Title: Random tensor networks and holographic coherent states

Date: Apr 21, 2017 09:00 AM

URL: http://pirsa.org/17040046

Abstract: Tensor network is a constructive description of many-body quantum entangled states starting from few-body building blocks. Random tensor networks provide useful models that naturally incorporate various important features of holographic duality, such as the Ryu-Takayanagi formula for entropy-area relation, and operator correspondence between bulk and boundary. In this talk I will overview the setup and key properties of random tensor networks, and then discuss how to describe quantum superposition of geometries in this formalism. By introducing quantum link variables, we show that random tensor networks on all geometries form an overcomplete basis of the boundary Hilbert space, such that each boundary state can be mapped to a superposition of (spatial) geometries. We discuss how small fluctuations around each geometry forms a â€œcode subspaceâ€• in which bulk operators can be mapped to boundary isometrically. We further compute the overlap between distinct geometries, and show that the overlap is suppressed exponentially in an area law fashion, in consistency with the holographic principle. In summary, random tensor networks on all geometries form an overcomplete basis of â€œholographic coherent statesâ€• which may provide a new starting point for describing quantum gravity physics.

References

[1] Patrick Hayden, Sepehr Nezami, Xiao-Liang Qi, Nathaniel Thomas, Michael Walter, Zhao Yang, JHEP 11 (2016) 009
[2] Xiao-Liang Qi, Zhao Yang, Yi-Zhuang You, arxiv:1703.06533

Random tensor networks, bulk gauge symmetry and geometry fluctuations

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04/20/2017

Outline

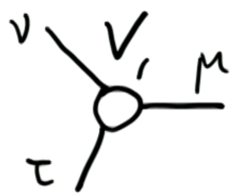
- Quantum entanglement and tensor networks
- Holographic duality
- Random tensor networks (RTN)
- Two new directions
  - Global symmetries
  - Fluctuation geometry (holographic coherent states)

Reference

- Patrick Hayden, Sepehr Nezami, Xiao-Liang Qi, Nathaniel Thomas, Michael Walter, Zhao Yang, arxiv: 1601.01694, JHEP 11 (2016) 009
- XLQ, Zhao Yang, Yi-Zhuang You, arxiv:1703.06533
- Xingshan Cui, XLQ, Zhao Yang, in progress

Tensor networks

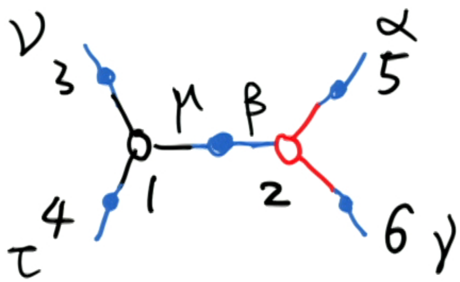
- Building many-body entangled states from few-qubit building blocks.
- Tensor contraction just like in Feynman diagrams



$T_{1\mu\nu\tau}|\mu\nu\tau\rangle = |V_1\rangle$

$T_{2\alpha\beta\gamma}|\alpha\beta\gamma\rangle = |V_2\rangle$

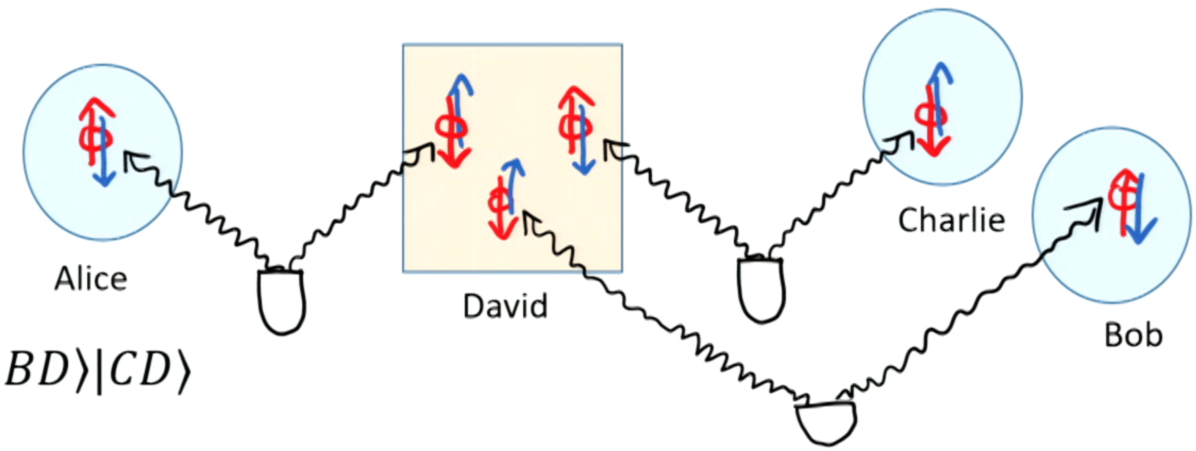
$|L\rangle = g^{\mu\beta}|\mu\beta\rangle$



$|\Psi\rangle = T_1^{\mu\nu\tau} T_{2\,\mu}^{\ \alpha\gamma}|\mu\nu\alpha\gamma\rangle$

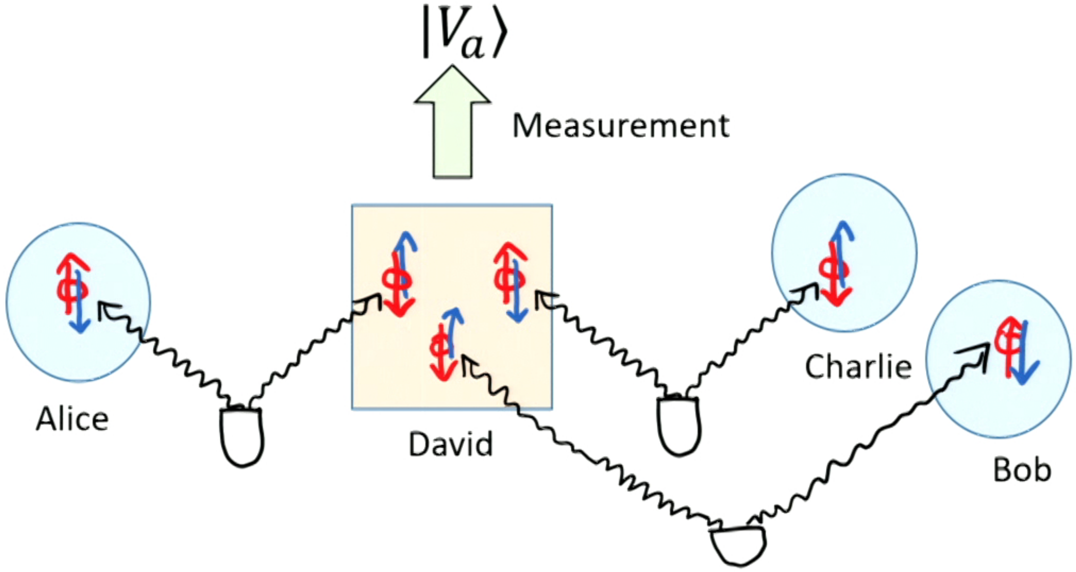
Tensor networks: Physical interpretation

- Projected Entangled Pair States (PEPS) F. Verstraete, J.I. Cirac, 04'
- 1. Prepare and distribute EPR pairs. Alice, Bob and Charlie are all entangled with David, but not with each other.

