Title: "Quantum advantage with shallow circuits"

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Abstract: <p>We prove that constant-depth quantum circuits are more powerful than their classical counterparts. We describe an explicit (i.e., non-oracular) computational problem which can be solved with certainty by a constant-depth quantum circuit composed of one- and two-qubit gates. In contrast, we prove that any classical probabilistic circuit composed of bounded fan-in gates that solves the problem with high probability must have depth logarithmic in the input size. This is joint work with Sergey Bravyi and Robert Koenig (arXiv:1704.00690).</p>

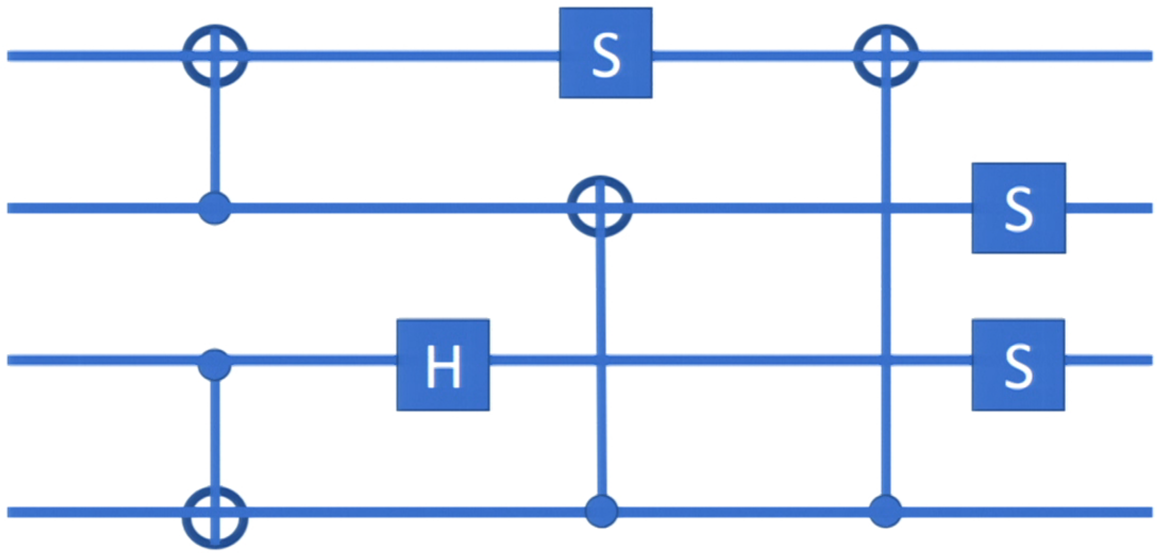
Quantum advantage with shallow circuits

arXiv:1704.00690

Sergey Bravyi (IBM)
David Gosset (IBM)
Robert Koenig (Munich)

I. Overview

A **depth-*d* quantum circuit** consists of d time steps.
Each time step contains one- and two-qubit gates acting on disjoint qubits.



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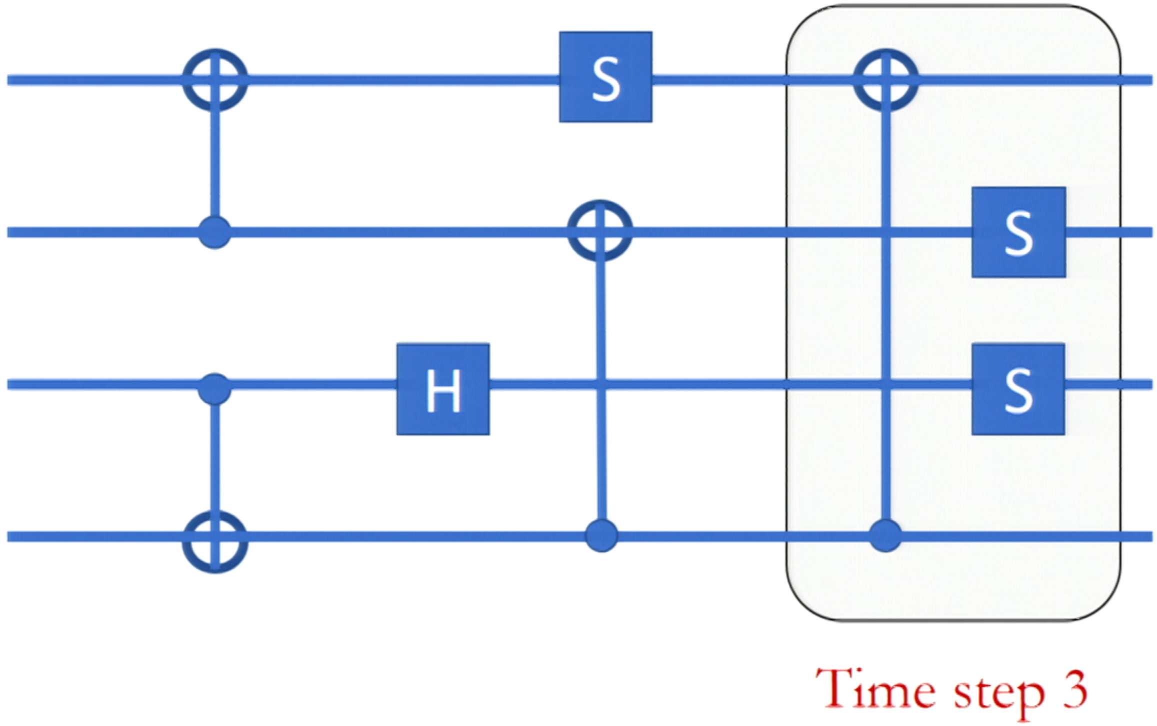

