

Title: Bootstrapping the spectral function

Date: Apr 04, 2017 02:30 PM

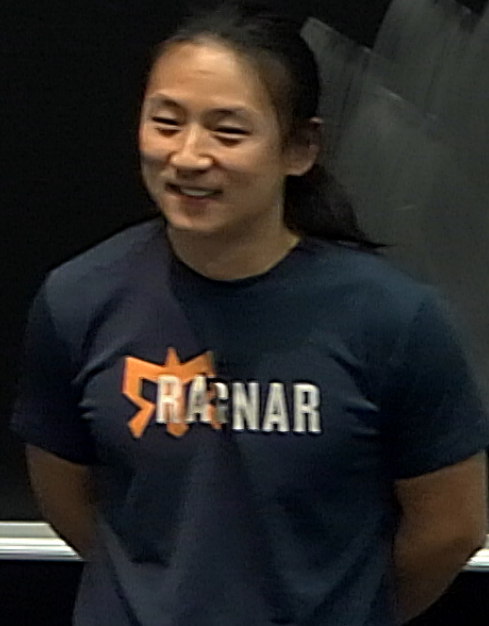
URL: <http://pirsa.org/17040028>

Abstract: <p>I will introduce the spectral function method in the context of conformal bootstrap. I will discuss some applications of this method in two dimensions: (1) substantial evidence for the conjecture that the only unitary $c > 1$ CFT with Virasoro primaries of bounded spin is Liouville theory, (2) detailed modular constraints on the spectrum of small c CFTs, (3) spectral density of large c CFTs with large gap, in connection to the (non-)universality of BTZ black hole entropy.</p>

Bootstrapping the Spectral Function

w/ S. Collier, P. Kravchuk, Y.-H. Lin

1702.00423



2D CFT

unitary, compact,

2D CFT

unitary, compact, no extra symmetry.

$c > 1$,

2D CFT
unitary,

- compact
- non compact

2D CFT
unitary,

compact
non compact

Claim: A 2D CFT (unitary)
with $c \geq 1$ and Virasoro primaries of bounded spin

$$(h, \tilde{h})$$
$$s = |h - \tilde{h}| \leq S_{\max}$$

2D CFT
unitary,

compact
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Claim: A 2D CFT (unitary)
with $c > 1$ and Virasoro primaries of bounded sp. n
must be Liouville CFT.

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$$(h, \tilde{h})$$
$$s = |h - \tilde{h}| \leq S_{\max}$$

modular inv. $Z(\tau, \bar{\tau})$

\Rightarrow bounded spin

\rightarrow

any scalar primaries, continuous spectrum,
Spectral density is that of Liouville.

$\in S_{max}$

CAUTION
DO NOT TOUCH THE BOARD SURFACE
OR THE BOARD SURFACE
OR THE BOARD SURFACE

modular inv. $Z(\tau, \bar{\tau})$

\Rightarrow bounded sp. n

\rightarrow

only scalar primaries, continuous spectrum,
spectral density is that of Liouville.

Liouville

$$C = 1 + 6Q^2,$$

$$V_{P \geq 0}$$

$$h = \tilde{h} = \frac{Q^2}{4} + P^2$$

modular inv. $Z(\tau, \bar{\tau})$

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Liouville

$$C = 1 + 6Q^2,$$

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$$\Delta = h + \tilde{h} \geq \frac{c-1}{12}$$

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\rightarrow

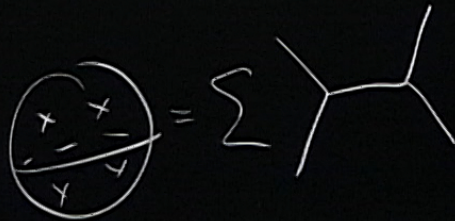
only scalar primaries, continuous spectrum,
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Liouville $C = 1 + 6Q^2$

$V_{P \geq 0}$

$h^2 + p^2$

$h \geq \frac{c-1}{12}$



CAUTION
DO NOT TOUCH THE BOARD WITH SHARP OBJECTS
OR WITH YOUR FEET
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modular inv. $Z(\tau, \bar{\tau})$

\Rightarrow bounded sp. n

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Spectral density is that of Liouville

Liouville $C = 1 + 6Q^2$

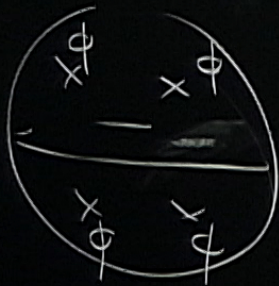
$$h = \tilde{h} = \frac{Q^2}{4} + P^2$$

$$\Delta = h + \tilde{h} \geq \frac{c-1}{12}$$

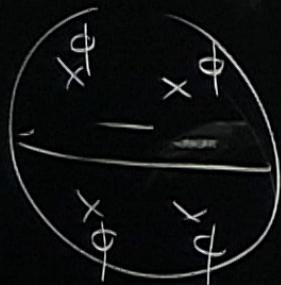
$\sum_{\text{diagrams}} = \sum_{P \geq 0} \text{diagrams}$

CAUTION
DO NOT TOUCH THE BOARD WITH SHARP OBJECTS
OR WITH YOUR FEET
OR IN ANY MANNER THAT COULD DAMAGE THE BOARD OR THE SURROUNDING AREA

$$\langle \phi(0) \phi(1) \phi(z) \phi(\infty) \rangle$$

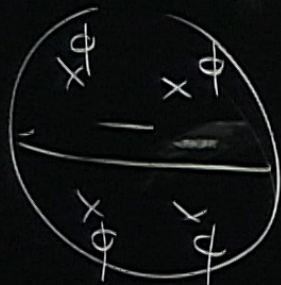


$$\langle \phi(0) \phi(1) \phi(z) \phi(\infty) \rangle$$

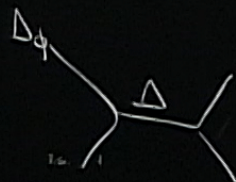


$$= f(z, \bar{z})$$

$$\langle \phi(0) \phi(1) \phi(z) \phi(\infty) \rangle$$



$$= f(z, \bar{z}) = \sum_{\Delta} C_{\Delta}^2 F_{\Delta}(z, \bar{z})$$



crossing eqn

$$\sum_{\Delta} C_{\Delta}^2 \left[F_{\Delta}(z, \bar{z}) - F_{\Delta}(1-z, 1-\bar{z}) \right] = 0$$



