Title: Tensor network renormalization and real space Hamiltonian flows

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Abstract: We will review the topic of tensor network renormalization, relate it to real space Hamiltonian flows, and discuss the emergence of matrix product operator algebras as symmetries of the renormalization fixed points.

joint work with Matthias Bal, Michael Marien and Jutho Haegeman

Outline

- Real Space Renormalization
  - Kadanoff, the tensor renormalization group
  - Tensor Netork Renormalization, MERA
  - TNR+

- Symmetries
  - Matrix Product Operator algebras and tensor fusion category

Real Space Renormalization

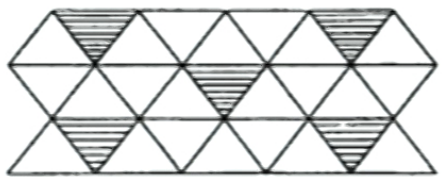
- Original ideas date back to Kadanoff, Fisher and Wilson in the 60's when studying critical phenomena in classical statistical mechanics settings
  - Consider 2D statistical mechanics model (e.g. Ising)

$$e^{-\beta F} = \sum_{s_1, s_2, \cdots} e^{-\sum_\alpha c_\alpha . H_\alpha(\{s_i\})}$$

  - We are interested in the thermodynamics or infrared physics of this model: specific heat, magnetization, critical exponents, …
  - Wilson: renormalize the spins (e.g. majority voting) and define flow of couplings $c'_\alpha = K_\alpha(c_\beta)$
  - RG fixed points are obtained when $c'_\alpha = c_\alpha$; critical exponents are then the eigenvalues of the matrix $T_{\alpha\beta} = \left.\frac{\partial K_\alpha}{\partial c_\beta}\right|_{c'=c}$

Example: Ising on Triangular lattice

- First successful real-space implementation of Wilson's RG was obtained by Niemeijer and van Leeuwen in '73 for Ising on the triangular lattice:



Decimation step: majority voting

Update of couplings is obtained by combinatorics: check how many times certain couplings appear, and only keep a certain subset of all possible interactions.
As the update rescales distances with $l = \sqrt{3}$, the eigenvalue of K should be

$$\lambda_T = l^1 = \sqrt{3}$$

$$\lambda_H = l^{15/8} = 3^{15/16}$$

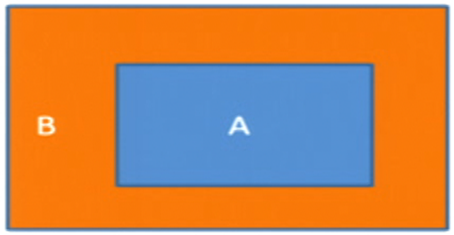
TABLE I. Values of the "thermal" and "magnetic" eigenvalues λ_T and λ_H and the value K_c for an Ising system as deduced from the fixed-point tangent plane.

Approximation	λ_T	λ_H	K_c
1st order perturbation	1.634	3.036	0.336
cluster	1.544	3.036	0.365
	1.501	2.501	0.255
	1.567	2.497	0.253
	1.782	3.186	0.281
	1.7590	2.8024	0.27416
exact (Ising)	1.73205	2.80092	0.27465

- Despite successes for simple models and several sophistications (e.g. bond-moving), such blocking methods have never worked systematically

Mutual Information Catastrophe for Kadanoff's real space RG

- Mutual information of any classical spin systems satisfies an area law for the mutual information:



$$F = \min_{\rho} \left(\mathrm{Tr}\,(\rho H) + \frac{1}{\beta} \mathrm{Tr}\,(\rho \log \rho) \right)$$

$$\Rightarrow S(A) + S(B) - S(AB) \simeq c.\beta.\delta A$$

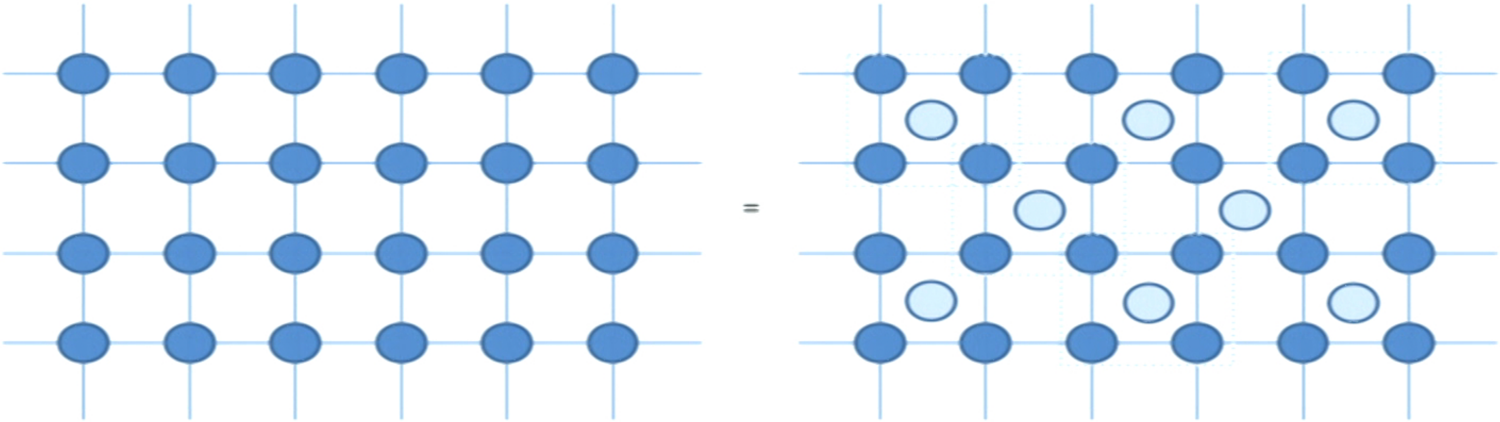
Cirac, Hastings, FV, Wolf '06

- By blocking spins and a judicious choice of A and B , the tensor product structure between the regions A, B does not change.
  - This implies that ever more terms with longer and longer interactions have to be included in the RG steps to be faithful: this is the only way to keep the "entanglement" satisfying an area law

Tensor Renormalization Group

- Way of overcoming mutual information obstacle: change the tensor product structure!

- Levin and Nave ('07): write partition function as a tensor network, and use singular value decomposition to renormalize this tensor network



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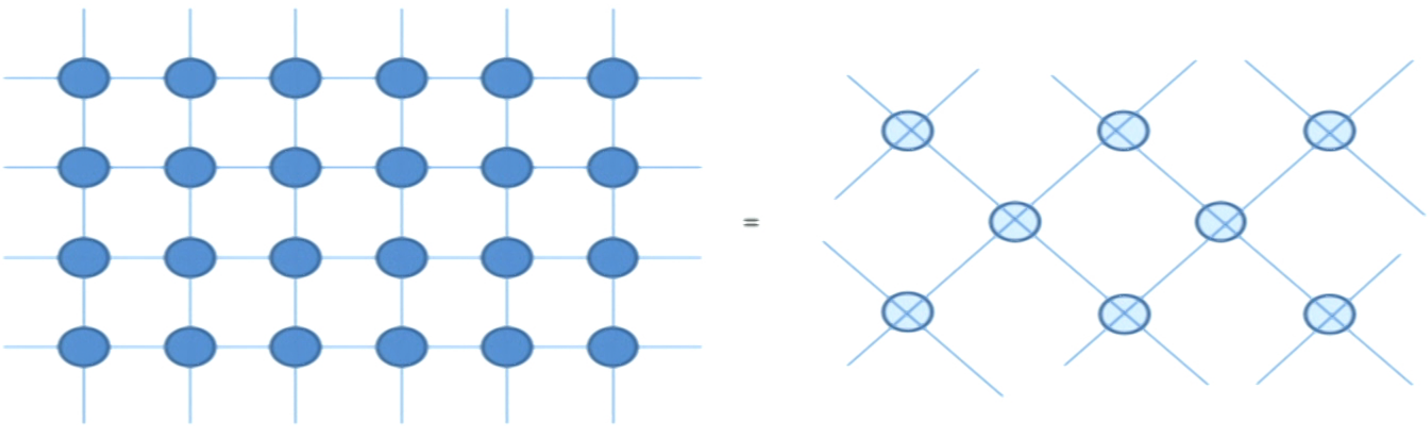
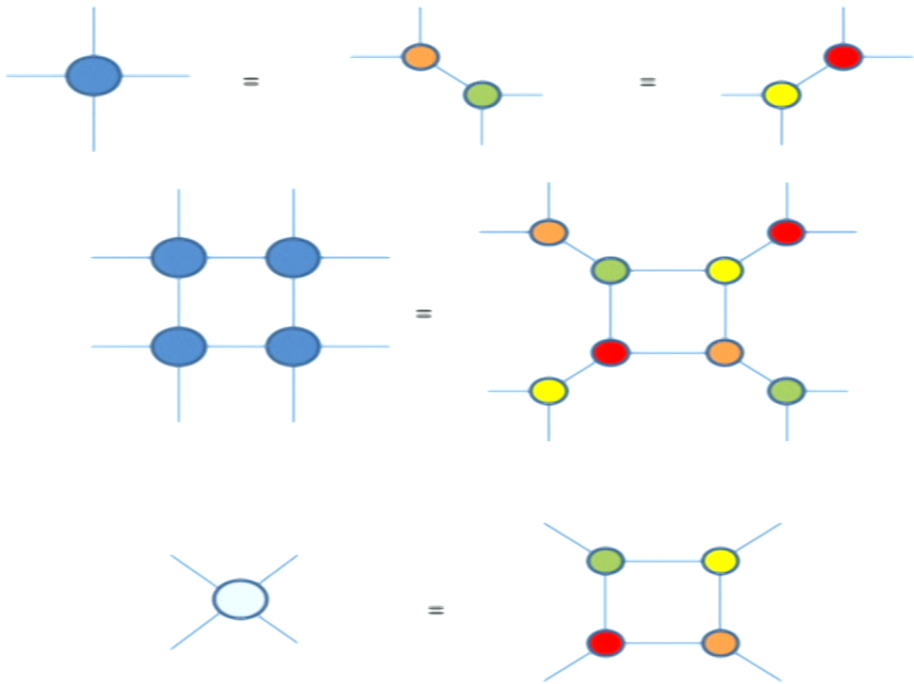
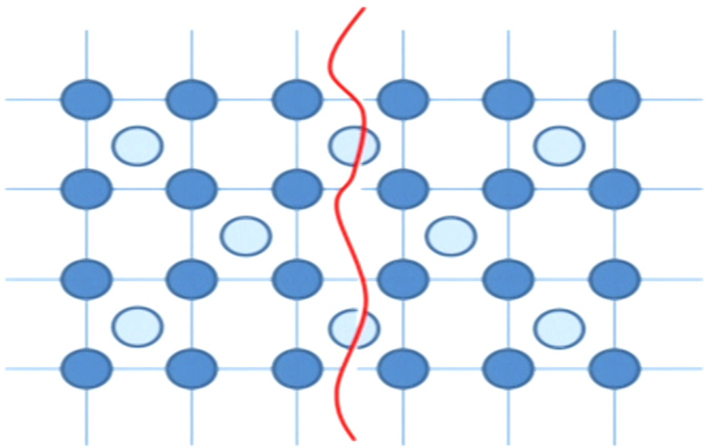


