Title: Tensor networks and Legendre transforms

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Abstract: Tensor networks have primarily, thought not exclusively, been used to the describe quantum states of lattice models where there is some inherent discreteness in the system. This raises issues when trying to describe quantum field theories using tensor networks, since the field theory is continuous (or at least the regulator should not play a central role). I'll present some work in progress studying tensor networks designed to directly compute correlation functions instead of the full state. Here the discreteness arises from our choice of where and how to probe the field theory. This approach is roughly analogous to studying a Legendre transform of the state. I'll discuss the properties of such networks and show how to construct them in some cases of interest, including non-interacting fermion field theories. Partly based on work with Volkher Scholz and Michael Walter.

Tensor Networks for Correlation Functions

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w/ Volkher Scholz and Michael Walter

Harvard

Stanford → MIT → UMD

Brandeis

It from Qubit

Simons Collaboration on
Quantum Fields, Gravity and Information

Overview

- Conceptually different approach to what it means to describe a quantum field theory with a tensor network
- Tensor networks for correlation functions (instead of for the “bare” quantum state)
- Various examples and properties
- Main result: A rigorous construction of a MERA-like network for 1+1 Dirac field with provable error bounds (use wavelets, c.f. Evenbly-White)

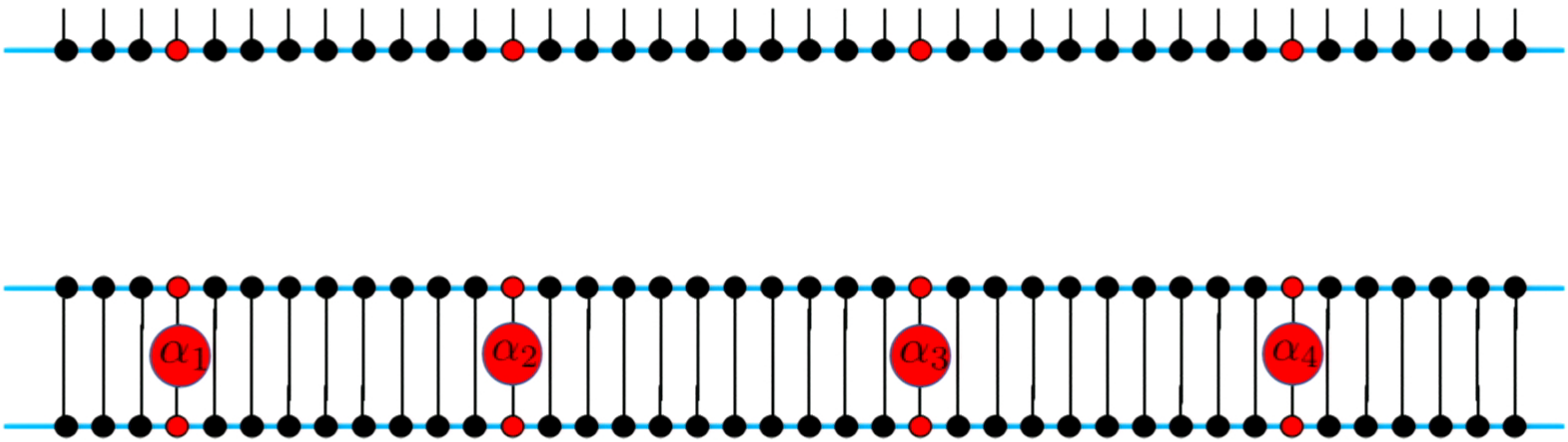
Tensor network states for lattice models

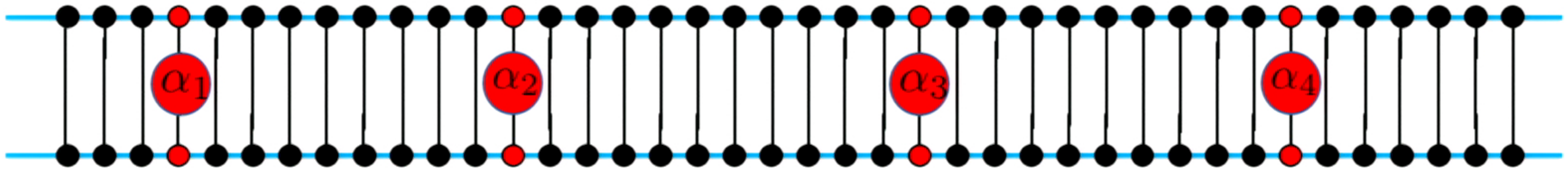
Fine grained lattice model:

Fine grained tensor network:

Perhaps we are interested only in marked points:

Tensor networks for correlation functions




α_1
α_2
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α_1
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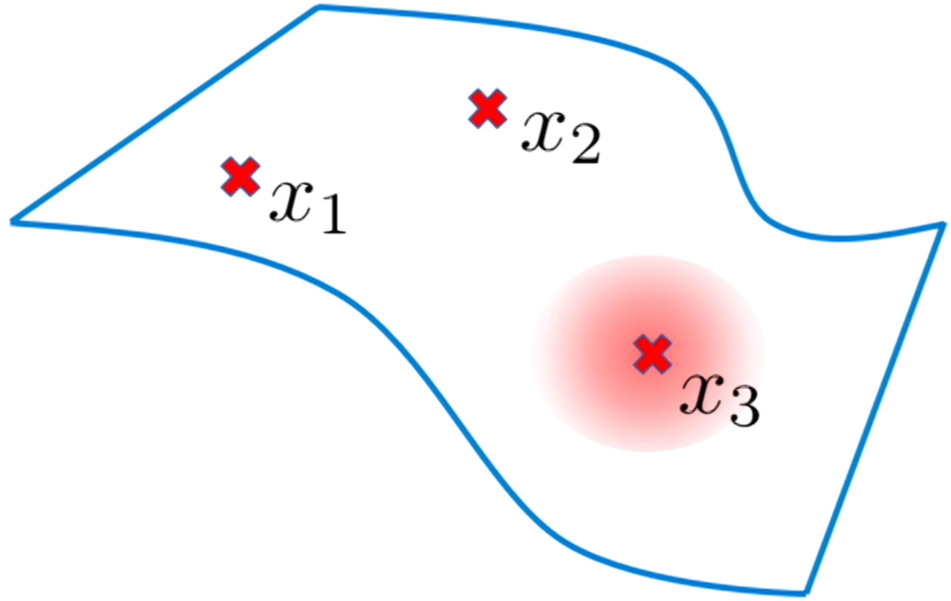
General definition