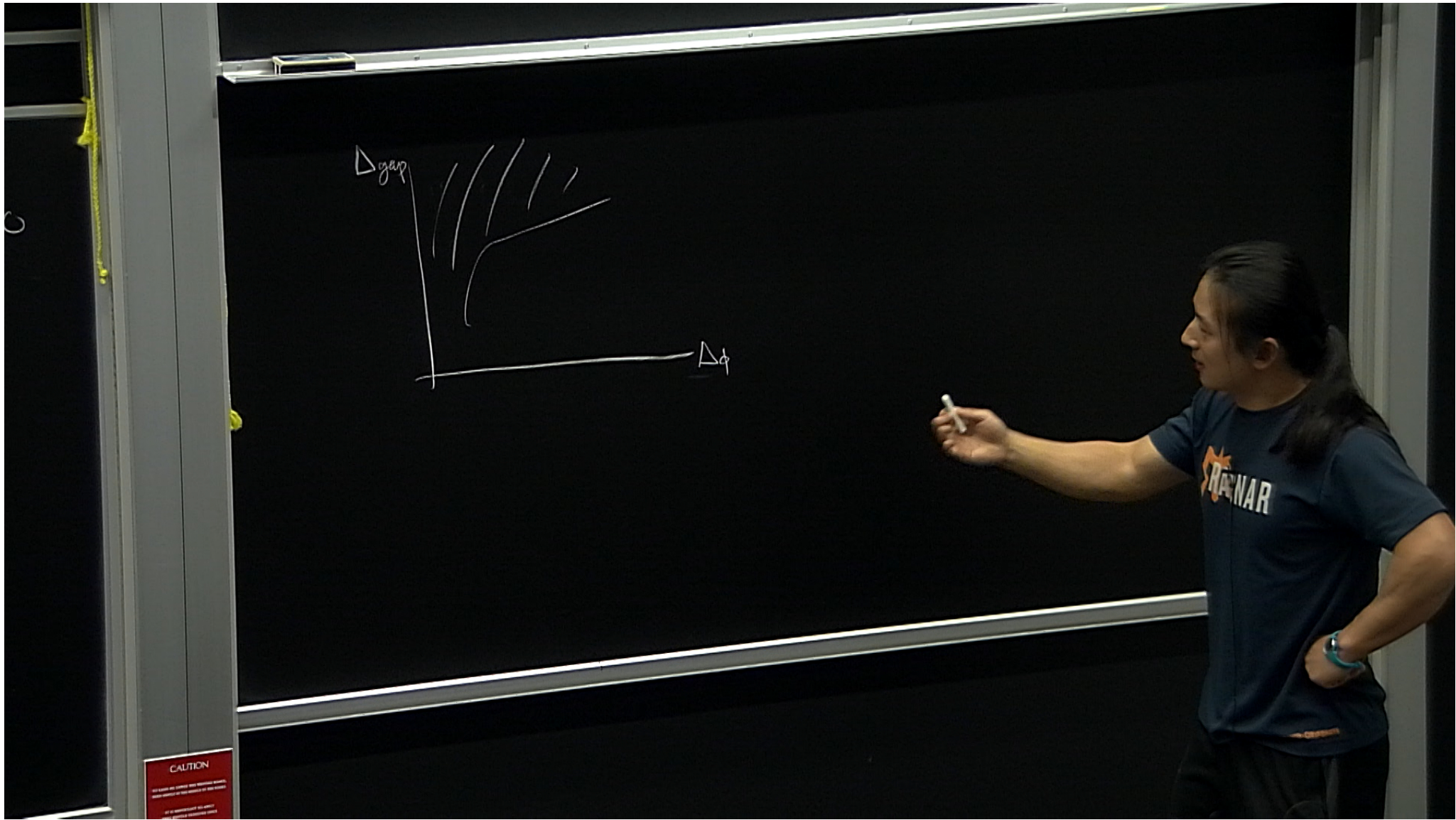
crossing eqn

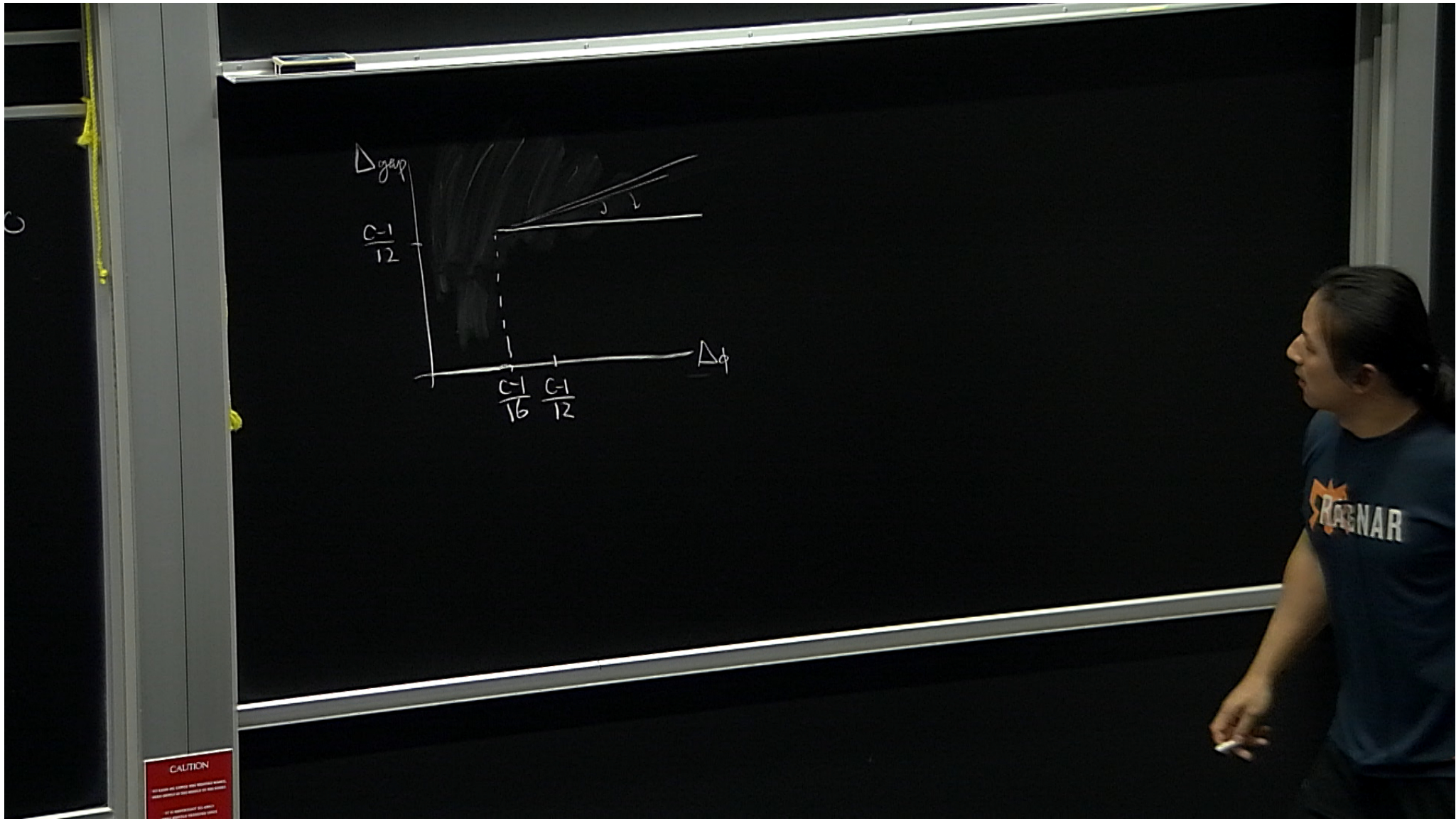
$$\sum_{\Delta} C_{\Delta}^2 \left[F_{\Delta}(z, \bar{z}) - F_{\Delta}(1-z, 1-\bar{z}) \right] = 0$$

\Updownarrow

$$\sum_{\Delta} C_{\Delta}^2 F_{m,n}(\Delta) = 0, \quad \begin{array}{l} z = \bar{z} = \frac{1}{2} \\ m+n \text{ odd} \end{array}$$

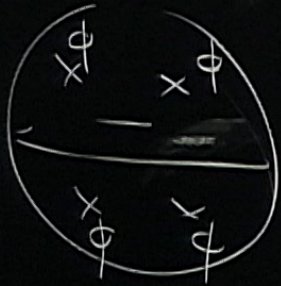
$$F_{m,n}(\Delta) \equiv \left. \partial_z^m \partial_{\bar{z}}^n F_{\Delta}(z, \bar{z}) \right|_{z=\bar{z}=\frac{1}{2}}$$



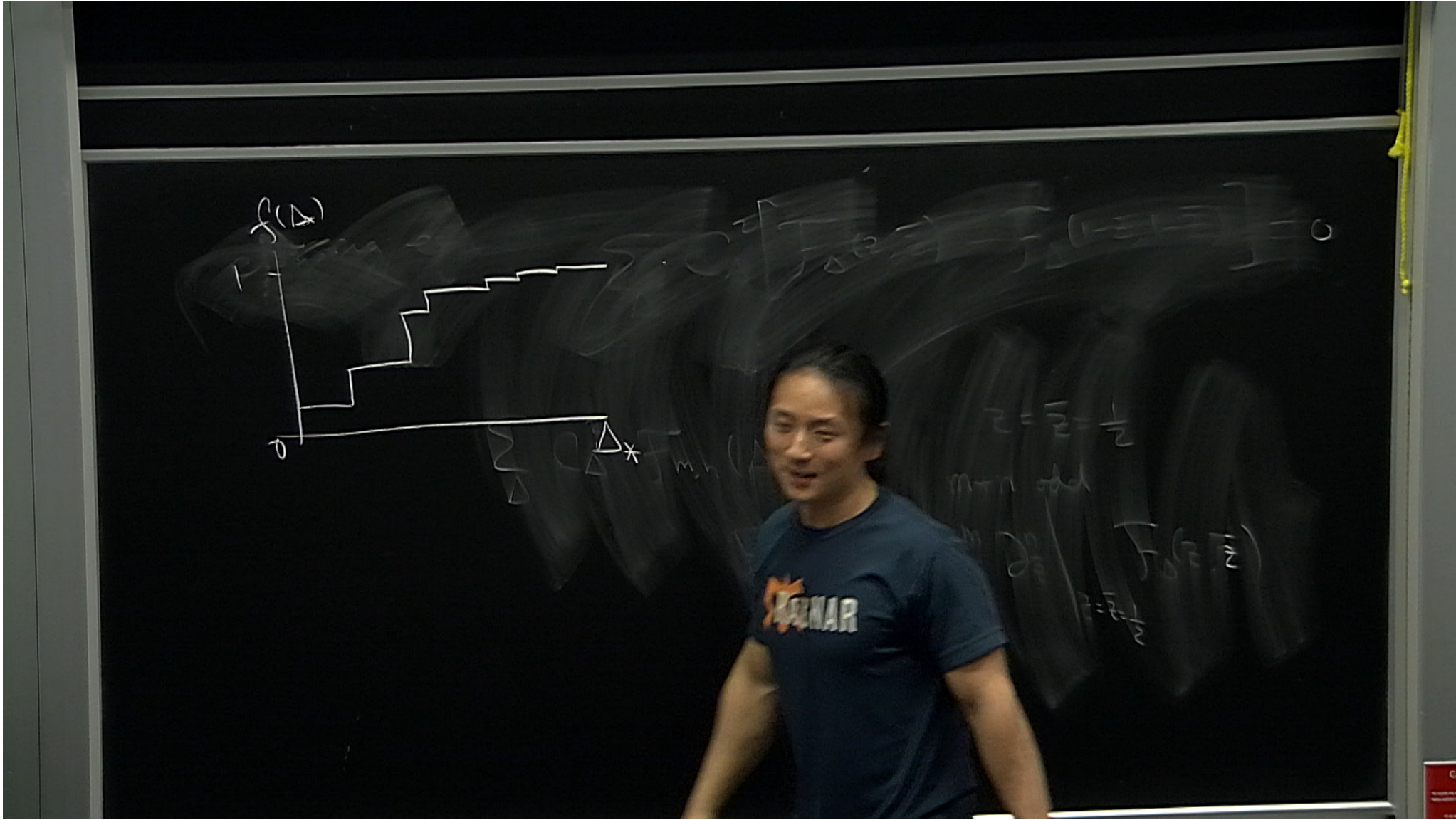


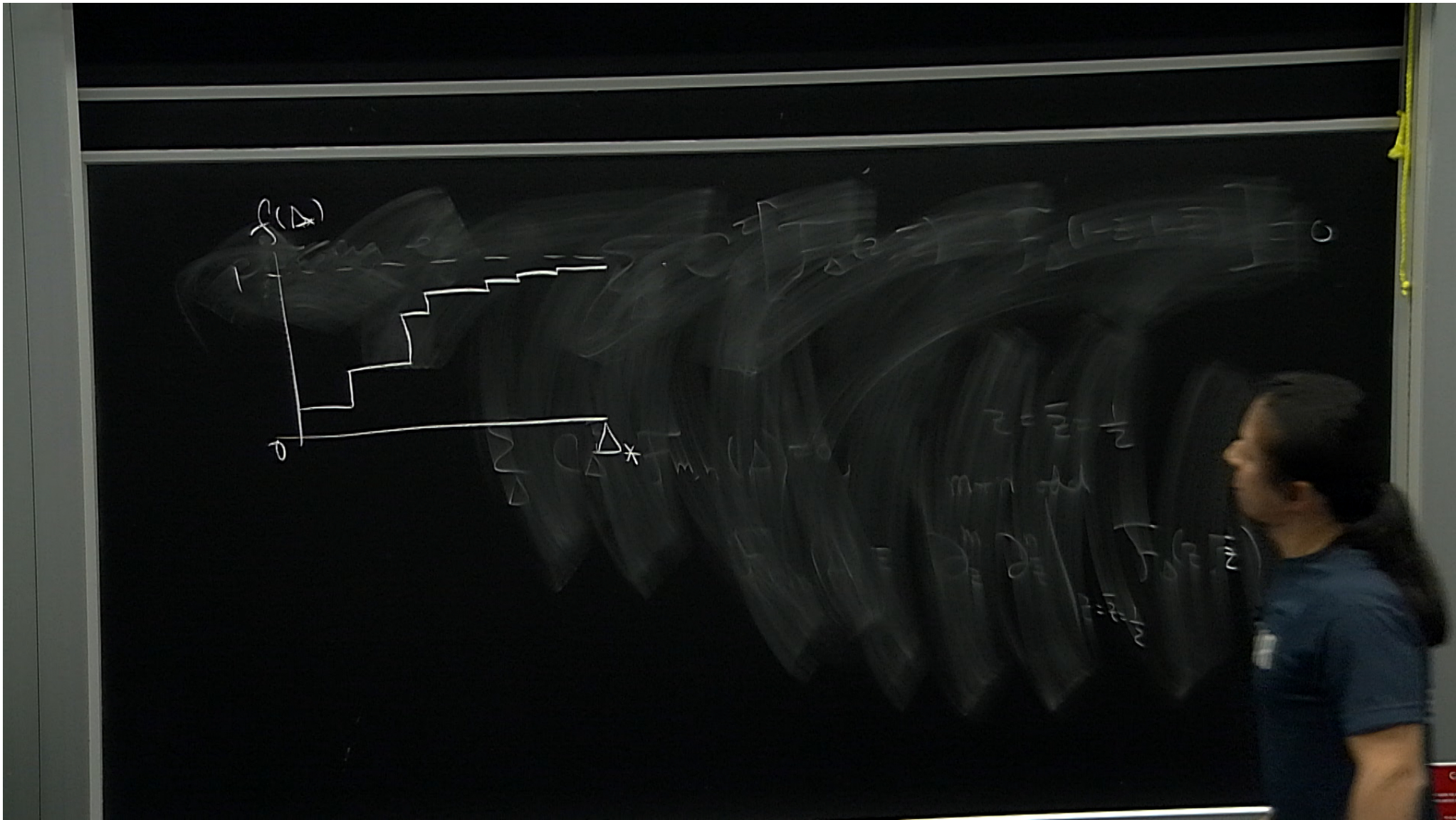
CAUTION
ALL GASES ARE TOXIC AND SOME ARE EXTREMELY TOXIC.
NEVER OPERATE OR REPAIR EQUIPMENT FOR THIS SYSTEM.
SEE INSTRUCTIONS FOR SAFETY.
(Please read instructions carefully)

