Title: Abstract quantum theory

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Abstract: <p>What is the essence of quantum theory? In the present talk I want to approach this question from a particular operationalist perspective. I take advantage of a recent convergence between operational approaches to quantum theory and axiomatic approaches to quantum field theory. Removing anything special to particular physical models, including underlying notions of space and (crucially) time, what remains is what I shall call "abstract quantum theory". This embeds into a more general framework that also includes classical theory, with classical and quantum theory representing two extremes in a spectrum of possible theories. I shall present this within a "hierarchy of abstraction", where adding structure leads to more specialized frameworks. In particular, adding topological spacetime and locality leads to the recently proposed positive formalism and recovers quantum field theory. Further rigidifying time and adding causality recovers the standard formulation of quantum theory. There are other promising specializations such as Hardy's proposal to work in op-space.</p>

Abstract quantum theory

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Origins

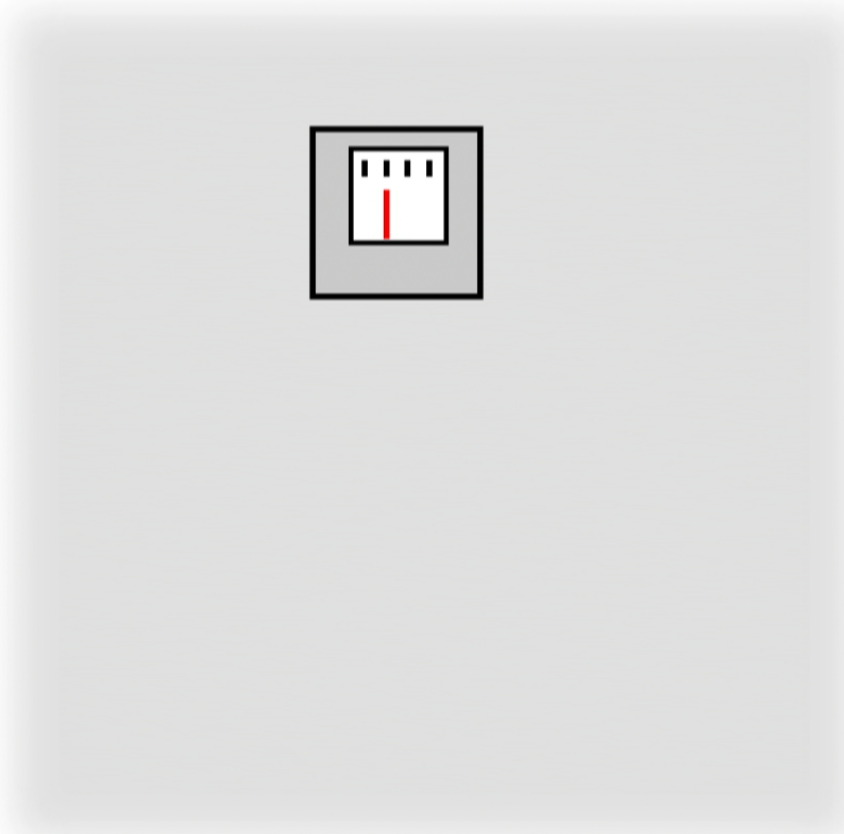
Quantum Field Theory

- Dirac (1933): *generalized transformation functions*
- Feynman (1948): *path integral*
- Witten, Segal, Atiyah (1988): *TQFT* aka. *FQFT*

Foundations of QT

- *convex operational framework*
- quantum information theory: *quantum operation, channel*
- specific approaches:
  - Hardy: *causaloid* (2005), *formalism locality* (2010)
  - Brukner, Oreshkov, ...: *process matrix* (2011), *timeless approach* (2014)
  - Selinger, ...: *categorical approach* (2005)
  - ...

Processes and interfaces



Operational approach:
Fundamental notions

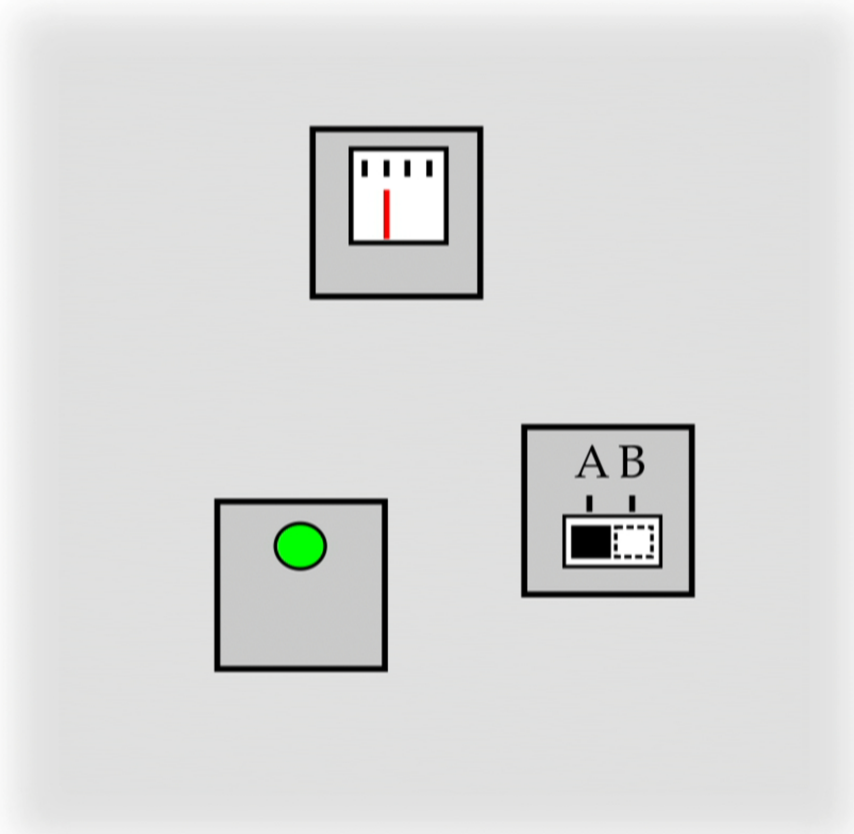
- experiment
- measurement
- observation
- preparation
- intervention

Subsume instance as:

- **process**

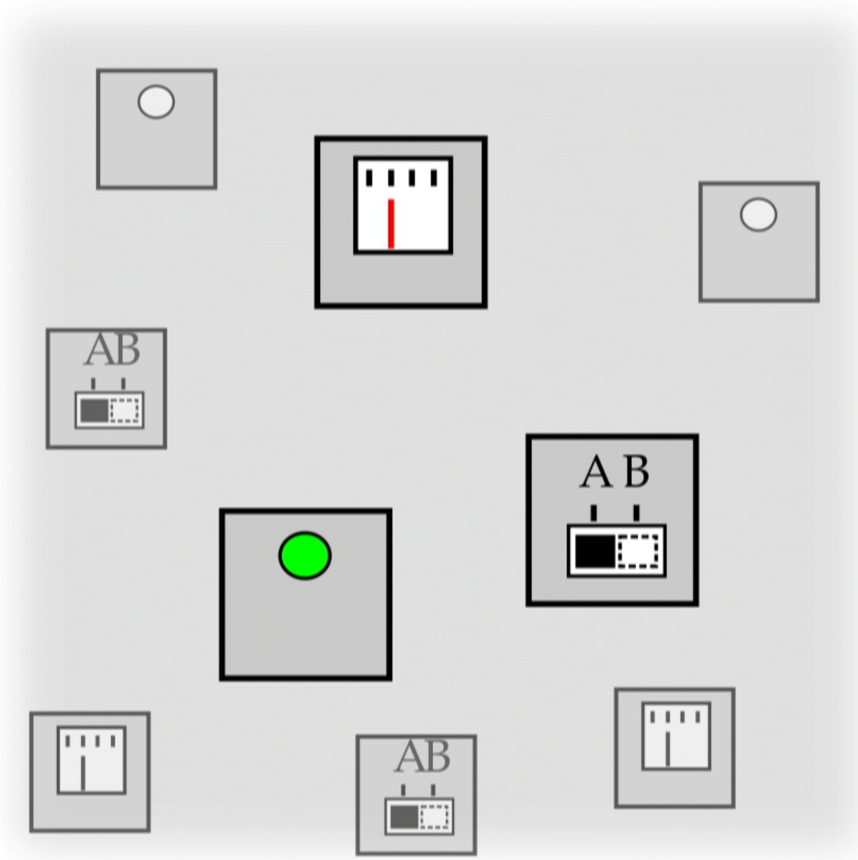
Processes have **outcomes**.
Represent processes as **boxes**.

Processes and interfaces



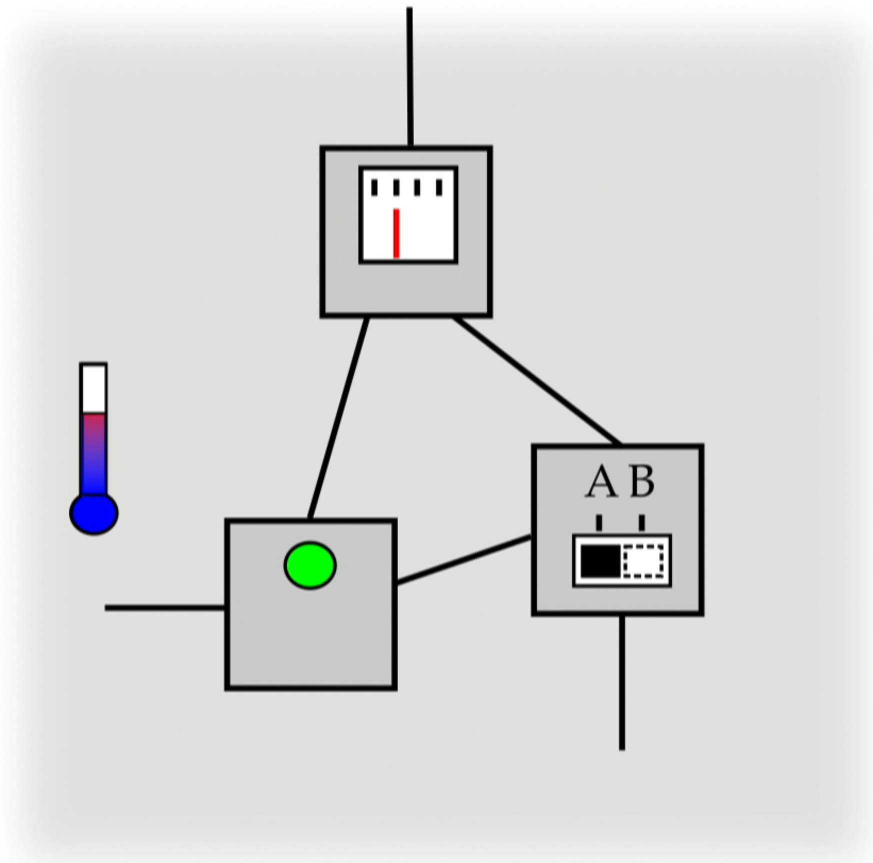
Processes are not isolated. Outcomes depend on other processes. We want to predict **correlations**.