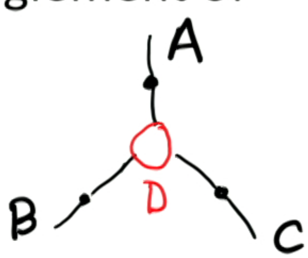


$|AD\rangle|BD\rangle|CD\rangle$

- 2. David measures the qubits he has.
- $\rho_D \to |V_a\rangle\langle V_a|$ with probability $p_a = \langle V_a|\rho_D|V_a\rangle$

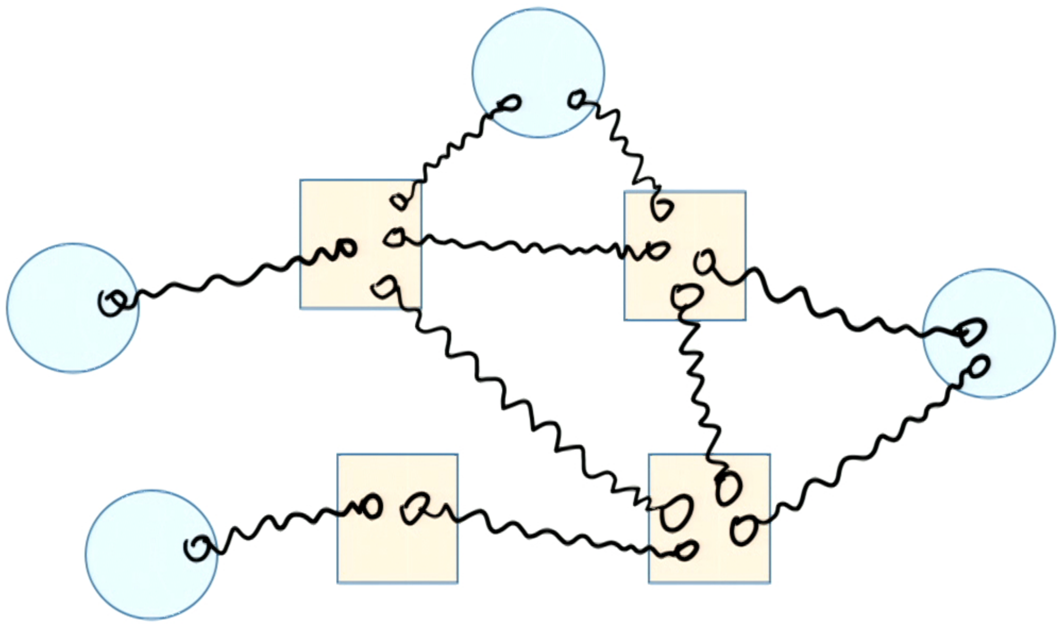


- For a given output a, David now has a pure state, but Alice, Bob and Charlie are entangled. (Entanglement of assistance, D.P. DiVincenzo et al. '99)
- $|\Psi_{ABC}\rangle = \langle V_a|AD\rangle|BD\rangle|CD\rangle$



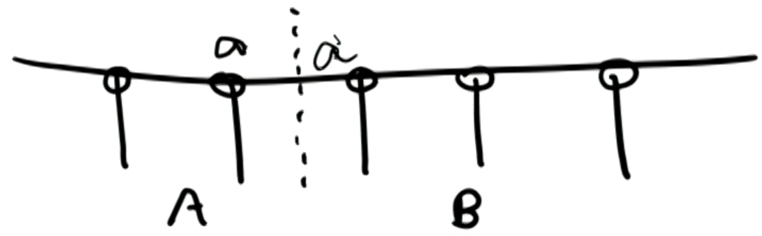
Tensor networks: Physical interpretation

- More generally, measurements occur on multiple parties, creating a complicated entangled state of the remaining parties that are not measured.



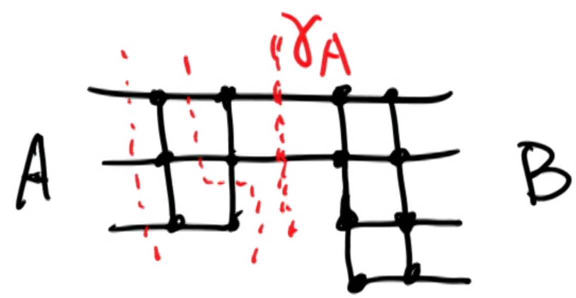
Geometry constrains entanglement

- Entanglement structure encoded in geometry (and the vertex tensors $|V_x\rangle$)
- For example, for any region A, $S_A \leq \log D_{min}(A)$
- or $S_A \leq |\gamma_A| \log D$. $|\gamma_A|$ is the minimal surface area bounding A region.

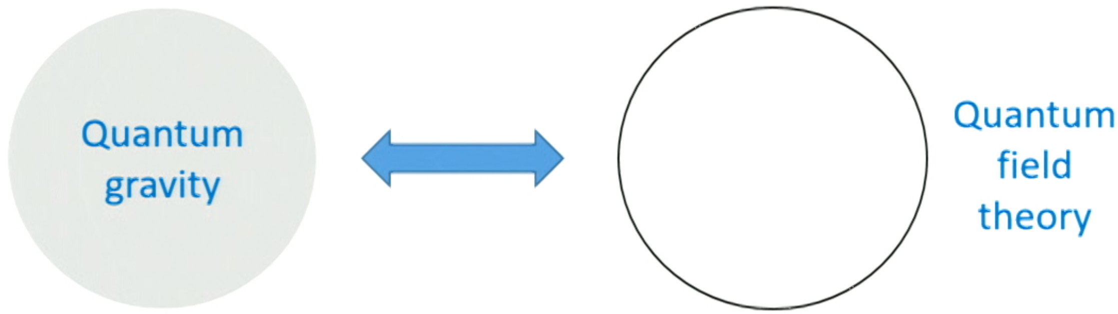


$$|\Psi\rangle = \sum_{a=1}^{D} |\Psi_a^A\rangle|\Psi_a^B\rangle$$

$$\Rightarrow rank(\rho_A) \leq D, S_A \leq \log D$$



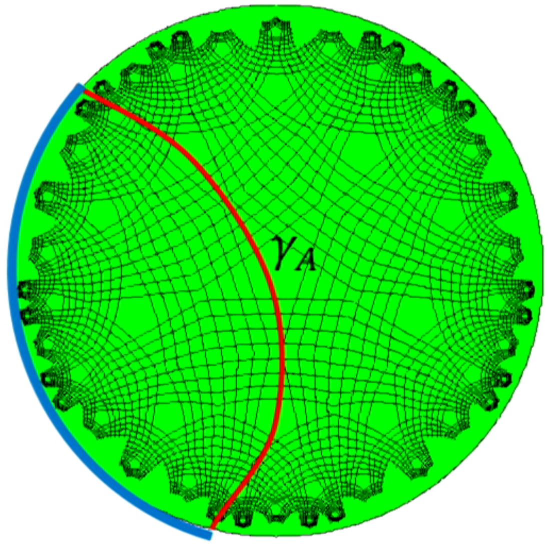
Holographic duality

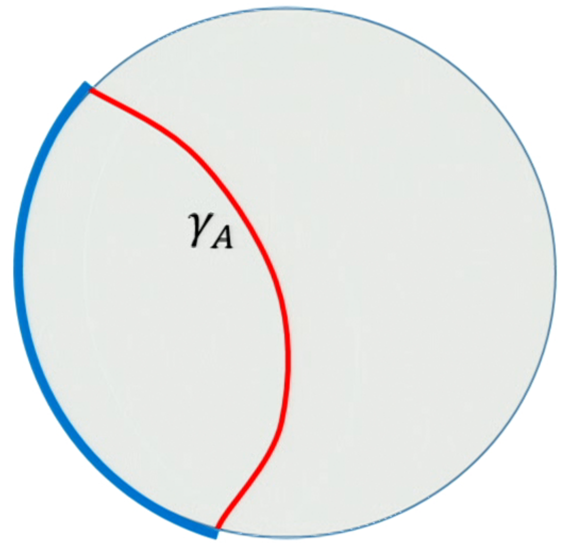


AdS	CFT
Black hole	Thermal state
$Z = \int D\phi e^{-S}$	$Z = \int D\psi e^{-S}$
Boundary condition of ϕ	$\langle O \rangle$ local operators
Classical equation of motion	RG flow