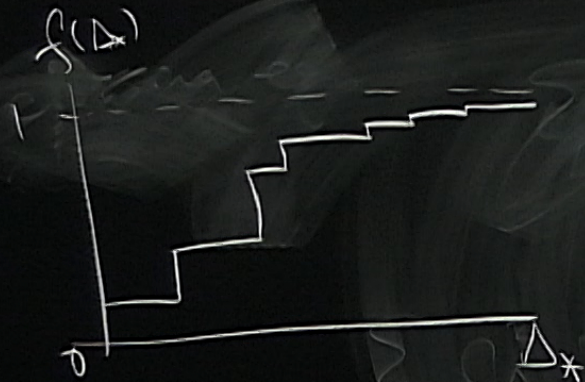
Spectral function



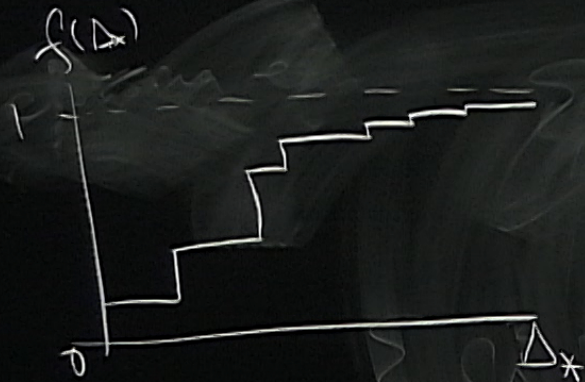
$$f(\Delta_*) := \frac{\sum_{\Delta \leq \Delta_*} C_{\Delta}^2 F_{\Delta}(z = \bar{z} = \frac{1}{2})}{\sum_{\Delta'} C_{\Delta'}^2 F_{\Delta'}(z = \bar{z} = \frac{1}{2})}$$



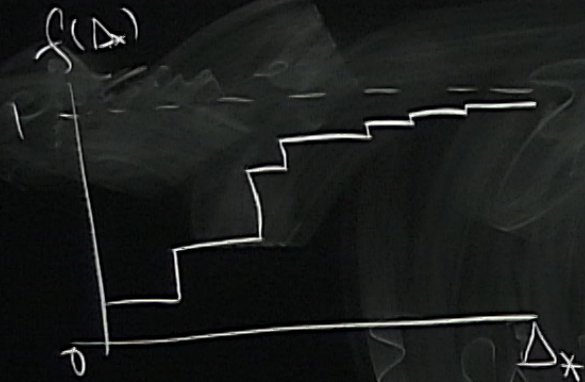




$$\theta(\Delta_* - \Delta) \int_{0,0}^{\Delta} f(\Delta) d\Delta$$



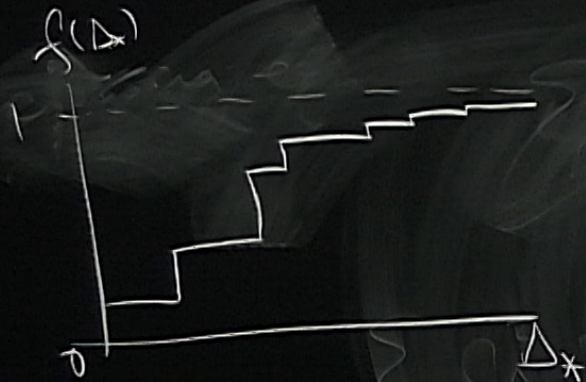
$$\theta(\Delta_* - \Delta) F_{0,0}(\Delta) + \gamma_{0,0} F_{0,0}(\Delta) + \sum_{m+n \text{ odd}} \gamma_{m,n} F_{m,n}(\Delta) \geq 0$$



Seek $y_{0,0}, y_{m,n}$ such that

$$\theta(\Delta_* - \Delta) F_{0,0}(\Delta) + y_{0,0} F_{0,0}(\Delta) + \sum_{m+n \text{ odd}} y_{m,n} F_{m,n}(\Delta) \geq 0$$