Given: ρ state

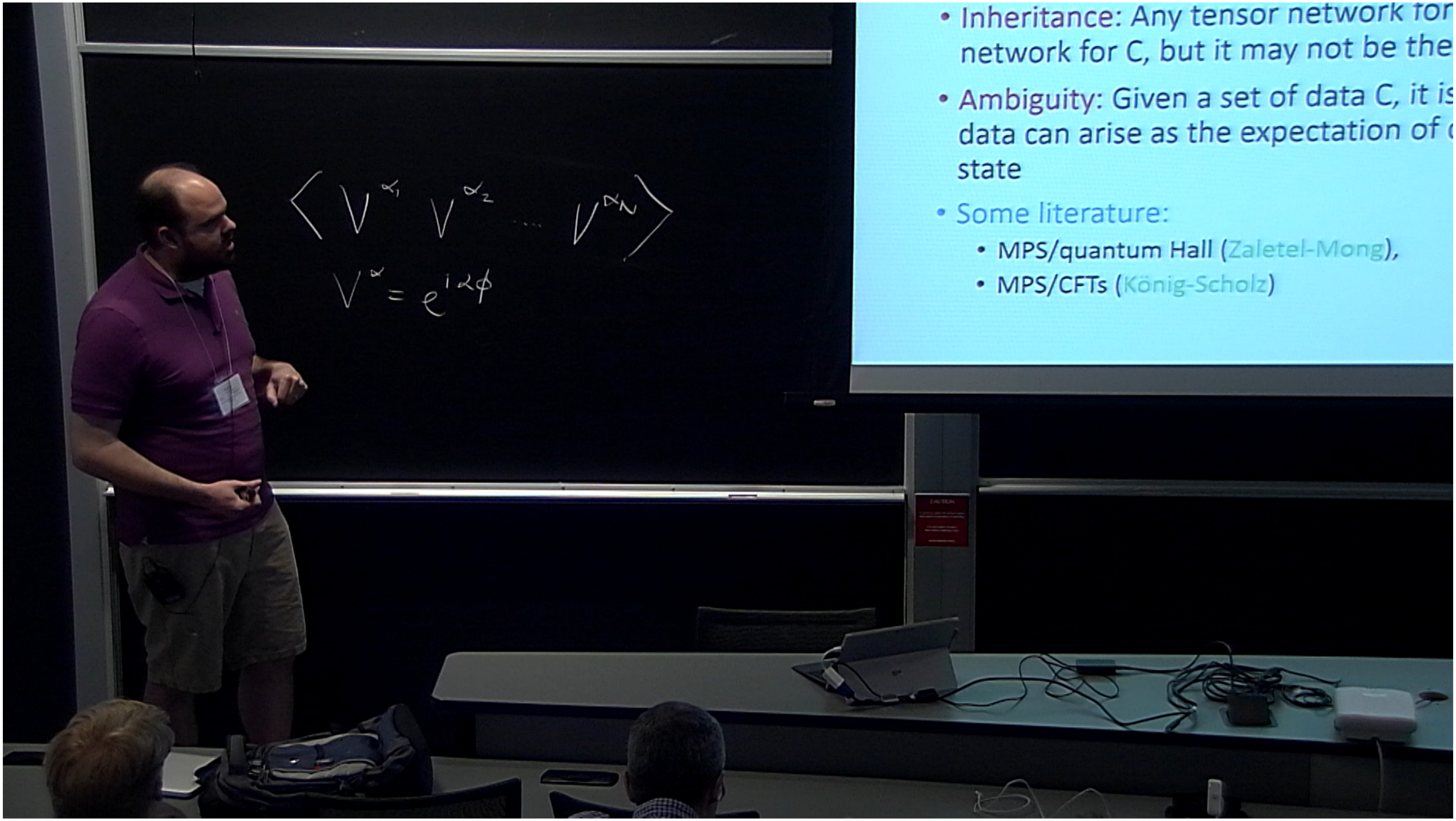
$\{O_{x_i,\alpha_i}\}$ operator choice

$$C(\alpha_1, ..., \alpha_n) = \mathrm{Tr}(\rho\, O_{x_1,\alpha_1} ... O_{x_n,\alpha_n})$$

Underlying system can be continuous; discreteness is imposed in our choice of how to probe the system

Comments

- Completeness: In a lattice model, given a sufficiently dense and complete set of operators, the object C is equivalent to the state
- Inheritance: Any tensor network for the state also gives a tensor network for C, but it may not be the most efficient network
- Ambiguity: Given a set of data C, it is not always the case that the data can arise as the expectation of operators in a positive quantum state
- Some literature:
  - MPS/quantum Hall (Zaletel-Mong),
  - MPS/CFTs (König-Scholz)



$\langle V^{\alpha_1} V^{\alpha_2} \cdots V^{\alpha_N} \rangle$
$V^{\alpha} = e^{i\alpha\phi}$
• Inheritance: Any tensor network for
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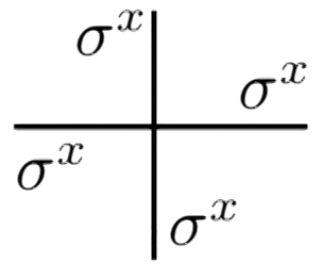
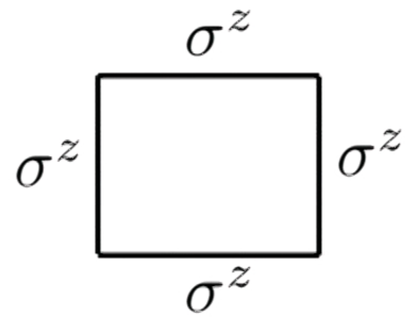
Example 1: Topological order

$$H = -\sum_v \prod_{\ell \in v} \sigma_\ell^x - \sum_p \prod_{\ell \in p} \sigma_\ell^z$$





ρ toric code ground state

$\{O_{x_i,\alpha_i}\}$ any operators separated all separated by multiple lattice constants

$$C(\alpha_1, ..., \alpha_n) = C_1(\alpha_1)...C_n(\alpha_n)$$

Example 2: Bosonic free field (d+1)

$$I = \int \left[\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2\right]$$

