

Title: Analytic approaches to tensor networks for field theories

Date: Apr 19, 2017 02:40 PM

URL: <http://pirsa.org/17040036>

Abstract: I will discuss analytic approaches to construct tensor network representations of quantum field theories, more specifically conformal field theories in 1+1 dimensions. A key insight is that we should understand how well the tensor network can reproduce the correlation functions of the quantum field theory. Based on this measure of closeness, I will present rigorous results allowing for explicit error bounds which show that both Matrix product states (MPS) as well as the multiscale renormalization Ansatz (MERA) do approximate conformal field theories. In particular, I will discuss the case of Wess-Zumino-Witten models.

based on joint work with Robert Koenig (MPS), Brian Swingle and Michael Walter (MERA)

# Goal

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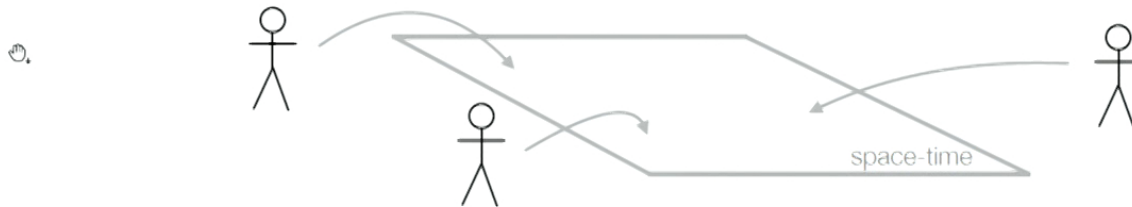
## Tensor networks for Quantum Fields

Develop new classes of Ansatz states which model the entanglement structure of quantum fields. We are interested in analytic solutions with rigorous error bounds.

### **Results presented:**

Tensor network approximations (MPS, MERA) to conformal field theories in  $1 + 1$  space-time dimensions.

# Quantum fields



- ▶ A quantum theory possessing a continuum of degree of freedoms representing the variables of space and time
- ▶ defined in terms of its correlation functions with respect to the vacuum state  $|\Omega\rangle$ ,

$$\langle \Omega | \phi(x_1) \phi(x_2) \cdots \phi(x_n) \Omega \rangle$$

$\phi(x_i)$ : measurement performed at point  $x_i$  in space-time

- ▶ The Hilbert space of the theory is necessary infinite-dimensional

# Approximating quantum field theories



What does approximate mean?

How to approximate a continuous quantum field theory by objects on discrete spatial structures?

## Idea:

If Tensor networks can approximately reproduce correlation functions of quantum field theories, then we can use them to understand the entanglement structure of quantum field theories.







# Conformal Field Theories and MPS

## Result

• Theorem (R.Koenig, VBS, PRL 2016, Nucl. Phys. B 2017)

Correlation functions of Conformal Field Theories can be arbitrarily well approximated by correlations functions of Matrix Product Tensor networks,

$$| \langle \Omega | \phi(x_1) \cdots \phi(x_n) \Omega \rangle - \langle \Psi_{MPS}^D | \hat{\phi}(x_1) \cdots \hat{\phi}(x_n) \Psi_{MPS}^D \rangle | \leq f(D, n, c_{CFT}) .$$

The function  $f(D, n, c_{CFT})$  leads to the following scaling of the bond dimension:

	fixed $n, d = \min  x_i - x_j $	fixed $\varepsilon$
Scaling of $D$	$\sim \left(\frac{1}{\varepsilon}\right)^{K c_{CFT} \frac{n}{d}}$	$\sim \gamma(\varepsilon) e^{2\pi \sqrt{\frac{1}{6} c_{CFT} n}}$

## Result



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Remarks:

- Holds for general CFTs, proof uses the theory of Vertex operator algebras

## Intuition via OPE

- Consider the Operator Product Expansion (OPE) of a CFT:

$$\Phi_\alpha \Phi_\beta = \sum_\gamma C_{\alpha,\beta,\gamma} \Phi_\gamma$$

Here  $\{\Phi_\gamma\}_\gamma$  are primary fields as well as their descendants.

### Idea:

Define the MPS tensor in terms of the coefficients of the OPE:

$$\begin{array}{c} \alpha \text{ --- } \text{---} \beta \\ | \\ \gamma \end{array} := C_{\alpha,\beta,\gamma}$$

This tensor needs to be truncated to get a finite bond dimension.

