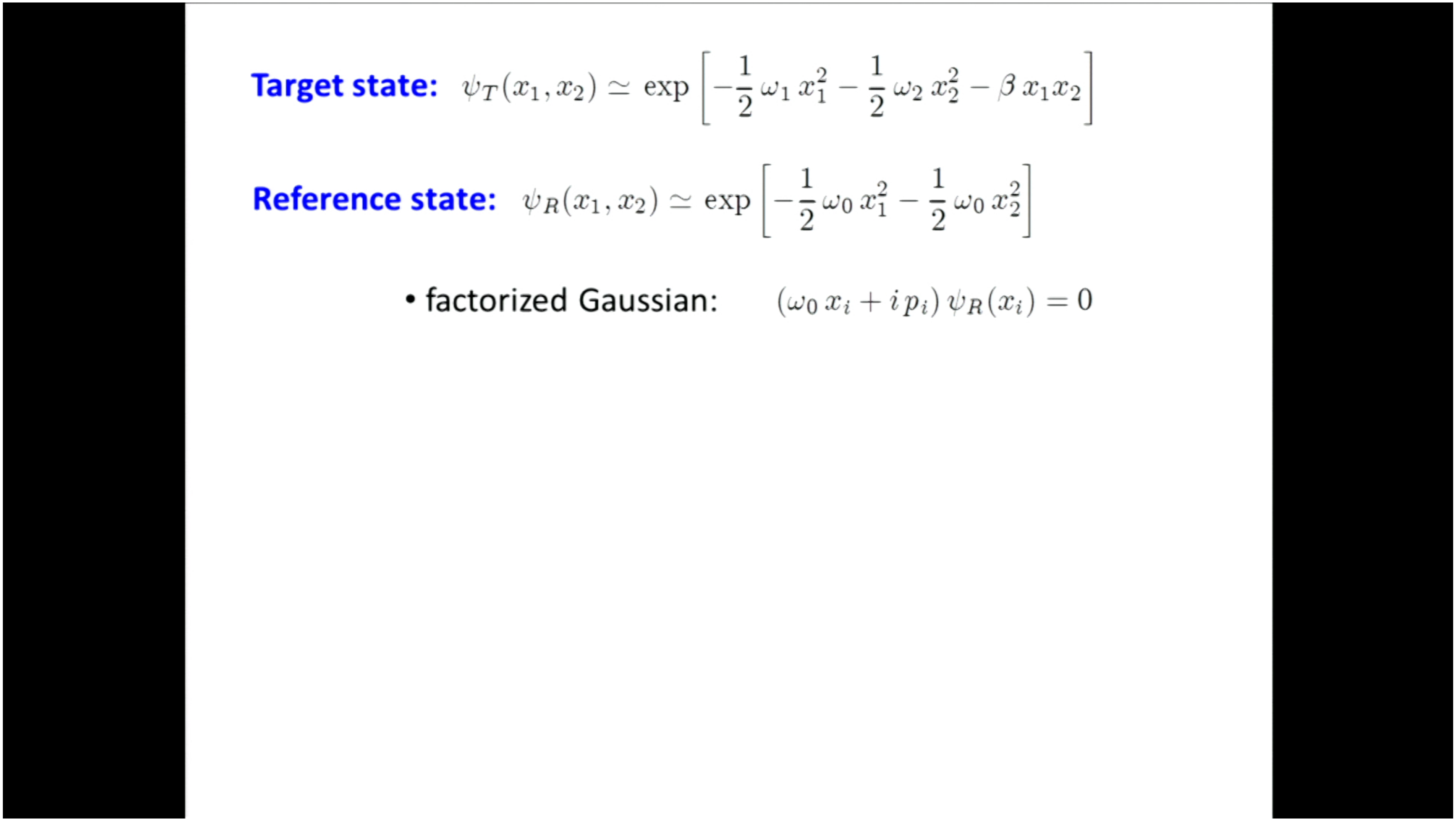
Target State!!

$$= \frac{(\omega_+\omega_-)^{1/4}}{\sqrt{\pi}}\exp\left[-\frac{1}{2}\omega_1 x_1^2 - \frac{1}{2}\omega_2 x_2^2 - \beta x_1 x_2\right]$$

with $\omega_1 = \omega_2 = \frac{\omega_+ + \omega_-}{2}$ and $\beta = \frac{\omega_+ - \omega_-}{2} < 0$

Target state: $\psi_T(x_1, x_2) \simeq \exp\left[-\frac{1}{2}\,\omega_1\,x_1^2 - \frac{1}{2}\,\omega_2\,x_2^2 - \beta\,x_1 x_2\right]$

Reference state: ??????



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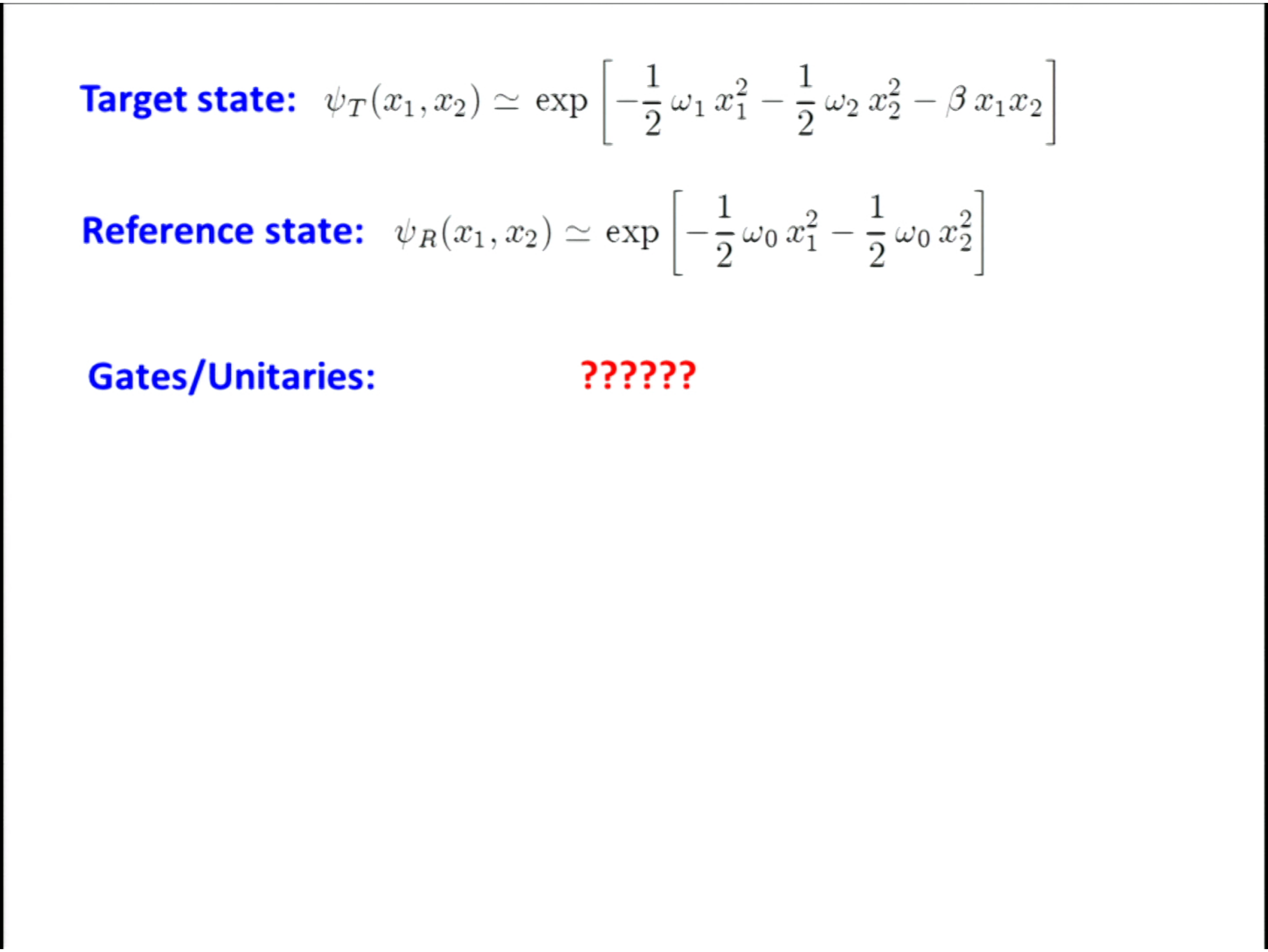
Reference state: $\psi_R(x_1, x_2) \simeq \exp\left[-\frac{1}{2}\,\omega_0\,x_1^2 - \frac{1}{2}\,\omega_0\,x_2^2\right]$

- factorized Gaussian: $(\omega_0\,x_i + i\,p_i)\,\psi_R(x_i) = 0$



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Gates/Unitaries: ??????

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Gates/Unitaries:

- natural operators: $x_1\,,\quad x_2\,,\quad p_1\,,\quad p_2$ $\qquad [x_i, p_j] = i\,\delta_{ij}$

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→ $Q_{00} = \exp[i\epsilon\,x_0\,p_0]$ "add a small phase"

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infinitesimal parameter $\epsilon \ll 1$

c-numbers

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$Q_{0i} = \exp[i\epsilon\,x_0\,p_i]$ "shift x_i by $\epsilon\,x_0$"

$Q_{i0} = \exp[i\epsilon\,x_i\,p_0]$ "shift p_i by $\epsilon\,p_0$"

(mulitply by small plane wave component)

Target state: $\psi_T(x_1, x_2) \simeq \exp\left[-\frac{1}{2}\,\omega_1\,x_1^2 - \frac{1}{2}\,\omega_2\,x_2^2 - \beta\,x_1 x_2\right]$

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$Q_{ij} = \exp[i\epsilon\, x_i\, p_j] \quad (i \neq j)$ "shift x_j by $\epsilon\, x_i$" (entangling)

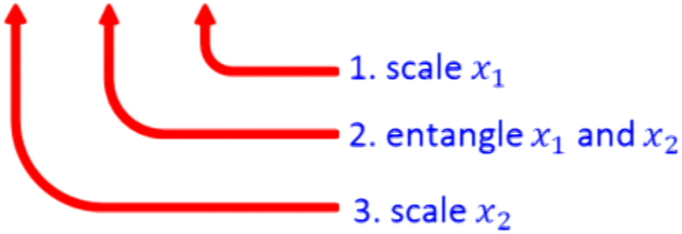
$$Q_{ii} = \exp\left[i\frac{\epsilon}{2}(x_i\, p_i + p_i\, x_i)\right] = e^{\epsilon/2}\exp[i\epsilon\, x_i\, p_i]$$

"rescale x_i to $e^{\epsilon}x_i$" (scaling)

Target state: $\psi_T(x_1,x_2) \simeq \exp\left[-\frac{1}{2}\,\omega_1\,x_1^2 - \frac{1}{2}\,\omega_2\,x_2^2 - \beta\,x_1 x_2\right]$

Reference state: $\psi_R(x_1,x_2) \simeq \exp\left[-\frac{1}{2}\,\omega_0\,x_1^2 - \frac{1}{2}\,\omega_0\,x_2^2\right]$

- one path/circuit: $\psi_T(x_1,x_2) = Q_{22}^{\alpha_3}\ Q_{21}^{\alpha_2}\ Q_{11}^{\alpha_1}\ \psi_R(x_1,x_2)$



1. scale x_1
2. entangle x_1 and x_2
3. scale x_2

Target state: $\psi_T(x_1,x_2) \simeq \exp\left[-\frac{1}{2}\,\omega_1\,x_1^2 - \frac{1}{2}\,\omega_2\,x_2^2 - \beta\,x_1 x_2\right]$