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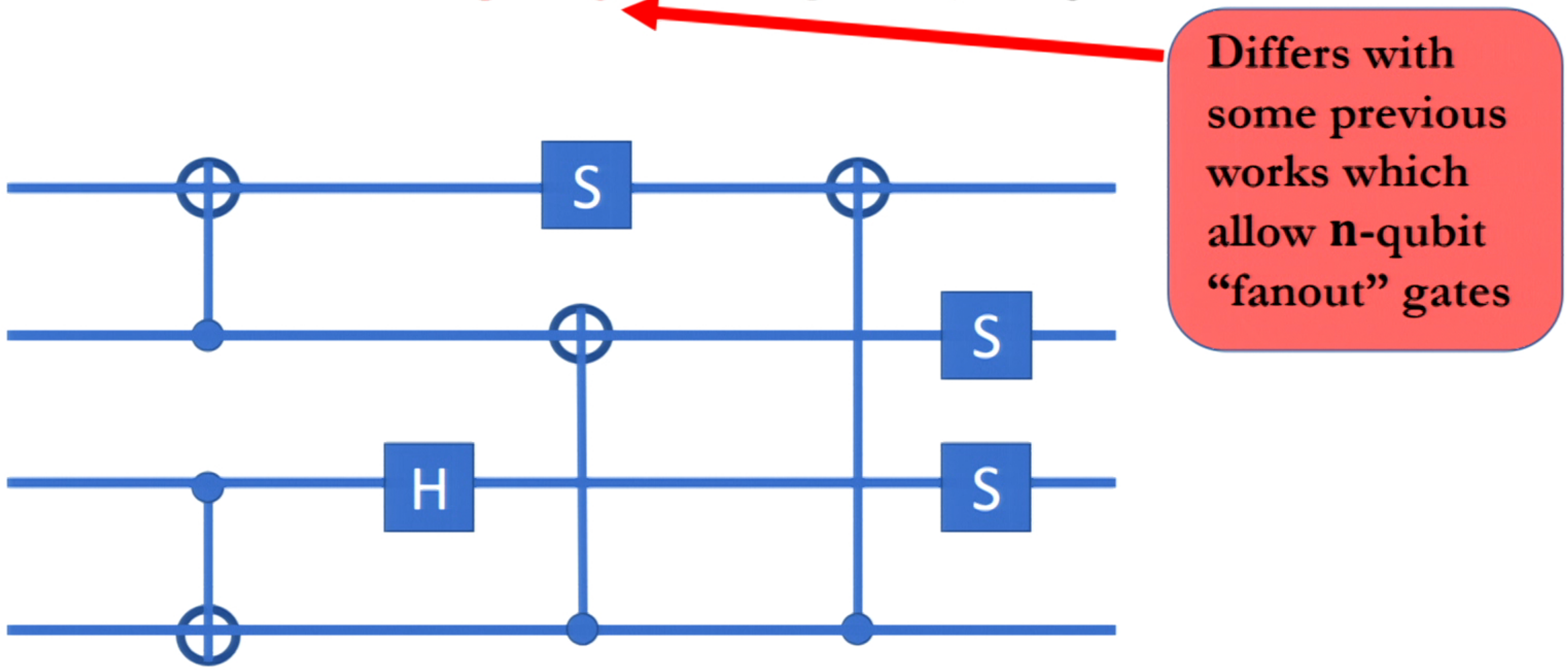


We are interested in **constant-depth quantum circuits**, for which $d = O(1)$.

Constant-time quantum computation

How much can we gain with parallelism if we only have a fixed computation time?

Structure/Simulation

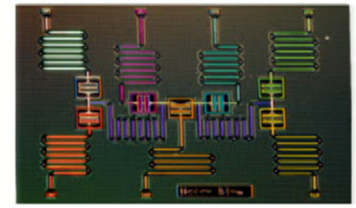
Cannot prepare codewords of good quantum codes
[Eldar, Harrow 2015]

Efficient classical simulation of depth-2 circuits
[Terhal, Divincenzo 2002]

General simulation algorithms (superpolynomial)
[Aaronson, Chen 2016]

Constant-depth quantum circuits

Quantum computers without error correction



Quantum supremacy?

Constant-depth unlikely to be classically simulable.
[Terhal, Divincenzo 02]

Beat the best classical computers for some task?
[Gao et al. 17]
[Bermejo-Vega et al. 17]
…uses IQP results…
[Bremner, Montanaro, Shepherd 16]

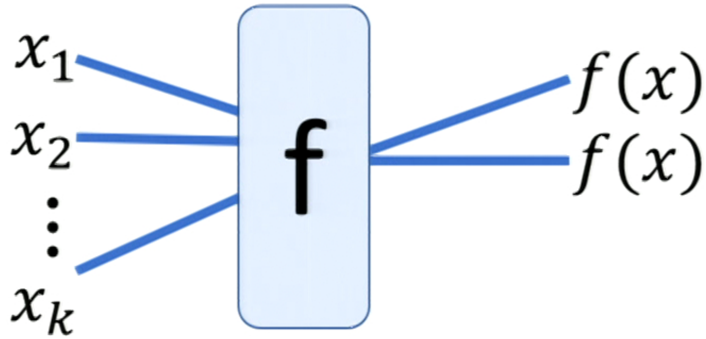
This talk: Can constant-depth quantum circuits solve a computational problem that constant-depth classical circuits cannot?

This talk: Can constant-depth quantum circuits solve a computational problem that constant-depth classical circuits cannot?

Classical circuits

A classical gate computes a boolean function $f\colon \{0,1\}^k \to \{0,1\}$

Number of input bits k is called the **fan-in**

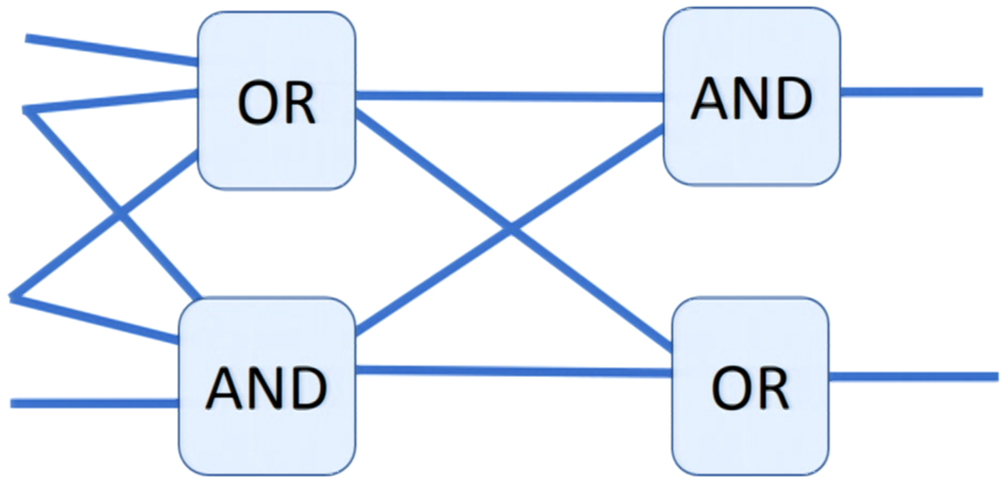


Number of times the output is used is called the **fan-out**

We consider circuits composed of **bounded fan-in gates**, i.e., $k = O(1)$.
We do not restrict the fan-out.