# Geometric Intuition



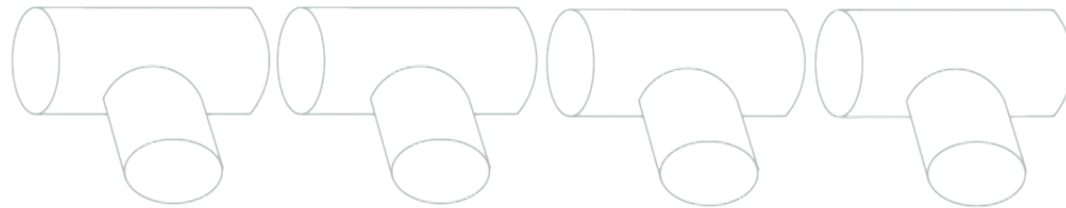
Segal's definition of a CFT: functor from two-dimensional Riemannian surfaces with (oriented) holes to trace-class operators.

## **Example: Genus 1**

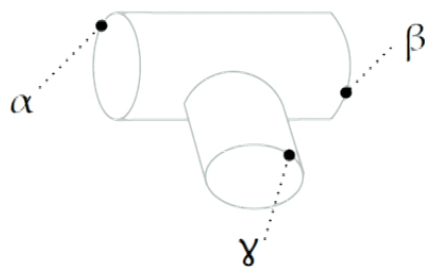
Take a sphere and stretch it. Field insertions correspond to holes which are parametrized by circles. Deformations of the holes correspond to the action of the Virasoro algebra.

## Geometric Intuition

Decompose into pants:



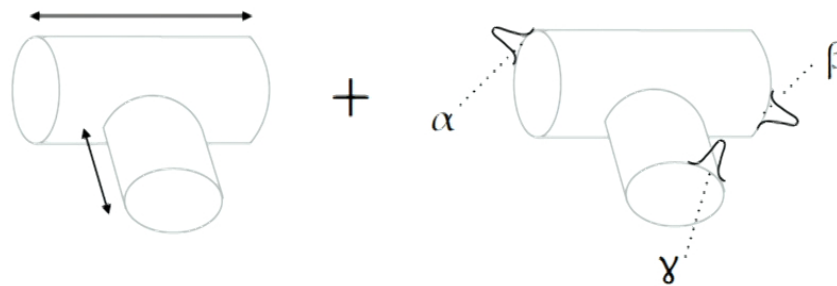
Each pant is parametrized by three circles:



Labelled points on the circles give rise to field insertions (Pant = OPE).

The exact representation of arbitrary smooth deformations requires infinite-dimensional Hilbert spaces.

## Proof idea: regularization



- ▶ We first require the pant to have finite width in each direction. This corresponds to requiring a minimal distance between the insertion points.
- ▶ We get a finite bond dimensions by truncating in the Fourier like basis. This corresponds to considering smoothed fields.



# Wess-Zumino-Witten Theories and MERA



## Result

👉 Theorem (B. Swingle, M. Walter, VBS, 2017)

Correlation functions of smoothed fields of  $\mathfrak{su}(m)_k$  ( $k$ : level) Wess-Zumino-Witten theories can be arbitrarily well approximated by those of a MERA Tensor network state,

$$|\langle \Omega | \phi(g_1) \cdots \phi(g_n) \Omega \rangle - \langle \Psi_{\text{MERA}} | \tilde{\phi}(g_1) \cdots \tilde{\phi}(g_n) \Psi_{\text{MERA}} \rangle| \leq f(n, L, d, n, k).$$

$L$ : locality,  $d$ : depth.

Remarks:

- ▶ The smoothed fields  $\phi(g) = \int_{\mathbb{R}} dx g(x) \phi(x)$  can be both currents as well as intertwiners.

## $\mathfrak{su}(m)_k$ -WZW models from free fermions

- Our result is based on the work of Witten, who proved that  $\mathfrak{su}(m)_k$  WZW theories are faithfully represented on the Fock space of  $n \cdot k$  free Dirac fermions  $\psi_{i,l}$ ,  $i = 1, \dots, m$ ,  $l = 1, \dots, k$ .

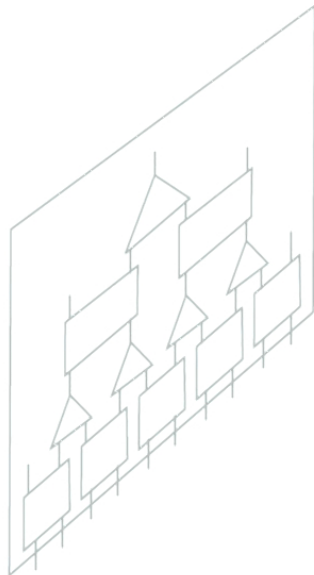
The current associated with an element  $A \in \mathfrak{su}(m)$ ,  $A = \sum_{ij} a_{ij} E_{ij}$  is

$$J_A(x) = \sum_l \sum_{ij} A_{ij} : \psi_{i,l}^\dagger(x) \psi_{j,l}(x) :$$

### **Idea:**

Take  $m \cdot k$  copies of the free fermion MERA (cf. Brian's talk) and show that smoothed versions of the currents  $J_A(g) = \int_{\mathbb{R}} dx g(x) J_A(x)$ ,  $g$  sufficiently smooth, correspond approximately to local operators  $\tilde{J}_A(g)$  which act on the MERA.