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"circuit depth": $\mathcal{D}_1$

$$\begin{aligned}\mathcal{D}_1 &= |\alpha_1| + |\alpha_2| + |\alpha_3| \\ &= \frac{1}{2\epsilon}\log\left[\frac{\omega_1\omega_2 - \beta^2}{\omega_0^2}\right] + \frac{|\beta|}{\epsilon}\sqrt{\frac{\omega_0}{\omega_1}}\left(\omega_1\omega_2 - \beta^2\right)^{-1/2}\end{aligned}$$

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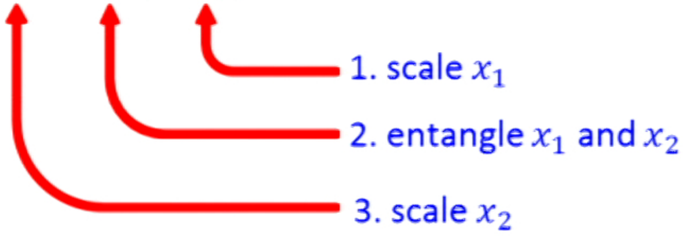

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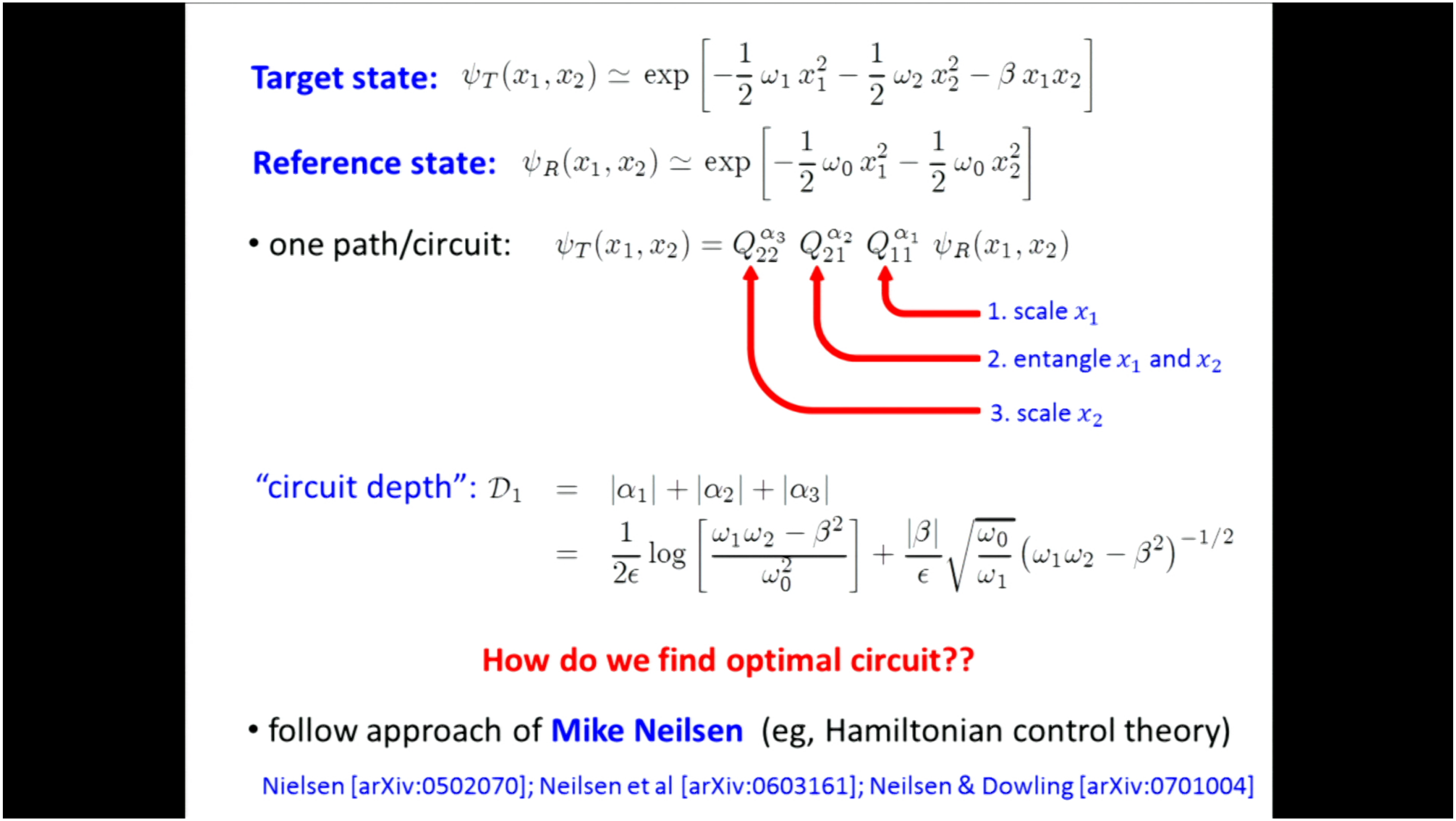
$$\begin{aligned} \mathcal{D}_1 &= |\alpha_1| + |\alpha_2| + |\alpha_3| \\ &= \frac{1}{2\epsilon}\log\left[\frac{\omega_1\omega_2 - \beta^2}{\omega_0^2}\right] + \frac{|\beta|}{\epsilon}\sqrt{\frac{\omega_0}{\omega_1}}\left(\omega_1\omega_2 - \beta^2\right)^{-1/2} \end{aligned}$$

How do we find optimal circuit??



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How do we find optimal circuit??

- follow approach of **Mike Neilsen** (eg, Hamiltonian control theory)

Nielsen [arXiv:0502070]; Neilsen et al [arXiv:0603161]; Neilsen & Dowling [arXiv:0701004]

“What is the minimal size quantum circuit required to exactly implement
a specified n-qubit unitary operation, U, without the use of ancilla qubits?”

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- consider trajectories:

$$U(s) = \mathcal{P}\exp\left[\int_0^s d\tilde{s}\; Y^I(\tilde{s})\,M_I\right] \quad \text{where} \quad U(s=0) = 1\,, \quad U(s=1) = U_{fin}$$

velocity: $Y^I(s) = \mathrm{Tr}\left[\partial_s U(s)\,U^{-1}(s)\,M_I\right]$

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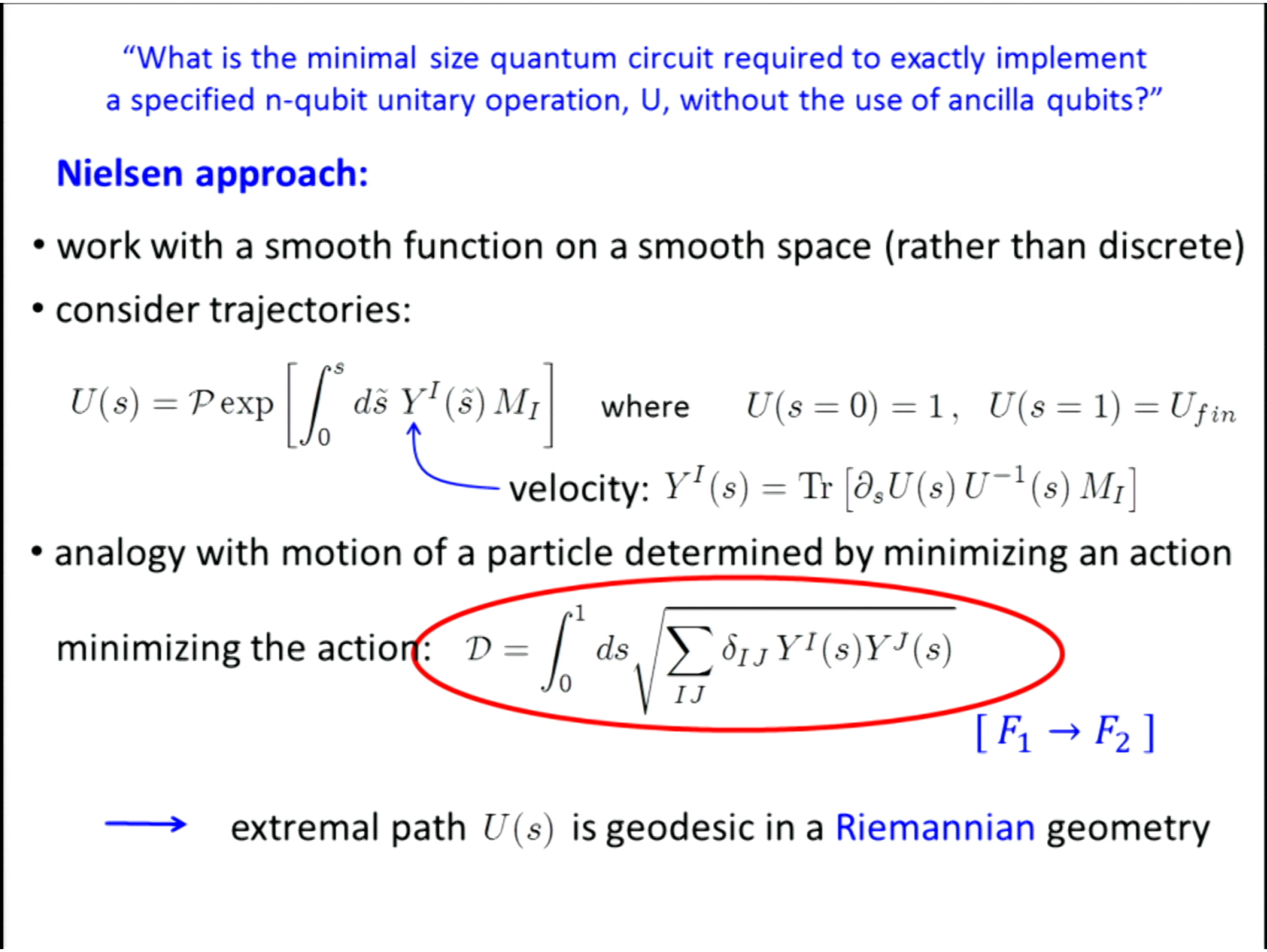
- analogy with motion of a particle determined by minimizing an action

minimizing the action: $\mathcal{D} = \int_0^1 ds \sum_I |Y^I(s)|$

⟶ extremal path $U(s)$ is geodesic in a Finsler geometry



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minimizing the action: $\mathcal{D} = \int_0^1 ds \sqrt{\sum_{IJ} \delta_{IJ}\, Y^I(s) Y^J(s)}$

$[\, F_1 \rightarrow F_2 \,]$

$\longrightarrow$ extremal path $U(s)$ is geodesic in a Riemannian geometry

Nielsen approach: to find optimal circuit

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- defining $Y^I(s) = \mathrm{Tr}\left[\partial_s U(s)\,U^{-1}(s)\,M_I\right]$ still seems relatively difficult?