holds for all $\Delta (> 0)$

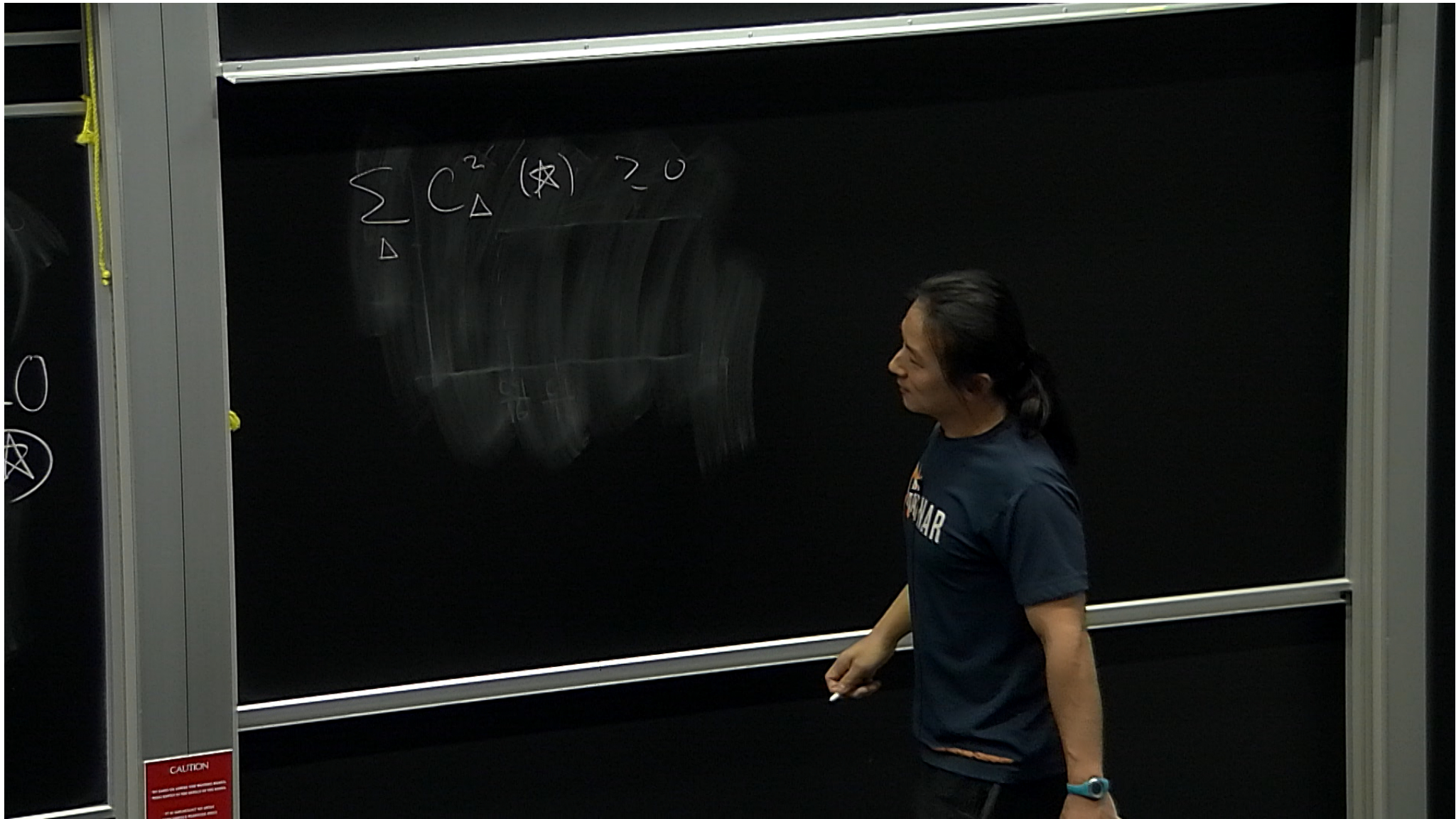


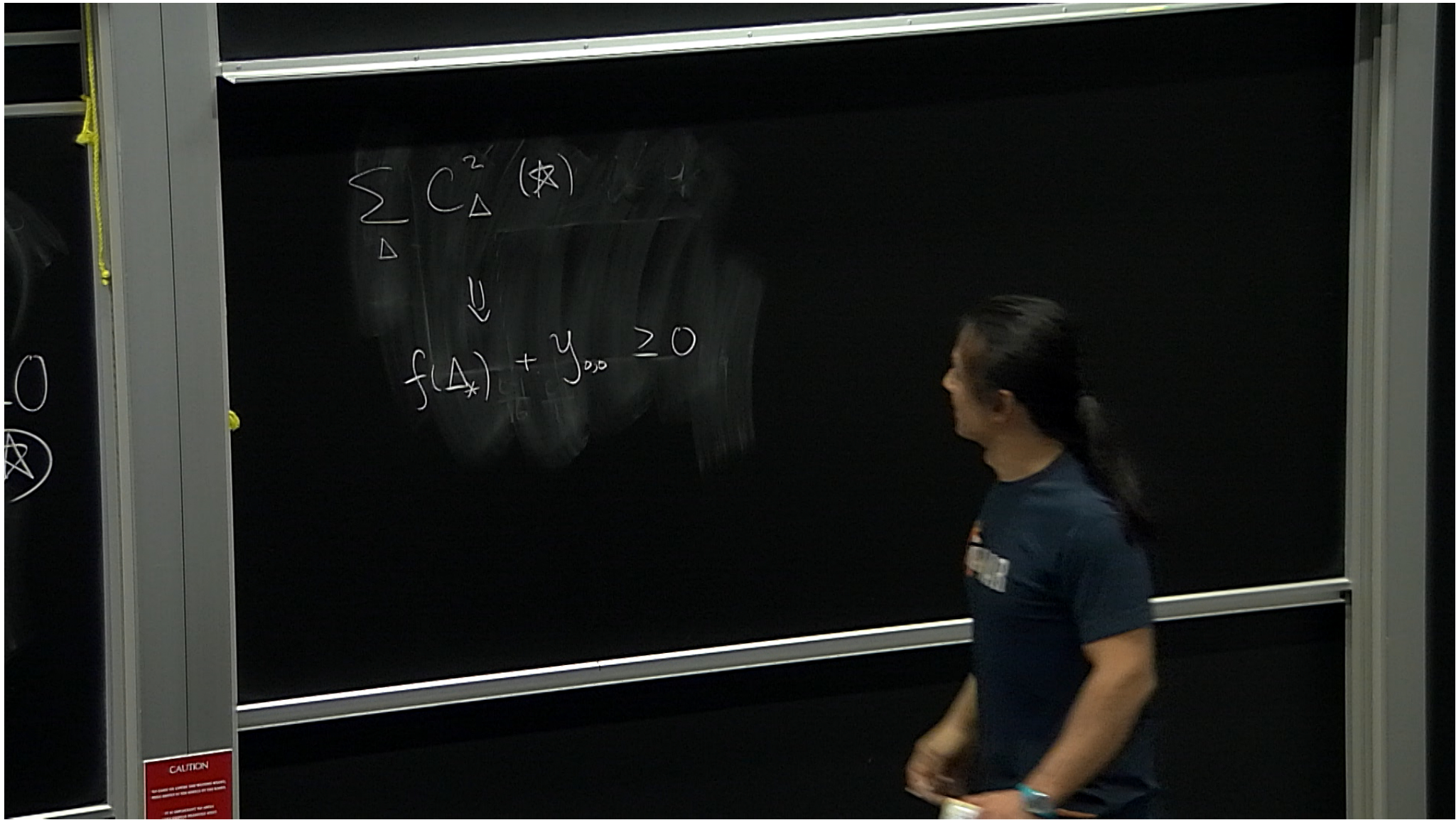
Seek $y_{0,0}, y_{m,n}$ such that

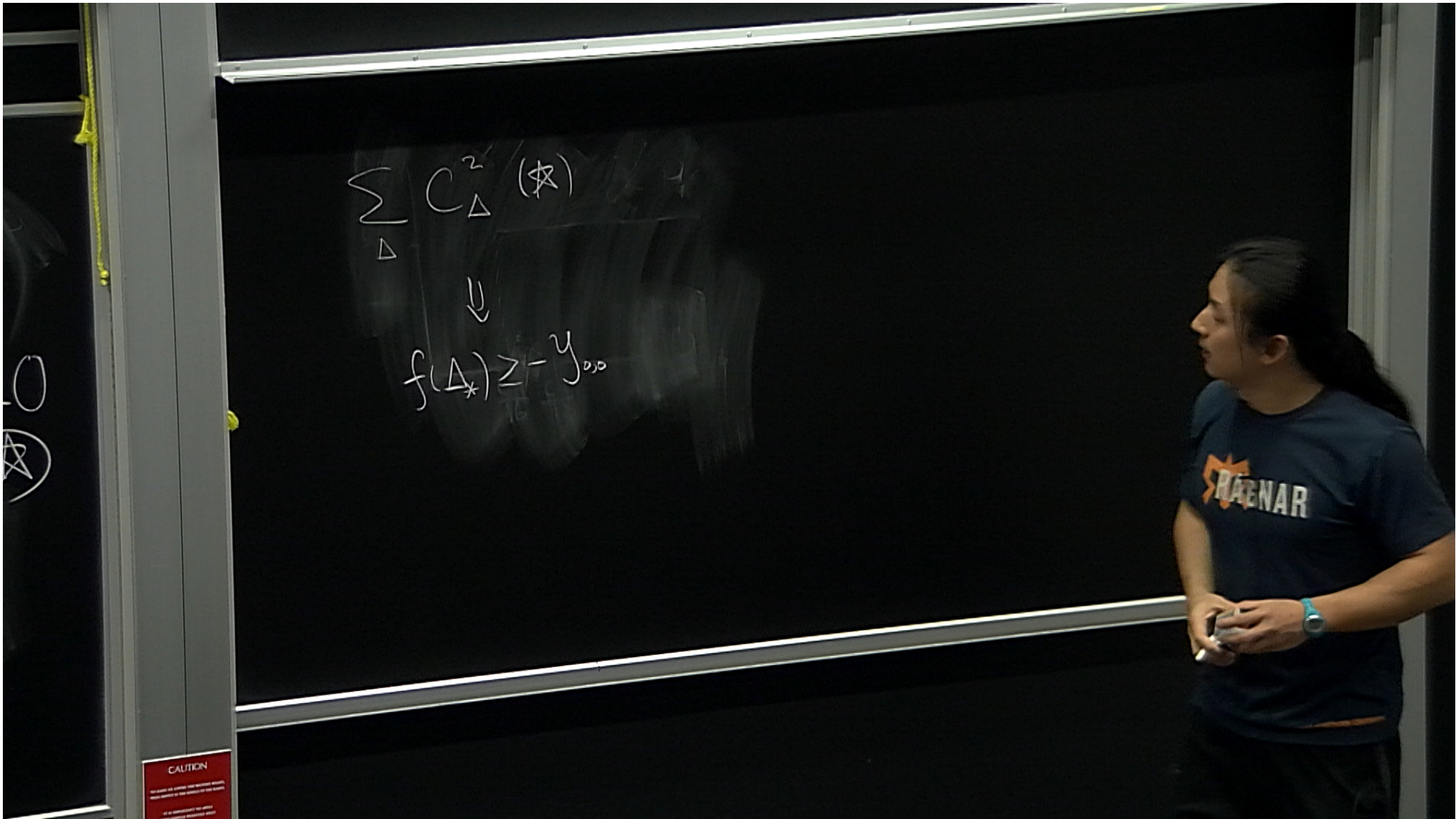
$$\theta(\Delta_* - \Delta) F_{0,0}(\Delta) + y_{0,0} F_{0,0}(\Delta)$$

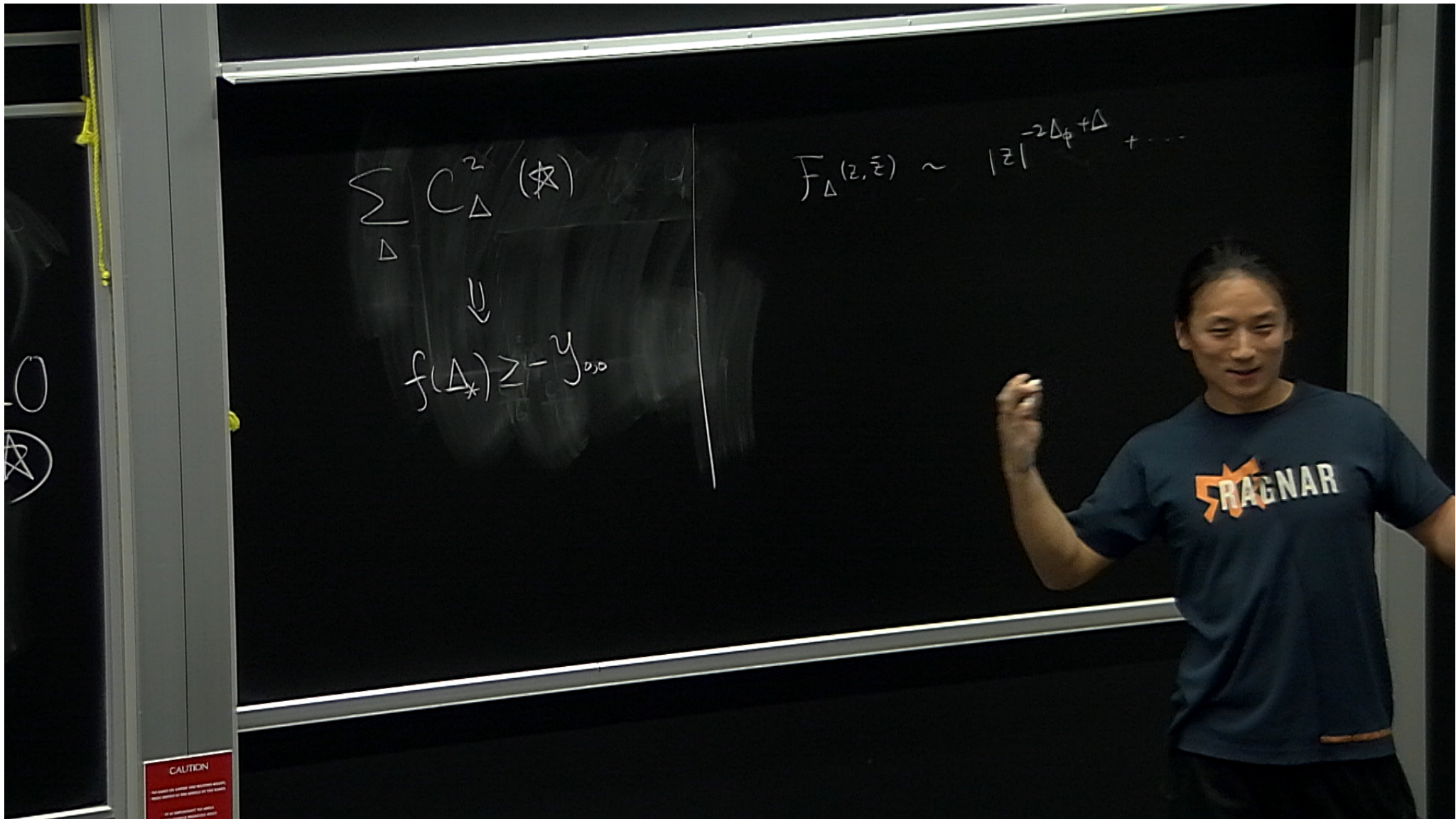
$$+ \sum_{m+n \text{ odd}} y_{m,n} F_{m,n}(\Delta) \geq 0$$

holds for all $\Delta (> 0)$ \star









$$\sum_{\Delta} C_{\Delta}^2 (\star)$$

↓

$$f(\Delta_{\star}) \geq -\gamma_{00}$$

$$F_{\Delta}(z, \bar{z}) \sim |z|^{-2\Delta_{\phi} + \Delta} + \dots$$

RAGNAR

CAUTION
Do not touch the surface with bare hands. Always wear gloves when handling the board.
Do not use sharp objects for writing.

$$\sum_{\Delta} C_{\Delta}^2 (\star)$$



$$f(\Delta_*) \geq -y_{0,0}$$

$$F_{\Delta}(z, \bar{z}) \sim |z|^{-2\Delta_{\phi} + \Delta} + \dots$$

$$\frac{F_{1,0}}{F_{0,0}} \sim (\Delta - 2\Delta_{\phi}) + \dots$$

CAUTION
Do not touch the surface with sharp objects.
Always clean up after use.
Do not use for other purposes.
Do not use for other purposes.