Processes and interfaces



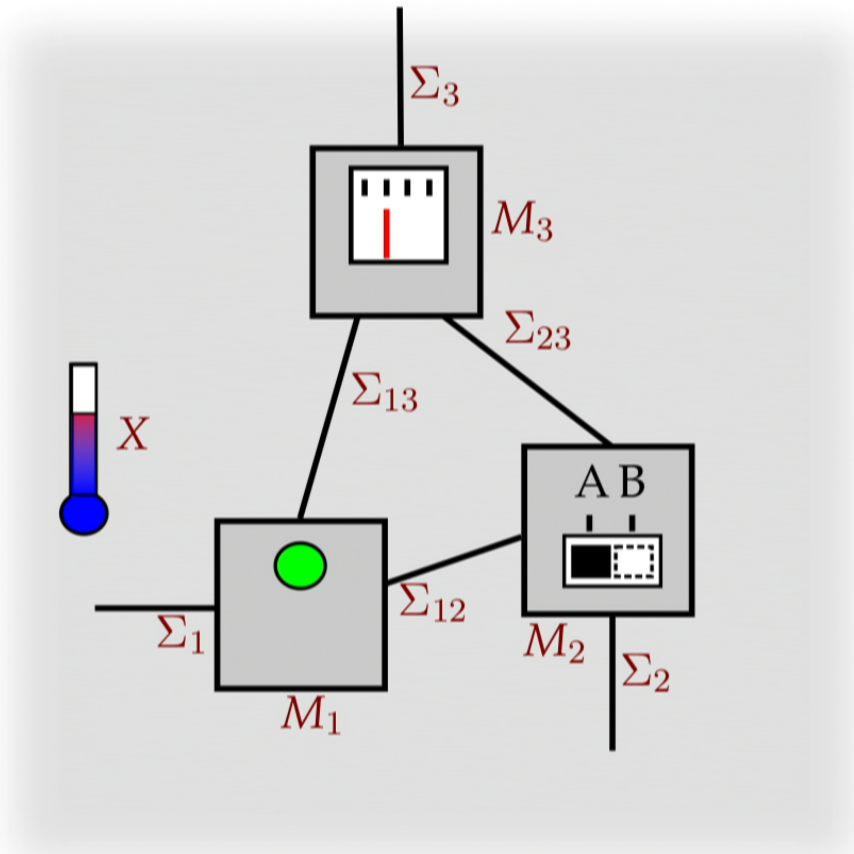
The outcome of a given set of processes depends generally on a large number of other processes.

Processes and interfaces



We introduce the notion of **interface** to model interaction between processes. An interface encodes communication or information exchange between processes. we depict this as a **link**.

Processes and interfaces



Processes are of specific **types**.

Interfaces are of a specific **type** depending on the processes which they connect.

We indicate this with **labels**.

Processes and probes

Associated to each process of type M is a space $\mathcal{P}_M$ of **probes**.

A **probe** provides a finer description of a process, specifying e.g. a

- specific experimental outcome
- specific apparatus settings

There is always a **null-probe** $\boxslash \in \mathcal{P}_M$, representing the absence of any apparatus, observation or intervention.

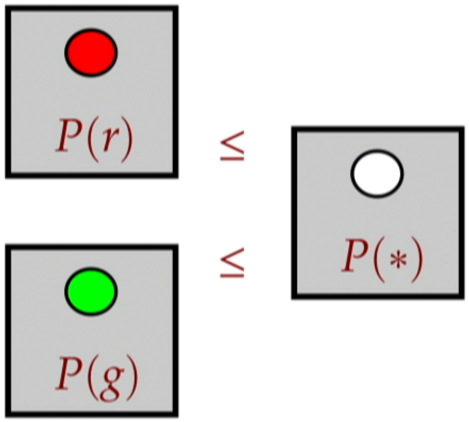
Hierarchies of probes

Probes form hierarchies of generality. This induces a **partial order** on the space of probes $\mathcal{P}_M$.

Consider an apparatus with one light that shows either `red` or `green`, encoded in three different probes:

- $P(r)$ for outcome `red`
- $P(g)$ for outcome `green`
- $P(*)$ for an unspecified outcome

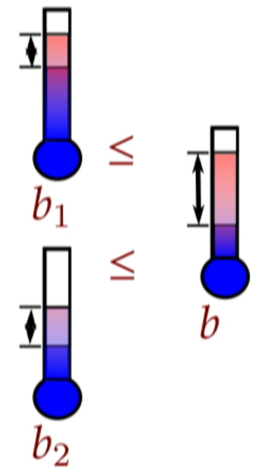
The unspecified state is more **general** than the others. Encode this in a **partial order** on $\mathcal{P}_M$, setting $P(r) \leq P(*)$ and $P(g) \leq P(*)$.



Hierarchies of boundary conditions

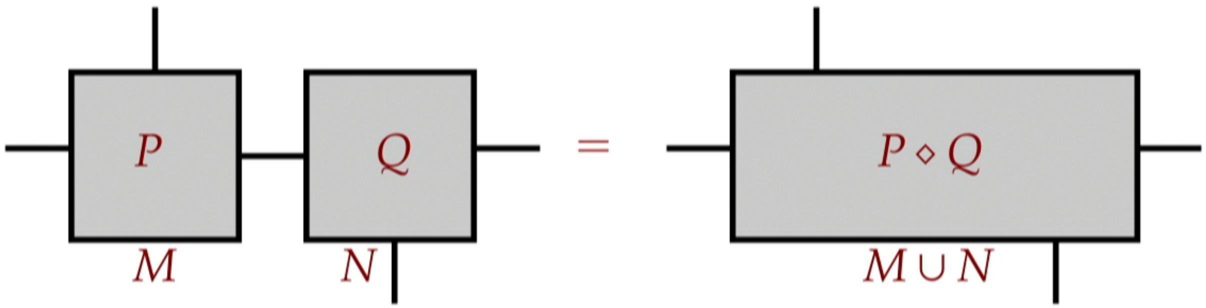
Boundary conditions also form **hierarchies of generality**. This gives rise to a **partial order** on $\mathcal{B}_M^+$. Here:

$$b_1 \leq b$$
$$b_2 \leq b$$



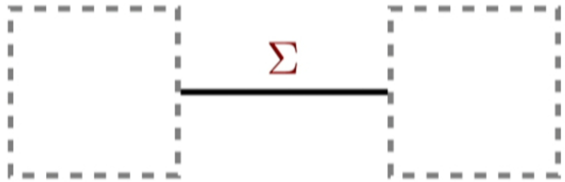
Composition

A set of processes joined by interfaces may be considered itself a process. Say we have a process of type M and one of type N. We say the **composite** process has type $M \cup N$.



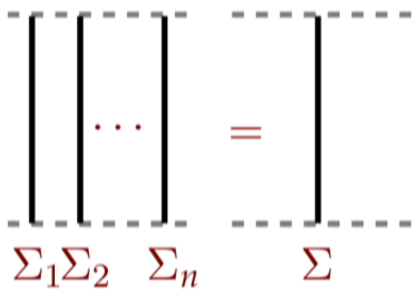
This induces a composition of associated probes $P \in \mathcal{P}_M$ with $Q \in \mathcal{P}_N$.
We write for the composite probe $P \diamond Q \in \mathcal{P}_{M \cup N}$.
This yields a **composition map** $\diamond : \mathcal{P}_M \times \mathcal{P}_N \to \mathcal{P}_{M \cup N}$.