Alternative picture:

- at this stage thinking of Gaussian states, so consider space of states as space of (positive) quadratic forms

$$\psi \simeq \exp\left[-\frac{1}{2} x_i\, A_{ij}\, x_j\right] \quad \longrightarrow \quad A_T = \begin{bmatrix} \omega_1 & \beta \\ \beta & \omega_2 \end{bmatrix}$$

$$A_R = \begin{bmatrix} \omega_0 & 0 \\ 0 & \omega_0 \end{bmatrix}$$

- unitary gates: $Q_{ij} = \exp[\epsilon\, \mathcal{O}_{ij}]$ with $\mathcal{O}_{ij} = i\, x_i\, p_j + \frac{1}{2}\delta_{ij}$

$$\longrightarrow \quad Q_{ij} = \exp[\epsilon\, M_{ij}] \quad \text{with} \quad [M_{ij}]_{ab} = \delta_{ia}\, \delta_{jb}$$

eg, $M_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

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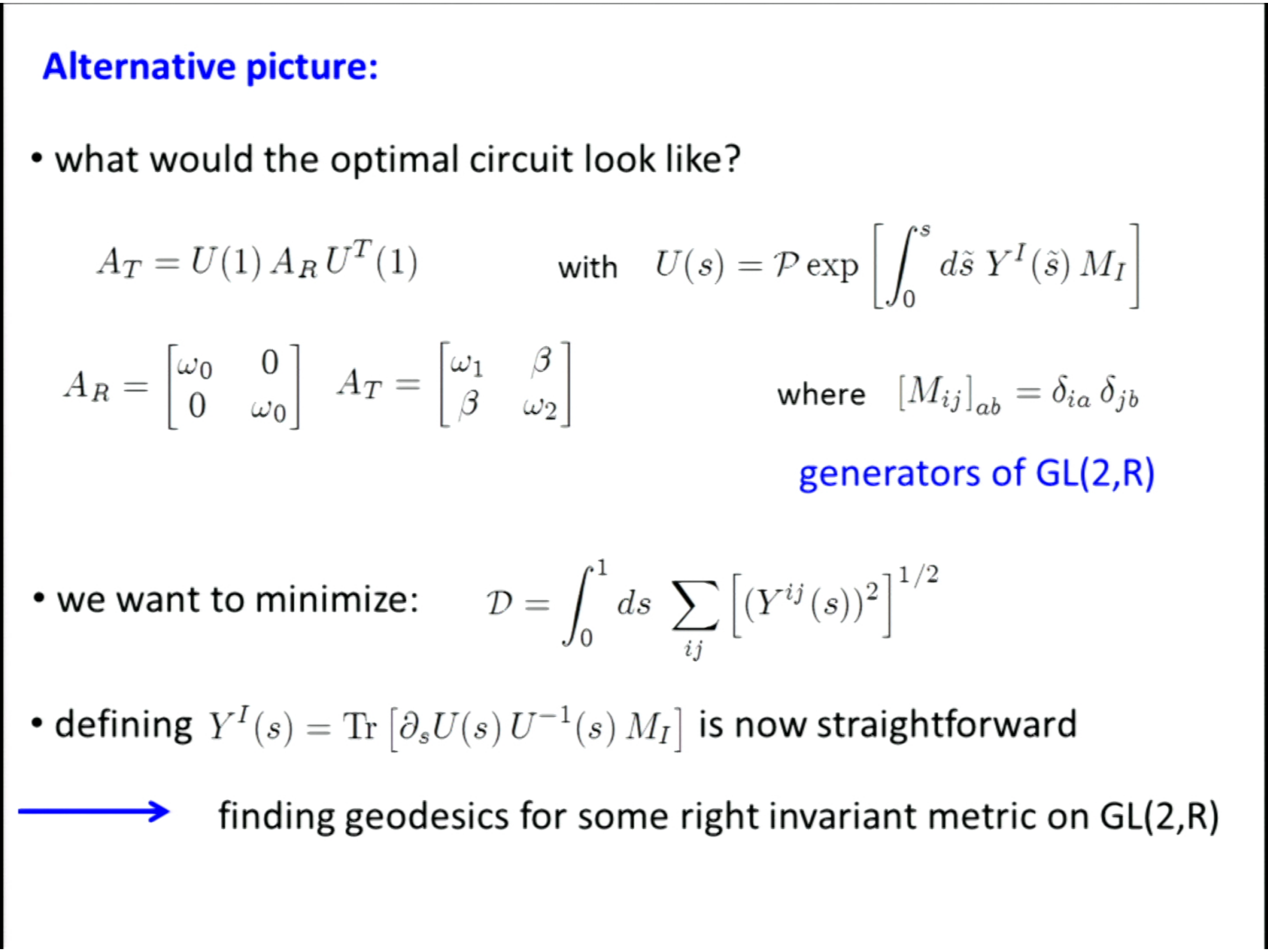
$\uparrow$

generators of GL(2,R)



Alternative picture:

- what would the optimal circuit look like?

$$A_T = U(1)\, A_R\, U^T(1) \qquad \text{with} \quad U(s) = \mathcal{P}\exp\left[\int_0^s d\tilde{s}\; Y^I(\tilde{s})\, M_I\right]$$

$$A_R = \begin{bmatrix} \omega_0 & 0 \\ 0 & \omega_0 \end{bmatrix} \quad A_T = \begin{bmatrix} \omega_1 & \beta \\ \beta & \omega_2 \end{bmatrix} \qquad \text{where} \quad [M_{ij}]_{ab} = \delta_{ia}\,\delta_{jb}$$

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- we want to minimize: $\mathcal{D} = \int_0^1 ds \sum_{ij} \left[(Y^{ij}(s))^2\right]^{1/2}$
- defining $Y^I(s) = \mathrm{Tr}\left[\partial_s U(s)\, U^{-1}(s)\, M_I\right]$ is now straightforward

→ finding geodesics for some right invariant metric on GL(2,R)

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(simplify with $\omega_0 = 1$)

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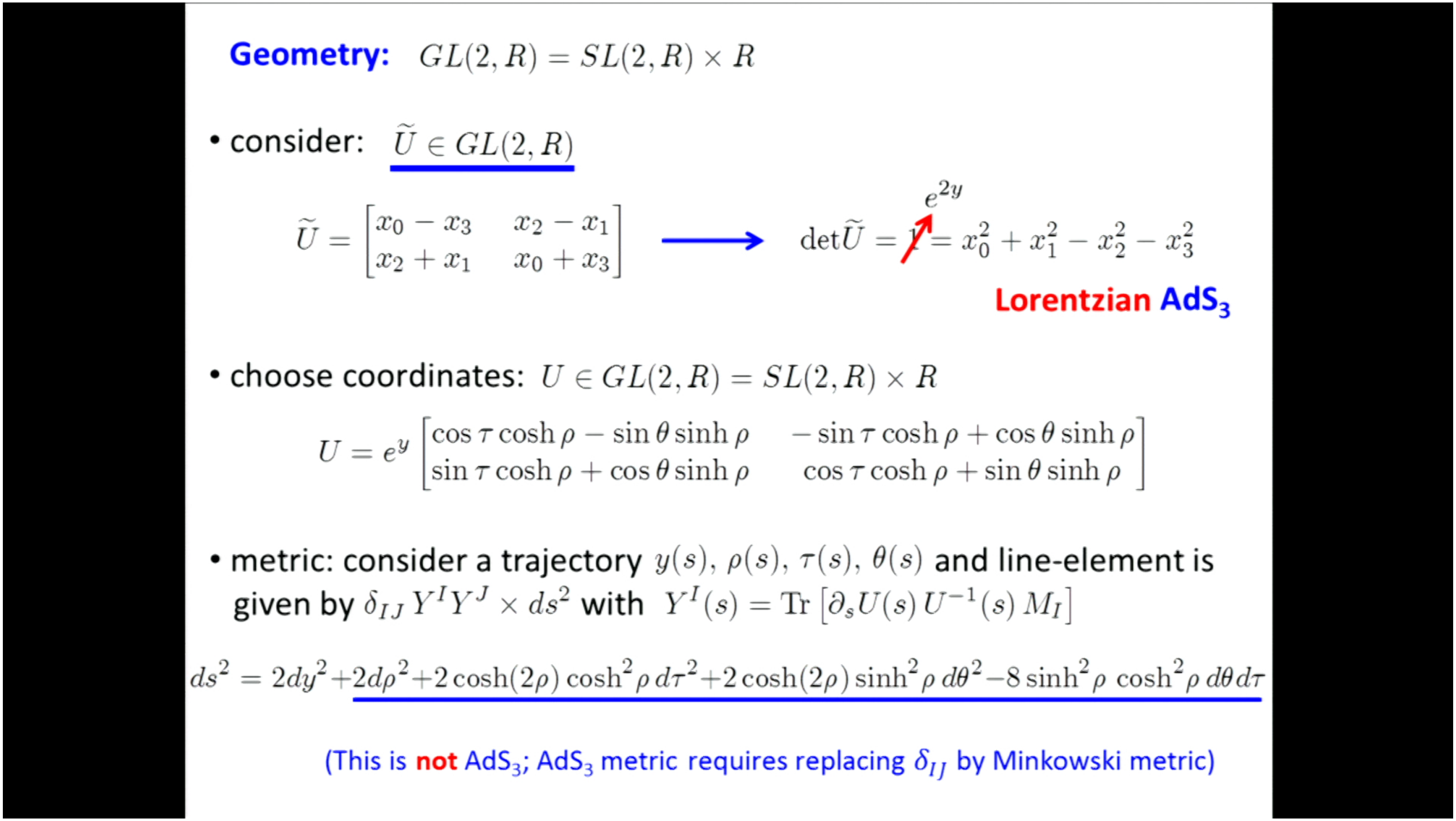


Geometry: $GL(2,R) = SL(2,R) \times R$

- consider: $\underline{\widetilde{U} \in GL(2,R)}$

$$\widetilde{U} = \begin{bmatrix} x_0 - x_3 & x_2 - x_1 \\ x_2 + x_1 & x_0 + x_3 \end{bmatrix} \longrightarrow \det\widetilde{U} = \overset{e^{2y}}{\cancel{1}} = x_0^2 + x_1^2 - x_2^2 - x_3^2$$

Lorentzian AdS$_3$

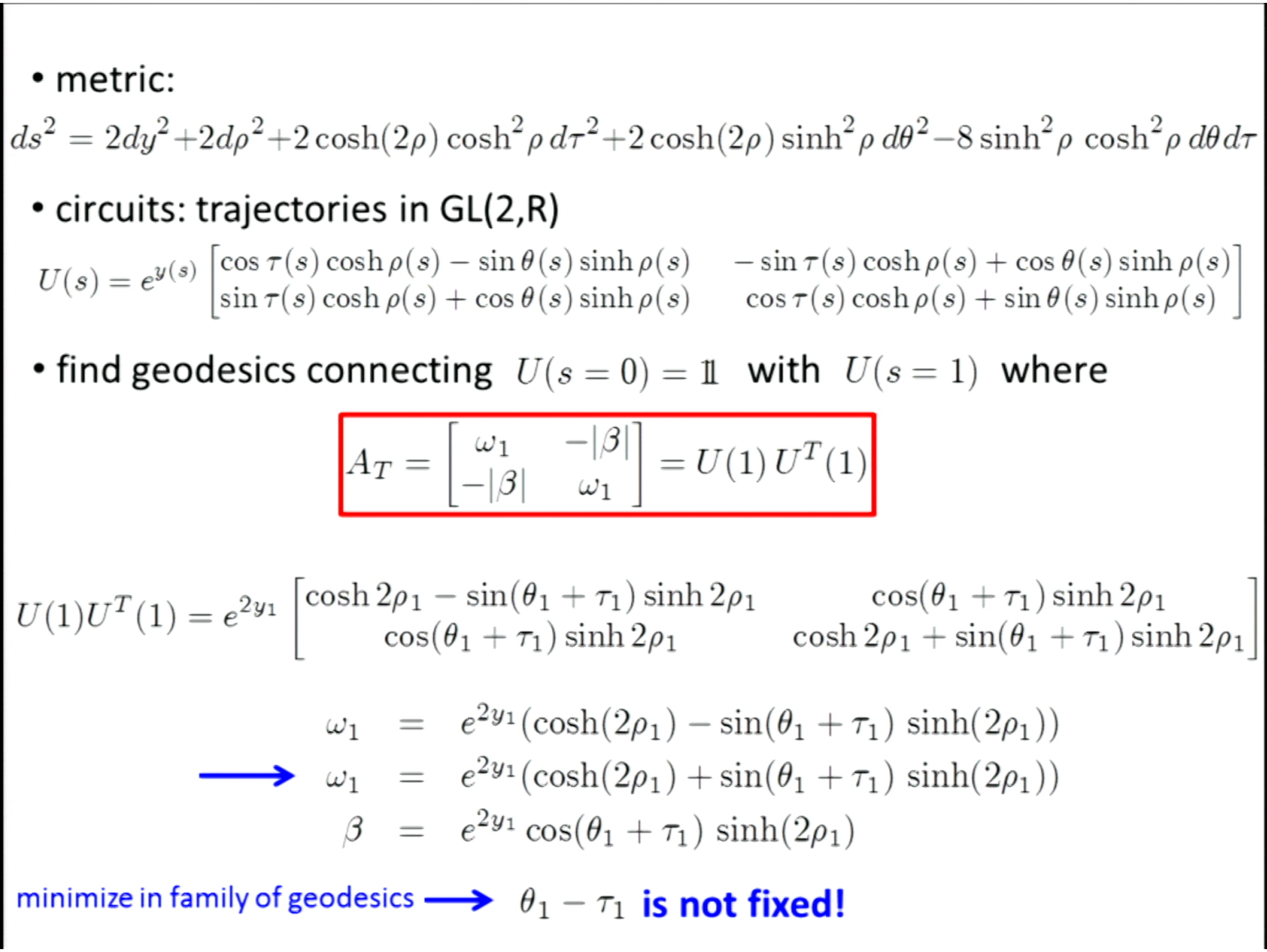
- choose coordinates: $U \in GL(2,R) = SL(2,R) \times R$