ρ ground state $\qquad O_{x_i,\alpha_i} = e^{i\alpha_i\phi(x_i)}$ vertex operators

$$C(\alpha_1, ..., \alpha_n) = \exp\left[-\frac{1}{2}\sum_{i,j}\alpha_i\alpha_j G(x_i - x_j)\right]$$

G is the 2-point function; if G is short ranged then C is approximately a PEPS

Example 3: Quench dynamics

$$\rho(t) = e^{-iHt}\rho_0 e^{iHt}$$

H is a good chaotic Hamiltonian

$O_{x,\alpha} =$ hydrodynamic variables, e.g. energy density, currents

$$C(\alpha_1, ..., \alpha_n) \begin{cases} \text{Simple at early times (assumption)} \\ \text{Simple at late times (thermalization)} \\ \text{Simple for all times?} \end{cases}$$

Example 4: Fermionic free field (1+1)

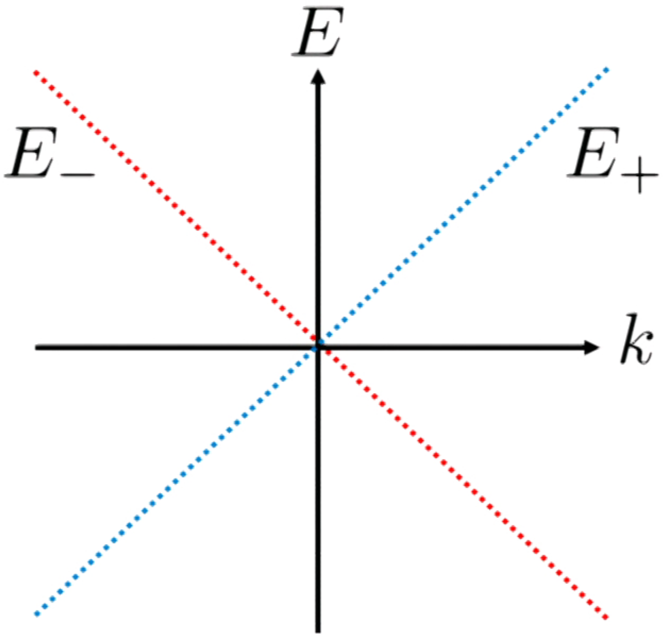
$$I = \int \bar{\psi} i \gamma^{\mu} \partial_{\mu} \psi$$

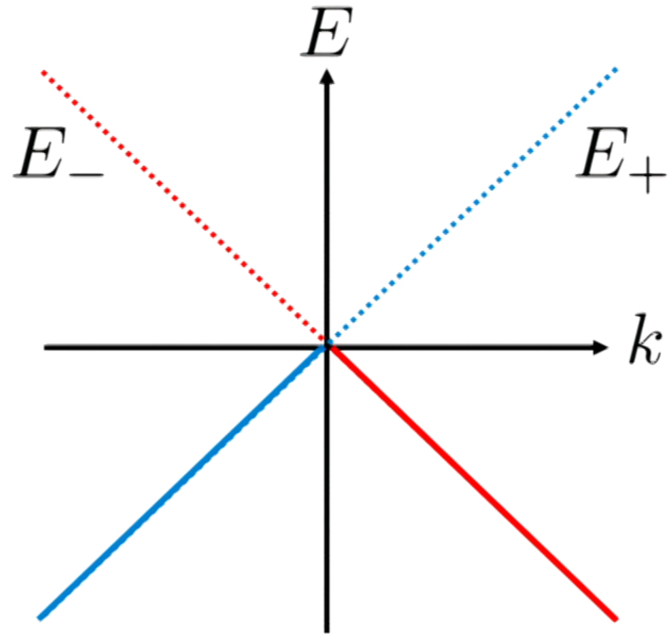
$$\{\gamma_{\mu}, \gamma_{\nu}\} = 2g_{\mu\nu}$$
$$\gamma_0 = \sigma^x, \; \gamma_1 = i\sigma^y$$

$$\begin{pmatrix} i\partial_t + i\partial_x & 0 \\ 0 & i\partial_t - i\partial_x \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = 0$$

massless!