(Maldacena ’97, Witten ’98, Gubser, Klebanov & Polyakov ’98)

Area measures entanglement





Minimal surface area

$$\frac{1}{4G_N}|\gamma_A|$$

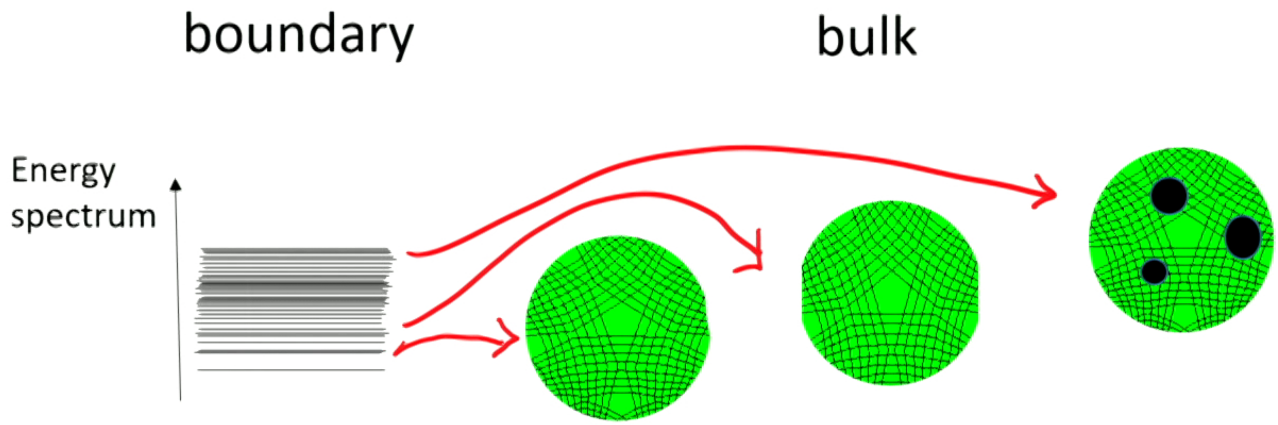
$\Leftrightarrow$

Entanglement entropy

$$S_A$$

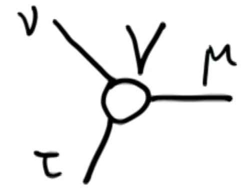
Ryu&Takayanagi '06

Tensor networks and holography

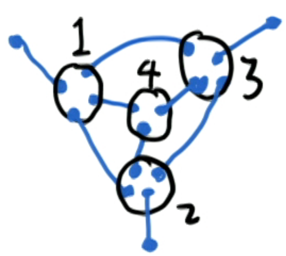


- TNW in holography: geometry emerges from the entanglement structure of quantum states (Swingle '09)
- Various tensor network related proposals (Nozaki et al '12, XLQ '13, Hartman&Maldacena '13, Maldacena & Susskind '13, Czech et al '14-15, Pastawski et al '15, Yang et al '15)
- Goal: an explicit holographic mapping between bulk and boundary

Random tensor networks



- Entanglement quantities are hard to compute for a given tensor networks.
- Random average greatly simplifies entanglement calculations.
- A random tensor $V_{\mu\nu\tau}$ corresponds to a (Haar) random state in the Hilbert space $|V\rangle = V_{\mu\nu\tau}|\mu\rangle|\nu\rangle|\tau\rangle$.
- Random tensor network state



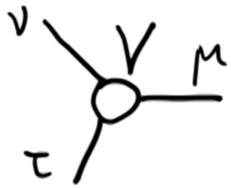
$$|\Psi\rangle = \prod_x \langle V_x| \, |P\rangle$$

Random vertex states

Link states. EPR pairs $\prod|xy\rangle$ or more general

- The link state can be EPR pairs $|\mathrm{P}\rangle = \prod_{xy} |L_{xy}\rangle$ but can also be more general

Random tensor networks



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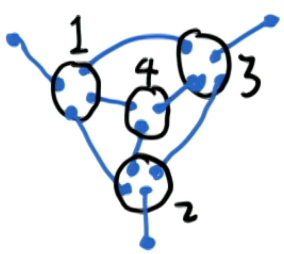

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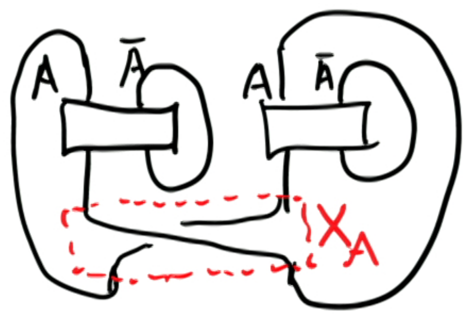
Renyi entropy calculation

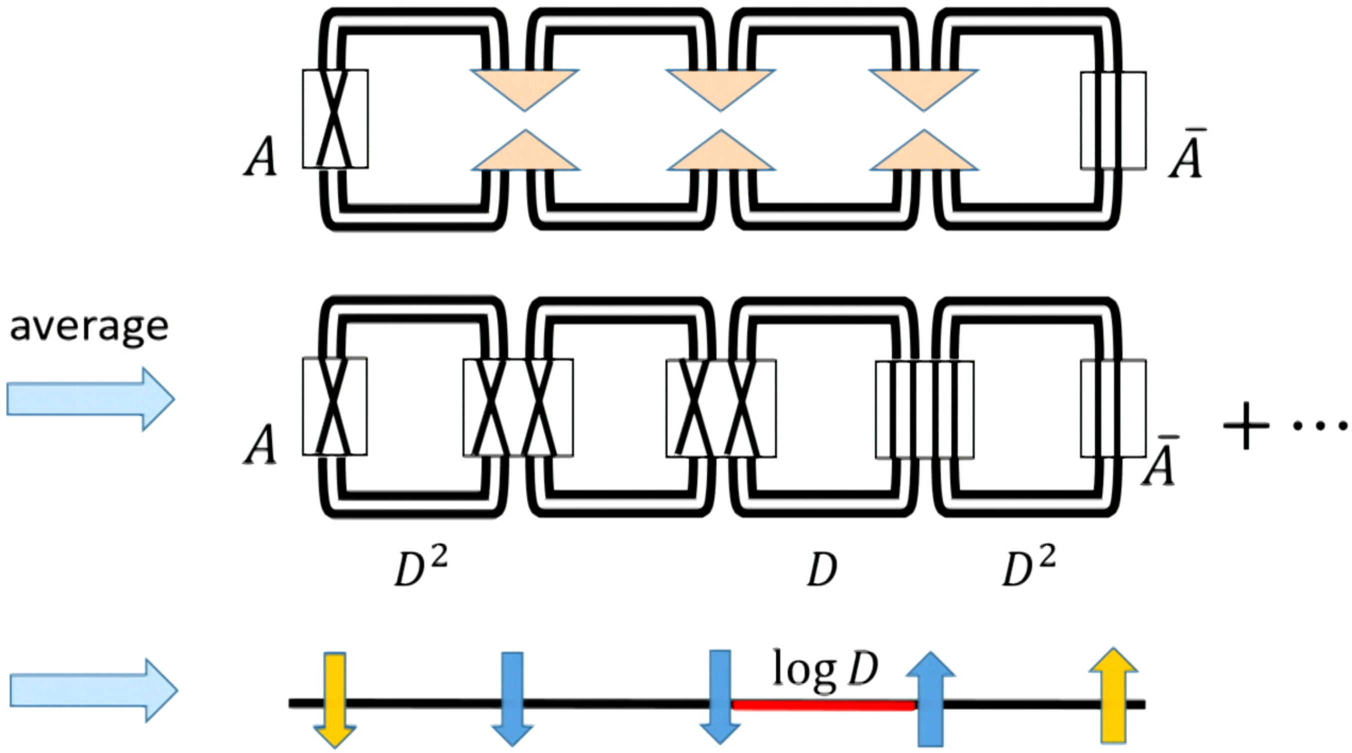
- $\rho = |\Psi\rangle\langle\Psi| = tr(\prod_x |V_x\rangle\langle V_x|\, \rho_P)$ is a linear function of $|V_x\rangle\langle V_x|$.
- Renyi entropies $S_A = \frac{1}{1-n} \log \frac{tr(\rho_A^n)}{tr(\rho)^n}$
- For any quantity that is polynomial in ρ, such as $tr(\rho_A^n)$, the random average can be easily obtained.