$$U = e^y \begin{bmatrix} \cos\tau\cosh\rho - \sin\theta\sinh\rho & -\sin\tau\cosh\rho + \cos\theta\sinh\rho \\ \sin\tau\cosh\rho + \cos\theta\sinh\rho & \cos\tau\cosh\rho + \sin\theta\sinh\rho \end{bmatrix}$$

- metric: consider a trajectory $y(s),\ \rho(s),\ \tau(s),\ \theta(s)$ and line-element is given by $\delta_{IJ}\, Y^I Y^J \times ds^2$ with $Y^I(s) = \mathrm{Tr}\left[\partial_s U(s)\, U^{-1}(s)\, M_I\right]$

$$ds^2 = 2dy^2 + \underline{2d\rho^2 + 2\cosh(2\rho)\cosh^2\rho\, d\tau^2 + 2\cosh(2\rho)\sinh^2\rho\, d\theta^2 - 8\sinh^2\rho\,\cosh^2\rho\, d\theta\, d\tau}$$

(This is **not** AdS$_3$; AdS$_3$ metric requires replacing δ_{IJ} by Minkowski metric)



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- circuits: trajectories in GL(2,R)

$$U(s) = e^{y(s)}\begin{bmatrix} \cos\tau(s)\cosh\rho(s) - \sin\theta(s)\sinh\rho(s) & -\sin\tau(s)\cosh\rho(s) + \cos\theta(s)\sinh\rho(s) \\ \sin\tau(s)\cosh\rho(s) + \cos\theta(s)\sinh\rho(s) & \cos\tau(s)\cosh\rho(s) + \sin\theta(s)\sinh\rho(s) \end{bmatrix}$$

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$$\begin{aligned} \omega_1 &= e^{2y_1}(\cosh(2\rho_1) - \sin(\theta_1 + \tau_1)\,\sinh(2\rho_1)) \\ \longrightarrow \quad \omega_1 &= e^{2y_1}(\cosh(2\rho_1) + \sin(\theta_1 + \tau_1)\,\sinh(2\rho_1)) \\ \beta &= e^{2y_1}\cos(\theta_1 + \tau_1)\,\sinh(2\rho_1) \end{aligned}$$

minimize in family of geodesics $\longrightarrow$ $\theta_1 - \tau_1$ **is not fixed!**

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$$\underline{0>\beta} = e^{2y_1}\cos(\theta_1+\tau_1)\,\sinh(2\rho_1) \longleftarrow \theta_1+\tau_1=\pi$$

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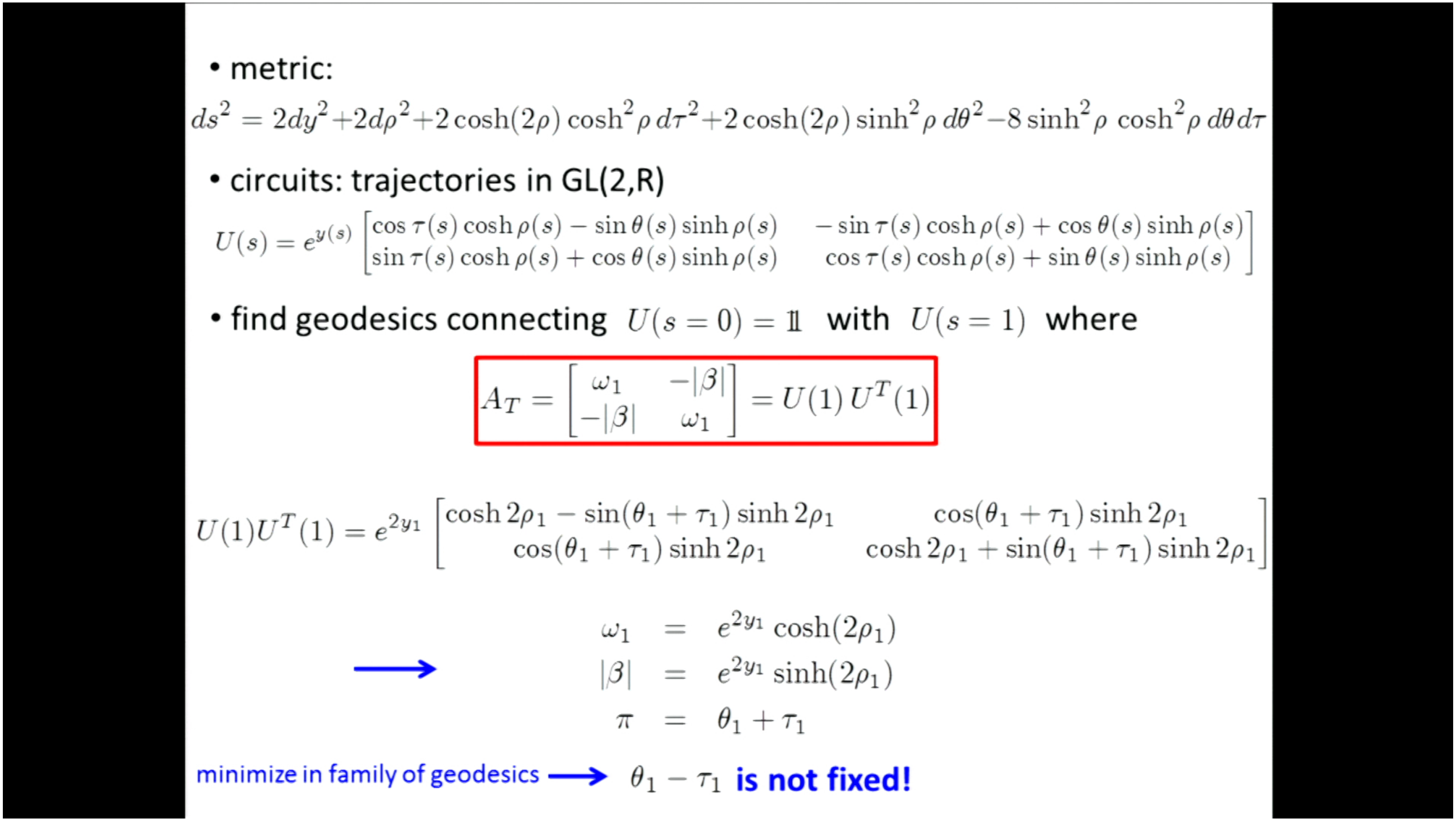

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$$\longrightarrow \quad \begin{aligned}\omega_1 &= e^{2y_1}\cosh(2\rho_1)\\ |\beta| &= e^{2y_1}\sinh(2\rho_1)\\ \pi &= \theta_1 + \tau_1\end{aligned}$$

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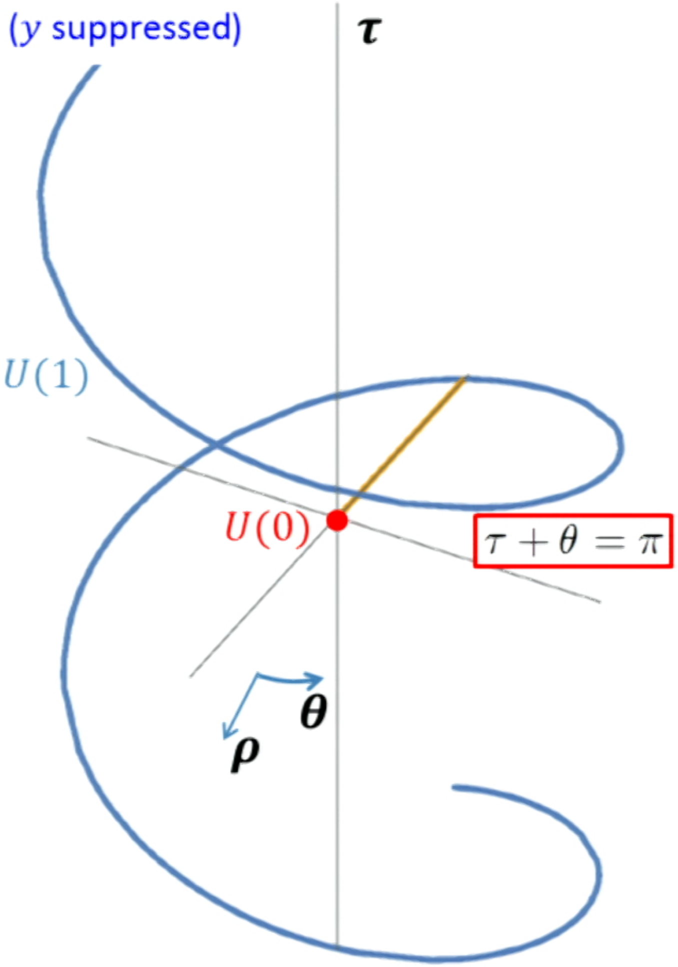