Interfaces and boundary conditions



We associate to each interface Σ a space of **boundary conditions** $\mathcal{B}_{\Sigma}^{+}$. This parametrizes possible signals/information exchange between adjacent processes.

Interfaces between the same pair of processes can be combined arbitrarily. Write: $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_n$.

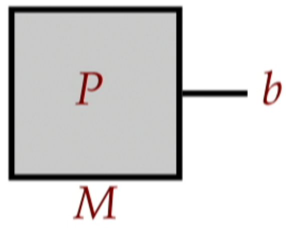


Induces a map between spaces of boundary conditions:
$\mathcal{B}_{\Sigma_1}^{+} \times \mathcal{B}_{\Sigma_2}^{+} \times \cdots \times \mathcal{B}_{\Sigma_n}^{+} \to \mathcal{B}_{\Sigma}^{+}$

We denote the joint interface for a process of type M by ∂M.

Values

Consider a process of type M.

$[\![P,b]\!]_M =$ 

To a **probe** $P \in \mathcal{P}_M$ and a **boundary condition** $b \in \mathcal{B}^+_{\partial M}$ we associate a **value** $[\![P,b]\!]_M$. We shall take this to be a **real number**. Formally, there is a **pairing** $[\![\cdot,\cdot]\!]_M : \mathcal{P}_M \times \mathcal{B}^+_{\partial M} \to \mathbb{R}$.

$[\![P,b]\!]_M \in \mathbb{R}^+$ quantifies **compatibility** between the apparatus or outcome represented by the (primitive) probe $P \in \mathcal{P}^+_M$ and the **boundary condition** $b \in \mathcal{B}^+_{\partial M}$.

Pairing and partial order structures are compatible:

$$P \le Q \quad \Longleftrightarrow \quad [\![P,b]\!]_M \le [\![Q,b]\!]_M \quad \forall b \in \mathcal{B}^+_{\partial M}$$
$$b \le c \quad \Longleftrightarrow \quad [\![P,b]\!]_M \le [\![P,c]\!]_M \quad \forall P \in \mathcal{P}^+_M$$

From values to measurements

A measurement is encoded by at least two **probes**:

- One **non-selective probe** Q encodes the measurement apparatus.
- One **selective probe** P encodes the measurement apparatus with a selected outcome.

$[\![Q,b]\!]_M \in \mathbb{R}^+$ quantifies **compatibility** of the **boundary condition** $b \in \mathcal{B}^+_{\partial M}$ with the presence of the **apparatus**.

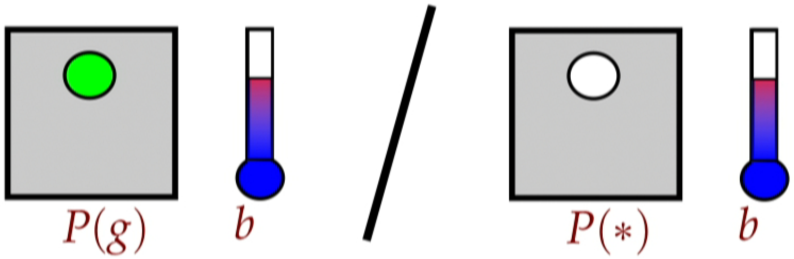
$[\![P,b]\!]_M \in \mathbb{R}^+$ quantifies **compatibility** of the **boundary condition** $b \in \mathcal{B}^+_{\partial M}$ with the presence of the apparatus with **selected outcome**.

Measurement probabilities

In M consider the probe $P(*) \in \mathcal{P}_M^+$ encoding a measurement device and $P(g)$ encoding in addition a selected outcome.

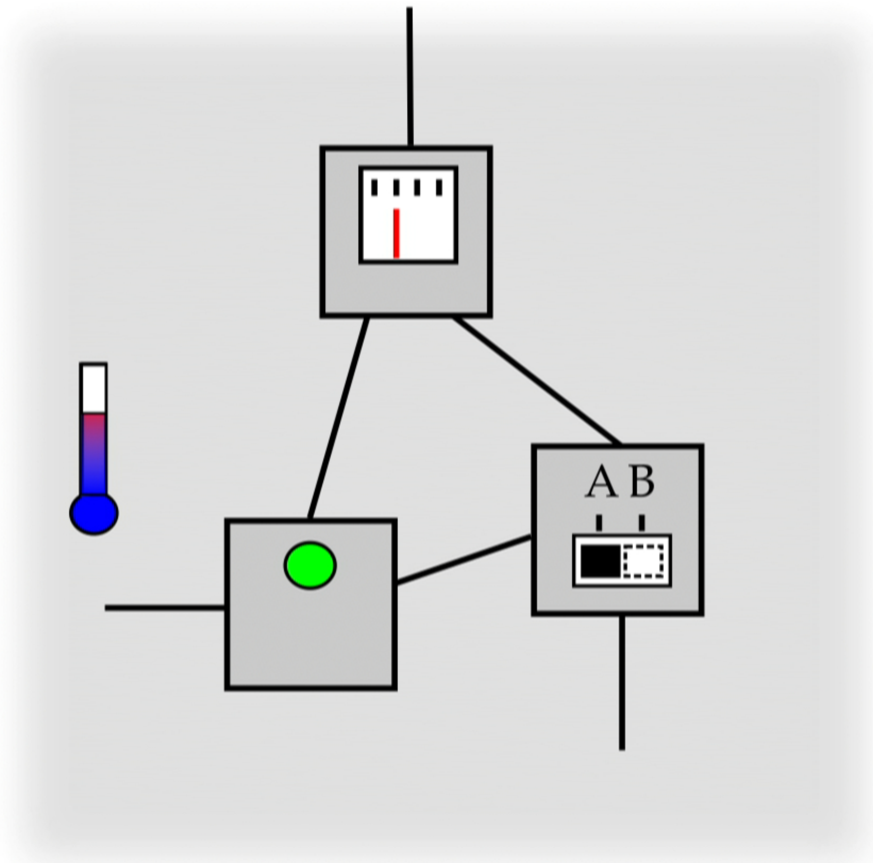
Given $b \in \mathcal{B}_{\partial M}^+$ the **probability** Π for an affirmative outcome is:

$$\Pi = \frac{[\![P(g), b]\!]_M}{[\![P(*), b]\!]_M}$$



Since $0 \leq P(g) \leq P(*)$ we have $0 \leq \Pi \leq 1$ (if $[\![P(*), b]\!]_M \neq 0$).