Levin and Nave's tensor renormalization group:



TRG

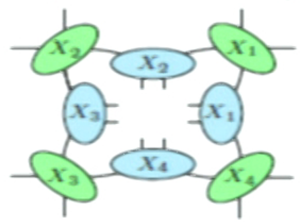
- The key novelty of Levin and Nave RG is the fact that the tensor product structure is changed:
  - Each new “site” consists of four “halves” of previous sites
  - The “entanglement” cut which satisfied the area law for mutual information does not exist anymore as the tensor product structure changed!

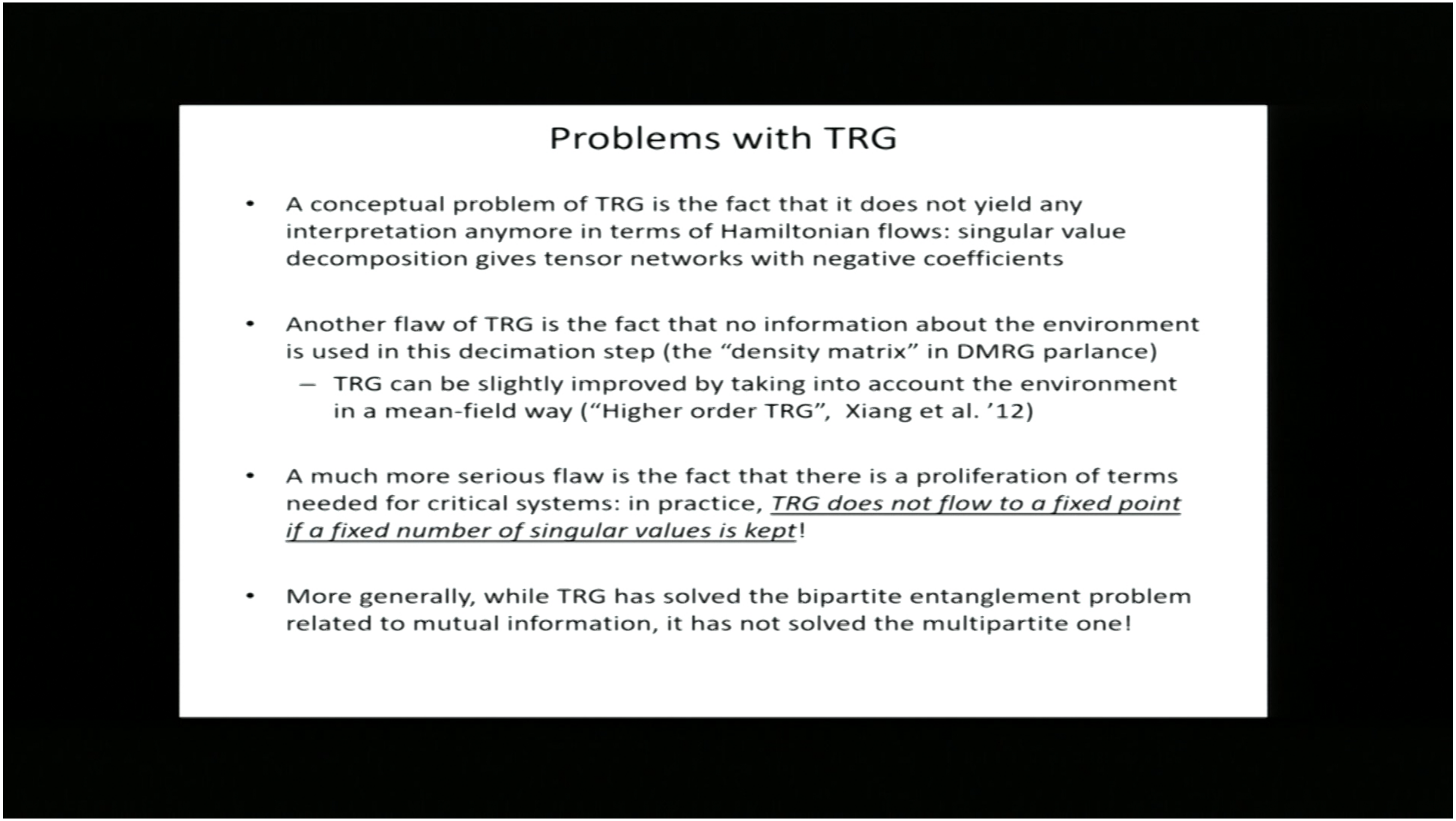


TRG

- TRG has been successfully applied to a variety of classical statistical mechanical problems (Potts model, XY-model, ...) and to the evaluation of expectation values in PEPS
  - Decimation is done by throwing away the smallest singular values
  - The number of singular values kept is the new effective dimension of the spins

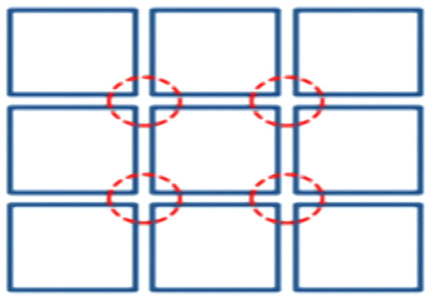
- Conceptually beautiful:
  - cost to contract translational invariant tensor network is logarithmic in size of the lattice!
  - We can start renormalizing impurity tensors, ... and *IF* we obtain fixed point tensors of this RG decimation step, we can envision conformal transformations and obtain critical exponents as eigenvalues of matrices of the form

$R =$  .


Problems with TRG
A conceptual problem of TRG is the fact that it does not yield any interpretation anymore in terms of Hamiltonian flows: singular value decomposition gives tensor networks with negative coefficients
Another flaw of TRG is the fact that no information about the environment is used in this decimation step (the "density matrix" in DMRG parlance)
– TRG can be slightly improved by taking into account the environment in a mean-field way ("Higher order TRG", Xiang et al. '12)
A much more serious flaw is the fact that there is a proliferation of terms needed for critical systems: in practice, TRG does not flow to a fixed point if a fixed number of singular values is kept!
More generally, while TRG has solved the bipartite entanglement problem related to mutual information, it has not solved the multipartite one!