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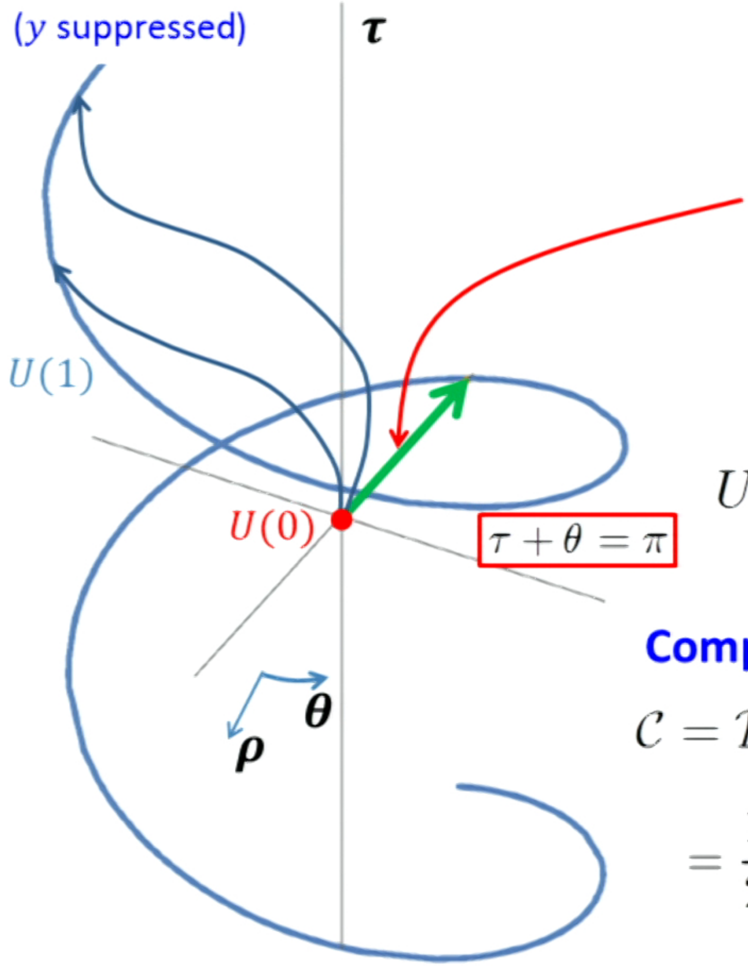

minimal geodesic:

$$\tau(s) = 0,\ \theta(s) = \pi,$$

$$y(s) = y_1\, s,\ \rho(s) = \rho_1\, s$$

$$U(s) = \mathcal{P} \exp \left[\begin{bmatrix} y_1 & -\rho_1 \\ -\rho_1 & y_1 \end{bmatrix} s \right]$$

Complexity:

$$\mathcal{C} = \mathcal{D}_{min} = \sqrt{2\rho_1^2 + 2y_1^2}$$

$$= \frac{1}{2}\sqrt{\log^2\left(\frac{\omega_+}{2}\right) + \log^2\left(\frac{\omega_-}{2}\right)}$$

with $\omega_+^2 = \omega^2, \quad \omega_-^2 = \omega^2 + 2\Omega^2$

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What else?

- examine normal circuit in normal mode basis

$$\tilde{x} = R\,x \quad \longleftrightarrow \quad \begin{bmatrix} x_- \\ x_+ \end{bmatrix} = R \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{with} \quad R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\longrightarrow \tilde{A}_T = R\,A_T R^\dagger = \begin{bmatrix} \omega_- & 0 \\ 0 & \omega_+ \end{bmatrix}, \qquad \tilde{A}_R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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- in normal mode basis, minimal circuit simply scales up diagonal entries

$$U(s) = \mathcal{P}\exp\left[M_{--}\left(y_1 + \rho_1\right) s + M_{++}\left(y_1 - \rho_1\right) s\right]$$

$$\text{with} \quad M_{\pm\pm} = \frac{1}{2}\left(M_{11} + M_{22} \pm M_{12} \pm M_{21}\right)$$

- minimal circuit expressed with two commuting n.m. generators

$\longrightarrow$ normal mode subspace is flat!

$$ds^2 = \underline{2dy^2 + 2d\rho^2} + 2\cosh(2\rho)\cosh^2\rho\,d\tau^2 + 2\cosh(2\rho)\sinh^2\rho\,d\theta^2 - 8\sinh^2\rho\,\cosh^2\rho\,d\theta d\tau$$

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$$\longrightarrow \quad A_T = m\,\mathbb{1}\,, \qquad A_R = \omega_0\,\mathbb{1}$$ "lower bound"

move on to
"Dumbo, the Flying Elephant"

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"lower bound"

$$\longrightarrow \mathcal{C} = \mathcal{D}_{min} = \frac{1}{2}\sqrt{\sum \log^2(m/\omega_0)} = \frac{\boxed{N^{\frac{d-1}{2}}}}{2}\log(\omega_0/m)$$

- **problem!?!?** $\mathcal{C} \sim N^{\frac{d-1}{2}} = \left[\frac{V}{\delta^{d-1}}\right]^{1/2}$ compare $\mathcal{C}_{holo} \sim \frac{V}{\delta^{d-1}}$

$\longrightarrow$ change the normalization: $\mathcal{D} = \underline{N^{\frac{d-1}{2}}}\int_0^1 ds\sqrt{\sum_{IJ}\delta_{IJ}\,Y^I(s)Y^J(s)}$

(Brown & Susskind)

$\longrightarrow$ each gate carries cost of $\left[\frac{V}{\delta^{d-1}}\right]^{1/2} + \cdots$

(problematic to compare complexities if V or δ change??)

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(penalty factors!)

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→ $A_T = m\,\mathbb{1}\,, \qquad A_R = \omega_0\,\mathbb{1}$ "lower bound"

→ $$\mathcal{C} = \mathcal{D}_{min} = \frac{1}{2}\sqrt{\sum \log^2\left(m/\omega_0\right)} = \frac{\boxed{N^{\frac{d-1}{2}}}}{2}\log\left(\omega_0/m\right)$$

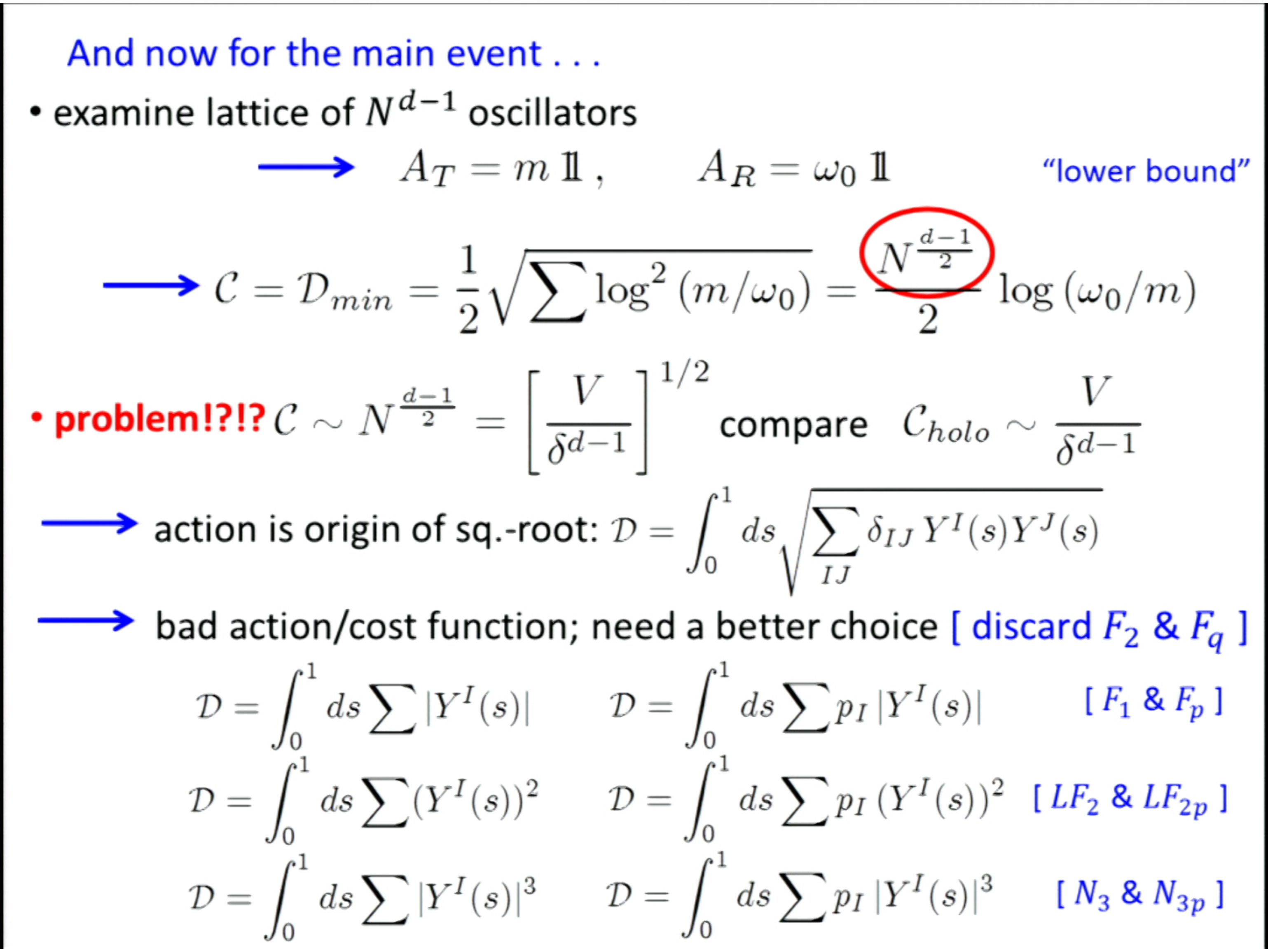
- **problem!?!?** $\mathcal{C} \sim N^{\frac{d-1}{2}} = \left[\frac{V}{\delta^{d-1}}\right]^{1/2}$ compare $\mathcal{C}_{holo} \sim \dfrac{V}{\delta^{d-1}}$

→ change the normalization: $\mathcal{D} = \underline{N^{\frac{d-1}{2}}}\int_0^1 ds\sqrt{\sum_{IJ}\delta_{IJ}\,Y^I(s)Y^J(s)}$

(Brown & Susskind)

→ each gate carries cost of $\left[\frac{V}{\delta^{d-1}}\right]^{1/2} + \cdots$

(problematic to compare complexities if V or δ change??)



And now for the main event . . .

- examine lattice of N^{d-1} oscillators

$$\longrightarrow \quad A_T = m\,\mathbb{1}\,, \qquad A_R = \omega_0\,\mathbb{1}$$ "lower bound"

$$\longrightarrow \quad \mathcal{C} = \mathcal{D}_{min} = \frac{1}{2}\sqrt{\sum \log^2(m/\omega_0)} = \frac{\boxed{N^{\frac{d-1}{2}}}}{2}\log(\omega_0/m)$$

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$\longrightarrow$ action is origin of sq.-root: $\mathcal{D} = \int_0^1 ds \sqrt{\sum_{IJ} \delta_{IJ}\, Y^I(s) Y^J(s)}$

$\longrightarrow$ bad action/cost function; need a better choice [discard F_2 & F_q]

$$\mathcal{D} = \int_0^1 ds \sum |Y^I(s)| \qquad \mathcal{D} = \int_0^1 ds \sum p_I\, |Y^I(s)|$$ [F_1 & F_p]

$$\mathcal{D} = \int_0^1 ds \sum (Y^I(s))^2 \qquad \mathcal{D} = \int_0^1 ds \sum p_I\, (Y^I(s))^2$$ [LF_2 & LF_{2p}]

$$\mathcal{D} = \int_0^1 ds \sum |Y^I(s)|^3 \qquad \mathcal{D} = \int_0^1 ds \sum p_I\, |Y^I(s)|^3$$ [N_3 & N_{3p}]

And now for the main event . . .

- examine lattice of N^{d-1} oscillators ⟶ for simplicity, $\mathbf{\Omega \rightarrow 0}$

$$\longrightarrow \quad A_T = m\,\mathbb{1}\,, \qquad A_R = \omega_0\,\mathbb{1}$$

$$\longrightarrow \mathcal{C} = \mathcal{D}_{min} = \frac{1}{2}\sqrt{\sum \log^2\left(m/\omega_0\right)} = \frac{N^{\frac{d-1}{2}}}{2}\boxed{\log\left(\omega_0/m\right)}$$

- **there be log's here!!** $\mathcal{C} \sim \log\left(\omega_0/m\right)$ **!!**

And now for the main event . . .