Filled Fermi sea





Ground state

Single particle

$$\Theta(e^{ikx}) = e^{ikx}\theta(k)$$

$$\psi_+ : 1 - \Theta$$
$$\psi_- : \Theta$$

ground state projectors

Many particle

$$\psi_+(k > 0)|\Omega\rangle = 0$$
$$\psi_+^\dagger(k < 0)|\Omega\rangle = 0$$
$$\psi_-(k < 0)|\Omega\rangle = 0$$
$$\psi_-^\dagger(k > 0)|\Omega\rangle = 0$$

Many particle “state” and creation/annihilation operators

Eigenstates

$$i\partial_t \to E, \; -i\partial_x \to k$$

$$\psi_+ : E - k = 0$$
$$\psi_- : E + k = 0$$

Infinite line:

$$k \in \mathbb{R}$$

Circle:

$$k \in \frac{2\pi}{L}\mathbb{Z}$$

Approximating smooth correlation functions?

$$\psi_+[f] = \int dx\, f(x)\psi_+(x)$$

$$\langle \psi_+[f]\psi_+^\dagger[g]\rangle = (f, \Theta g)$$

$$\langle \psi_+(x)\psi_+^\dagger(y)\rangle_\Omega = \int \frac{dk}{2\pi} e^{ik(x-y)}\theta(k)$$

f and g = delta functions

Suppose f and g are smooth functions; can we approximate the ground state correlation function using an approximation of the projector?

Crucial: We will never get all the fine grained information, but we can get correlators of smoothed and separated operators!

Multiresolution analysis (MRA)

$$V_0 \subset V_1 \subset V_2 \ldots$$

1. Union of spaces is dense
2. Spaces generated by scaling and translating a single function

Scaling function (father wavelet): $s(x)$

$$V_j = \mathrm{span}\{s_{j,a}(x)\}_{a=1,\ldots,2^j} \quad s_{j,a}(x) = s(2^j x - a2^{-j}L)$$

Wavelet function (mother wavelet): $w(x)$

$$V_j = V_{j-1} \oplus W_j \qquad w_{j,a}(x) = w(2^j x - a2^{-j}L)$$

generates the wavelet space

Approximating smooth correlation functions

- Smooth functions can be well approximated using MRA tools

$$\left\| \sum_a f(a2^{-j}L)s_{j,a} - f \right\|_2 \sim O(2^{-j}) \times \text{Sobolev norm}(f)$$

- Hence, if we can approximate the ground state projector on MRA spaces, then we can approximate smooth correlation functions
- Roughly speaking, one approximates the continuum with a discretuum with a finite dimensional Hilbert space whose correlators approximate those of the continuum system

Approximating the projector

Goal 1: Find U $$U^\dagger \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U = \begin{pmatrix} \Theta & 0 \\ 0 & 1-\Theta \end{pmatrix}$$

Goal 2: Approximate U $$U \approx U_M$$

approximation should be local; can be used to build a MERA-like circuit

Proceed via the Hilbert transform:

$$\mathcal{H}(e^{ikx}) = i\,\mathrm{sign}(k)e^{ikx} \qquad \Theta = \frac{1-i\mathcal{H}}{2}$$

Hilbert pairs

- Suppose we had a pair of wavelets that were Hilbert transform pairs

$$w^{(2)} = \mathcal{H}(w^{(1)})$$

- Discrete wavelet transform

$$\mathcal{D}^{(i)} : V_j \to V_0 \oplus W_1^{(i)} \oplus \ldots \oplus W_j^{(i)}$$

- Idea: $\mathcal{H} = (\mathcal{D}^{(2)})^\dagger \mathcal{D}^{(1)}$

Calculation

$$U = \frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} \mathcal{D}^{(1)} & 0 \\ 0 & \mathcal{D}^{(2)} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\mathcal{H} = (\mathcal{D}^{(2)})^\dagger \mathcal{D}^{(1)}$$
$$\mathcal{H} = -(\mathcal{D}^{(1)})^\dagger \mathcal{D}^{(2)}$$

$$U^\dagger \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & i\mathcal{H} \\ i\mathcal{H} & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