- For example, second Renyi $tr\left(\rho_A^2\right) = tr(\rho \otimes \rho X_A)$ $= tr(\rho_P \otimes \rho_P [X_A \otimes \prod_x |V_x\rangle\langle V_x| \otimes |V_x\rangle\langle V_x|])$
- Random average $\overline{|V_x\rangle\langle V_x| \otimes |V_x\rangle\langle V_x|} = \frac{1}{D_x^2 + D_x}(I_x + X_x)$
- $tr\left(\rho_A^2\right) = const. \times tr(\rho_P \otimes \rho_P X_A \otimes \prod_x (I_x + X_x))$
- Expanding the product gives 2^N terms

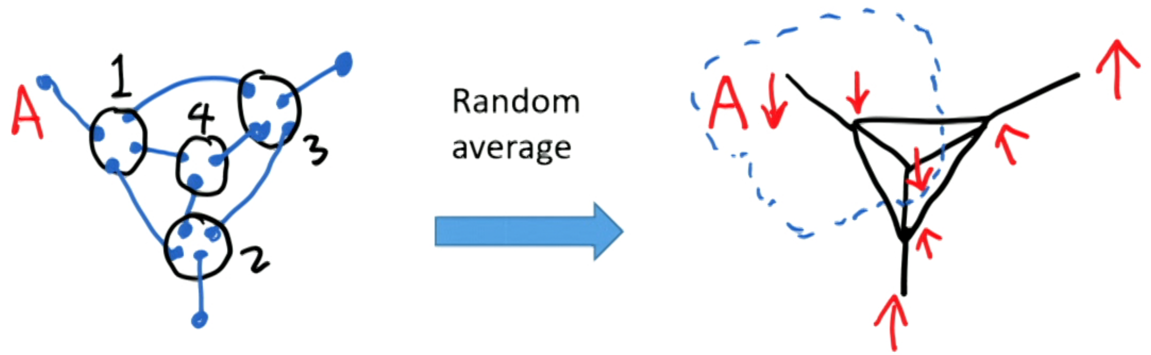
- $tr(\rho_A^2) = tr(\rho_P \otimes \rho_P X_A \otimes \prod_x (I_x + X_x))$

$$= \sum_{R \subseteq bulk} tr(\rho_P \otimes \rho_P X_A \otimes X_R) = \sum_{R \subseteq bulk} e^{-S_P^{(2)}(RA)}$$



Purity = Ising partition function

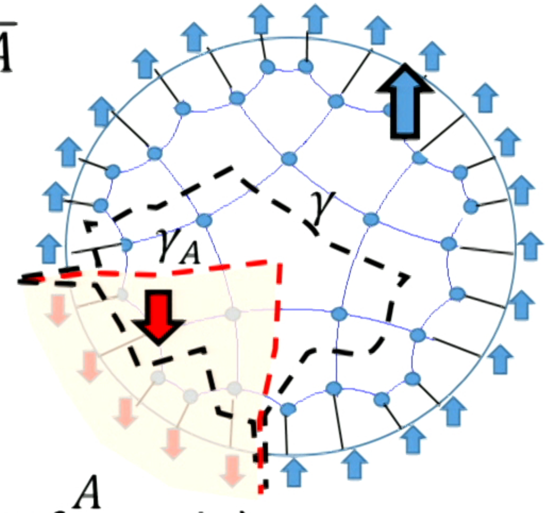
- For a random tensor network
- $\overline{tr(\rho_A^2)} = Z_A = \sum_{\{\sigma_x = \pm 1\}} e^{-\mathcal{A}[\{\sigma_x\}]}$
- $\mathcal{A}[\{\sigma_x\}] = S(\{\sigma_x = -1\}; \rho_P)$ "the second Renyi entropy of $\sigma_x = -1$ domain for state $\rho_P = |P\rangle\langle P|$"
- Boundary condition: spin $\downarrow$ in A and $\uparrow$ elsewhere



- The second Renyi entropy $S_A \simeq -\log \frac{Z_A}{Z_\emptyset}$ is the "cost of free energy" of flipping spins in A from $\uparrow$ to $\downarrow$.

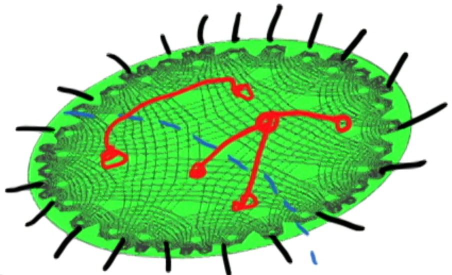
RT formula

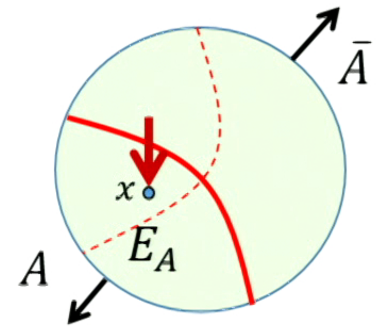
- If $|P\rangle = \prod_{xy} |xy\rangle$ consists of maximally entangled EPR pairs with rank D,
- $\mathcal{A}[\{\sigma_x\}] = -\frac{1}{2}\log D \sum_{xy} \sigma_x \sigma_y$
- Boundary cond. $\sigma_x = \begin{cases} -1, x \in A \\ +1, x \in \bar{A} \end{cases}$
- The action is proportional to the domain wall area.
- $\overline{tr(\rho_A^2)} = \sum_{\gamma \sim A} e^{-\log D |\gamma|}$
- $D \to \infty \Rightarrow$
  low T limit of Ising model
- $\overline{S} \simeq -\log \overline{tr(\rho_A^2)} \simeq \log D\, |\gamma_A|$ (RT formula)

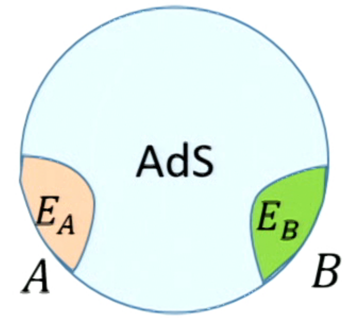


Other properties of RTN

- The random average technique applies to more general networks
- Other properties of RTN:
- $\overline{tr(\rho_A^n)} \Rightarrow S_n$ spin model partition function
- RT formula with quantum correction (agree with Faulkner, Lewkowycz & Maldacena'13)
  $S_A \simeq \log D\,|\gamma_A| + S(\mathrm{E}_A, |\Psi_b\rangle\langle\Psi_b|)$.
- RT formula for higher Renyi entropies
- Quantum error correction properties (Almheiri, Dong, Harlow '14)
  ϕ_x can be reconstructed on boundary region A if $x \in E_A$.
- Scaling dimension of operators
- $\langle O_A O_B\rangle \propto \langle \phi_x \phi_y\rangle_{bulk}$