Constant-depth classical circuits

A depth-d classical circuit consists of d layers (time steps) of gates.



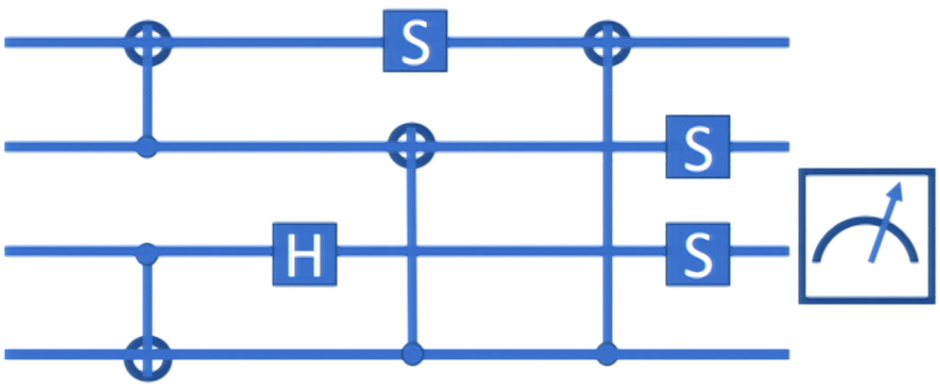
We consider constant-depth circuits composed of bounded fan-in gates.
This class of circuits is known as NC^0.
We also allow the circuit to be probabilistic (random input bits are provided).

Can constant-depth quantum circuits solve a **computational problem** that constant-depth classical circuits cannot?

	Input	Output
❌ Decision problem	Bit-string x	$b_x \in \{0,1\}$

Reduced density matrix of any output qubit is determined by a constant-sized subcircuit (containing at most 2^d qubits).

Example:



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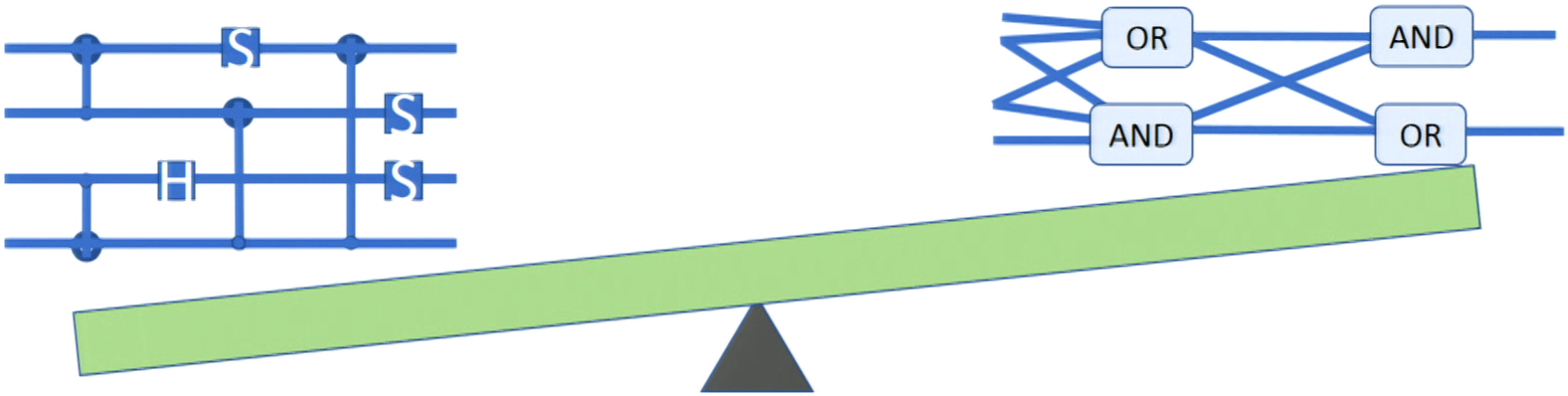
	Input	Output	
✗ **Decision problem**	Bit-string x	$b_x \in \{0,1\}$	
✗ **Search problem**	Bit-string x	$z_x \in \{0,1\}^n$	(unique solution)
✓ **Relation problem**	Bit-string x	$z \in S_x \subseteq \{0,1\}^n$	(non-unique)



Our result:

We describe a (relation) problem that is solved with certainty by a constant-depth quantum circuit.

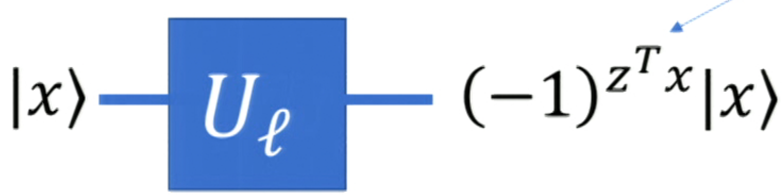
We prove that any probabilistic classical circuit composed of bounded fan-in gates which solves the problem with high probability must have depth increasing logarithmically with input size.

II. Hidden Linear Function Problems

Hiding a linear function in an oracle [Bernstein-Vazirani 1993]

Goal: Find $z \in \{0,1\}^n$ using few queries to a quantum oracle:

Linear Boolean function parameterized by a "secret" bit string z



We only need to use the quantum oracle once: $|z\rangle = H^{\otimes n} U_\ell H^{\otimes n} |0^n\rangle$.

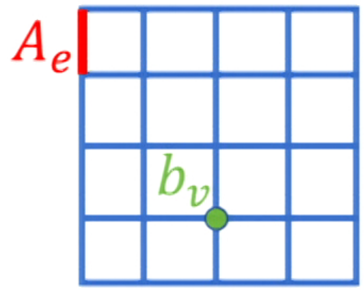
In contrast, a classical algorithm needs n queries to a classical oracle computing ℓ.