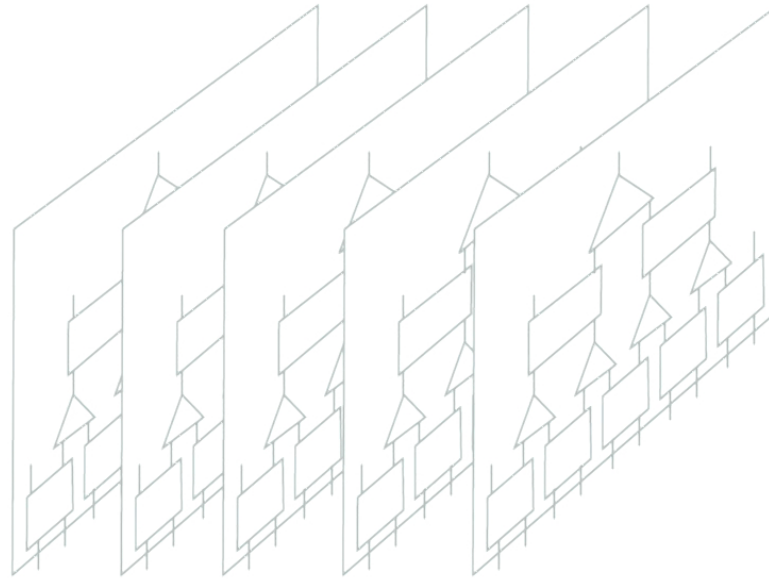
## Review on MERA for free fermions (cf. Brian's talk)



- ▶ MERA constructed from continuous Wavelet transform associated with Selesnick's wavelet based Hilbert transform
- ▶ Gives rise to **provable error bounds** for correlation functions based on smoothed creation/annihilation operators
- ▶ Same MERA approximates the ground state of free fermions on a 1D lattice and can be used to build a branching MERA for the 2D hopping Hamiltonian (joint work with J. Haegeman, M. Bal, J. Cotler, B. Swingle, M. Walter)

## From the free fermion MERA to WZW MERA

- Take  $m \cdot k$  copies of the free fermion MERA. Each site now has dimension  $2 \cdot m \cdot k$ .



## $\mathfrak{su}(m)_k$ -WZW models from free fermions

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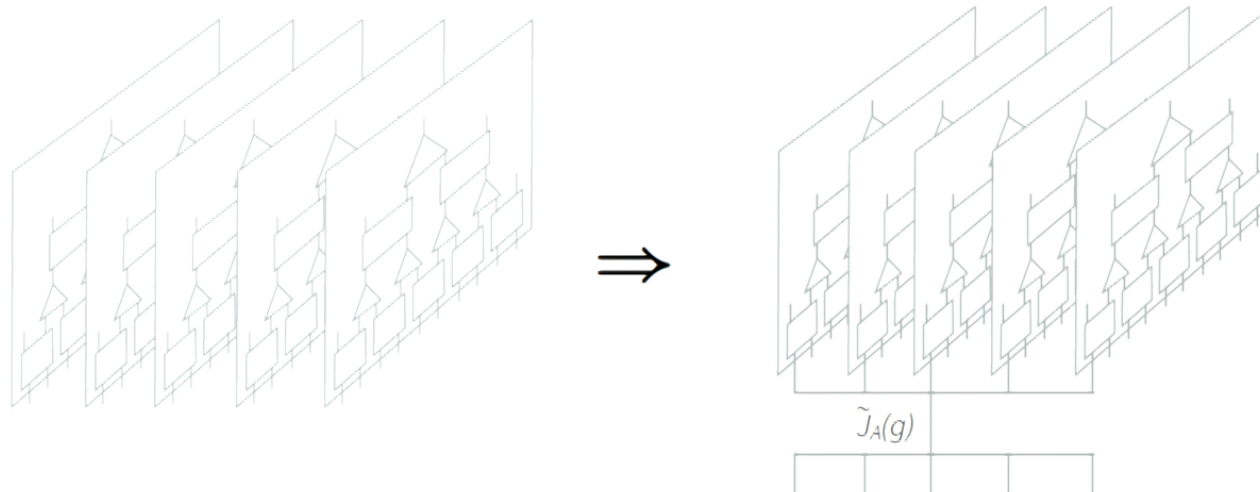
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## From the free fermion MERA to WZW MERA