Processes and interfaces



We introduce the notion of **interface** to model interaction between processes. An interface encodes communication or information exchange between processes. we depict this as a **link**.

Convexity

In a probabilistic setting it makes sense to combine different probes probabilistically, even when they correspond to different experimental situations. Say we have probes $P_1, \ldots, P_n$ and probabilities $p_1, \ldots, p_n$ such that $\sum_k p_k = 1$. Then we can consider $P := \sum_k p_k P_k$ as a probe.

Since an arbitrary real multiple of a probe is a probe, this equips the space $\mathcal{P}_M$ of probes with the structure of a **real vector space**. The subset of primitive probes $\mathcal{P}_M^+ \subset \mathcal{P}_M$ is a **positive cone** making $\mathcal{P}_M$ into a **partially ordered vector space**.

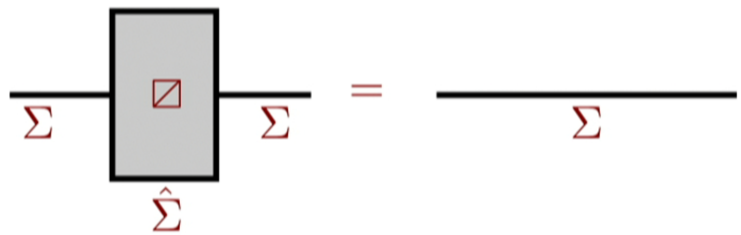
Similarly, the set of boundary conditions $\mathcal{B}_\Sigma^+$ forms a **positive cone** in the **partially ordered vector space** $\mathcal{B}_\Sigma$ generated by it. We call this the space of **generalized boundary conditions**.

We extend the pairing, $[\![\cdot,\cdot]\!]_M : \mathcal{P}_M \times \mathcal{B}_{\partial M} \to \mathbb{R}$

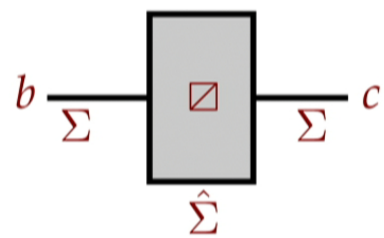
Slice processes and inner product

For any type Σ of **interface** we postulate a type of **slice process** $\hat{\Sigma}$:

- $\partial\hat{\Sigma} = \Sigma \cup \Sigma$
- the **null probe** "passes signals through"



Putting **boundary conditions** on the two sides allows evaluation. This yields an **inner product** $\mathcal{B}_\Sigma \times \mathcal{B}_\Sigma \to \mathbb{R}$ on the space of boundary conditions.



$$(\!|b, c|\!)_\Sigma := [\![\boxslash, b \otimes c]\!]_{\hat{\Sigma}}$$

This should be **symmetric** and **positive-definite**.

Values

Consider a process of type M.

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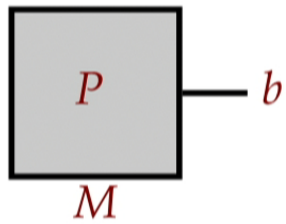

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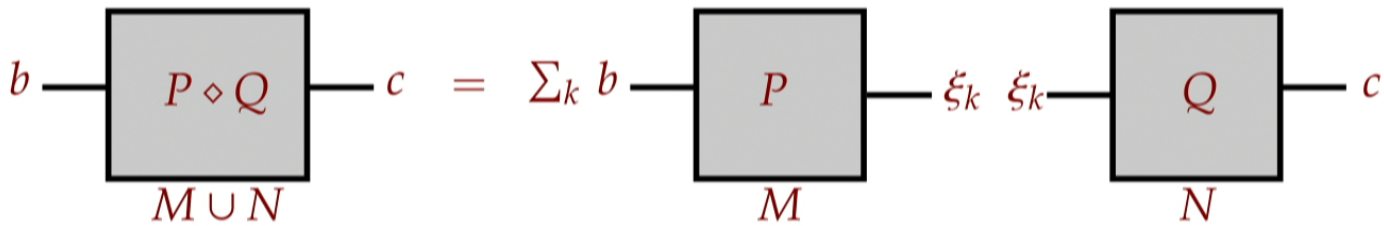
Composition rule for probes

As a generalization we obtain the **composition rule for probes**.



$$[\![P \diamond Q, b \otimes c]\!]_{M \cup N} = \sum_{k \in I} [\![P, b \otimes \xi_k]\!]_M \, [\![Q, \xi_k \otimes c]\!]_N$$

Here, $\{\xi_k\}_{k \in I}$ is an ON-basis of $\mathcal{B}_\Sigma$.

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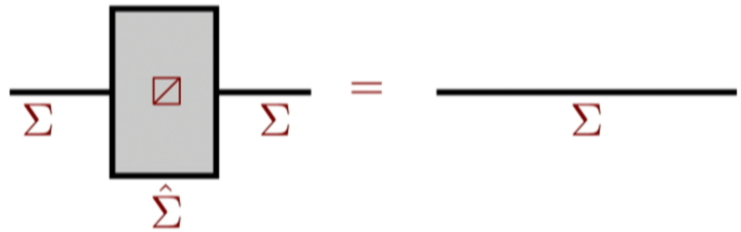

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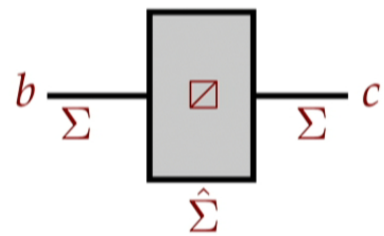