Double loop tensors

- An especially interesting class of tensor networks is represented by the double loop tensor network (which Chen, Gu, Wen et al. used to construct the first nontrivial bosonic SPT phases):



As can easily be demonstrated, this tensor network is a fixed point of TRG

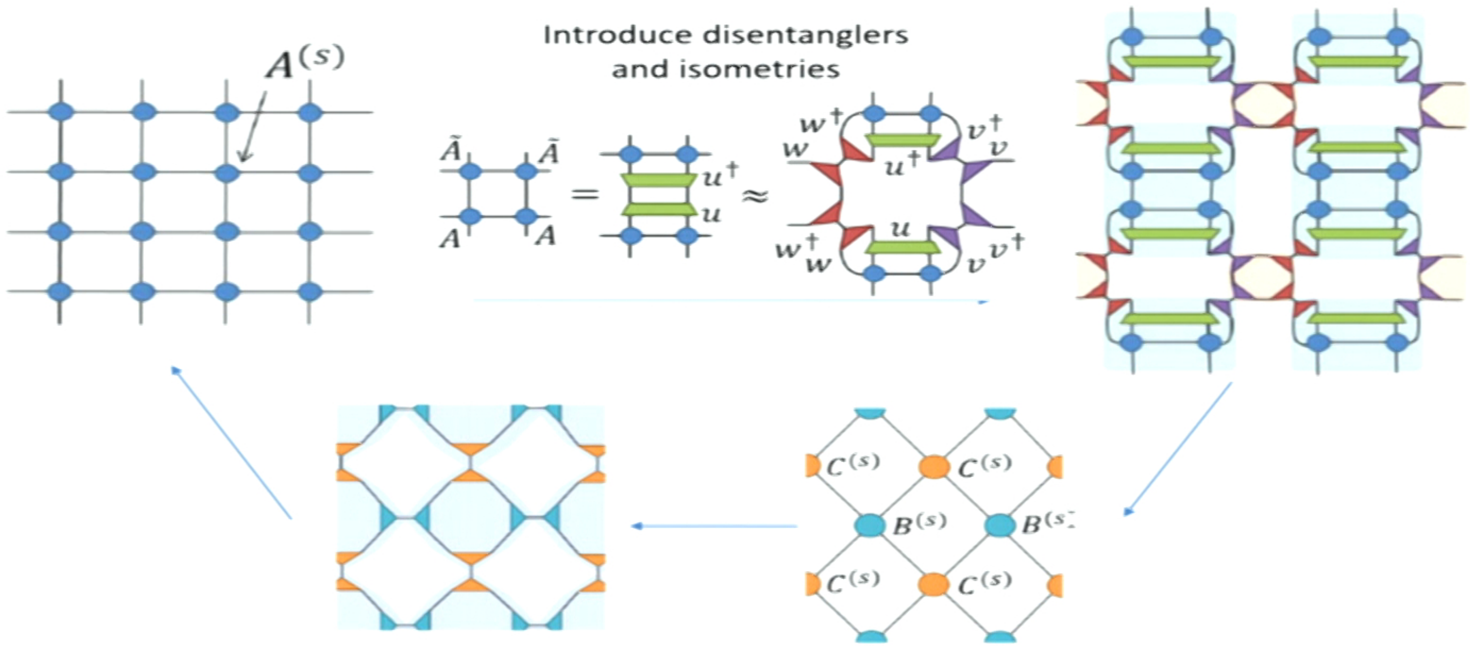
Even though all correlations are strictly “local”, TRG is not able to remove this 4-partite “entanglement”, but on the contrary promotes it to next RG scale

- This fundamental problem can be traced back to the bipartite decimation step

- Gu and Wen suggested to remove such double line entanglement using a filtering approach, but their procedure was not powerful enough to lead to fixed point tensors at criticality

Tensor Network Renormalization

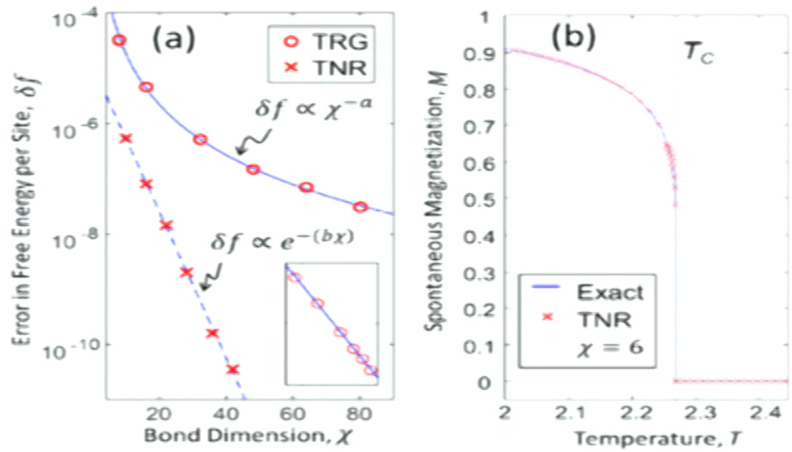
- Evenbly and Vidal (PRL '15) cured all those problems by combining the formalism of TRG with the firepower of the multiscale entanglement renormalization ansatz:



Tensor Network Renormalization

- TNR hence captures both the bipartite and some multipartite entanglement; is this enough?

- Simulations on Ising model:



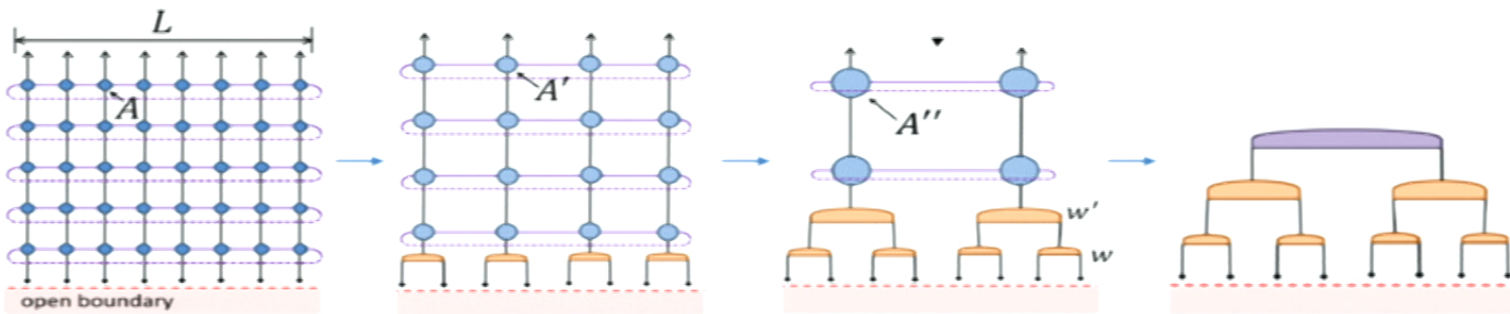
Evenbly, Vidal '15

- Most important point: TNR yields nontrivial fixed points at criticality!

Eigenvectors of transfer matrices: TRG vs TNR

- A beautiful way of distinguishing TRG and TNR is to consider the representations that they generate of the leading eigenvectors of the transfer matrices of the original lattice

- TRG yields a tree tensor network representation of the leading eigenvector:

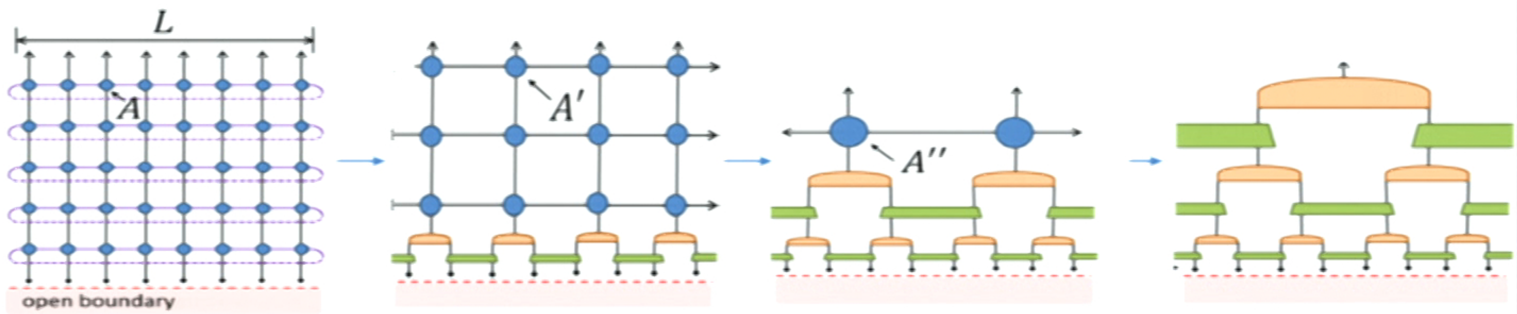


Evenbly, Vidal '16

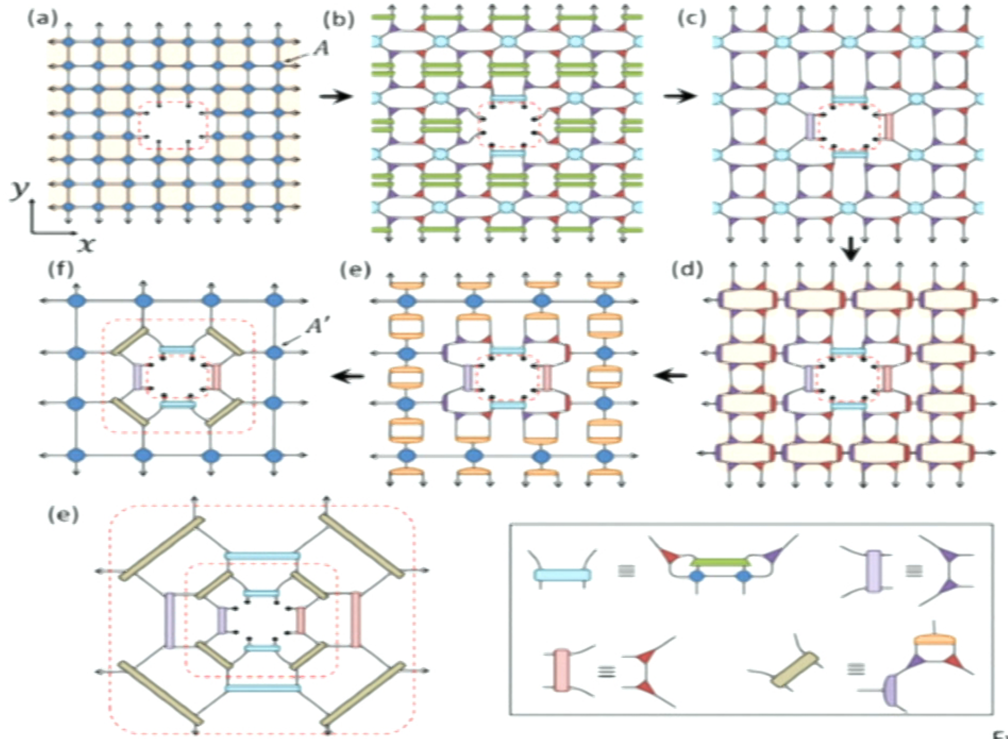
Eigenvectors of transfer matrices: TRG vs TNR

- A beautiful way of distinguishing TRG and TNR is to consider the representations that they generate of the leading eigenvectors of the transfer matrices of the original lattice

- TNR yields a much better MERA representation of the leading eigenvector:



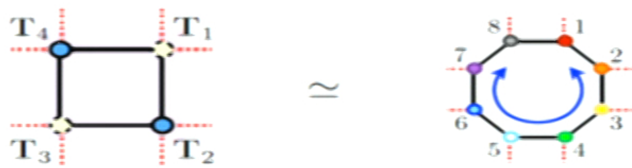
- Because TNR yields fixed points, it is possible to realize the original dream of TRG and construct conformal transformations on the lattice!