The Bernstein-Vazirani speedup is relative to an oracle and is not guaranteed to translate into a real-world advantage.

Where else can we hide a linear function?

Quadratic form on a grid

Let $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ be an $\mathrm{N} \times \mathrm{N}$ grid graph. Write $n = N^2 = |V|$



Choose coefficients $A_e \in \{0,1\}$ for each edge and $b_v \in \{0,1\}$ for each vertex.

Any choice of coefficients defines a quadratic form $q\colon \{0,1\}^n \to \mathbb{Z}_4$

$$q(x) = \sum_{e=(v,w)\in E} 2A_e x_v x_w + \sum_{v\in V} b_v x_v$$

The quadratic form hides a linear function

Define a set

$$\mathcal{L}_q = \{x \in \mathbb{F}_2^n \colon \; q(x \oplus y) = q(x) + q(y) \text{ for all } y \in \mathbb{F}_2^n\}$$

Lemma
The set $\mathcal{L}_q$ is a linear subspace of $\mathbb{F}_2^n$. Furthermore, there is a "secret" bit string $z \in \{0,1\}^n$ such that

$$q(x) = 2z^T x \qquad \forall x \in \mathcal{L}_q$$

The 2D Hidden Linear Function Problem

Input: Coefficients $A \in \{0,1\}^{|E|}$ and $b \in \{0,1\}^{|V|}$.

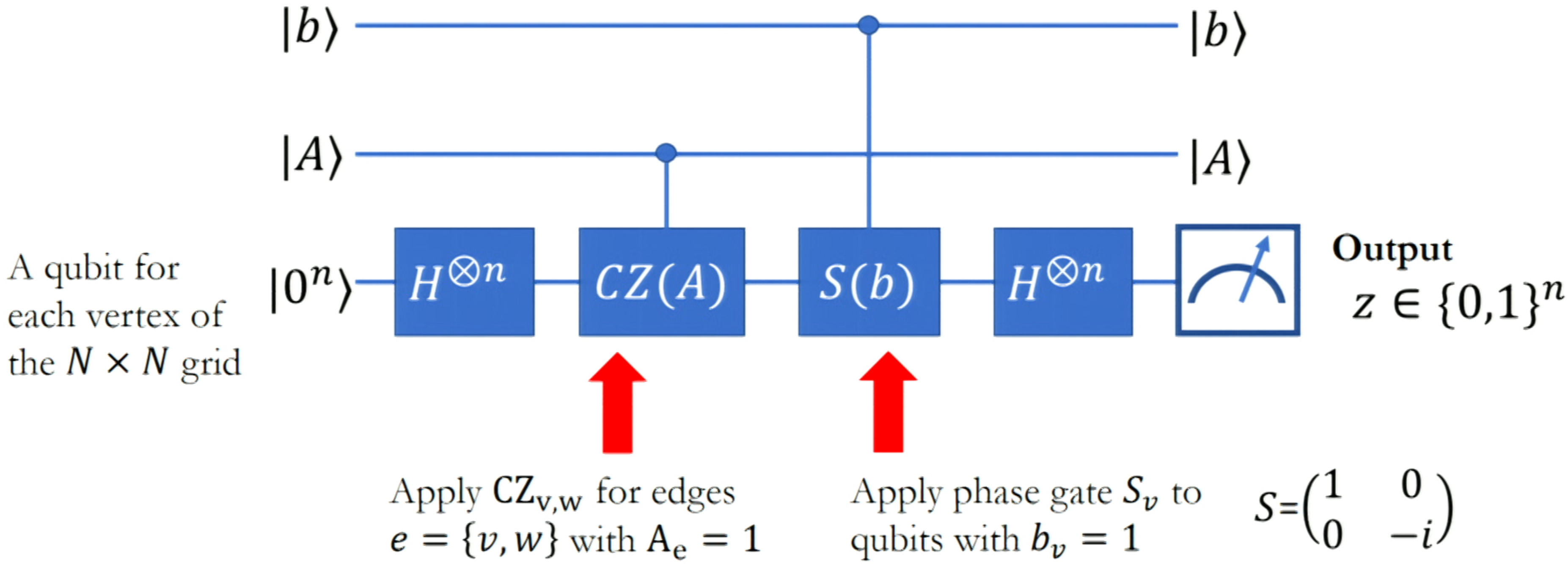
Specifies a quadratic form $q(x)$ and a subspace $\mathcal{L}_q \subseteq \mathbb{F}_2^n$

Output: A "secret" bit string $z \in \{0,1\}^n$ such that

$$q(x) = 2z^T x \quad \forall x \in \mathcal{L}_q$$

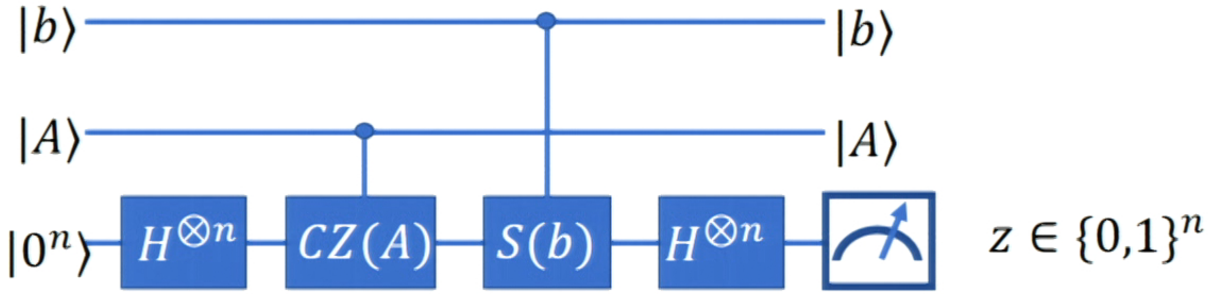
In general each instance has many valid solutions z.

Quantum algorithm



Next we'll show that:

1. **This algorithm solves the 2D Hidden Linear Function Problem.**
2. **It can be implemented in constant-depth.**



Lemma: The output z is a uniformly random solution to the 2D HLF Problem.