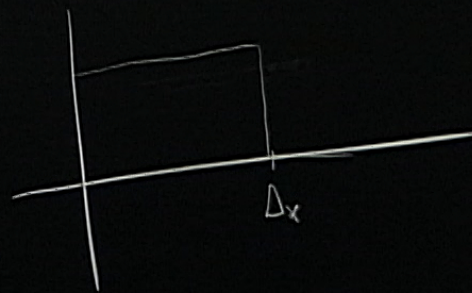
$$\sum_{\Delta} C_{\Delta}^2 (\star)$$

\Downarrow

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CAUTION
No glass or other brittle objects
should be used on this board.
If in contact with the board
the board should be used.

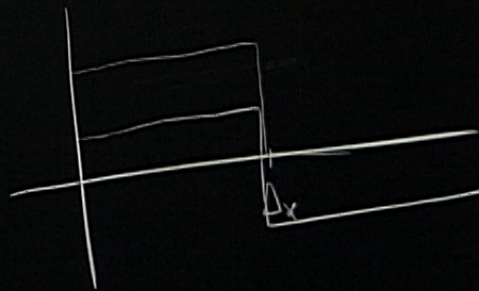
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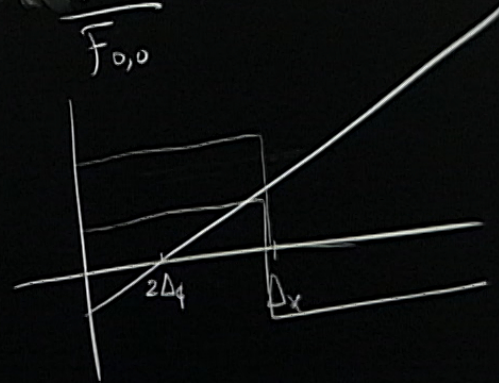
$$\sum_{\Delta} C_{\Delta}^2 (\star)$$

$$\Downarrow$$

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$$\frac{F_{1,0}}{F_{0,0}} \sim (\Delta - 2\Delta_{\phi}) + \dots$$



CAUTION
 No sharp or pointed ends
 Please contact us for more information
 or to report a problem

$$\sum_{\Delta} C_{\Delta}^2 \quad (\star)$$

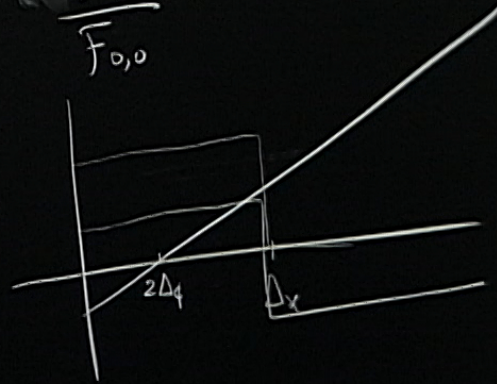
↓

$$f(\Delta_*) \geq -y_{0,0}$$

optimal:
 minimize $y_{0,0}$
 subject to (\star)

$$F_{\Delta}(z, \bar{z}) \sim |z|^{-2\Delta_{\phi} + \Delta} + \dots$$

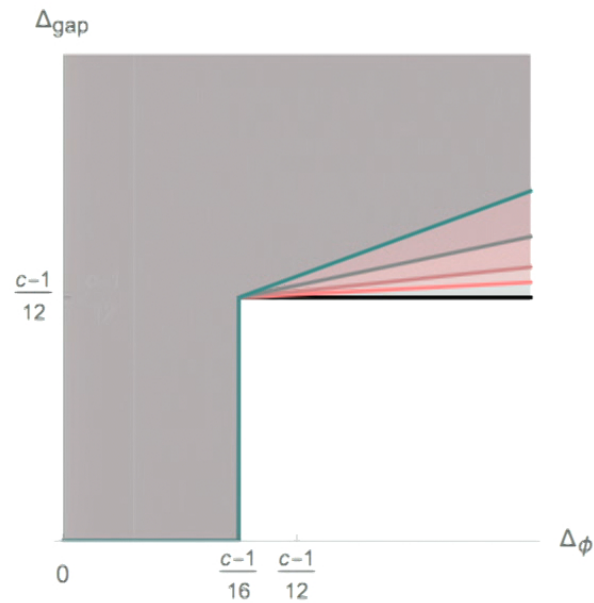
$$\frac{F_{1,0}}{F_{0,0}} \sim (\Delta - 2\Delta_{\phi}) + \dots$$



CAUTION
 No sharp edges. Do not touch. Do not use. Do not use. Do not use.

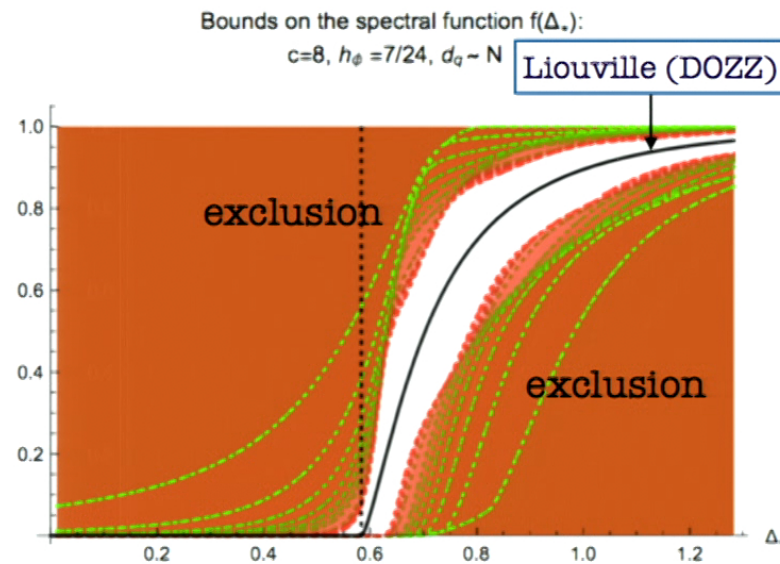
Example of a bound on the OPE gap

Assuming **only scalar** Virasoro primaries in OPE



[van Rees, unpublished; Collier, Lin, XY, unpublished]

Claim: assuming unitarity, $c > 1$, **only scalar** primaries, DOZZ structure constants of Liouville theory are the only solution to the crossing equation.



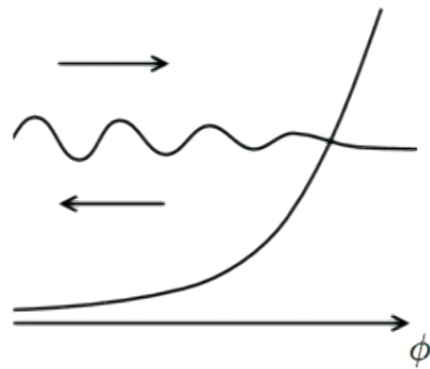
[Collier, Kravchuk, Lin, XY, 1702.00423]

Liouville CFT - a brief recap for millennials

[Seiberg '91, Dorn-Otto '94, Zamolodchikov², '95, Teschner '95, Ponsot-Teschner '99]

$$S_{\text{Liouville}} = \frac{1}{4\pi} \int d^2z \sqrt{g} (g^{mn} \partial_m \phi \partial_n \phi + QR\phi + 4\pi\mu e^{2b\phi})$$

$$c = 1 + 6Q^2 \quad Q = b + b^{-1}$$



Virasoro primary operators take the form

$$\mathcal{V}_\alpha \sim S(\alpha)^{-\frac{1}{2}} e^{2\alpha\phi} + S(\alpha)^{\frac{1}{2}} e^{2(Q-\alpha)\phi}$$

$\phi \rightarrow -\infty$

$$\alpha = \frac{Q}{2} + iP$$

Liouville CFT - a brief recap for millennials

[Seiberg '91, Dorn-Otto '94, Zamolodchikov², '95, Teschner '95, Ponsot-Teschner '99]

Reflection coefficient:

$$S(\alpha) = - (\pi\mu\gamma(b^2))^{(Q-2\alpha)/b} \frac{\Gamma(1 - (Q - 2\alpha)/b)\Gamma(1 - (Q - 2\alpha)b)}{\Gamma(1 + (Q - 2\alpha)/b)\Gamma(1 + (Q - 2\alpha)b)}$$

DOZZ structure constants:

$$\langle \mathcal{V}_{\alpha_1} \mathcal{V}_{\alpha_2} \mathcal{V}_{\alpha_3} \rangle = \prod_{j=1}^3 S(\alpha_j)^{-\frac{1}{2}} \left[\pi\mu\gamma(b^2) b^{2-2b^2} \right]^{\frac{Q-\sum\alpha_i}{b}}$$

$$\times \frac{\Upsilon'_b(0)\Upsilon_b(2\alpha_1)\Upsilon_b(2\alpha_2)\Upsilon_b(2\alpha_3)}{\Upsilon_b(\sum\alpha_i - Q)\Upsilon_b(\alpha_1 + \alpha_2 - \alpha_3)\Upsilon_b(\alpha_2 + \alpha_3 - \alpha_1)\Upsilon_b(\alpha_3 + \alpha_1 - \alpha_2)}$$

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$$

$$\log \Upsilon_b(x) = \int_0^\infty dt t^{-1} \left[\left(\frac{Q}{2} - x \right)^2 e^{-t} - \frac{\sinh^2 \left[\left(\frac{Q}{2} - x \right) \frac{t}{2} \right]}{\sinh \frac{bt}{2} \sinh \frac{t}{2b}} \right], \quad 0 < \operatorname{Re}(x) < \operatorname{Re}(Q)$$

$$\theta(\Delta_* - \Delta) + y_{0,0} + \sum_{\substack{m+n \leq d \\ \leq N}} y_{m,n} \frac{F_{m,n}(\Delta)}{F_{0,0}(\Delta)} \geq 0$$