Evenbly, Vidal PRL '16

Loop Tensor Network Renormalization

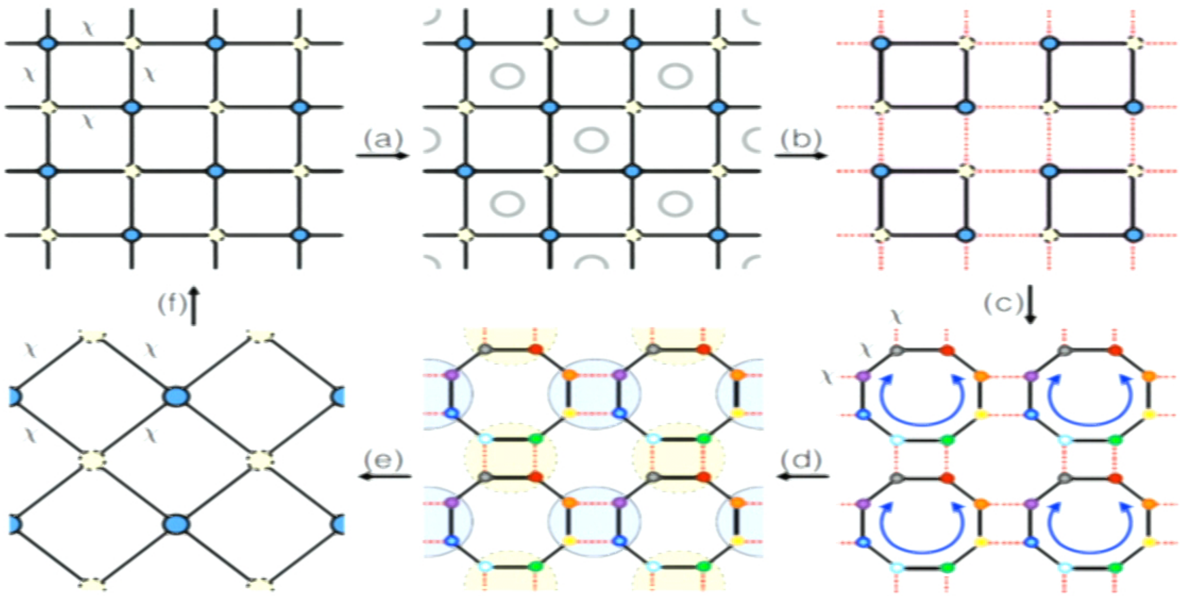
- An alternative TNR scheme has been proposed by Yang, Gu and Wen (PRL '17), which yields very similar results in practice
- The difference lies in the decimation step where a multipartite generalization of the SVD is used to approximate a ring of 4 tensors by a matrix product state with periodic boundary conditions with 8 sites



  - Such a step can be done efficiently using standard MPS algorithms (FV, Porras, Cirac '04)
  - The multipartite SVD is of course able to figure out how to deal with double line tensors

Loop Tensor Network Renormalization

Yang, Gu, Wen PRL '17



- As in TNR, this scheme yields fixed points for critical systems with a similar accuracy as the works of Evenbly and Vidal; same games can be played

Real space RG and TNR schemes

Kadanoff '14: "… the more recent tensor-style work often employs indices which are summed over hundreds of values, each representing a sum of configurations of multiple spinlike variables. All these indices are generated and picked by the computer. The analyst does not and cannot keep track of the meaning of all these variables. Therefore, even if a fixed point were generated, it would not be very meaningful to the analyst. In fact, the literature does not seem to contain much information about the values and consequences of fixed points for the new style of renormalization…"

TNR yields a flow of tensors, but it is indeed not clear how this flow is related to an RG flow of Hamiltonians as envisioned by Kadanoff and Wilson: the tensors contain negative values, and have therefore no interpretation anymore in terms of Hamiltonians (same would be true in the quantum case, where the tensors should be positive definite)

Can we devise a TNR-like scheme which works equally well, but gives rise to tensors which are elementswise positive and hence yield a fixed point Hamiltonian?

TNR_+

M. Bal et al.:arXiv:1703.00365

- TNR_+ is a TNR scheme which produces similar results as the other TNR schemes, but preserves positivity
  - The Hamiltonian obtained during the flow represents a classical system with nearest neighbour interactions (e.g. 4-point on the square lattice), and with a number of levels equal to the decimation level D
  - The resulting Hamiltonian is one with constraints (such as in the 6-vertex problem); the constraints arise due to the fact that optimization over nonnegative tensors yields a lot of zero components
  - The scheme is very stable and no tricks nor symmetries (as e.g. symmetric tensors) are needed to get it to work

Nonnegative Matrix Factorization

- Key element in all TRG/TNR schemes is the singular value decomposition or its tensor generalization. How can this be generalized such that no negative coefficients are obtained?
  - The Nonnegative Matrix Factorization (NMF) does exactly that, and is an elementswise positive generalization of the SVD:

$$A \in \mathbb{R}_+^{m \times n} \qquad X \in \mathbb{R}_+^{m \times k} \qquad Y \in \mathbb{R}_+^{k \times n}$$

$$\mathrm{argmin}_{X,Y} \, \|A - XY\|_F^2$$

  - NMF is popular in machine learning community: in many cases, we are interested in extracting features which are non-negative!
  - Problem is in principle NP-hard, but in practice very good algorithms exist

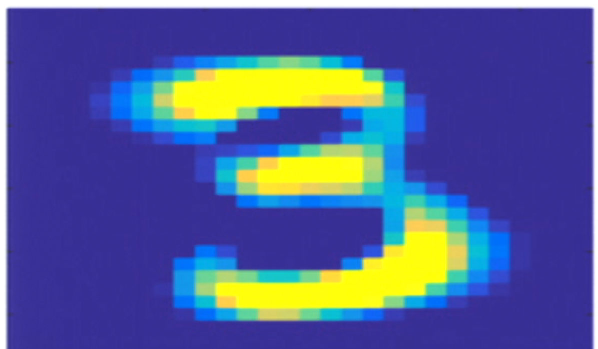
Interludum: unsupervised digit recognition using NMF

- Given a 42000x785 matrix A representing 42000 images of the digit 0..10, compress this matrix in the form

$$A \in \mathbb{R}_+^{m\times n} \qquad X \in \mathbb{R}_+^{m\times k} \qquad Y \in \mathbb{R}_+^{k\times n}$$

$$\mathrm{argmin}_{X,Y} \, \|A - XY\|_F^2$$

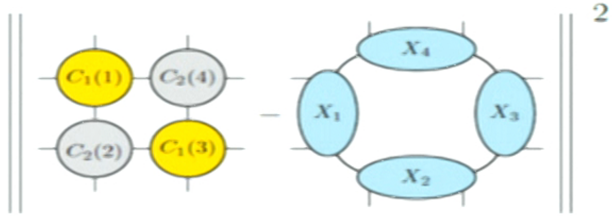
For k=10, a typical “singular vector” looks like



NMF and TNR_+

- The key technical tool in using the nonnegative matrix factorization in the TNR setting is the following optimization problem:

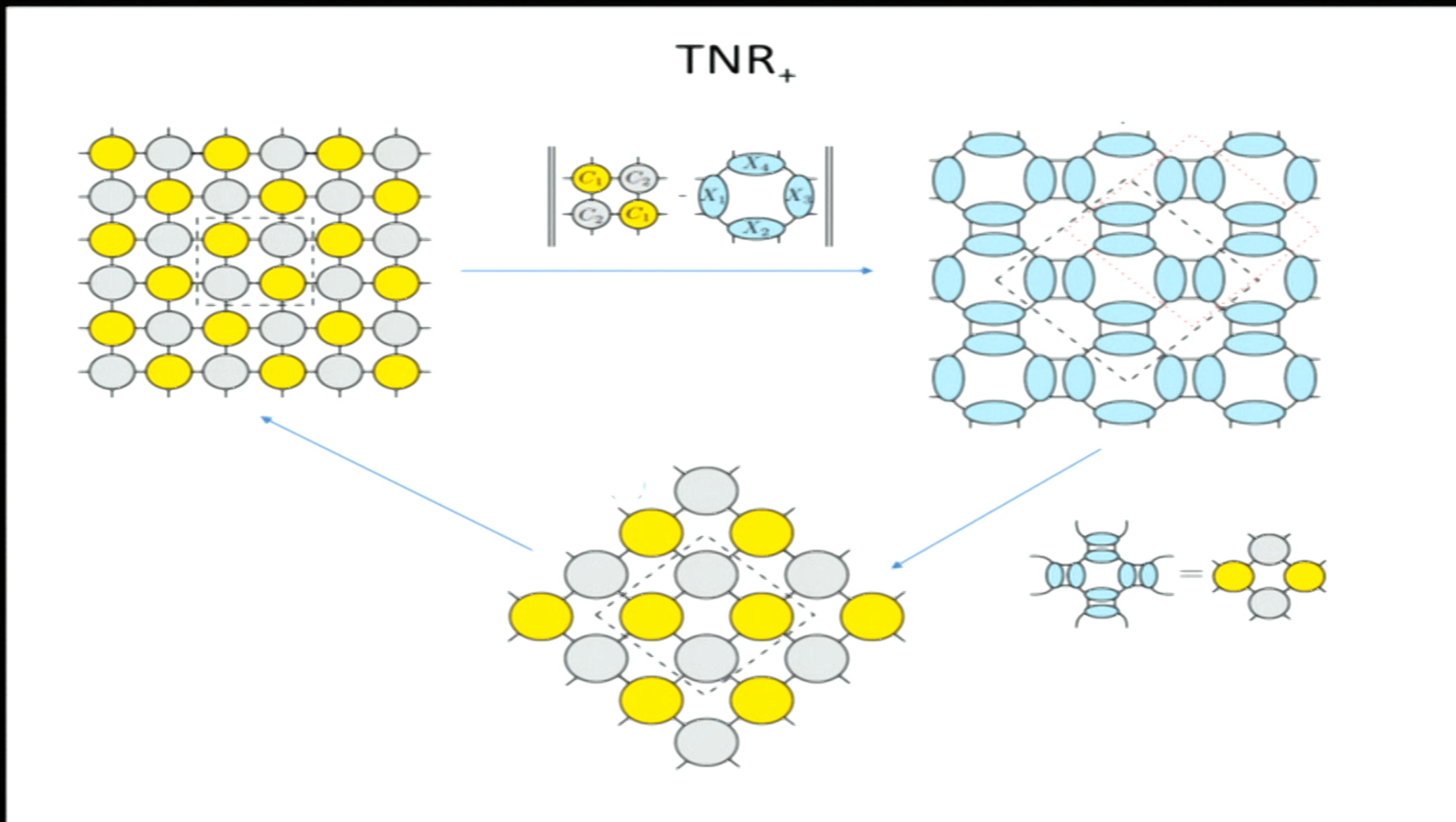
$\text{argmin}_{X_1, X_2, X_3, X_4}$



$, \quad X_1, X_2, X_3, X_4 \geq 0$

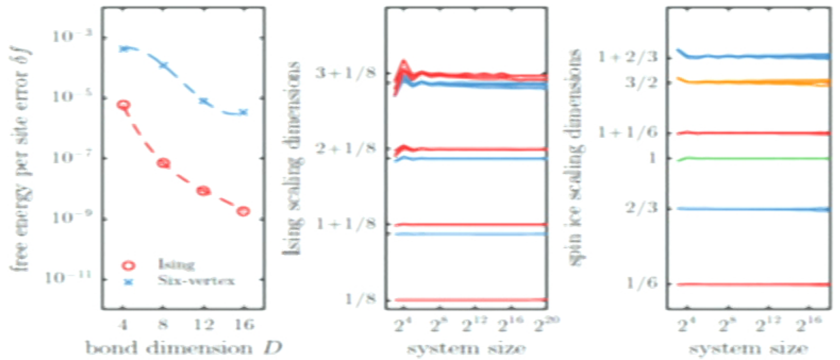
- This is equivalent to the approximation problem of a multipartite probability distribution by a stochastic matrix product state (Temme, FV, PRL '10)
  - In [Bal et al '17], we have devised efficient algorithms for dealing with this, crucially relying on gauge degrees of freedom which bring positive matrices to stochastic matrices (as opposed to isometries)

- The non-negativity constraints typically lead to sparse solutions (a lot of zeros), which is great as the zero coefficients encode constraints, while the non-zero coefficients denote coupling constants


TNR+
C1 C2
C2 C1
X4
X1
X3
X2

Numerical Results for TNR_+

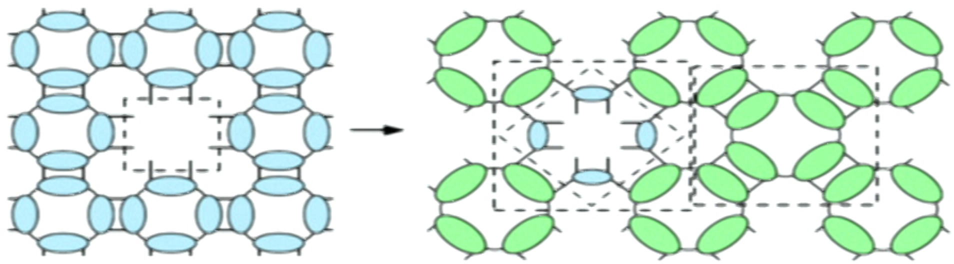
- We studied the critical Ising Hamiltonian (c=1/2) as well as spin ice (c=1) on the square lattice, and observed clear convergence to Hamiltonian fixed points.



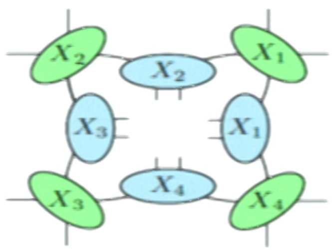
- Smallest Scaling Dimensions:

exact	Ising $\text{TNR}_+(6)$	exact	Spin ice $\text{TNR}_+(10)$
0.125	0.125236	1/6	0.163117
1	0.999282	1/6	0.167204
1.125	1.123883	2/3	0.659684
1.125	1.123883	2/3	0.662008
2	1.998575	1	0.997413
2	1.992823	1	0.997286
2	1.996882	7/6	1.163503
2	1.994090	7/6	1.163503

Critical exponents using the radial transfer matrix

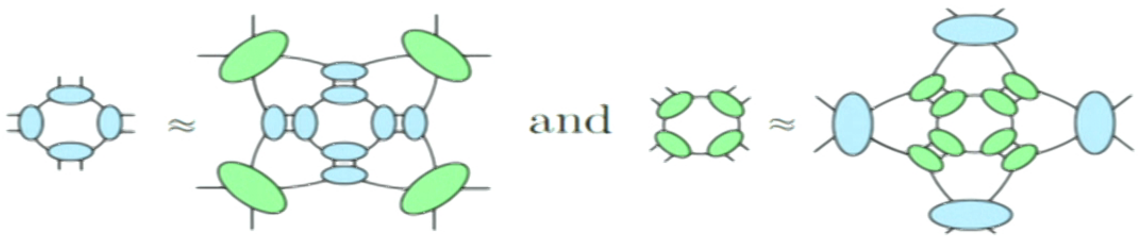


- Critical exponents correspond to eigenvalues of the radial transfer matrix R:

$R =$ 

Fixed Point Equations for TNR_+

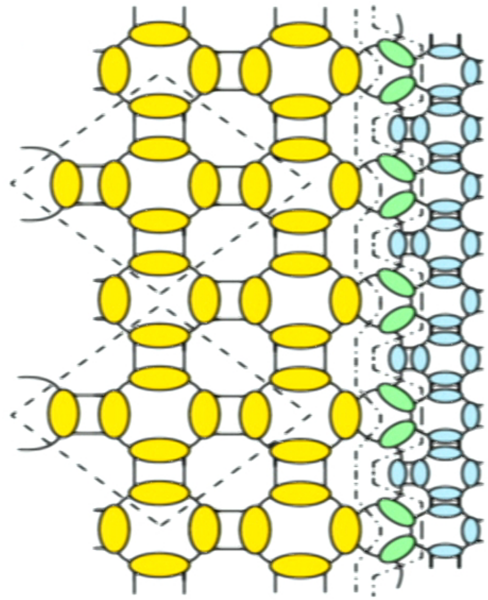
- TNR_+ yields an algebraic relation for fixed point tensors:



- Solutions of those equations yield scale-invariant theories.
  - Is it possible to find all solutions of those algebraic equations?
  - If we let the dimension D go to infinity, do we recover all CFT's, or do we get less or more?