Proof Sketch:

Define $U_q = S(b)CZ(A)$. It satisfies $U_q|y\rangle = i^{q(y)}|y\rangle$

Output distribution: $p(z) = \left|\langle z|H^{\otimes n}U_q H^{\otimes n}|0^n\rangle\right|^2 = \frac{1}{4^n}\left|\sum_{y\in\mathbb{F}_2^n}(-1)^{z^T y} i^{q(y)}\right|^2$

Square of Fourier Transform $\mathcal{F}[i^{q(y)}, \mathbb{F}_2^n](z)$

Write $\mathbb{F}_2^n = \mathcal{L}_q + \mathcal{M}$ and write the FT as a product of "partial" FTs.

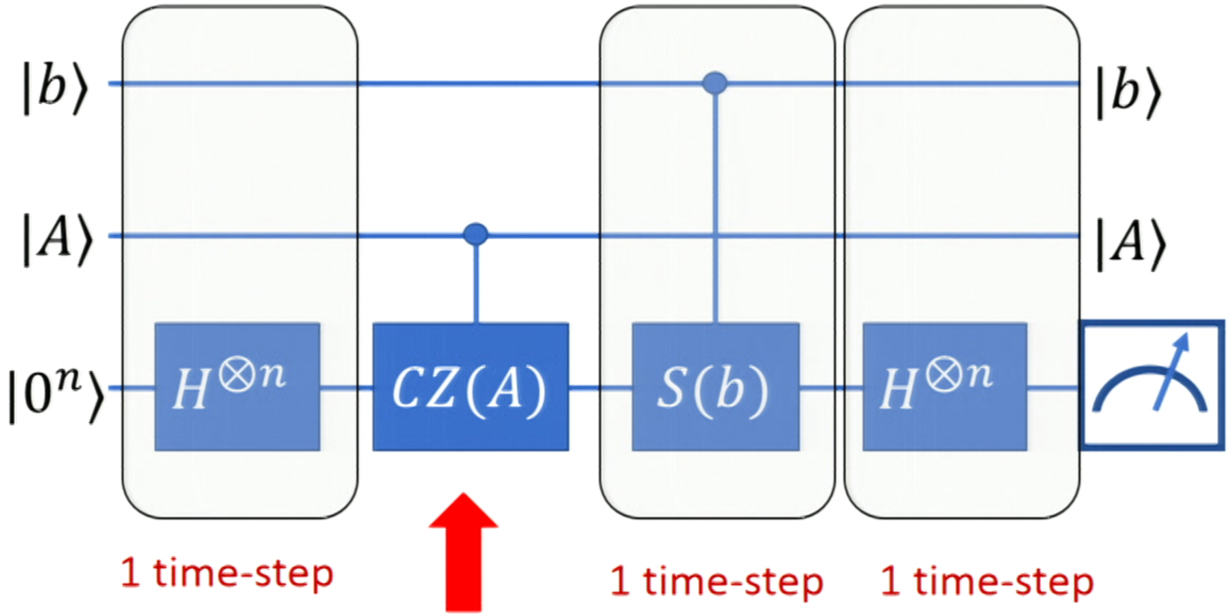
$$\mathcal{F}[i^{q(y)}, \mathbb{F}_2^n](z) = \underbrace{\mathcal{F}[i^{q(y)}, \mathcal{L}_q](z)} \cdot \underbrace{\mathcal{F}[i^{q(y)}, \mathcal{M}](z)}$$

Use basic properties of FT and quadratic forms:

Nonzero iff z is a solution
Constant over solution set.

Constant
(independent of z)

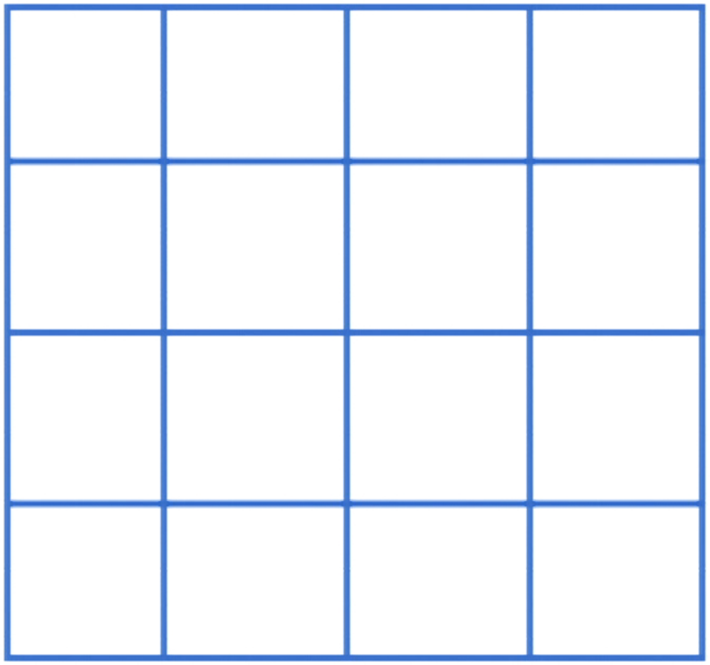
The algorithm can be implemented in constant-depth



Four layers of **CCZ** gates.
(even/odd vertical/horizontal edges)
Decompose **CCZ** gates into 1- and 2-qubit gates.

…it only requires classically controlled Clifford gates between nearest neighbor qubits on a 2D grid.

Example:



Place a qubit at each vertex

Place input bits on vertices and edges:

(red line) : Edge with $A_e = 1$

(green circle) : Vertex with $b_v = 1$

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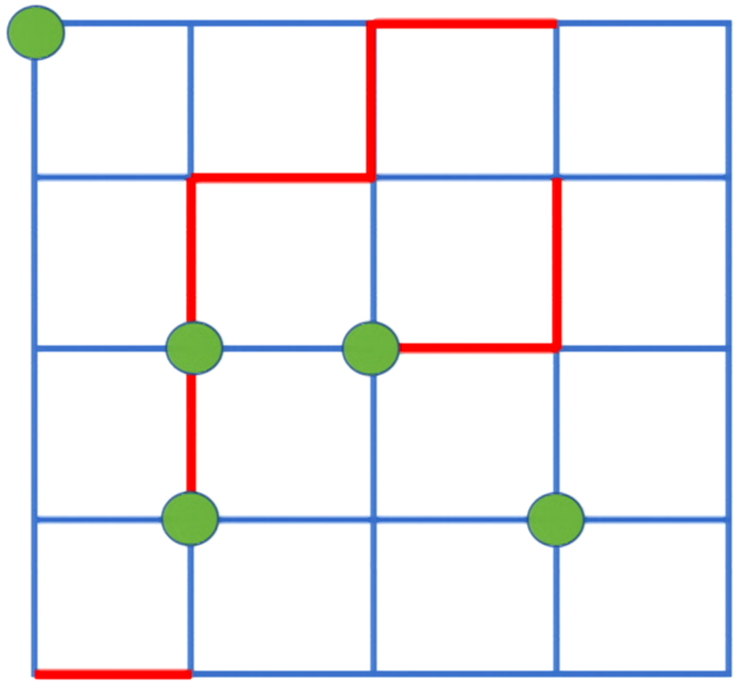

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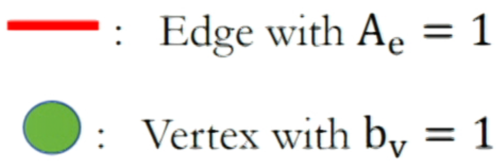

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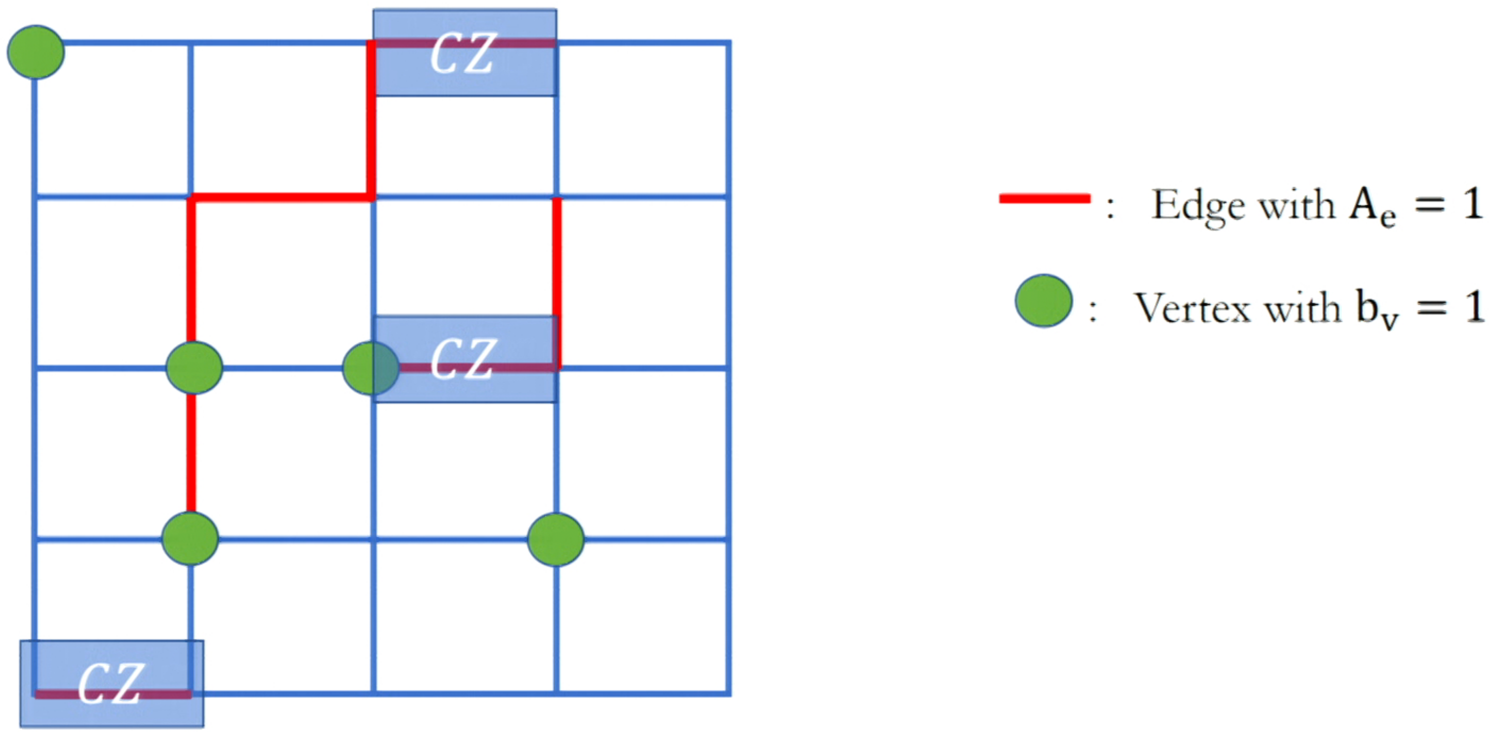