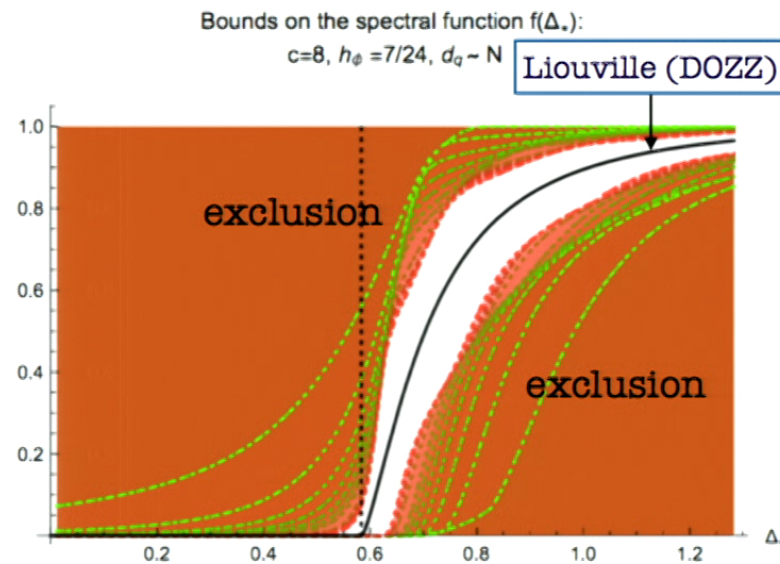
$$\theta(\Delta_* - \Delta) + y_{0,0} + \sum_{\substack{m+n \leq d \\ \leq N}} y_{m,n} \frac{F_{m,n}(\Delta)}{F_{0,0}(\Delta)} \geq 0$$

$$\theta(\Delta_* - \Delta) + y_{0,0} + \sum_{m+n \leq d} y_{m,n} \frac{F_{m,n}(\Delta)}{F_{0,0}(\Delta)} = 0$$

$$\theta(\Delta_* - \Delta) + y_{0,0} + \sum_{m+n \leq d} y_{m,n} \frac{F_{m,n}(\Delta)}{F_{0,0}(\Delta)} = 0$$

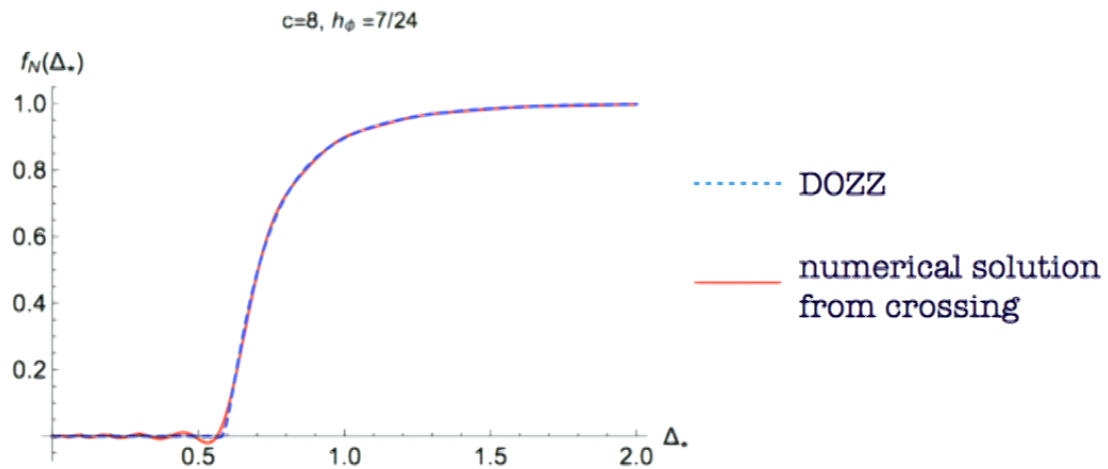
for all Δ

Claim: assuming unitarity, $c > 1$, **only scalar** primaries, DOZZ structure constants of Liouville theory are the only solution to the crossing equation.



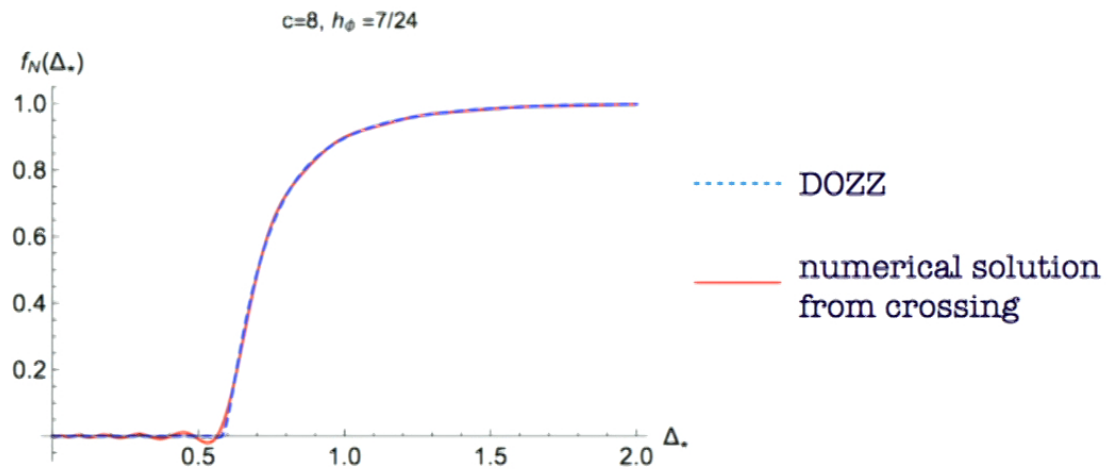
[Collier, Kravchuk, Lin, XY, 1702.00423]

Direct numerical solution for the scalar-only spectral function from truncated crossing equation:



[Collier, Kravchuk, Lin, XY, 1702.00423]
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Direct numerical solution for the scalar-only spectral function from truncated crossing equation:



[Collier, Kravchuk, Lin, XY, 1702.00423]

$$\theta(\Delta_* - \Delta) + y_{0,0} + \sum_{m+n \geq 1} y_{m,n} \frac{F_{m,n}(\Delta)}{F_{0,0}(\Delta)} = 0$$

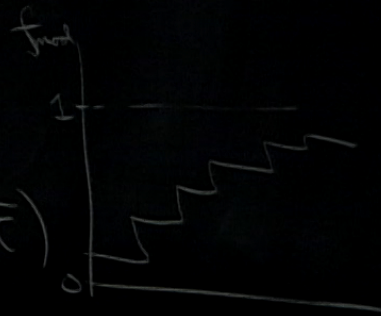
$$Z(\tau, \bar{\tau}) = \sum d_{h,\bar{h}} X_{h,\bar{h}}(\tau, \bar{\tau})$$

$$\tau \rightarrow -1/\tau \quad \tau = i e^{z-1/2}$$

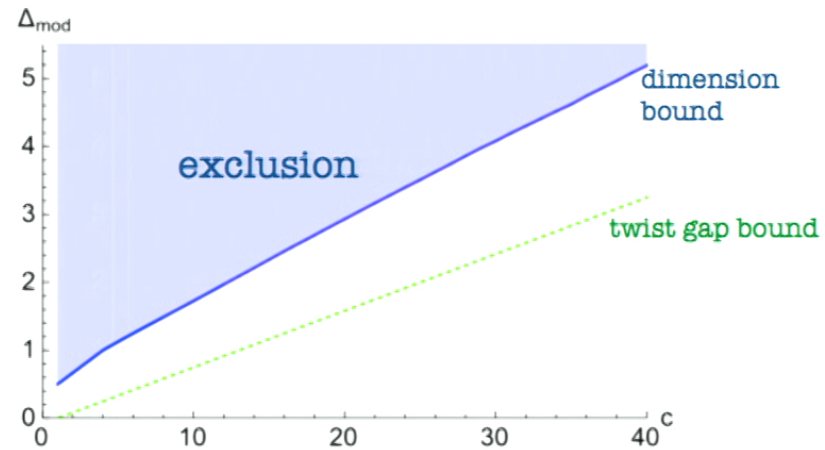
$$\theta(\Delta_* - \Delta) + y_{0,0} + \sum_{m+n \geq 1} y_{m,n} \frac{F_{m,n}(\Delta)}{F_{0,0}(\Delta)} = 0$$

$$Z(\tau, \bar{\tau}) = \sum d_{h,\bar{h}} X_{h,\bar{h}}(\tau, \bar{\tau})$$

$$\tau \rightarrow -1/\tau \quad \tau = i e^{z-1/2}$$



Upper bound on gap in operator dimension

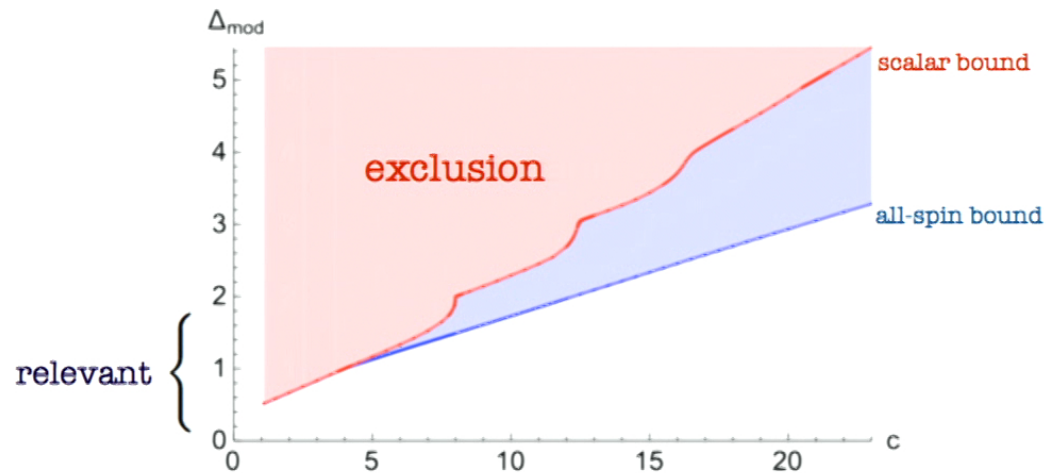


Conjecture: the slope $\frac{d\Delta_{\text{mod}}(c)}{dc}$ is monotonically non-increasing.