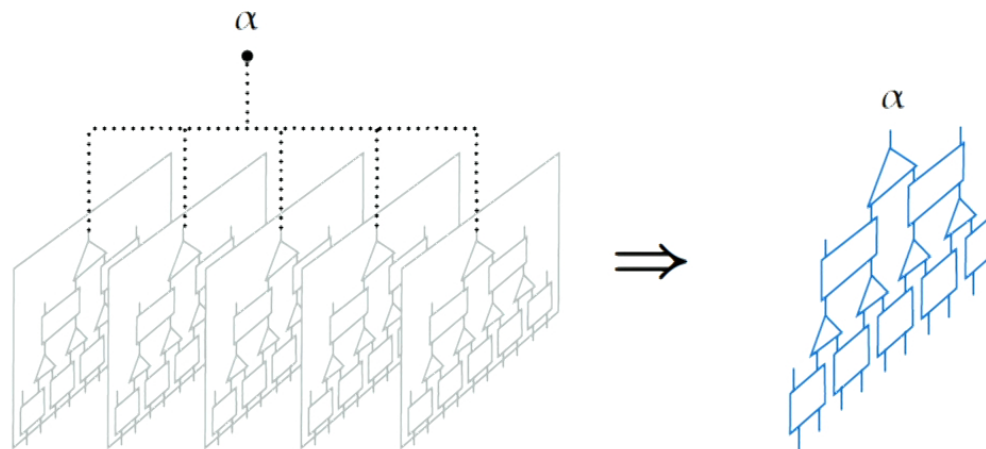
- The original action of WZW currents amounts to mixing the different fermions. This can be emulated:

$$J_A(g) \approx \tilde{J}_A(g) = \sum_l \sum_{ij} a_{ij} |\text{MERA}_{i,l}\rangle \langle \text{MERA}_{j,l}|$$



## Reducing to specific WZW modules

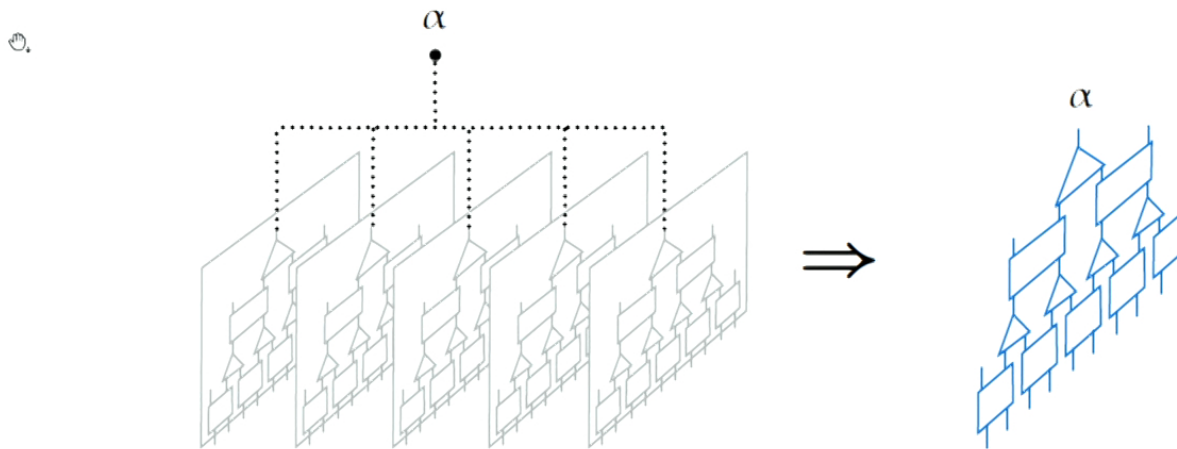
- As some fermions are not mixed,  $\mathfrak{su}(m)$  acts reducibly on each site. The  $m \cdot k$  free fermion MERAs decompose into irreducible components, which are associated with global superselection rules.



Selecting one of the components at the highest MERA level correspond to selecting a specific WZW module.



## Reducing to specific WZW modules



- ▶ We also obtain the correct fusion rules between the WZW modules.
- ▶ This projection however does not reduce the entanglement significantly,
- ▶ Is only possible if correlation functions of currents and not of intertwiners are of concern.



# Tensor networks for Quantum Fields



## Main statement

Tensor networks can faithfully represent Conformal Field Theories.

## Results presented:

- ▶ MPS for general CFTs by truncating the OPE,
- ▶ MERA for  $\mathfrak{su}(m)_k$  WZW models by using the free fermion MERA.

## Key idea:

Aim to approximate correlation functions of smoothed observables.

**Thank you!**