- examine lattice of N oscillators ⟶ restore $\boldsymbol{\Omega = 1/\delta}$ & $\boldsymbol{\omega = m}$

And now for
"Matterhorn Bobsleds"

And now for the main event . . .

- examine lattice of N oscillators $\longrightarrow$ restore $\boldsymbol{\Omega = 1/\delta}$ **&** $\boldsymbol{\omega = m}$

$$\longrightarrow \mathcal{C} = \mathcal{D}_{min} = \frac{1}{2^\alpha} \sum \left| \log\left(\omega_{\vec{k}}/\omega_0\right) \right|^\alpha$$

where for $(d-1)$-dimensional (periodic) square lattice:

$$\omega_k^2 = m^2 + \frac{4}{\delta^2} \sum_i \sin^2 \frac{\pi k_i}{N}\,, \qquad k_i = 0, 1, \cdots N-1$$

- dominated by UV modes, ie, $k_i \sim N/2$ • **need to choose $\boldsymbol{\omega_0}$**

1) $\omega_0 = \Lambda$ (IR) with ${}^1/_\delta \gg \Lambda$ $\longrightarrow \mathcal{C} \sim N^{d-1} \log^\alpha\left(\frac{1}{\Lambda\delta}\right) + \cdots$

- choose $\alpha = 1$: $\mathcal{C} \sim \# \frac{V}{\delta^{d-1}} \log\left(\frac{1}{\Lambda\delta}\right) + \# \frac{V}{\delta^{d-1}} + \cdots$ $d\mathcal{C}_A/dt$

2) $\omega_0 = \beta/\delta$ (UV) $\longrightarrow \mathcal{C} \sim \# \frac{V}{\delta^{d-1}} + \enclose{circle}{\#} \frac{m^2\, V}{\delta^{d-3}} \cdots$ (α unconstrained)

useful probe of α or β?

And now for the main event . . .

- examine lattice of N oscillators ⟶ restore $\boldsymbol{\Omega = 1/\delta}$ **&** $\boldsymbol{\omega = m}$

$$\longrightarrow \mathcal{C} = \mathcal{D}_{min} = \frac{1}{2^\alpha}\sum |\log(\omega_{\vec{k}}/\omega_0)|^\alpha$$

where for $(d-1)$-dimensional (periodic) square lattice:

$$\omega_0$$

- reference state introduces scale
- $\omega_0 \sim {}^1/_R$: complexity is superextensive
- $\omega_0 \sim {}^1/_\delta$: IR contributions depend on UV cutoff
- $\omega_0 = \omega_0$: complexity depends on new (unphysical) scale

$$\frac{k_i}{N}\,, \qquad k_i = 0, 1, \cdots N-1$$

$N/2$

- **need to choose $\boldsymbol{\omega_0}$**

$$\longrightarrow \mathcal{C} \sim N^{d-1}\log^\alpha\left(\frac{1}{\Lambda\delta}\right) + \cdots$$

$$\log\left(\frac{1}{\Lambda\delta}\right) + \#\,\frac{V}{\delta^{d-1}} + \cdots$$

$d\mathcal{C}_A/dt$

$$\frac{V}{\delta^{d-1}} + \boxed{\#}\,\frac{m^2\,V}{\delta^{d-3}}\cdots$$

(α unconstrained)

be of α or β?

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where for $(d-1)$-dimensional (periodic) square lattice:

ω_0	$\alpha = L/\ell$
• reference state introduces scale	• eliminates AdS scale ✓
• $\omega_0 \sim {}^1/_R$: complexity is superextensive	• $\ell \sim V^{1/(d-1)}$: complexity is superextensive
• $\omega_0 \sim {}^1/_\delta$: IR contributions depend on UV cutoff	• $\ell \sim \delta$: IR contributions depend on UV cutoff, eg,
• $\omega_0 =\ \omega_0$: complexity depends on new (unphysical) scale	• $\ell =\ \ell_0$: complexity depends on new (unphysical) scale


Disneyland
Holography
Quantum Field Theory
CMERA
Complexity
CRITTER COUNTRY
BIG THUNDER RANCH
FRONTIERLAND
RIVERS OF AMERICA
PLAZA GARDENS STAGE
PLAZA PAVILION
ADVENTURELAND
MAIN STREET, U.S.A.
CITY HALL
MICKEY & FRIENDS TRAM ROUTE
DOWNTOWN DISNEY DISTRICT AND HOTELS OF THE DISNEYLAND RESORT
Disney's CALIFORNIA ADVENTURE
LION KING TRAM ROUTE

Optimized circuit versus cMERA?

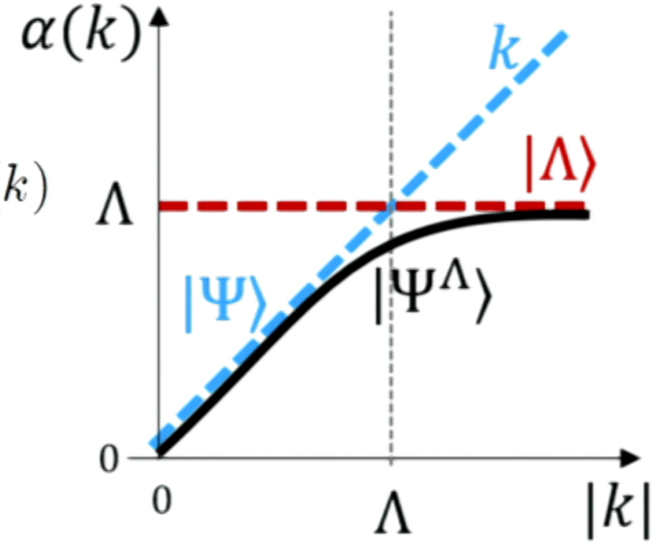
- optimized circuit/path was restricted to normal mode subspace

$\longrightarrow$ by design, cMERA lies within normal mode subspace ✓

- recall Guifre's plots of $\boldsymbol{\alpha(k)}$:

$$a^{\Lambda}(k)=\sqrt{\frac{\alpha(k)}{2}}\,\phi(k)+i\sqrt{\frac{1}{2\alpha(k)}}\,\pi(k)$$

$$\longrightarrow \quad a^{\Lambda}(k)\,|\Psi^{\Lambda}\rangle=0$$



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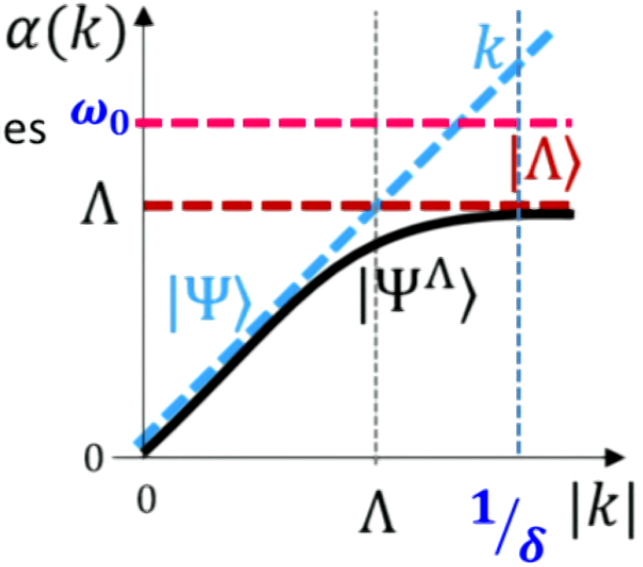

- optimized circuit discussion distinguishes UV cutoff $\boldsymbol{1/\delta}$ and reference scale $\boldsymbol{\omega_0}$

- cMERA has single scale for "UV cutoff" and reference scale $\boldsymbol{\Lambda}$

  ("UV cutoff"= entanglement cutoff)

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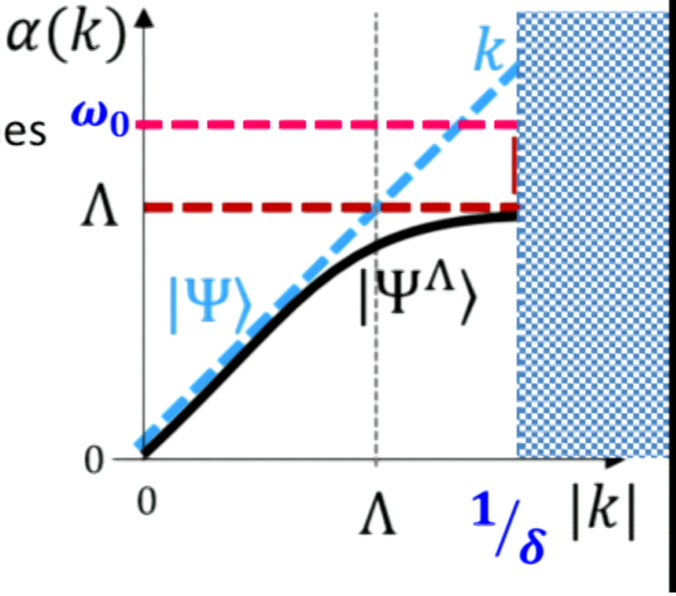

- optimized circuit discussion distinguishes UV cutoff $1/\delta$ and reference scale ω_0

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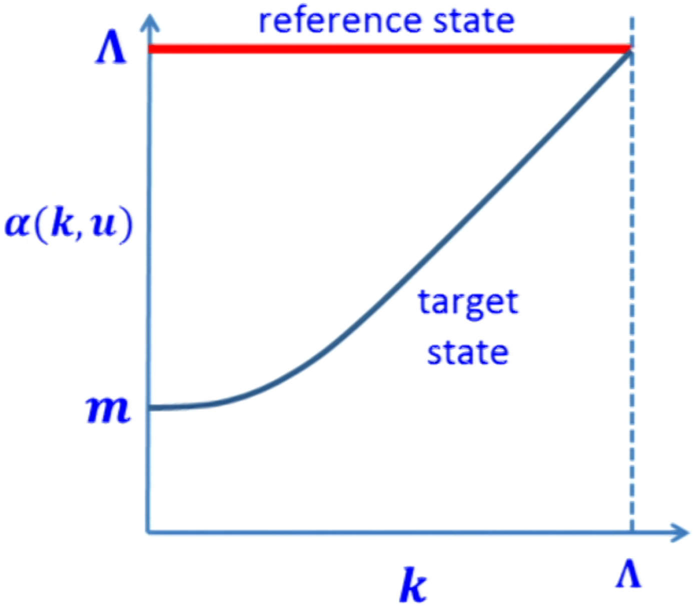
- consider scale evolution/circuit trajectory:

$$a^{\Lambda}(k,u) = \sqrt{\frac{\alpha(k,u)}{2}}\,\phi(k) + i\sqrt{\frac{1}{2\alpha(k,u)}}\,\pi(k) \longrightarrow a^{\Lambda}(k,u))\,|\Psi^{\Lambda}(u)\rangle = 0$$



cMERA

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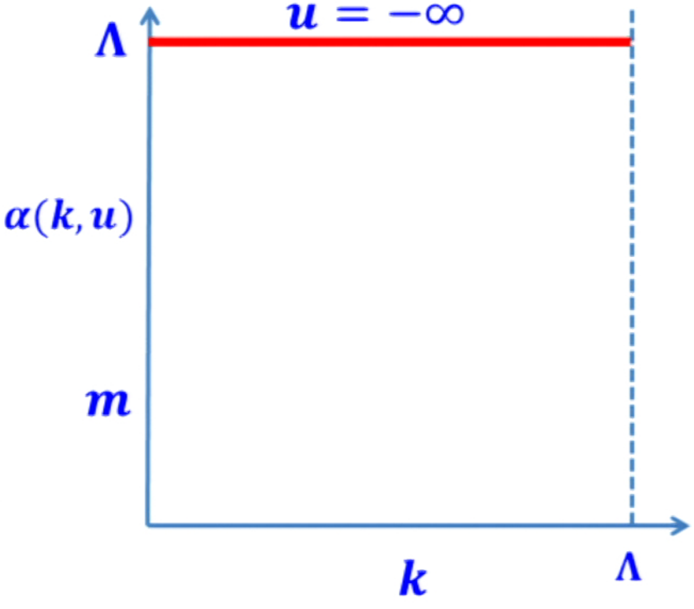