TNR$_+$ yields stochastic MERA

- The leading eigenvector of a partition function should be positive (Perron-Frobenius); this is reflected in the fact that we get a stochastic MERA as fixed point:



Symmetries in TNR_+ networks

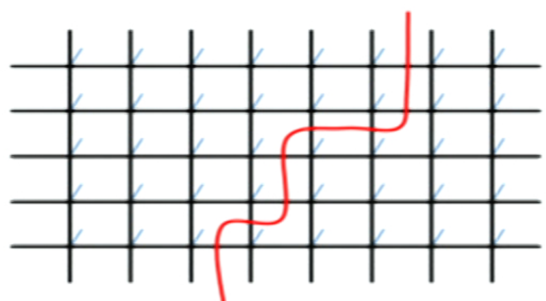
- The IR tensor network of the original Hamiltonian inherits all the symmetries of the Hamiltonian
  - Typical symmetries are of the form

  - The TNR_+ decimation steps involve approximating a tensor network with such symmetries by an sMPS; the fundamental theorem of MPS (Perez Garcia et al '08) implies that this symmetry is represented on the virtual level in the form of a projective representation.
  - Those virtual levels become the physical levels of the scaled Hamiltonian, so symmetries are preserved using the decimation steps (although they could be projective, cfr. Haldane chain)

Symmetries in tensor networks

- There are however also much more interesting ways in which symmetries can manifest themselves, namely in the form of matrix product operators (MPO):

- Such operators can be pulled through the lattice, and correspond to Wilson loops (cfr. Logical operators in toric code states)

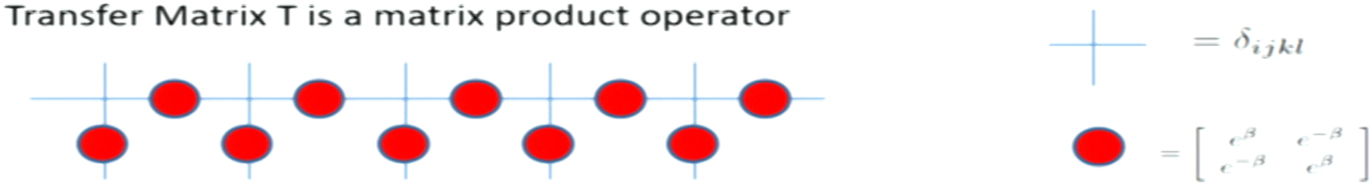


MPO symmetries and algebras

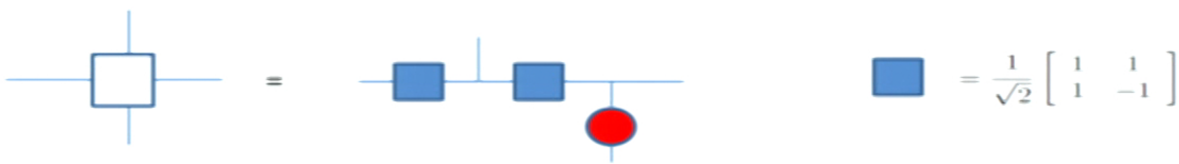
- Such symmetries have independently been discovered in tensor networks in different forms :
  - Mehmet Burak Şahinoğlu, Dominic Williamson, Nick Bultinck, Michael Mariën, Jutho Haegeman, Norbert Schuch, Frank Verstraete, "Characterizing Topological Order with Matrix Product Operators", arXiv:1409.2150
  - Nick Bultinck, Michael Mariën, Dominic J. Williamson, Mehmet B. Şahinoğlu, Jutho Haegeman, Frank Verstraete, "Anyons and matrix product operator algebras", arXiv:1511.08090
  - Markus Hauru, Glen Evenbly, Wen Wei Ho, Davide Gaiotto, Guifre Vidal: "Topological conformal defects with tensor networks", arXiv:1512.03846
  - David Aasen, Roger Mong, Paul Fendley: "Topological Defects on the Lattice", arXiv:1601.07185

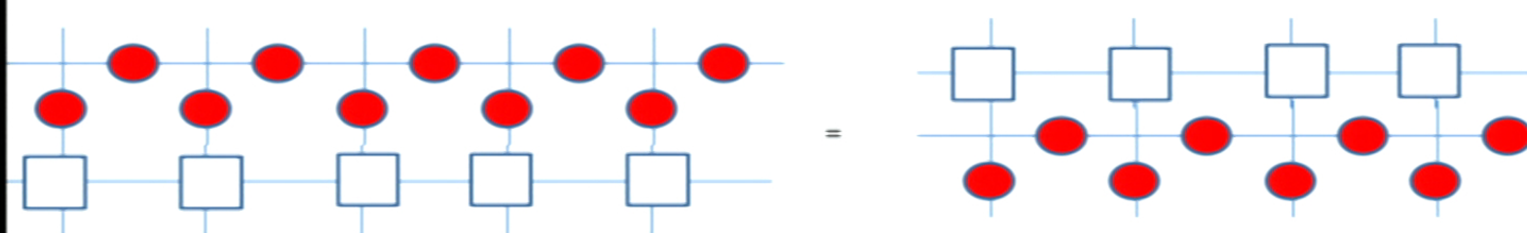
Example: Critical Ising model

- Transfer Matrix T is a matrix product operator



- At criticality (where the Ising model is self dual), an MPO symmetry emerges which corresponds to the duality defect, where the MPO σ is of the form





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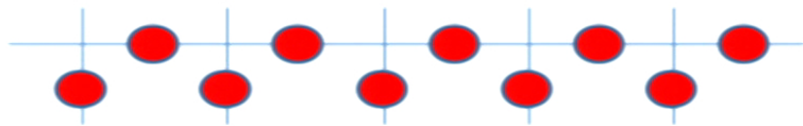


$= \delta_{ijkl}$

$= \begin{bmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{bmatrix}$

- At criticality (where the Ising model is self dual), an MPO symmetry emerges which corresponds to the duality defect, where the MPO σ is of the form

=

$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

- Other MPO-symmetries of the Ising model are much simpler:

$\Psi = X^{\otimes N}$

Shift Operator S

Transfer Matrix T

- Those MPOs (with PBC) obey the following algebraic relations:

$$\Psi.\sigma = \sigma = \sigma.\Psi \qquad \Psi^2 = I \qquad \sigma^2 = (1+\Psi).S.T$$

$$[\sigma, S] = 0 = [\sigma, T] = [\Psi, S] = [\Psi, T] = [T, S]$$

- We obtain a closed algebra for the set of MPOs

$$\{n, m \in \mathbb{Z} : \sigma_m^n = \sigma S^n T^m, \Psi_m^n = \Psi S^n T^m, I_m^n = S^n T^m\}$$

$$\begin{aligned} \sigma_{m_1}^{n_1}\sigma_{m_2}^{n_2} &= I_{m_1+m_2+1}^{n_1+n_2+1} + \Psi_{m_1+m_2+1}^{n_1+n_2+1} \\ \Psi_{m_1}^{n_1}\sigma_{m_2}^{n_2} &= \sigma_{m_1+m_2}^{n_1+n_2} \\ \Psi_{m_1}^{n_1}\Psi_{m_2}^{n_2} &= I_{m_1+m_2}^{n_1+n_2} \end{aligned}$$

$$\{n, m \in \mathbb{Z} : \sigma^n_m = \sigma S^n T^m, \Psi^n_m = \Psi S^n T^m, I^n_m = S^n T^m\}$$

$$\begin{aligned} \sigma^{n_1}_{m_1}\sigma^{n_2}_{m_2} &= I^{n_1+n_2+1}_{m_1+m_2+1} + \Psi^{n_1+n_2+1}_{m_1+m_2+1} \\ \Psi^{n_1}_{m_1}\sigma^{n_2}_{m_2} &= \sigma^{n_1+n_2}_{m_1+m_2} \\ \Psi^{n_1}_{m_1}\Psi^{n_2}_{m_2} &= I^{n_1+n_2}_{m_1+m_2} \end{aligned}$$

- We clearly observe the fusion rules for the Ising CFT

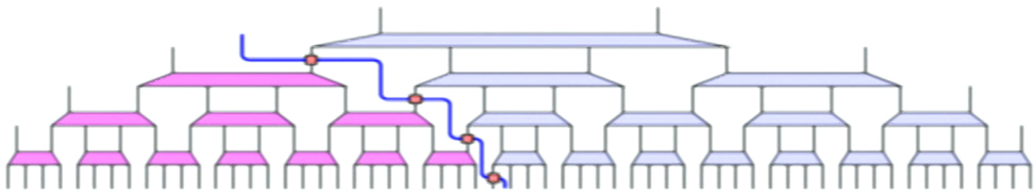
$$\sigma \times \sigma = 1 + \Psi \qquad \sigma \times \Psi = \sigma \qquad \Psi^2 = I$$

  - The I^n_m represent the shifts in "space and time"

- The MPOs colloquially correspond to the exponentials of the zero Fourier components of the primary fields

TNR, MERA and MPO algebras

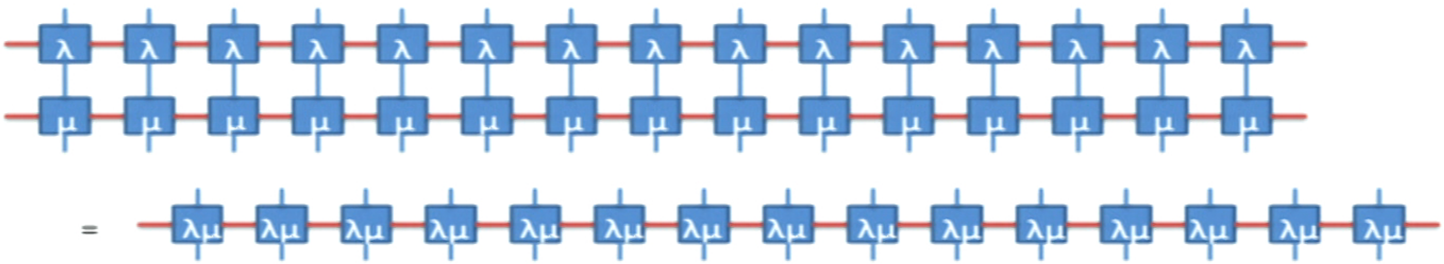
- The symmetries of the tensor network are hence represented by Matrix Product Operator algebras, and the TNR_+ schemes can explicitly be constructed such as to keep those symmetries
  - Because of the equivalence of TNR and MERA, this gives rise to MERA with MPO symmetries