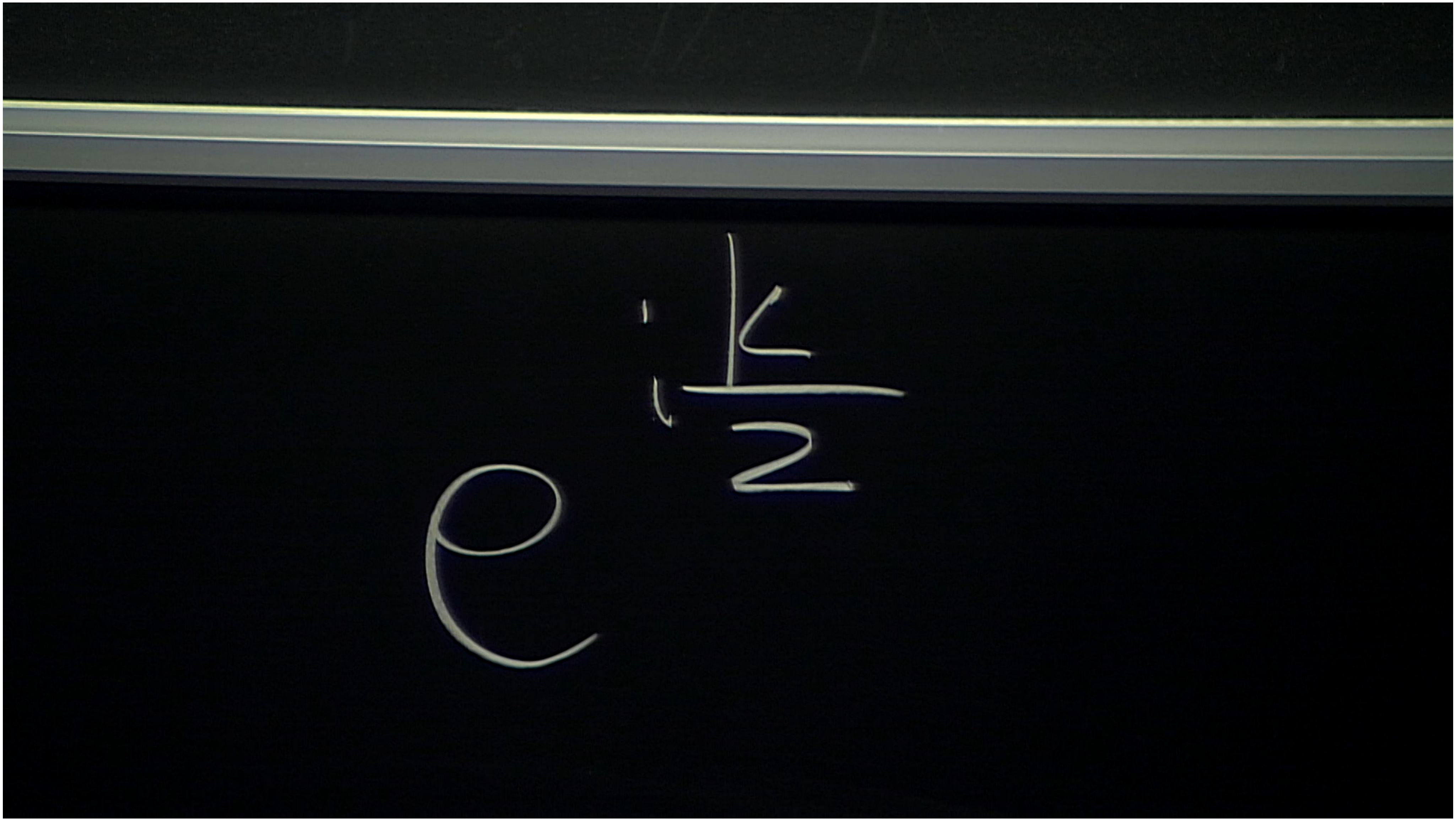
$$\boxed{U^\dagger \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U = \frac{1}{2}\begin{pmatrix} 1+i\mathcal{H} & 0 \\ 0 & 1-i\mathcal{H} \end{pmatrix}}$$