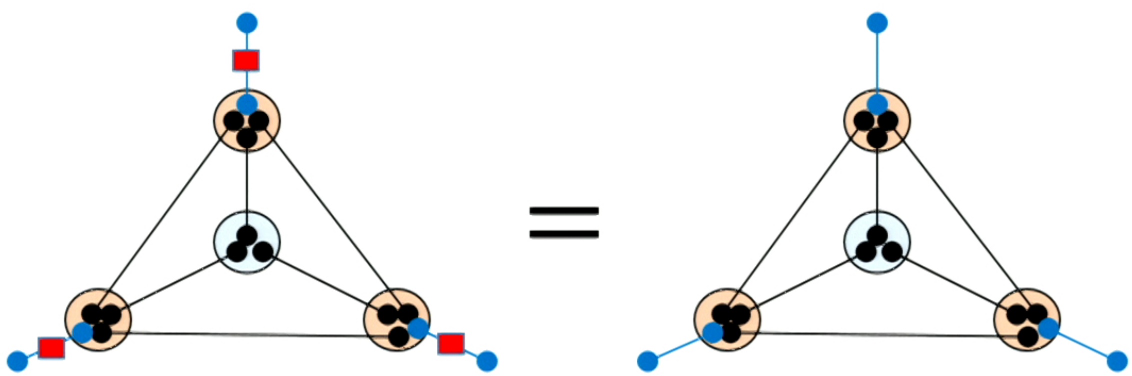




(For more details see our paper 1601.01694)

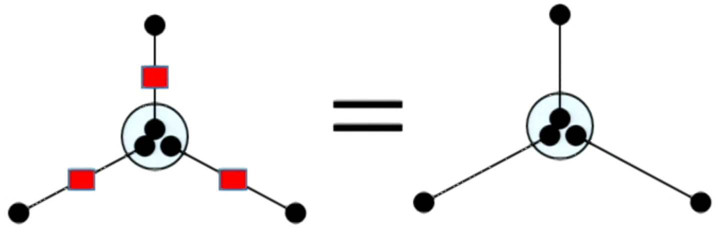
New direction I: Global symmetry

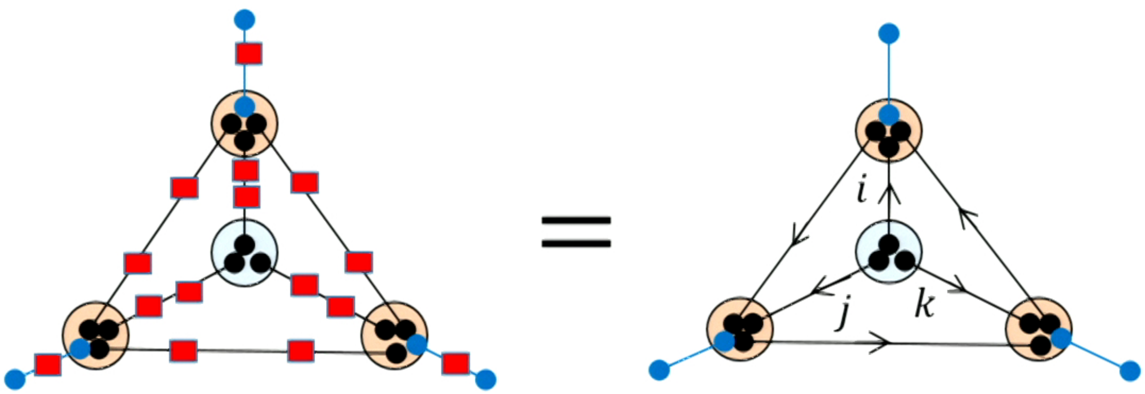
- Random tensor networks capture entanglement properties, but do not preserve any symmetry
- To describe states on the boundary with global symmetry, the tensors need to satisfy further constraints
- Consider on-site global symmetry
  $\prod_x g_x \, |\Psi\rangle = |\Psi\rangle$



Symmetric tensor networks

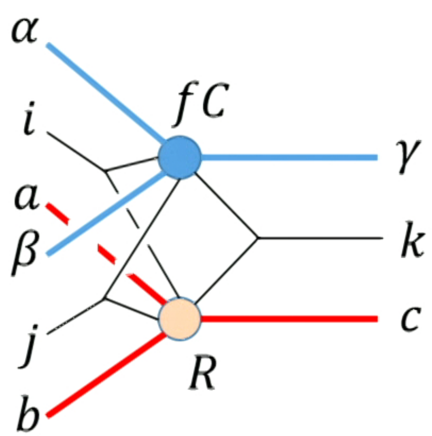
- Sufficient condition to preserve symmetry: requiring each tensor to be symmetric. (Singh, Pfeifer, Vidal '10)
- Fix representation for each (oriented) leg





Symmetric *random* tensor networks

- Generalization of RTN: Symmetric-but-otherwise-random tensor networks.
- Representation of each link $\mathbb{H} = \bigoplus_{i \in irrep} \mathbb{H}_i^{\otimes n_i}$
- Irreducible rep. i appears n_i times.
- General form of the tensor $T^{ijk}_{abc\ \alpha\beta\gamma} = C^{ijk}_{\alpha\beta\gamma} R^{ijk}_{abc} f_{ijk}$



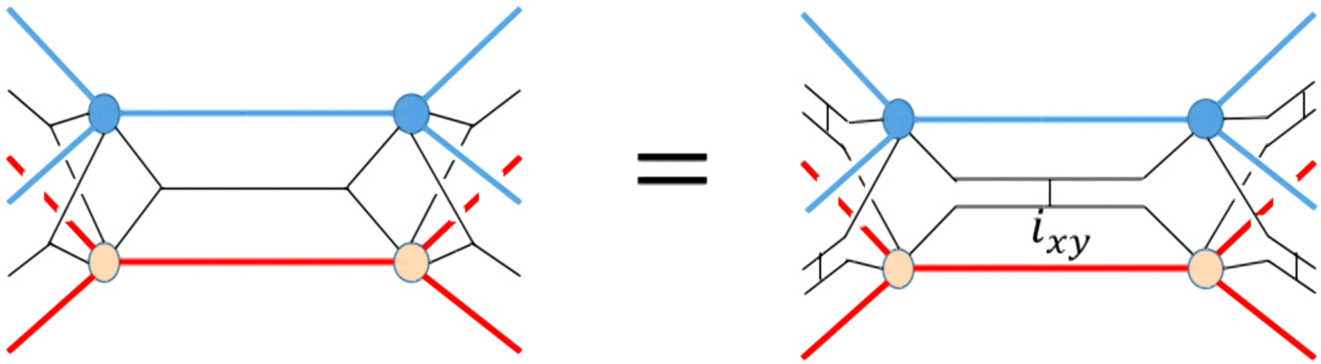
Clebush-Gordon coefficients

Normalized Random tensor in abc for fixed ijk

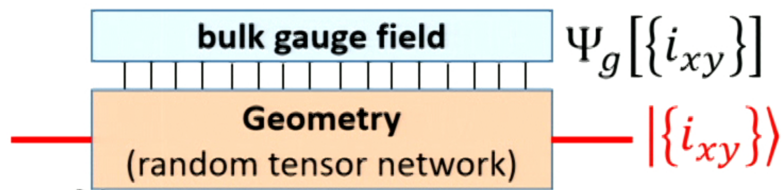
Normalization that is ijk dependent

Cui, XLQ, Yang, to appear '17

Bulk gauge field



- Contraction of α, β, γ ... leads to a bilayer structure of wavefunction

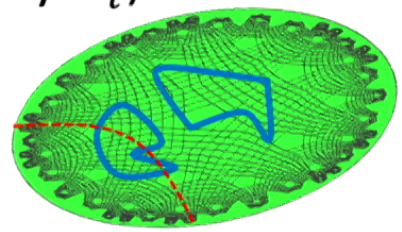


- $|\Psi_\partial\rangle = \sum_{\{i_{xy}\}} \Psi_g[\{i_{xy}\}] |\{i_{xy}\}\rangle$
- A bulk gauge field with wavefunction $\Psi_g[\{i_{xy}\}]$ (Levin-Wen '05)
- Maybe confined or deconfined, controlled by f_{ijk}