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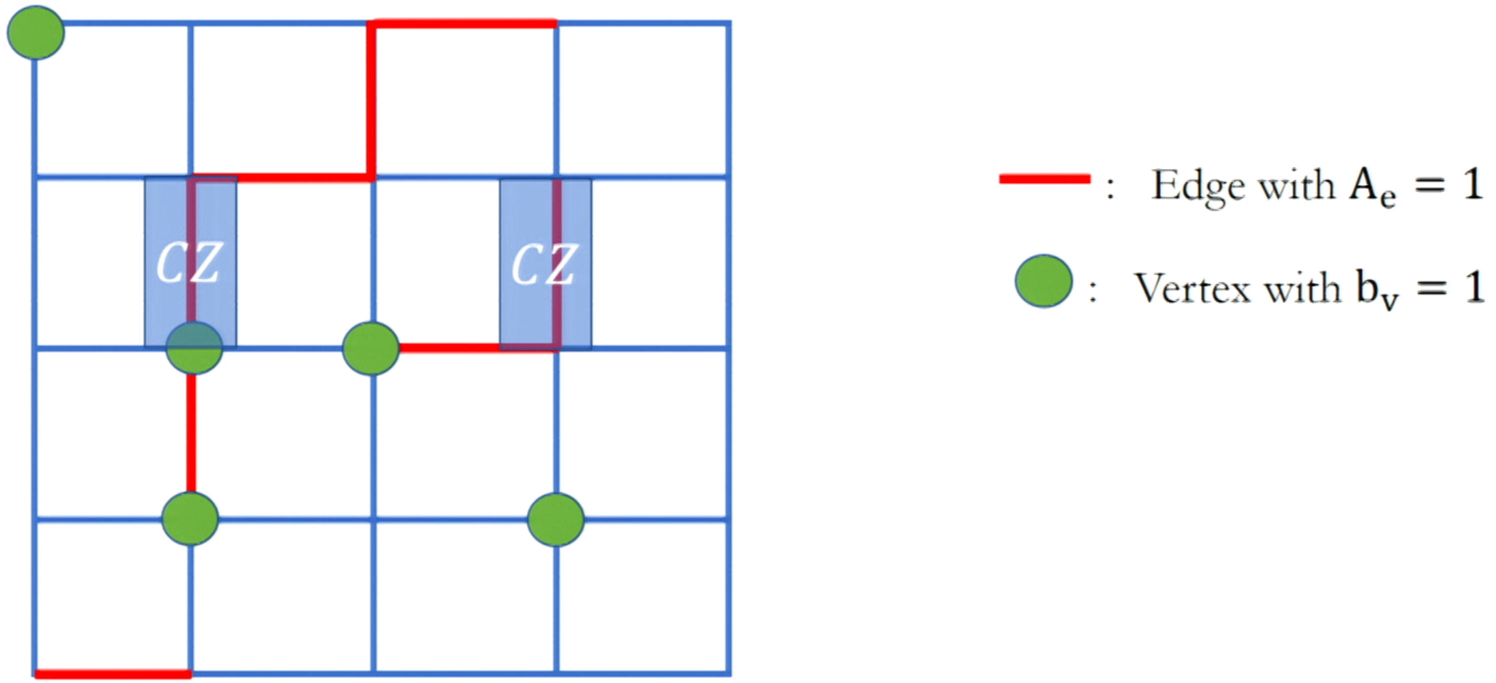

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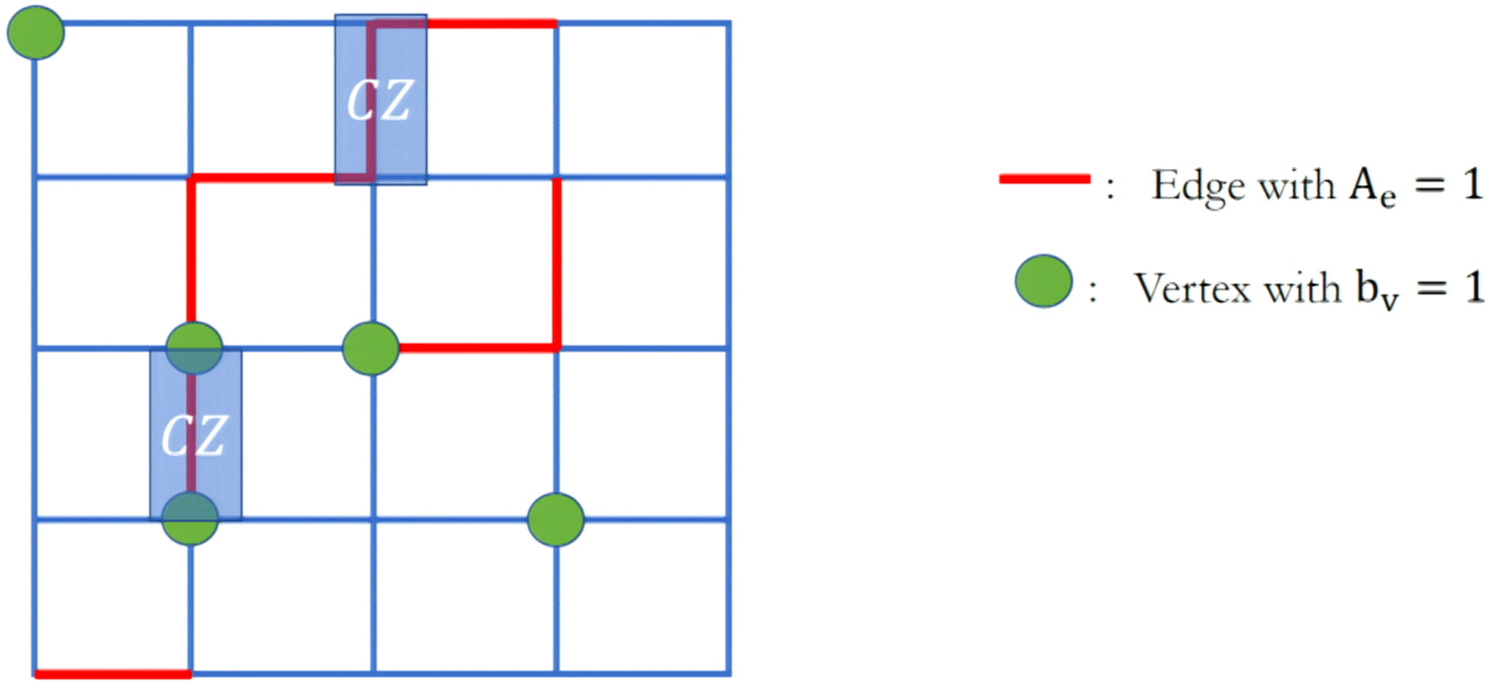