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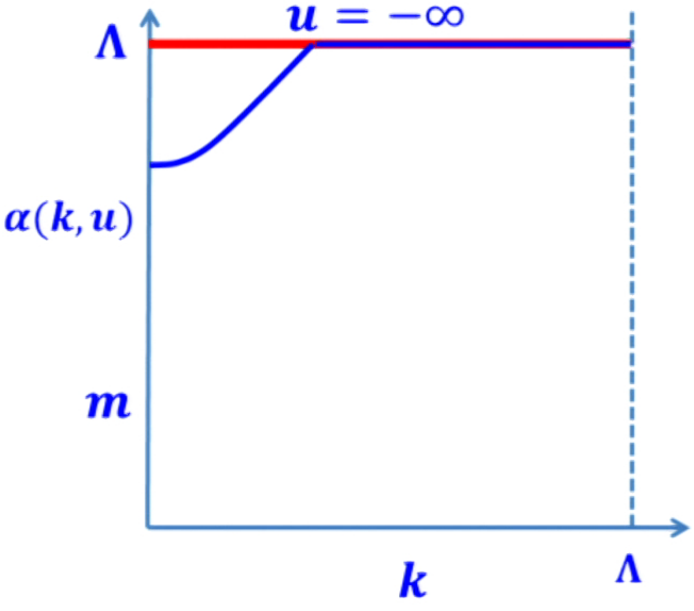

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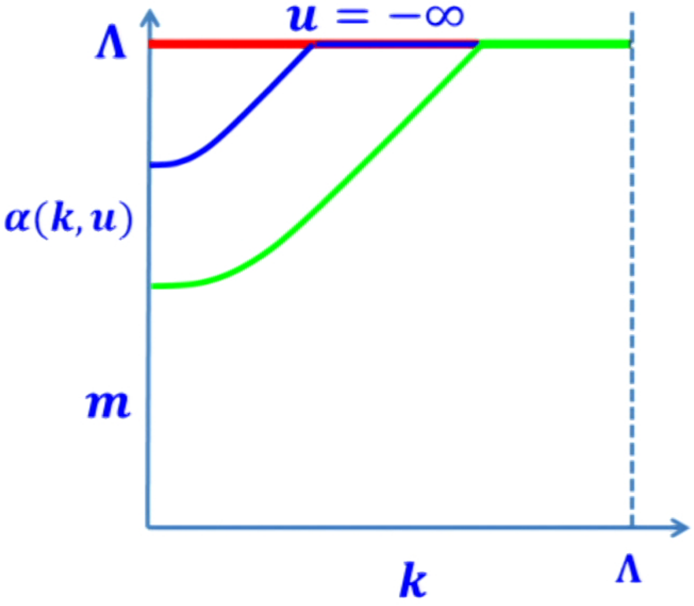

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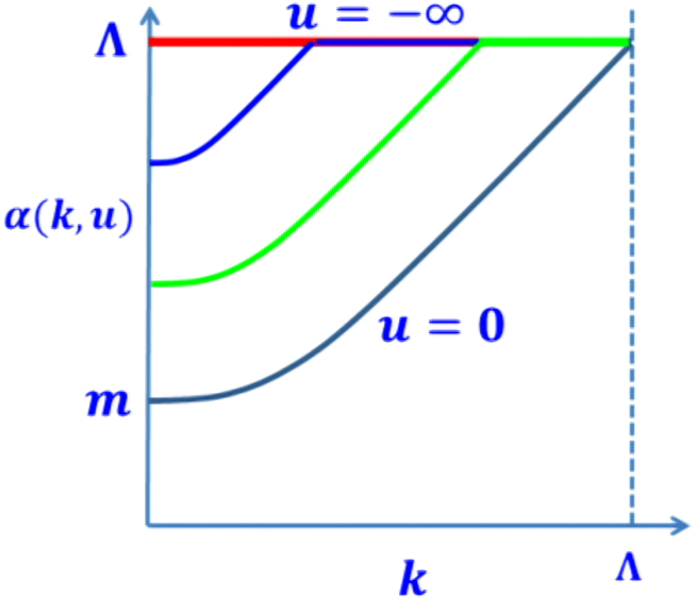

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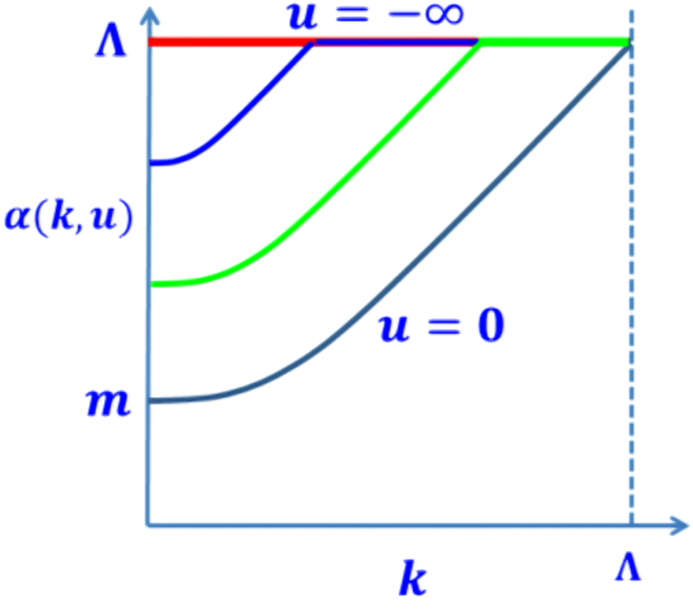
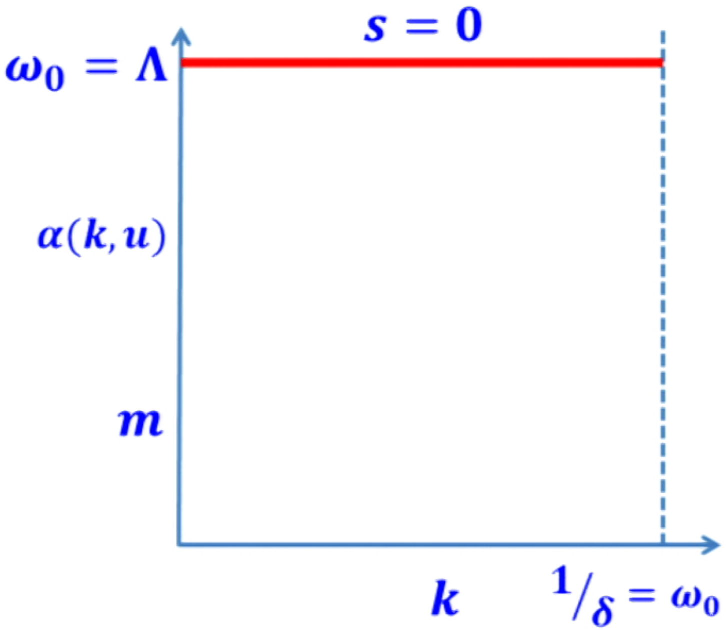


cMERA



Optimized circuit

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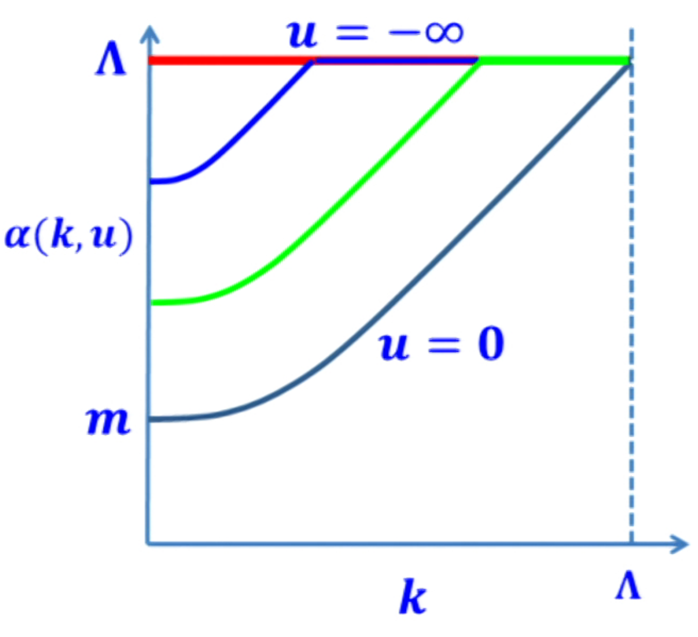
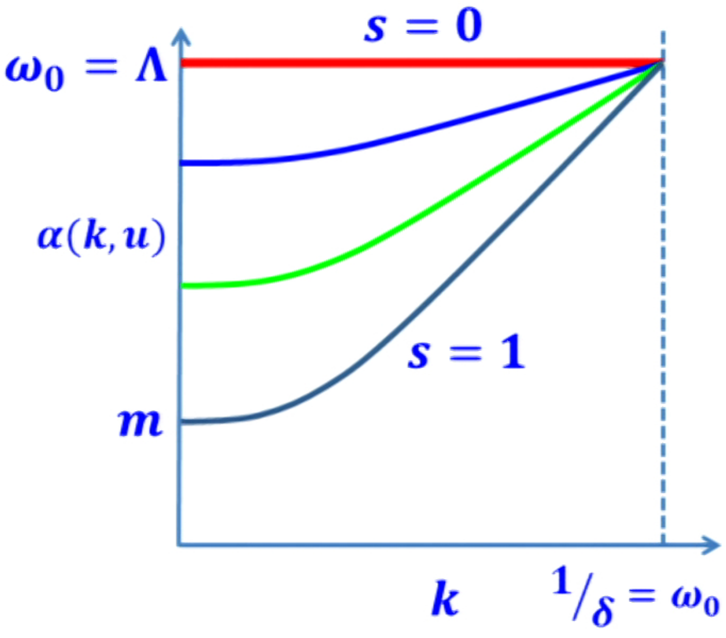
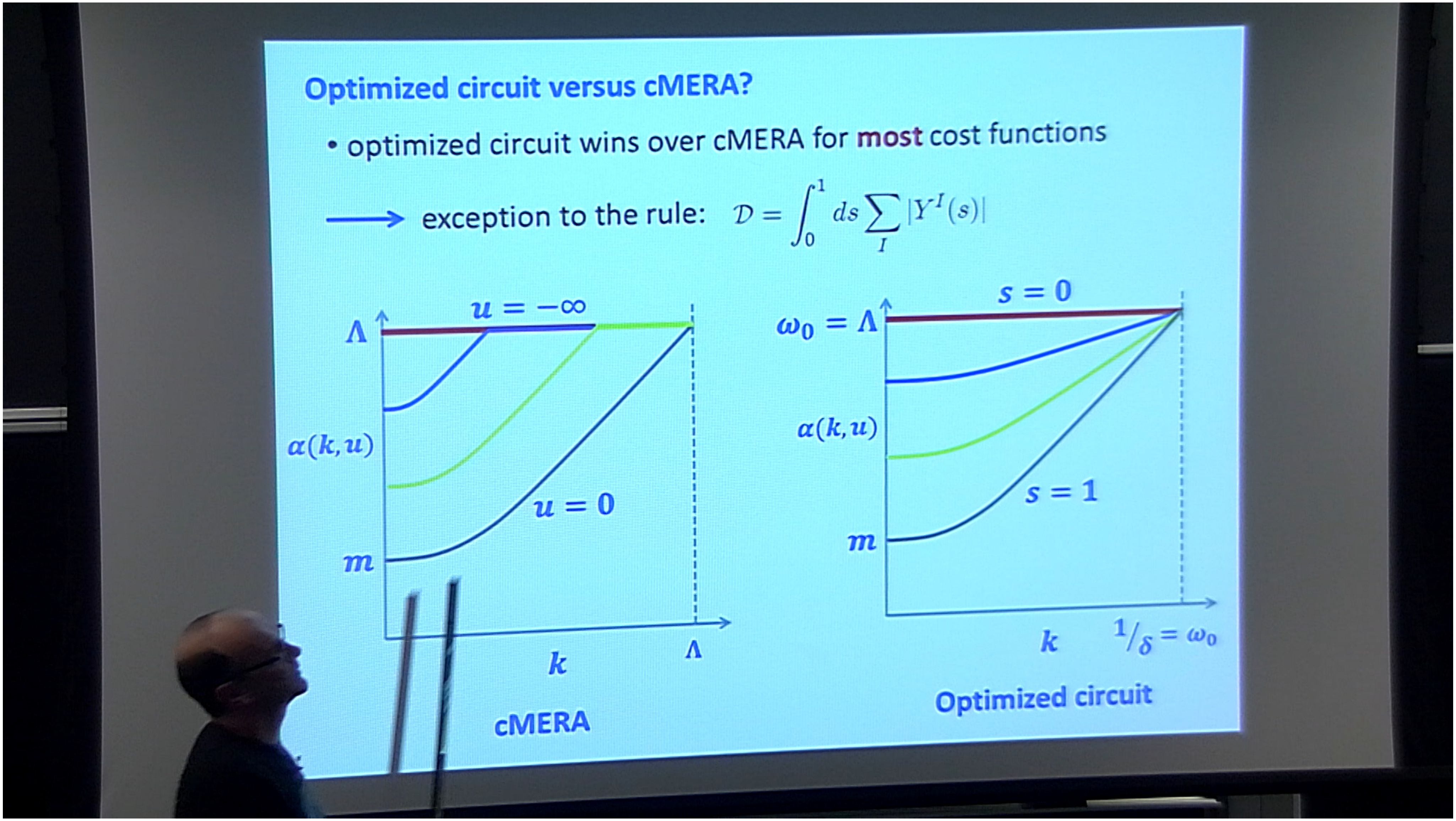