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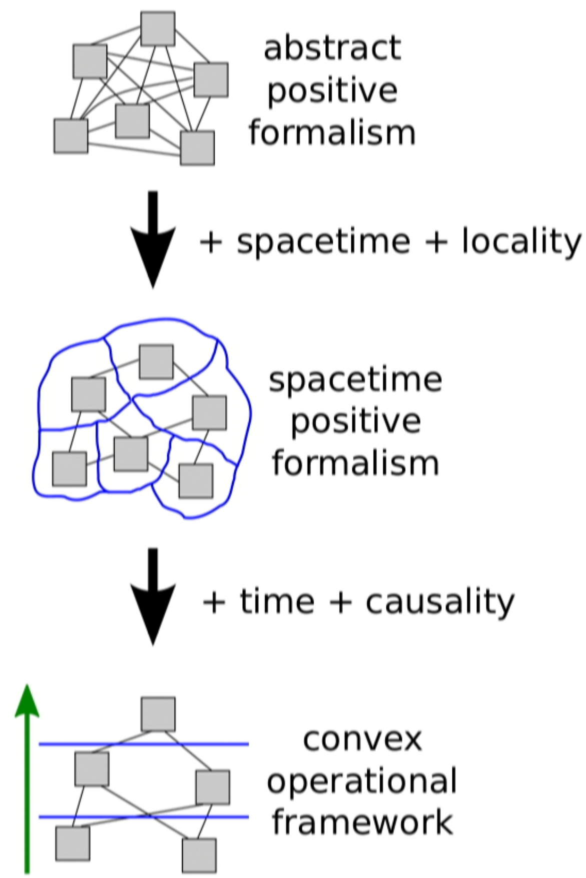
This should be **symmetric** and **positive-definite**.

(Abstract) Positive Formalism

We obtain an **axiomatic framework** for encoding physical theories that requires no notion of space or time.

Adding structure as necessary (but no new rules!) this specializes to:

- the **convex operational framework**
- the **spacetime positive formalism**
- "abstract" quantum theory
- **quantum field theory** (as axiomatized in the sense of Segal)
- the **standard formulation of quantum theory**


abstract
positive
formalism
+ spacetime + locality
spacetime
positive
formalism
+ time + causality
convex
operational
framework

classical
(lattices)

quantum
(anti-lattices)

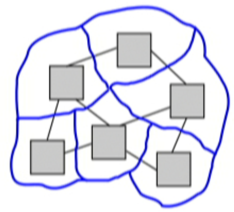
abstract classical statistical theory

abstract positive formalism

abstract quantum theory

+ spacetime + locality

spacetime statistical field theory

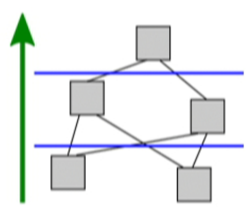


spacetime positive formalism

general boundary formulation / axiomatic QFT

+ time + causality

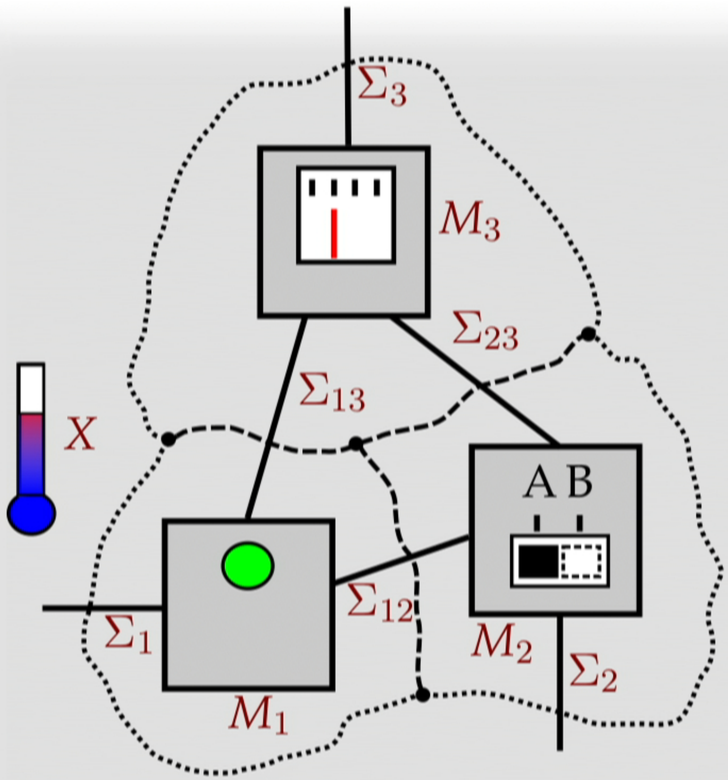
statistical mechanics



convex operational framework

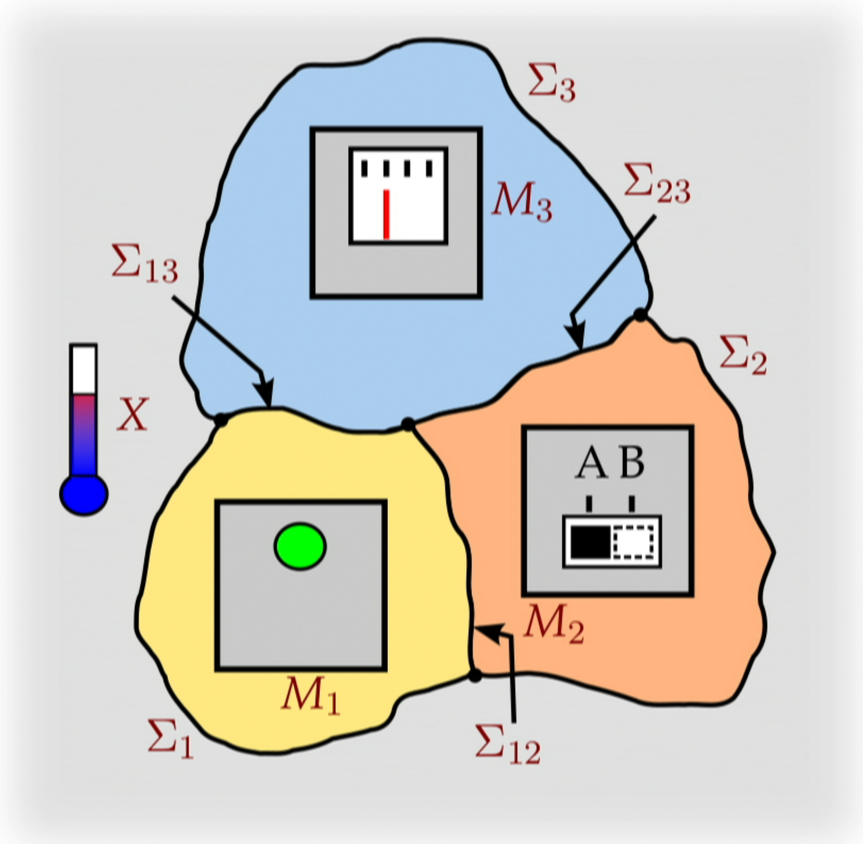
standard formulation of quantum theory

Adding spacetime and locality



Spacetime locality provides a powerful organizing principle. Processes only interface with adjacent processes. This decreases considerably the interconnectivity of the graph.