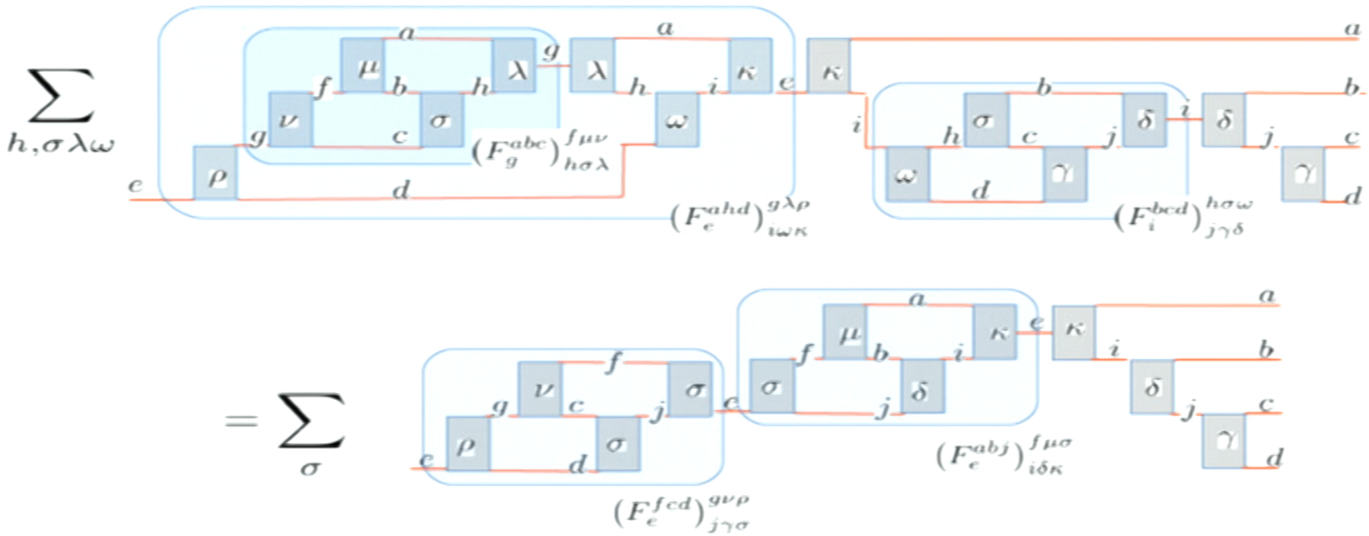
Bridgeman, Williamson
arXiv:1503.07782

MPO algebras and tensor fusion categories

- Basic premise: MPO's form a representation of the tensor fusion categories describing the algebraic content of TQFTs/CFTs
- Let us try to find all solutions of closed MPO algebras:

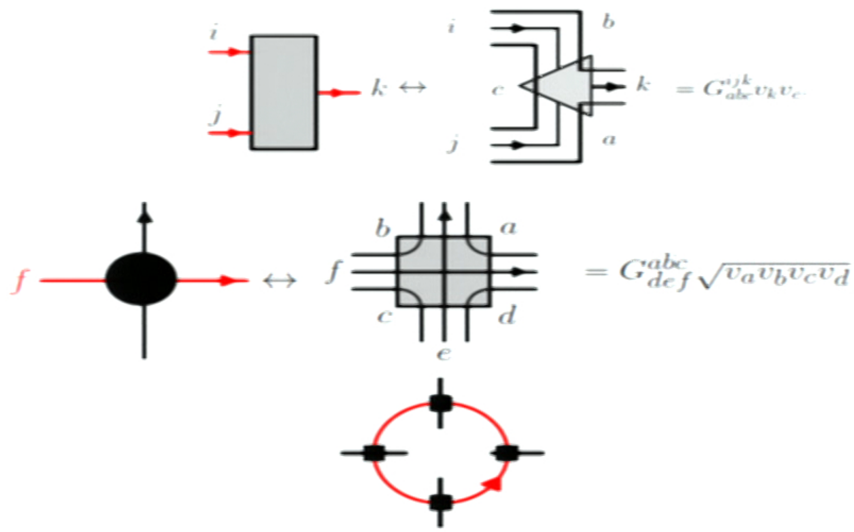


6. Two different ways of fusion leads to pentagon equation:



Hence MPO algebras are classified by the triple $(N_{ab}^{c}, F_{d}^{abc}, \kappa_a)$

7. Just as in the case of the Yang Baxter equation in Bethe ansatz, we can now construct a “fundamental representation” in terms of those F-symbols: all equations written out are satisfied by choosing all tensors in terms of F



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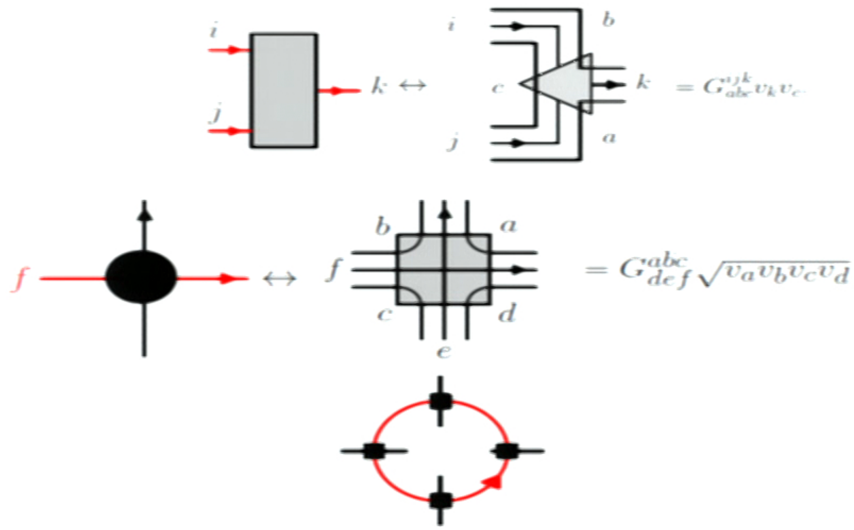
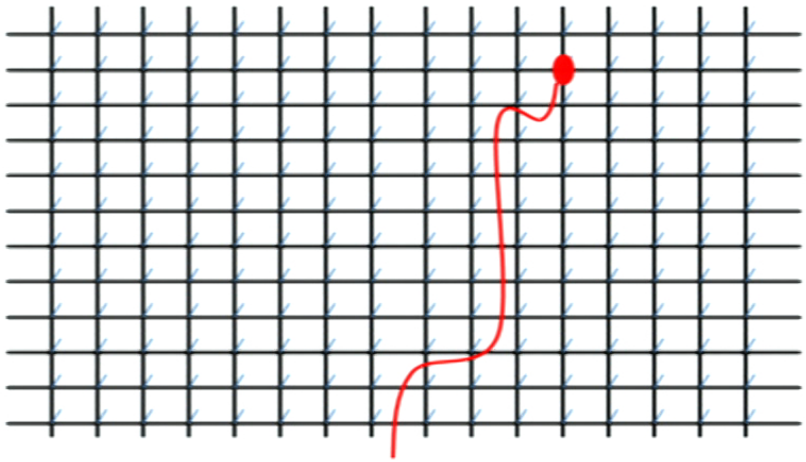
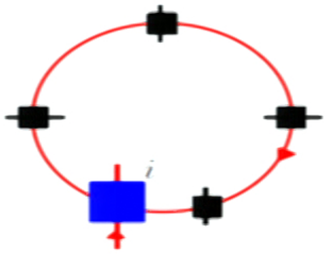
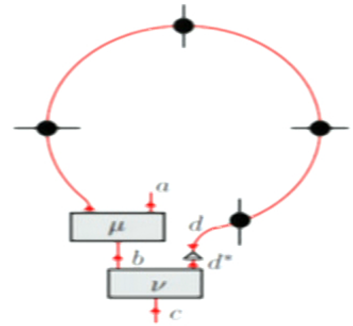


8. We now have to construct the topological sectors of the theory. For this we need to introduce an object which lives at the end of an MPO string; those will correspond to the primary fields