cMERA



Optimized circuit



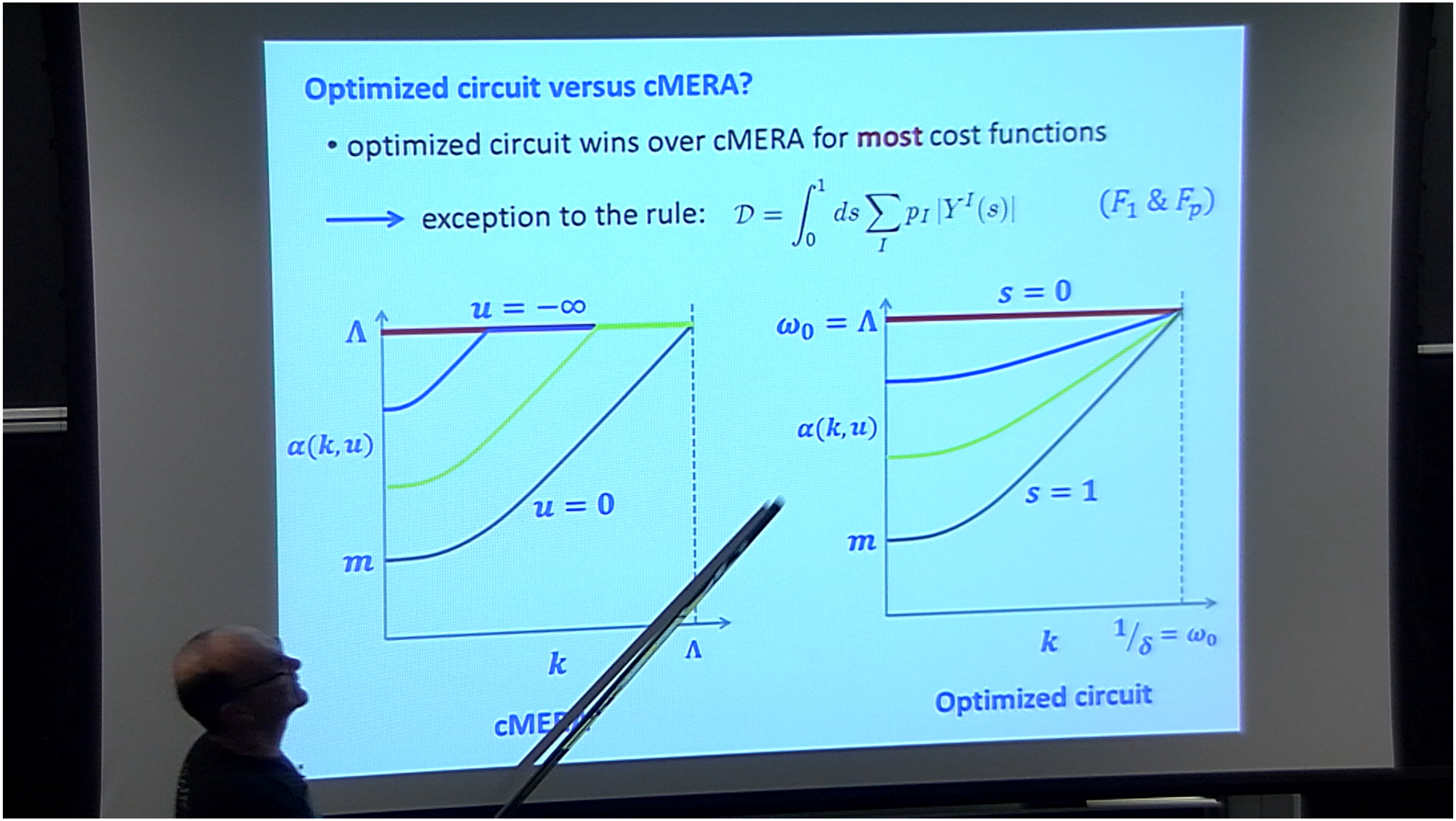
Optimized circuit versus cMERA?

- optimized circuit wins over cMERA for **most** cost functions

→ exception to the rule: $\mathcal{D} = \int_0^1 ds \sum_I |Y^I(s)|$

cMERA

Optimized circuit



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Manhattan metric* or taxicab distance:

- all routes have same cost as long as we don't retrace steps
- ***no metric!** in **Finsler geometry**, even diagonal distance is same

→ with these cost functions, optimized circuit & cMERA offer equal complexities (as would many other circuits)

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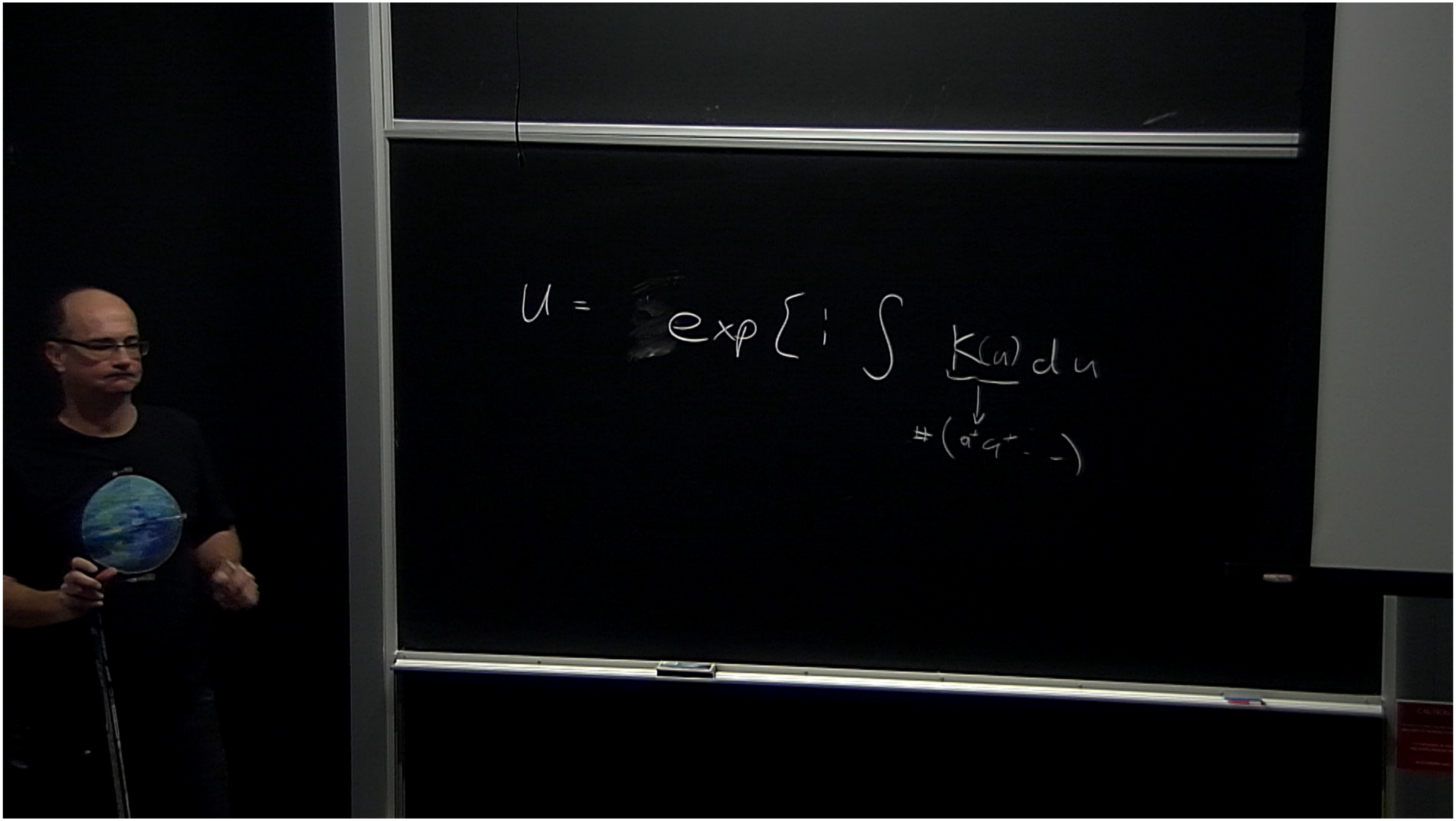
- there is room to add new physical principles to discussion:
  - ➢ entanglement evolution?
  - ➢ renormalization group flow?
  - ➢ parade route??
  - ➢ expanding universe model??
  - ➢ . . . .

Still lots to explore!



$$U = \exp\{ i \int \underbrace{K(u)}_{\downarrow \; \#(a^{+}a^{+} \cdots)} du$$