Cui, XLQ, Yang, to appear '17

Gauge field contribution to entanglement

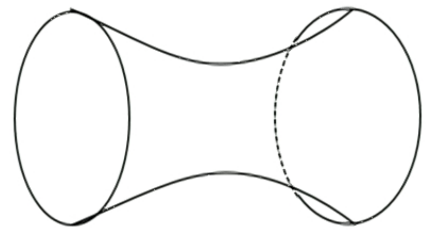
- Modified RT formula (for large multiplicity n_i)
- $S_A = \log D\,|\gamma_A| + S_{gauge}(E_A)$



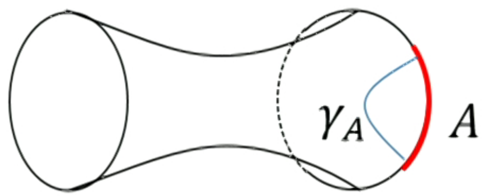
- For discrete gauge theory in deconfined phase,
  $S_{gauge}(C) = a|\partial C| - S_{topo}$, for bulk region C
  $S_{topo} = \log D_g$.
- D_g is the total quantum dimension.
- What is the consequence of S_{topo} on the boundary?
- Consider situation with topology change of γ_A.

Gauge field contribution to entanglement

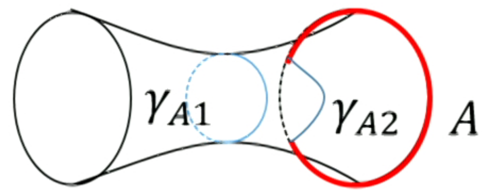
- Consider a thermal double state for *symmetric* states
$$|TFD\rangle = \sum_n' e^{-\frac{\beta E_n}{2}} |n\rangle_L |n\rangle_R.$$



- Dual to an eternal black hole.

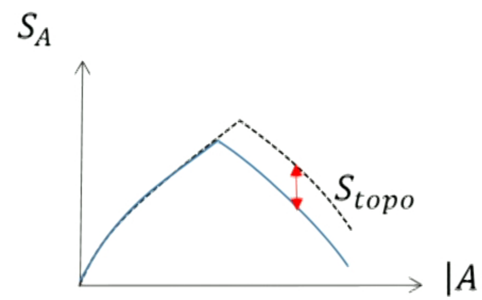


small region



large region

- Entropy reduction
- Physical interpretation: Only large region knows the whole state is symmetric (with zero charge).



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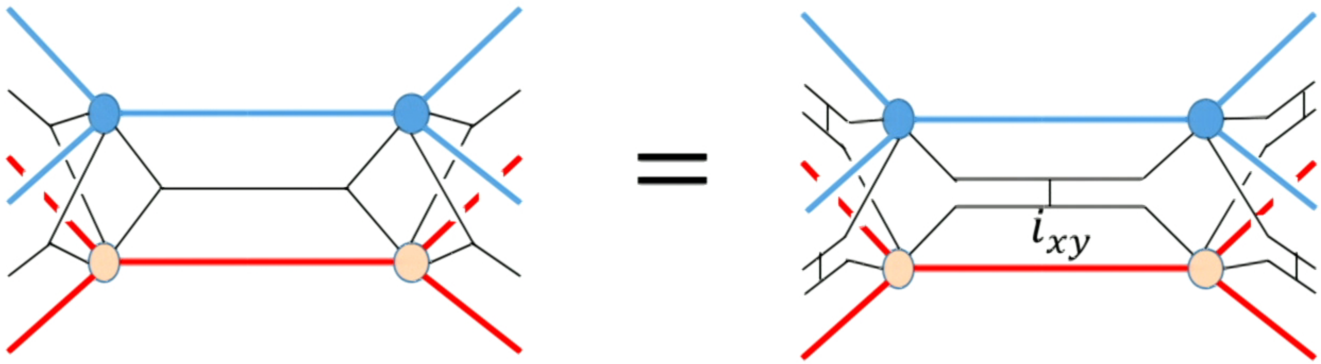
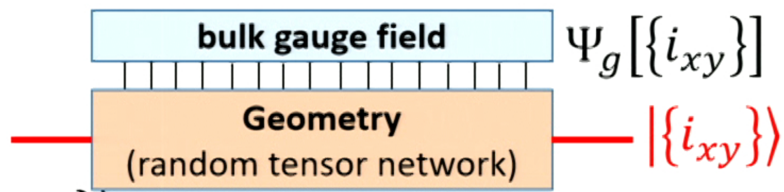
Bulk gauge field



- Contraction of α, β, γ ... leads to a bilayer structure of wavefunction

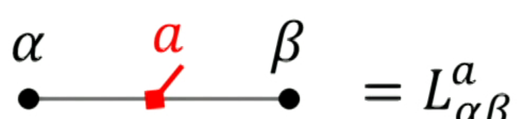


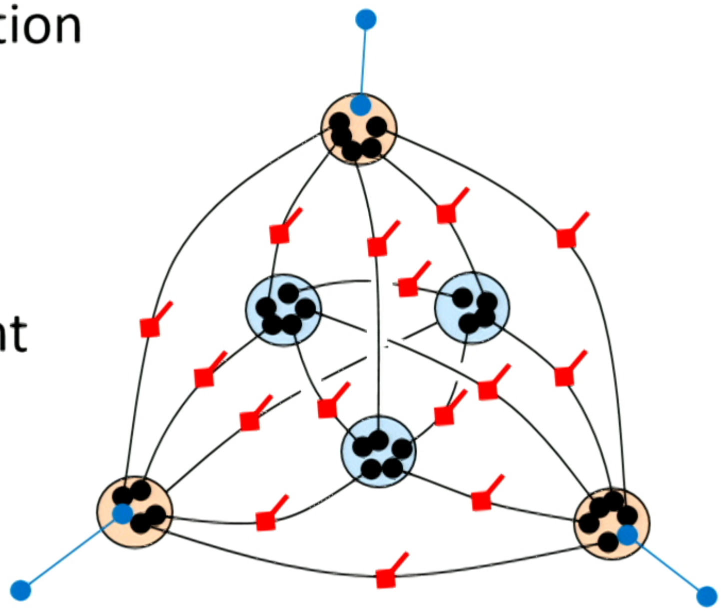
- $|\Psi_\partial\rangle = \sum_{\{i_{xy}\}} \Psi_g[\{i_{xy}\}]|\{i_{xy}\}\rangle$
- A bulk gauge field with wavefunction $\Psi_g[\{i_{xy}\}]$ (Levin-Wen '05)
- Maybe confined or deconfined, controlled by f_{ijk}