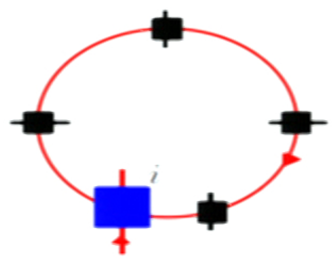


$$A_{abcd,\mu\nu} =$$



- This "anyon" tensor has 5 indices in and 5 out:
  - defines a C* algebra (similar to Ocneanu's "tube algebra") which contains all the content of the ***output*** tensor category
  - Anyons should be locally distinguishable by their charge. So we define them as central idempotents of this algebra:

$$\mathcal{P}_i = \sum_{abd,\mu\nu} c^i_{abd,\mu\nu} A_{abad,\mu\nu}$$



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Example: Ising case

$$F^{\sigma\sigma\sigma}_{\sigma 11} = \frac{1}{\sqrt{2}},\ F^{\sigma\sigma\sigma}_{\sigma\psi 1} = \frac{1}{\sqrt{2}},\ F^{\sigma\sigma\sigma}_{\sigma 1\psi} = \frac{1}{\sqrt{2}},\ F^{\sigma\sigma\sigma}_{\sigma\psi\psi} = -\frac{1}{\sqrt{2}},\ F^{\psi\sigma\psi}_{\sigma\sigma\sigma} = -1,\ F^{\sigma\psi\sigma}_{\psi\sigma\sigma} = -1$$

$$\mathcal{P}_1 = \frac{1}{4}\left(A_{1111} + 2^{3/4} A_{1\sigma 1\sigma} + A_{1\psi 1\psi}\right)$$
$$\mathcal{P}_2 = \frac{1}{4}\left(A_{\sigma\sigma\sigma 1} + 2^{1/4} e^{\frac{\pi i}{8}} A_{\sigma 1\sigma\sigma} + 2^{1/4} e^{-\frac{3\pi i}{8}} A_{\sigma\psi\sigma\sigma} + e^{\frac{\pi i}{2}} A_{\sigma\sigma\sigma\psi}\right)$$
$$\mathcal{P}_3 = \frac{1}{4}\left(A_{\sigma\sigma\sigma 1} + 2^{1/4} e^{-\frac{\pi i}{8}} A_{\sigma 1\sigma\sigma} + 2^{1/4} e^{\frac{3\pi i}{8}} A_{\sigma\psi\sigma\sigma} + e^{-\frac{\pi i}{2}} A_{\sigma\sigma\sigma\psi}\right)$$
$$\mathcal{P}_4 = \frac{1}{4}\left(A_{\psi\psi\psi 1} + 2^{3/4} e^{\frac{\pi i}{2}} A_{\psi\sigma\psi\sigma} - A_{\psi 1\psi\psi}\right)$$
$$\mathcal{P}_5 = \frac{1}{4}\left(A_{\psi\psi\psi 1} + 2^{3/4} e^{\frac{-\pi i}{2}} A_{\psi\sigma\psi\sigma} - A_{\psi 1\psi\psi}\right)$$
$$\mathcal{P}_6 = \frac{1}{4}\left(A_{\sigma\sigma\sigma 1} + 2^{1/4} e^{-\frac{7\pi i}{8}} A_{\sigma 1\sigma\sigma} + 2^{1/4} e^{\frac{5\pi i}{8}} A_{\sigma\psi\sigma\sigma} + e^{\frac{\pi i}{2}} A_{\sigma\sigma\sigma\psi}\right)$$
$$\mathcal{P}_7 = \frac{1}{4}\left(A_{\sigma\sigma\sigma 1} + 2^{1/4} e^{\frac{7\pi i}{8}} A_{\sigma 1\sigma\sigma} + 2^{1/4} e^{-\frac{5\pi i}{8}} A_{\sigma\psi\sigma\sigma} + e^{-\frac{\pi i}{2}} A_{\sigma\sigma\sigma\psi}\right)$$
$$\mathcal{P}_8 = \frac{1}{4}\left(A_{1111} - 2^{3/4} A_{1\sigma 1\sigma} + A_{1\psi 1\psi}\right)$$
$$\mathcal{P}_9 = \frac{1}{2}\left(A_{1111} + A_{\psi\psi\psi 1} - A_{1\psi 1\psi} + A_{\psi 1\psi\psi}\right)$$

Those idempotents define the topological sectors in the Ising CFT

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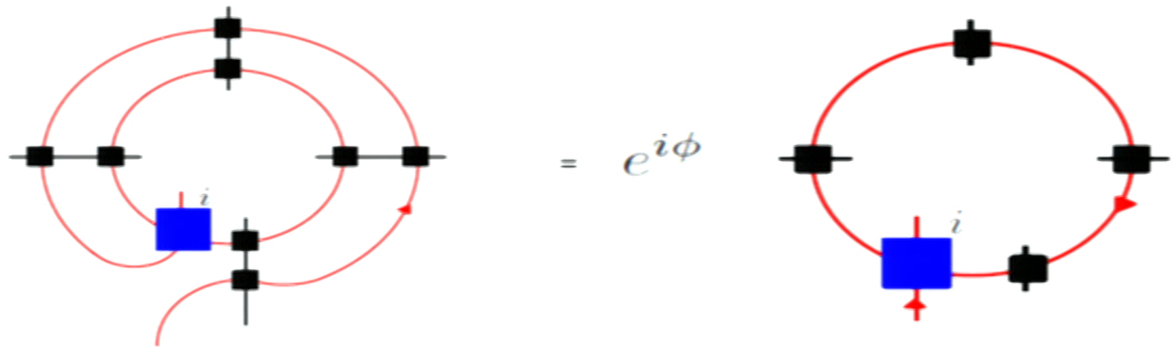
Topological / Conformal Spin:



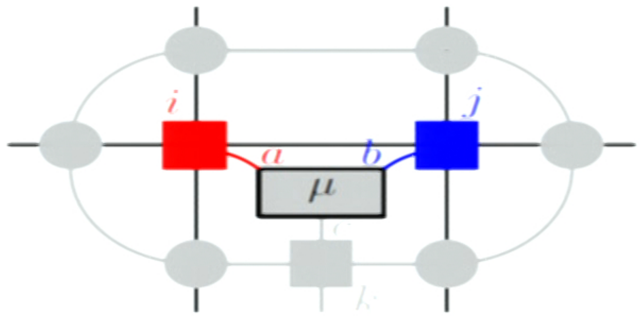
Ising CFT:

$$h_1 = 0,\ h_2 = \frac{1}{16},\ h_3 = -\frac{1}{16},\ h_4 = \frac{1}{2},\ h_5 = -\frac{1}{2}$$
$$h_6 = -\frac{7}{16},\ h_7 = \frac{7}{16},\ h_8 = 0,\ h_9 = 0.$$

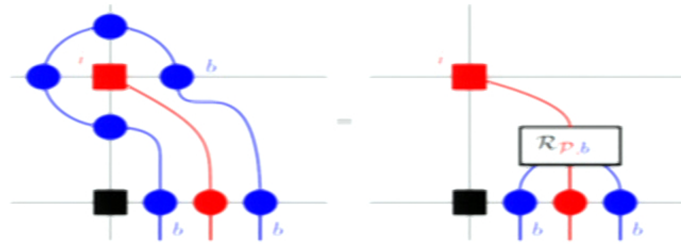
$$\mathcal{P}_1 = (1,1),\ \mathcal{P}_2 = (\sigma,1),\ \mathcal{P}_3 = (1,\bar{\sigma}),$$
$$\mathcal{P}_4 = (\psi,1),\ \mathcal{P}_5 = (1,\psi),\ \mathcal{P}_6 = (\sigma,\bar{\psi}),.$$
$$\mathcal{P}_7 = (\psi,\bar{\sigma}),\ \mathcal{P}_8 = (\psi,\bar{\psi}),\ \mathcal{P}_9 = (\sigma,\bar{\sigma})$$

Fusion:

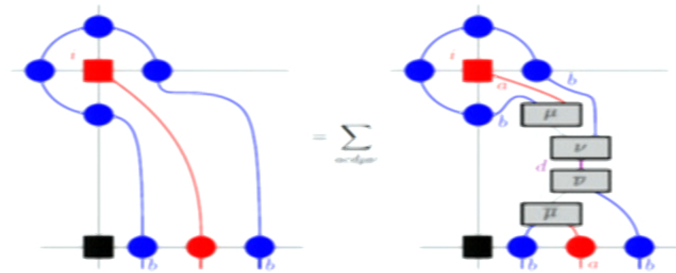
- Fusion rules of output category can readily be obtained from the idempotents



Braiding



- The braiding matrix R is itself determined by the central idempotent (argument by “teleportation”)



- Out of this, we can immediately calculate the relevant S and T matrices, …

MPO symmetries: summary

- We started from an MPO representation of a (modular) input tensor category

$$\sigma \times \sigma = 1 + \Psi \qquad \sigma \times \Psi = \sigma \qquad \Psi^2 = I$$

  - These MPOs corresponds to the exponential of the zero Fourier components of the chiral primary fields

- We then constructed the Drinfeld center of this input category, and obtained the chiral and anti-chiral part (full CFT) including topological spin and S and T matrices:

$$\mathcal{P}_1 = (1, 1),\ \mathcal{P}_2 = (\sigma, 1),\ \mathcal{P}_3 = (1, \bar{\sigma}),$$
$$\mathcal{P}_4 = (\psi, 1),\ \mathcal{P}_5 = (1, \psi),\ \mathcal{P}_6 = (\sigma, \bar{\psi}).,$$
$$\mathcal{P}_7 = (\psi, \bar{\sigma}),\ \mathcal{P}_8 = (\psi, \bar{\psi}),\ \mathcal{P}_9 = (\sigma, \bar{\sigma})$$

Topological conformal defects

- This MPO approach is related to the one pursued by
  - Markus Hauru, Glen Evenbly, Wen Wei Ho, Davide Gaiotto, Guifre Vidal: "Topological conformal defects with tensor networks", arXiv:1512.03846
  - David Aasen, Roger Mong, Paul Fendley: "Topological Defects on the Lattice", arXiv:1601.07185
- TNR/MERA/PEPS algorithms are taken to a different level by incorporating those MPO symmetries

Conclusion

- Real space RG: from Kadanoff to TRG to TNR to TNR_+
- Symmetries in those tensor networks are most concisely represented by matrix product operator algebras, from which all algebraic content of the underlying CFT can be obtained