Construction of wavelet transform

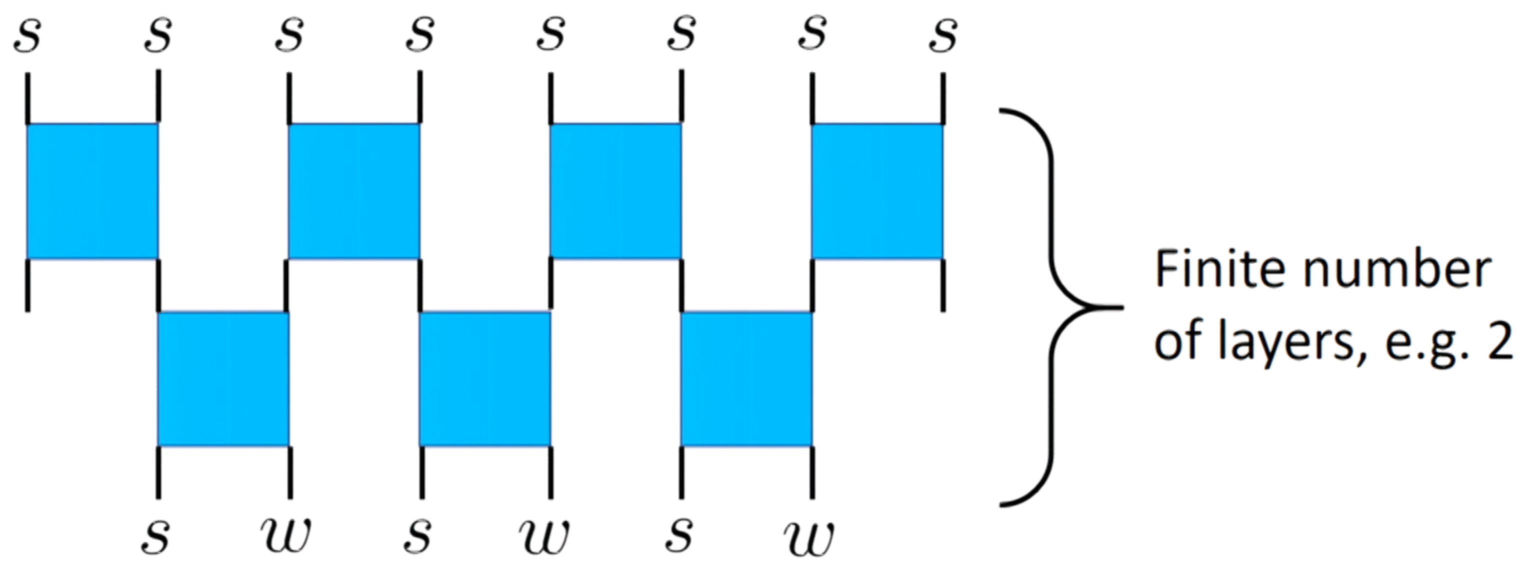
- We still need to construct an approximate Hilbert pair with locally implementable discrete wavelet transformation
- Fortunately, Selesnick and co-workers have already done this!
- Being Hilbert transform pairs translates into the corresponding filters having a "half-delay shift"

$$s(x) = \sqrt{2} \sum \mathcal{F}(n) s(2x - n) \qquad \mathcal{F}^{(2)}(n) = \mathcal{F}^{(1)}(n - 1/2)$$

- No exact solution with finite filter, but Selesnick also gave a way to design good approximations


$e^{i\frac{k}{2}}$

Result



$$
\begin{aligned}
&|\langle\Omega|\psi[f_1]...\psi[f_n]\psi^\dagger[g_1]...\psi^\dagger[g_m]|\Omega\rangle \\
&-\langle\text{circuit}|\psi[f_1]...\psi[f_n]\psi^\dagger[g_1]...\psi^\dagger[g_m]|\text{circuit}\rangle| \leq \epsilon(n,m,\text{wavelets})
\end{aligned}
$$

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Summary

- Tensor networks can (of course) be used to describe the physics of correlation functions directly
- Particularly well suited to quantum field theories, but also probably useful in other contexts
- Generalize beyond states at fixed time: operator insertions in Euclidean path integral, insertions on real time contours
- Rigorous RG network for a quantum field theory (Dirac 1+1); many generalizations are possible (lattice w/ J. Haegeman, M. Bal, J. Cotler, V. Scholz, M. Walter)