Adding spacetime and locality



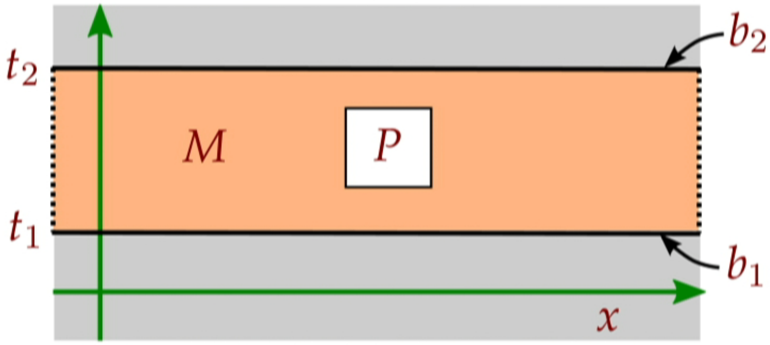
We may forget about the graph and identify **process types** with **regions** and **interface types** with **hypersurfaces**.

This framework is called the **spacetime version** of the **positive formalism**.

Time-evolution

Specialize to a global factorizing spacetime $\mathbb{R} \times \Sigma$ and restrict the spacetime system to **equal-time hyperplanes** Σ_t and **time-interval regions** $[t_1, t_2] = [t_1, t_2] \times \Sigma$.

Write $\mathcal{B}_t := \mathcal{B}_{\Sigma_t}$ and call this the (generalized) **state space** at time t.



Consider probe $P \in \mathcal{P}_{[t_1,t_2]}$. Define the **probe map** $\tilde{P} : \mathcal{B}_{t_1} \to \mathcal{B}_{t_2}$ via

$$(\!| b_2, \tilde{P}(b_1) |\!)_{t_2} = [\![P, b_1 \otimes b_2]\!]_{[t_1,t_2]}, \qquad \forall b_1 \in \mathcal{B}_{t_1}, b_2 \in \mathcal{B}_{t_2}.$$

Primitive probe maps and positivity

Probe maps for **primitive probes** are **positive**. They map proper states to proper states, $\mathcal{B}^+_{t_1} \to \mathcal{B}^+_{t_2}$. They even have the stronger property of **boundary positivity**,

$$\sum_i (\!| c_i, \tilde{P}(b_i) |\!)_{t_2} \geq 0 \qquad \text{if} \quad \sum_i b_i \otimes c_i \in \mathcal{B}^+_{\partial[t_1,t_2]} \supseteq \mathcal{B}^+_{t_1} \otimes \mathcal{B}^+_{t_2}$$

In classical theory, positivity and boundary positivity are equivalent.

In quantum theory, boundary positivity is **complete positivity**.

Time-evolution maps

The probe map associated to the **null-probe** is the **time-evolution map** $T_{[t_1,t_2]} : \mathcal{B}_{t_1} \to \mathcal{B}_{t_2}$,

$$(\!| b_2, T_{[t_1,t_2]}(b_1) |\!)_{t_2} = [\![\boxslash, b_1 \otimes b_2]\!]_{[t_1,t_2]}$$

The time-evolution maps compose for $t_1 \le t_2 \le t_3$ as,

$$T_{[t_1,t_3]} = T_{[t_3,t_2]} \circ T_{[t_1,t_2]}.$$

Usually, time-evolution preserves the state space. Thus, $\mathcal{B} = \mathcal{B}_t$. Probe maps become **operators** on $\mathcal{B}$. Assume this from now on.

Many systems are also **time-translation symmetric** meaning that $T_{[t_1,t_1+\Delta]} = T_{[t_2,t_2+\Delta]} = T_\Delta$. We then get a **one-parameter semigroup** of boundary positive operators,

$$T_{\Delta_1+\Delta_2} = T_{\Delta_1} \circ T_{\Delta_2}.$$

The state of maximal uncertainty

Recall that the **boundary conditions** form a **hierarchy of generality**.

We assume that there exists a state $\mathbf{e} \in \mathcal{B}^+$ that is maximally general, call this the **state of maximal uncertainty**. This encodes a complete lack of knowledge.

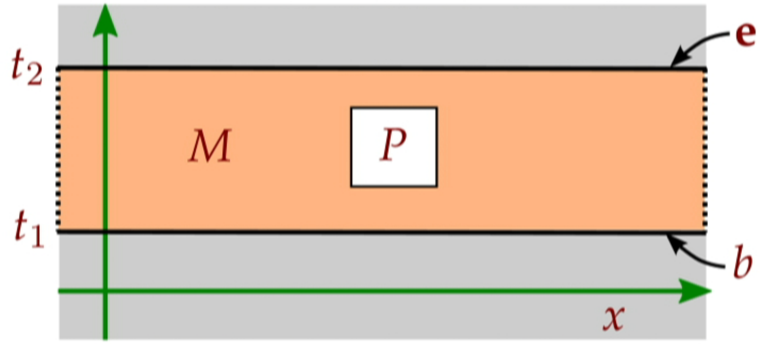
Mathematically, for any $b \in \mathcal{B}^+$ there exists $\lambda > 0$ such that $b \leq \lambda \mathbf{e}$. This is called an **order unit**.

Most often in a measurement, we are only interested in the outcome given a fixed initial state b_1, but do not care about the state after the measurement.

This is encoded by setting the final state $b_2 = \mathbf{e}$.

Measurement without post-selection

Consider a binary measurement in $[t_1, t_2]$ encoded by a **non-selective probe** Q and a **selective probe** P.