Cui, XLQ, Yang, to appear '17

New direction II: fluctuating geometries

- RTN represent ``ansatz states" with various holographic properties
- To describe quantum gravity, we need to allow superposition of geometries
- RTN with geometry fluctuation can be defined by considering link qudits
- $|a\rangle = L^a_{\alpha\beta}|\alpha\rangle|\beta\rangle$
- a controls the entanglement of this link

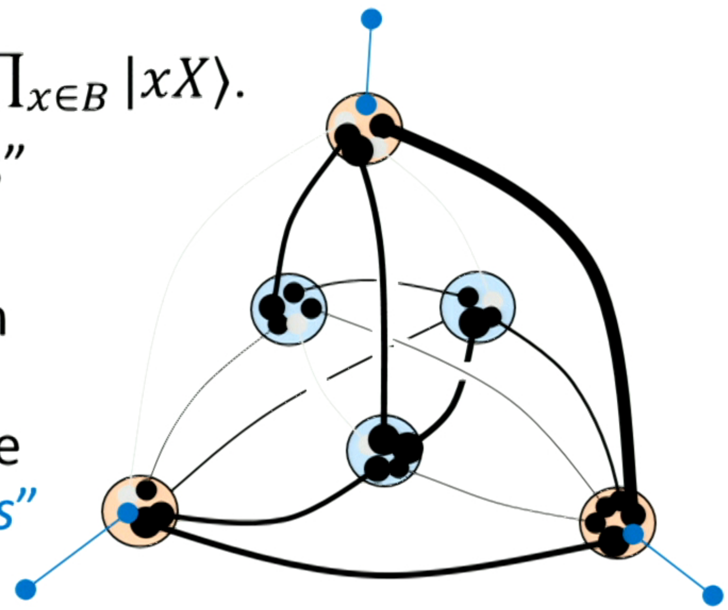




XLQ, Yang, You arxiv: 1703.06533

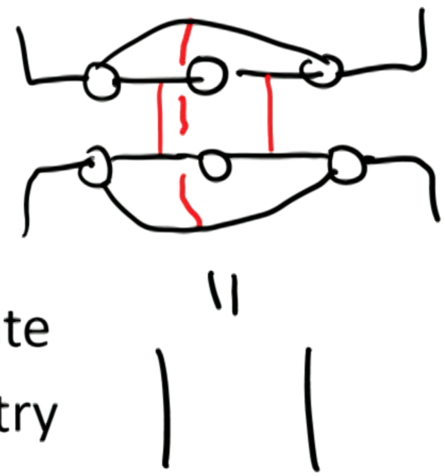
Superposition of geometries

- $S_a \propto a$ increases with a.
- $\langle a|b\rangle = \delta_{ab}$. $|a=0\rangle = |0\rangle|0\rangle$ corresponds to a disconnected link
- Random tensors map each weighted graph a_{xy} to a boundary state $|\Psi[\mathrm{a}]\rangle = \prod_x \langle V_x| \prod_{xy} |a_{xy}\rangle \prod_{x\in B} |xX\rangle$.
- $|\Psi[a]\rangle$ are ``geometry states" satisfying RT formula.
- Question: Do $|\Psi[a]\rangle$ form an (over-)complete basis?
  Short answer: Yes. $|\Psi[a]\rangle$ are *``holographic coherent states"*



Boundary-to-bulk isometry

- With enough number of bulk vertices, $|\Psi[a]\rangle$ is an overcomplete basis satisfying $\sum_a |\Psi[a]\rangle\langle\Psi[a]| = \mathbb{I}$
- Boundary-to-bulk isometry
- To prove this, define $\underline{\rho_B = \sum_a} |\Psi[a]\rangle\langle\Psi[a]|$, and calculate $tr\left(\rho_B^2\right)$. Maximal entropy ⇒ Isometry
- The random average maps to Ising model on the complete graph
- $\mathcal{A} = \underbrace{-\frac{J}{2}\sum_{xy} s_x s_y}_{\boxed{J = s_x - \frac{1}{2}\log D_L}} \underbrace{-\frac{h}{2}\sum_x s_x}_{\boxed{h = \frac{V-1}{2}\log D_L}} + \underbrace{\frac{1}{2}\log D \sum_{x\in B} s_x}_{\boxed{\text{Boundary pinning field}}}.$

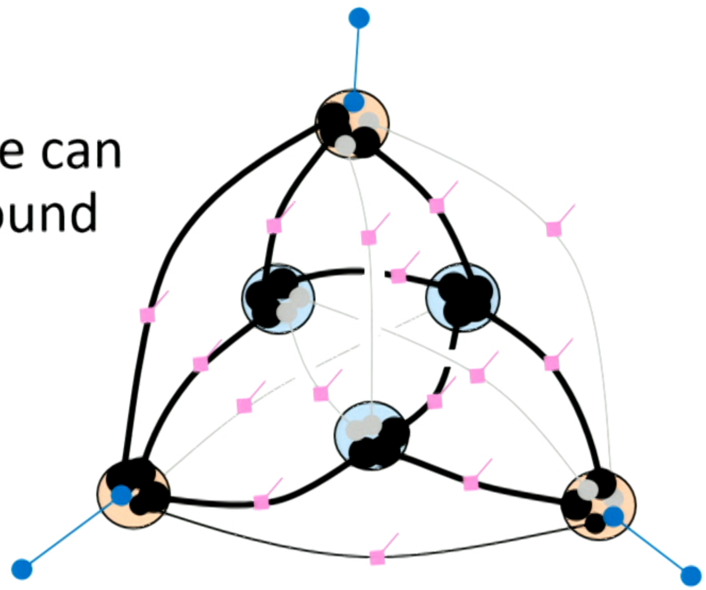


Boundary-to-bulk isometry

- Isometry condition: bulk pinning field wins.
- Minimal energy configuration "all up" $s_x = +1$
- Sufficient condition:
- $\frac{V-1}{2}\log D_L \gg \log D$
- $\left|tr\left[\rho_{xy}^{\otimes 4} g \otimes h\right]\right|^2 < tr\left[\rho_{xy}^{\otimes 4} g \otimes g\right] tr\left[\rho_{xy}^{\otimes 4} h \otimes h\right]$
  (for all $g \neq h$ elements of permutation group S_4)

- 4-th order permutations appear in order to bound the fluctuation $\overline{tr\left(\rho_B^2\right)^2} - \overline{tr\left(\rho_B^2\right)}^2$

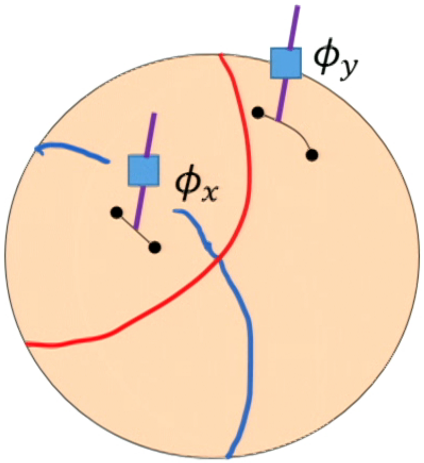
Small fluctuations



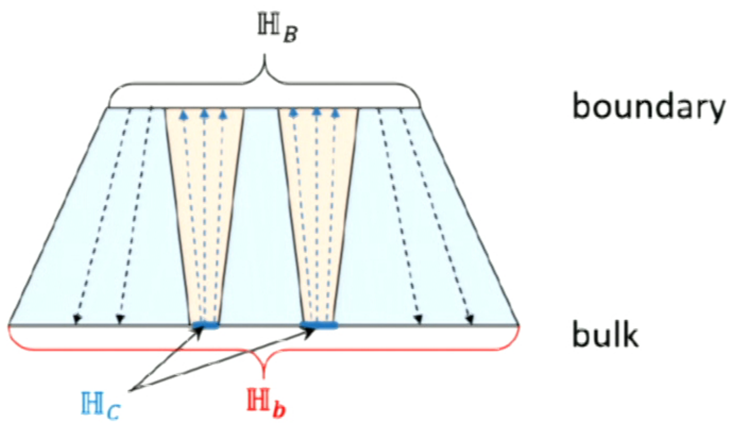
- If we take D_L to be large, we can define small fluctuation around a classical geometry.
- $|\Psi[a_0 + \delta a]\rangle$
- $|\delta a| \leq \Lambda$
- In the limit $D_L \to \infty$, finite Λ, there is an isometry from bulk to boundary.
- $\sigma_B = \sum_{\delta a} |\Psi[a_0 + \delta a]\rangle\langle\Psi[a_0 + \delta a\,]|$
- Isometry condition

$$tr\left(\sigma_B^2\right) = \frac{1}{D_{bulk}} = (2\Lambda + 1)^{-\frac{V(V-1)}{2}}.$$