Asymptotic slope? $\frac{1}{12} \leq \lim_{c \rightarrow \infty} \frac{d\Delta_{\text{mod}}(c)}{dc} < \frac{1}{9}$

[Collier-Lin-XY '16]

Upper bound on dimension of lightest scalar ($c < 25$)



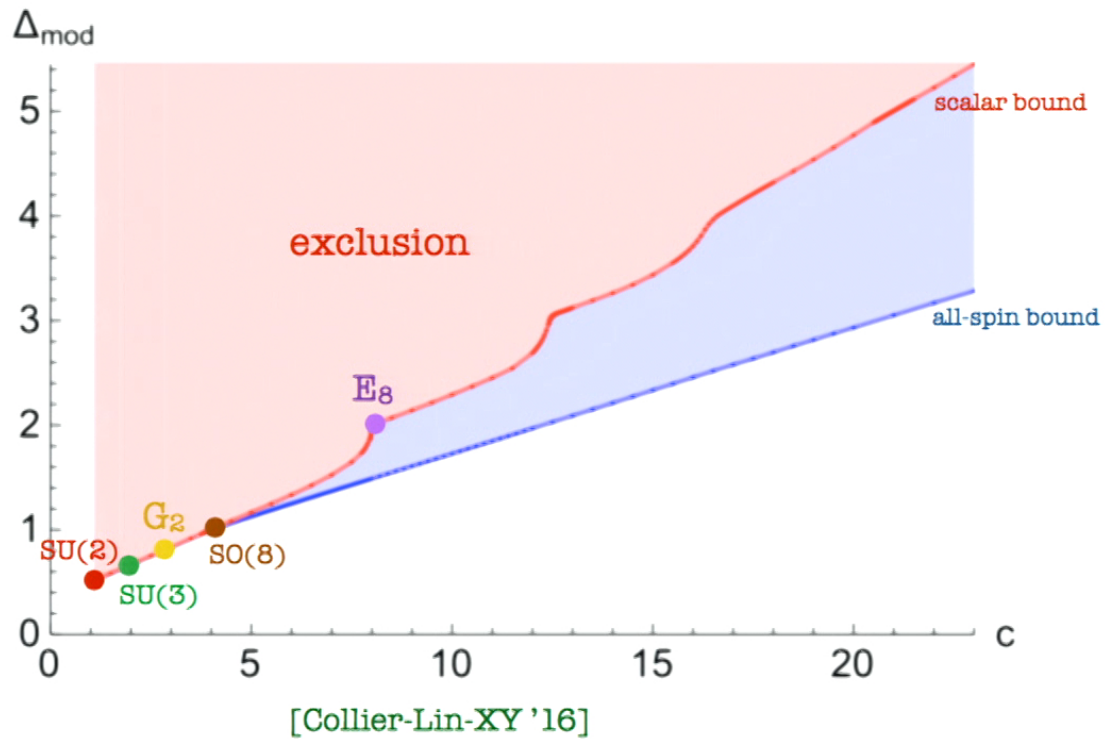
Implies all unitary CFTs with $c < 8$ must admit relevant deformation.

“Perfect metal” does not exist for $c \leq 8$.

[Collier-Lin-XY '16]

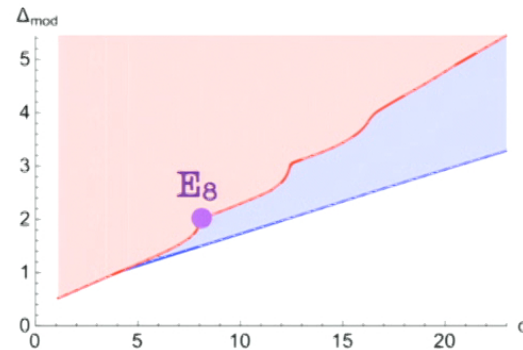
Upper bound on dimension of lightest scalar ($c < 25$)

For some values of c , bound saturated by WZW models at level 1.



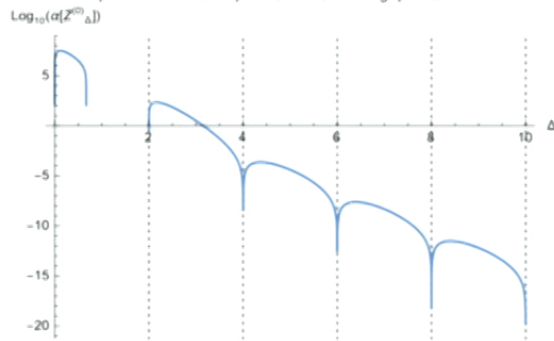
Extremal spectrum

[Collier-Lin-XY '16]



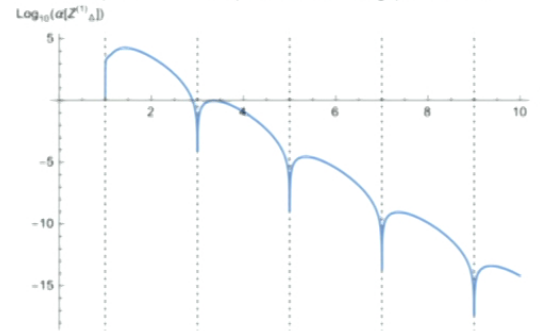
When bound is saturated, zeros of the optimal linear functional acting on Virasoro characters give the spectrum

Optimal functional: spin=0, $c=8$, scalar gap = 2, $N=31$



spin 0 spectrum

Optimal functional: spin=1, $c=8$, scalar gap = 2, $N=31$



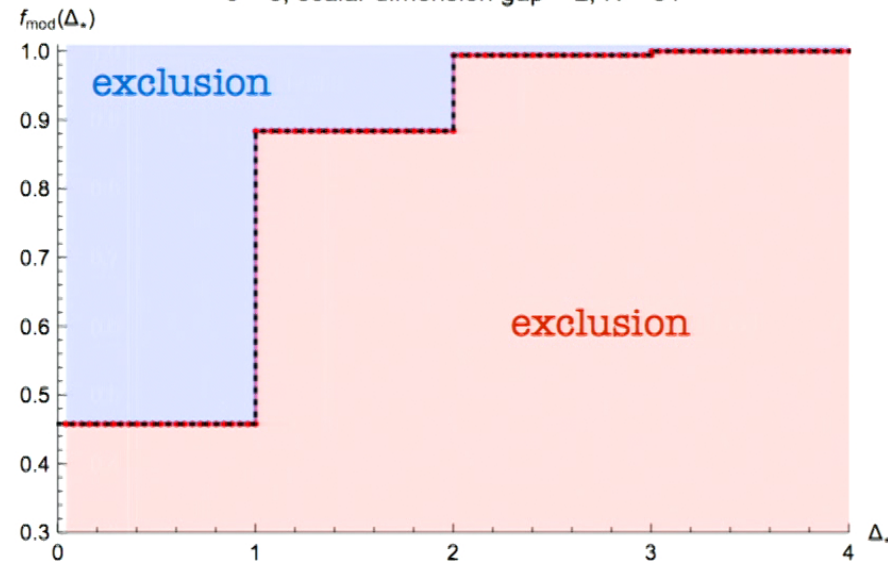
spin 1 spectrum

Modular spectral function

$$f_{\text{mod}}(x) = \frac{1}{Z(\tau = -\bar{\tau} = i)} \sum_{s, \Delta \leq x} d_{\Delta, s} \chi_{\Delta, s}(\tau = -\bar{\tau} = i)$$

(assuming maximal gap)

$c = 8$, scalar dimension gap = 2, $N = 31$



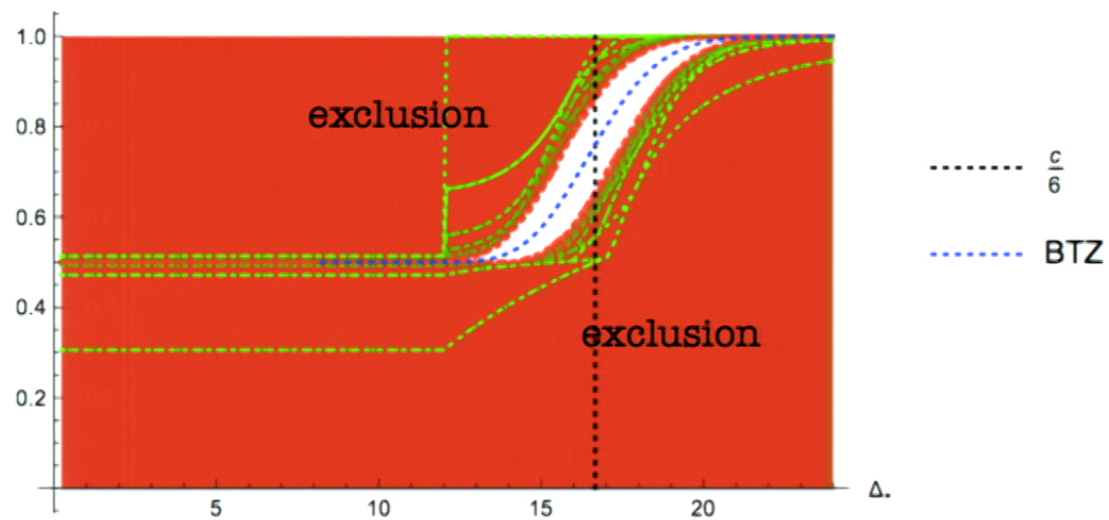
[Collier, Kravchuk, Lin, XY, 1702.00423]

Modular spectral function

Large c , large gap limit?

Bounds on the modular spectral function:

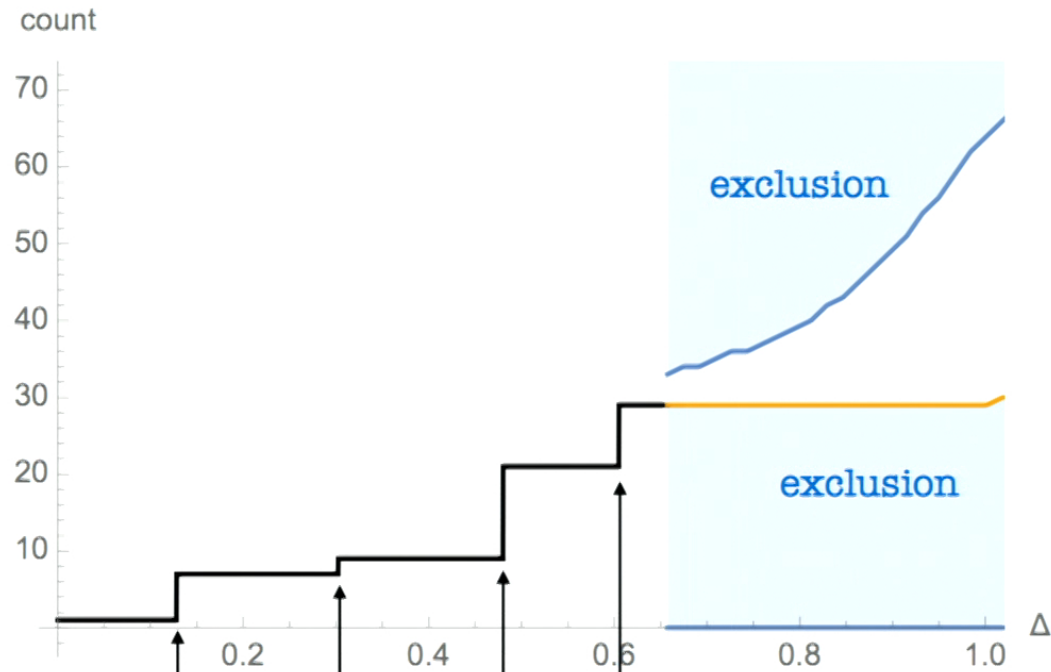
$c=100$, dimension gap = 12 (close to maximal gap)