The probability Π for an affirmative outcome given an initial state $b \in \mathcal{B}$, but disregarding the final fate of the system is thus,

$$\Pi = \frac{[\![P, b \otimes \mathbf{e}]\!]_{[t_1,t_2]}}{[\![Q, b \otimes \mathbf{e}]\!]_{[t_1,t_2]}} = \frac{(\!|\mathbf{e}, \tilde{P}(b)|\!)}{(\!|\mathbf{e}, \tilde{Q}(b)|\!)}.$$

One also says that this is a measurement **without post-selection**.

Normalization

The **positivity** and **positive-definiteness** of the inner product implies that for any $b \in \mathcal{B}^+$ with $b \neq 0$ we have $(\!|\mathbf{e}, b|\!) > 0$.

$b \in \mathcal{B}^+$ is **normalized** iff $(\!|\mathbf{e}, b|\!) = 1$.

Normalization

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$b \in \mathcal{B}^+$ is **normalized** iff $(\!|\mathbf{e}, b|\!) = 1$.

This suggests corresponding notions for probe maps $\tilde{P} : \mathcal{B} \to \mathcal{B}$.

$\tilde{P} : \mathcal{B} \to \mathcal{B}$ is **normalization preserving** iff $(\!|\mathbf{e}, \tilde{P}(b)|\!) = (\!|\mathbf{e}, b|\!)$ for all $b \in \mathcal{B}$.

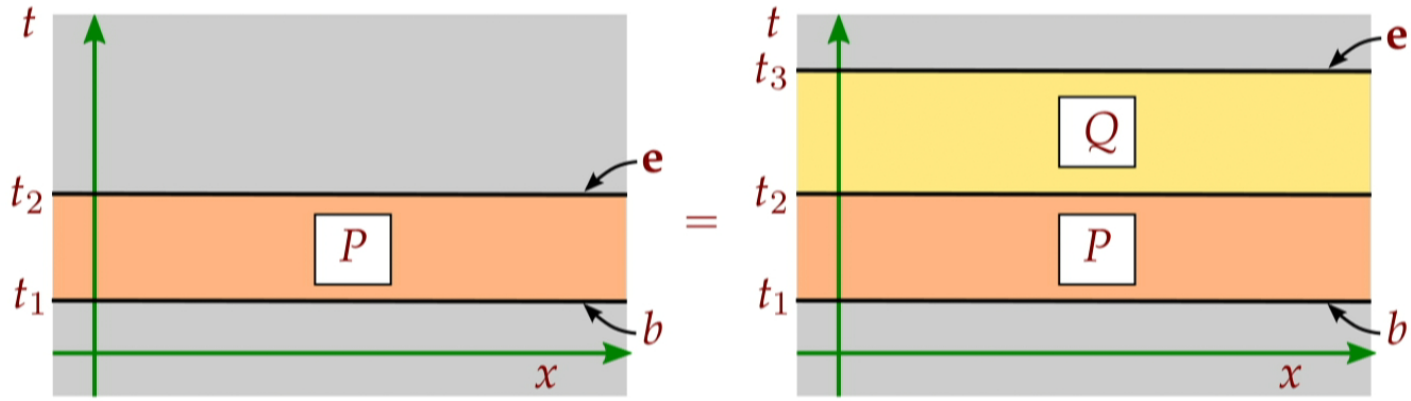
$\tilde{P} : \mathcal{B} \to \mathcal{B}$ is **normalization decreasing** iff $(\!|\mathbf{e}, \tilde{P}(b)|\!) \leq (\!|\mathbf{e}, b|\!)$ for all $b \in \mathcal{B}$.

Causality

Require that **non-selective probe maps** are **normalization preserving**.

This implies that **selective probe maps** are **normalization decreasing**.

Causality



Consider a binary measurement given by non-selective probe P_* and selective probe P_r. Possibly, a later measurement is performed given by non-selective probe Q.

The probability Π of an affirmative outcome of the first measurement does not depend on the second measurement being performed or not.

$$\Pi = \frac{(\!|\mathbf{e}, \tilde{P}_r(b)|\!)}{(\!|\mathbf{e}, \tilde{P}_*(b)|\!)} = \frac{(\!|\mathbf{e}, \tilde{Q}(\tilde{P}_r(b))|\!)}{(\!|\mathbf{e}, \tilde{Q}(\tilde{P}_*(b))|\!)}$$

Note: If b is normalized the denominators are equal to 1.

Time-asymmetry

The normalization conditions for probe maps are **time-asymmetric**. The normalization preserving condition,

$$(\!|\mathbf{e}, b|\!) = (\!|\mathbf{e}, \tilde{P}(b)|\!) \qquad \forall b \in \mathcal{B}$$

reads in the general formalism as,

$$[\![\boxslash, b \otimes \mathbf{e}]\!] = [\![P, b \otimes \mathbf{e}]\!] \qquad \forall b \in \mathcal{B}.$$

The time-reversed condition reads,

$$[\![\boxslash, \mathbf{e} \otimes b]\!] = [\![P, \mathbf{e} \otimes b]\!] \qquad \forall b \in \mathcal{B}.$$

Classical and quantum theory

If we take $\mathcal{B}$ to be a **lattice**, i.e., the space of real valued functions on a set (**phase space**) we obtain **classical statistical mechanics**.

- $\mathbf{e} = \mathbf{1}$ the constant function with value 1.
- $(\!|\cdot,\cdot|\!) = \int \cdot\, \cdot\, \mathrm{d}\mu$ the L^2 **inner product**
- certain **probe maps** describe **observables**, others describe modified dynamics

If we take $\mathcal{B}$ to be an **anti-lattice**, i.e., the space of self-adjoint operators on a Hilbert space $\mathcal{H}$, we obtain **quantum statistical mechanics**.

- $\mathbf{e} = \mathrm{id}_{\mathcal{H}}$ the **identity operator**
- $(\!|\cdot,\cdot|\!) = \mathrm{tr}(\cdot\,\cdot)$ the **Hilbert-Schmidt inner product**
- **primitive probe maps** are **quantum operations**

Main reference

R. O., *A local and operational framework for the foundations of physics*, `arXiv:1610.09052`.