Small fluctuations

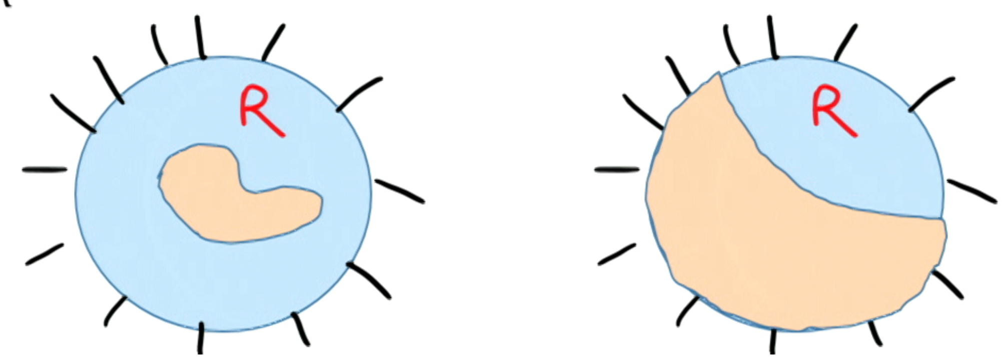


- The small fluctuations form a "code subspace" $\mathbb{H}_a$.
- Emergent bulk locality in the code subspace.
  ϕ_x is actually $P_a \phi_x P_a$ only acting on $\mathbb{H}_a$.
- Bidirectional holographic mapping:
  - boundary $\rightarrow$ bulk in entire Hilbert space
  - bulk$\rightarrow$ boundary in code subspaces
- Local reconstruction



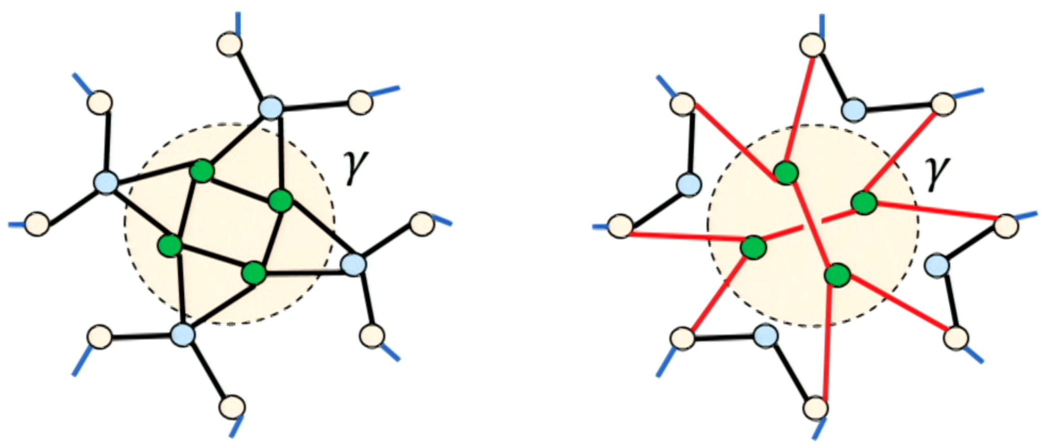
Classical geometries

- Overcompleteness $\Rightarrow$ boundary state $|\Phi\rangle = \sum_a \langle\Psi[a]|\Phi\rangle|\Psi[a]\rangle$ can be expanded in geometries
- However, it will be bad if the basis is "too overcomplete", s.t. significantly different geometries have a big overlap.
- $C_{ab} = \langle\Psi[a]|\Psi[b]\rangle$
- $\overline{|C_{ab}|^2} = \sum_{R\subseteq bulk} tr\left(\rho_R^a \rho_R^b\right) D^{|R\cap B|-V_B}$
  $\leq \sum_{R,(a=b)_R} e^{-\frac{1}{2}\left(S_a^{(2)}(\overline{R})+S_b^{(2)}(\overline{R})\right)}.$



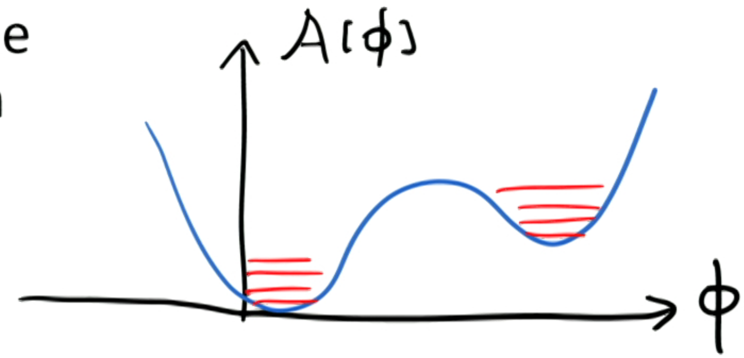
Classical geometries

- Example:
- $a \neq b$ for all links $\Rightarrow \overline{|C_{ab}|^2} = \mathrm{D}^{-\mathrm{V_B}}$
- For generic regions, Large D limit $\overline{|C_{ab}|^2} \leq e^{-\frac{1}{2}(|\gamma|_a + |\gamma|_b)}$
- γ: minimal area surface enclosing the region $a \neq b$.
- Inner product suppress exponentially with **area law.**



Comparison with boson coherent states

- Boson coherent state of a superfluid $|\phi(x)\rangle = e^{\int d^d x \phi(x) b^+(x)}|0\rangle$
- Overcomplete basis $\int D\phi|\phi\rangle\langle\phi| = \mathbb{I}$
- Overlap $|\langle\phi|\phi'\rangle| = \exp\left(-\int d^d x|\phi(x) - \phi'(x)|^2\right) \sim e^{-V}$ for distinct states
- Small fluctuations around a state $\phi(x)$ are subspaces of independent excitations (Goldstone modes)
- Key difference: volume law versus area law in overlap. $\langle a|b\rangle \sim e^{-A}$



Summary and open questions

- Random tensor networks encode entanglement properties geometrically
- Global symmetry is mapped to gauge symmetry in the bulk
- Random tensor networks form a basis of "holographic coherent states". A generic boundary states is mapped to a superposition of geometries $\sum_a \phi_a |\Psi[a]\rangle$
- Overlap between different geometries suppressed by exp of area law.
- Bulk to boundary isometries in code subspace of small fluctuations
- Open question:
  - Continuous symmetry. Gapless photons?
  - Optimization of geometry for a given boundary state. - Comparison with GR calculation? (e.g. D. Jafferis 1703.01519)
  - Einstein equation from boundary dynamics?