[Collier, Kravchuk, Lin, XY, 1702.00423]

Primary counting function

Coupled Potts model [Dotsenko et al '98] (preliminary)



low lying spectrum from conformal perturbation theory

$$\left(C = \frac{4}{5} \right)^{\otimes 3}$$

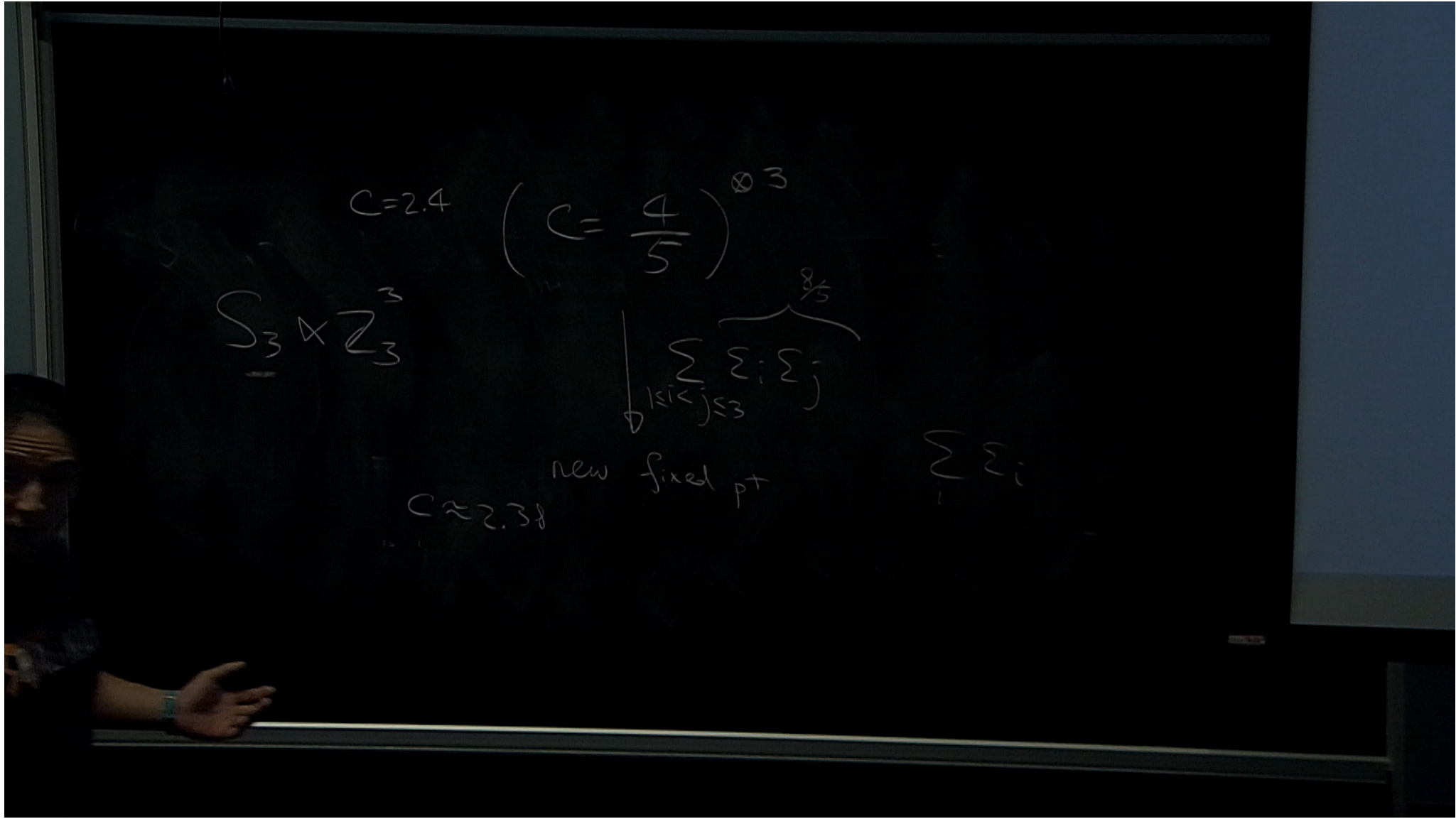
$$\downarrow \sum_{1 \leq i < j \leq 3} \sum_i \sum_j \frac{8}{5}$$

$$S_3 \times \mathbb{Z}_3$$

$$\left(c = \frac{4}{5} \right)^{\otimes 3}$$

$$\downarrow \sum_{1 \leq i < j \leq 3} \xi_i \xi_j$$

$$\sum_i \xi_i$$



$$C=2.4 \quad \left(C = \frac{4}{5} \right) \otimes 3$$

$$S_3 \times Z_3$$

$$\sum_{1 \leq i < j \leq 3} \xi_i \xi_j$$

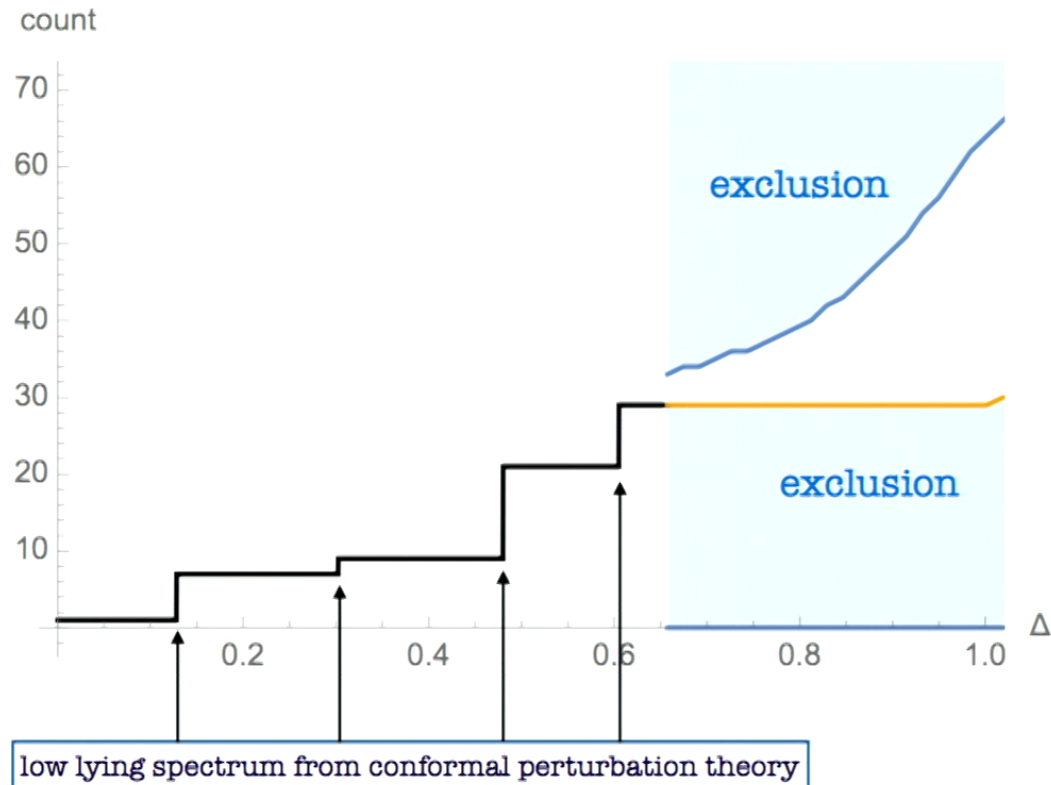
new fixed pt

$$C \approx 2.38$$

$$\sum_i \xi_i$$

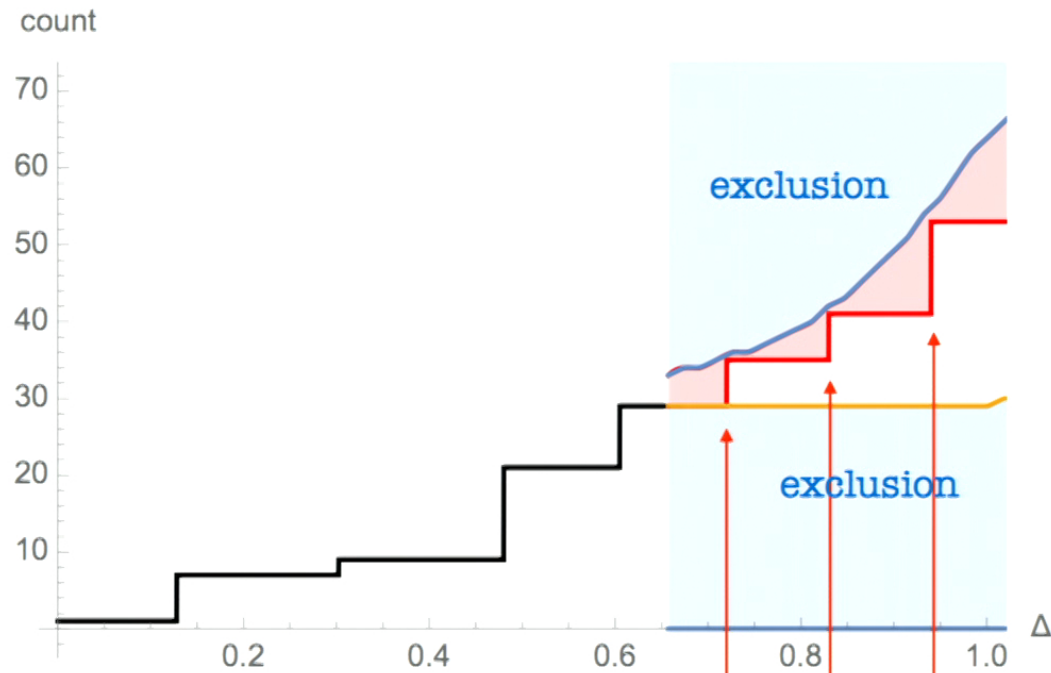
Primary counting function

Coupled Potts model [Dotsenko et al '98] (preliminary)



Primary counting function

Coupled Potts model [Dotsenko et al '98] (preliminary)



lower bounds on the next three scaling dimensions