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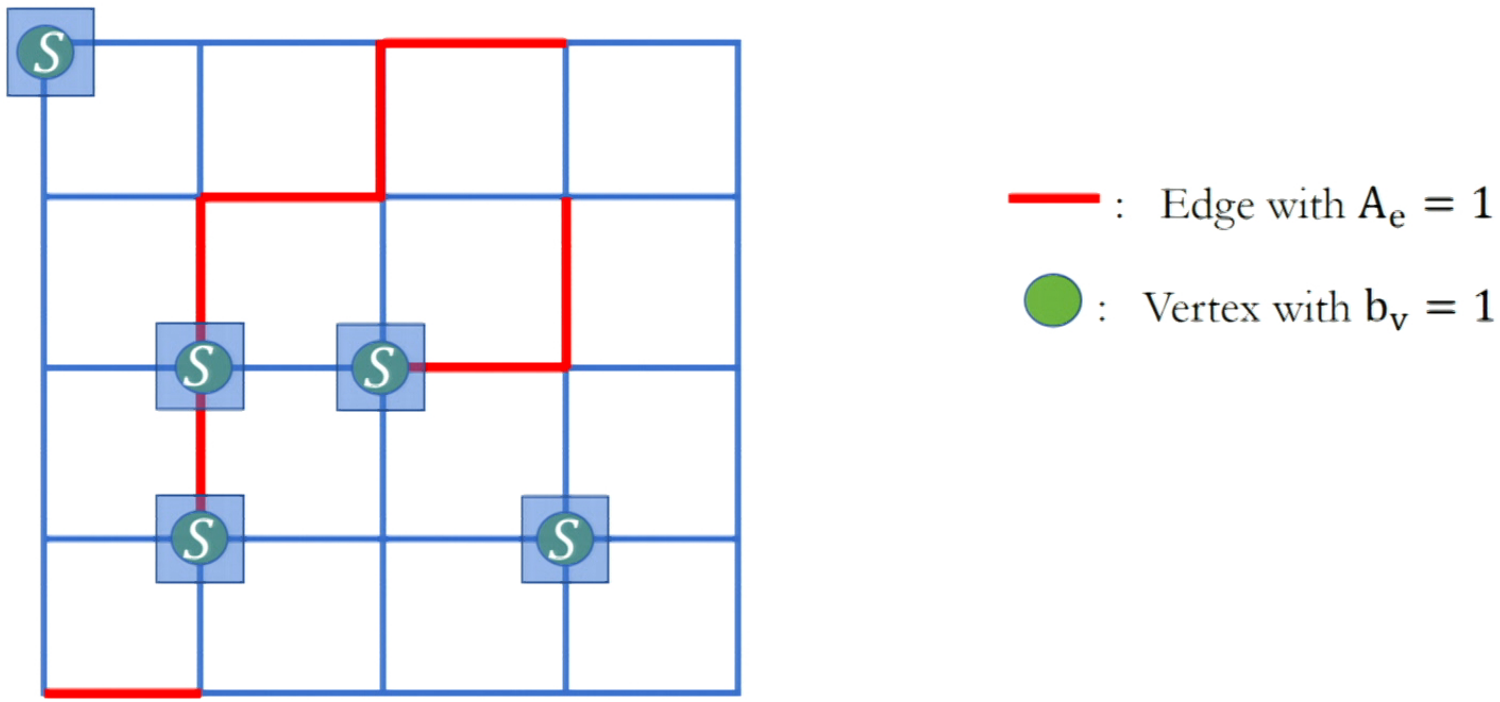

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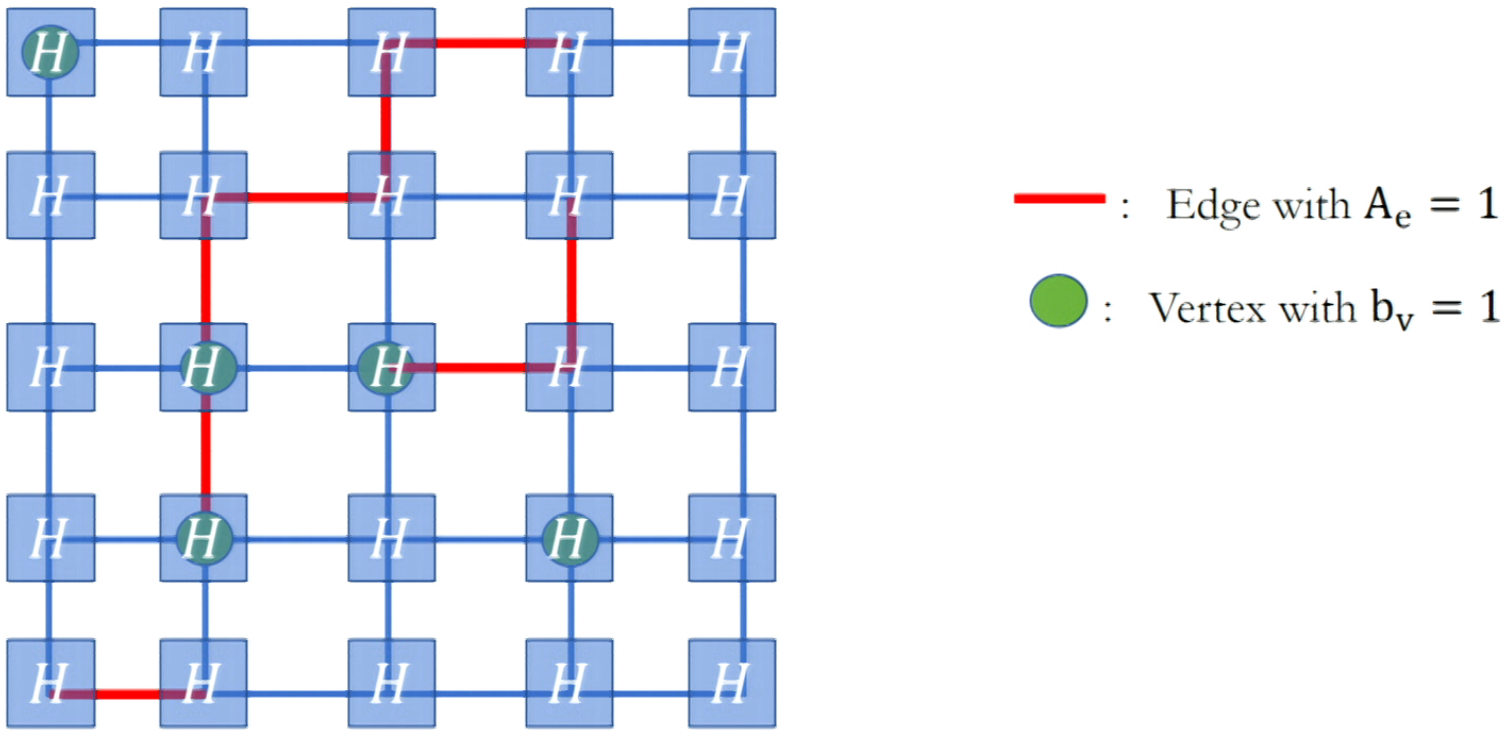

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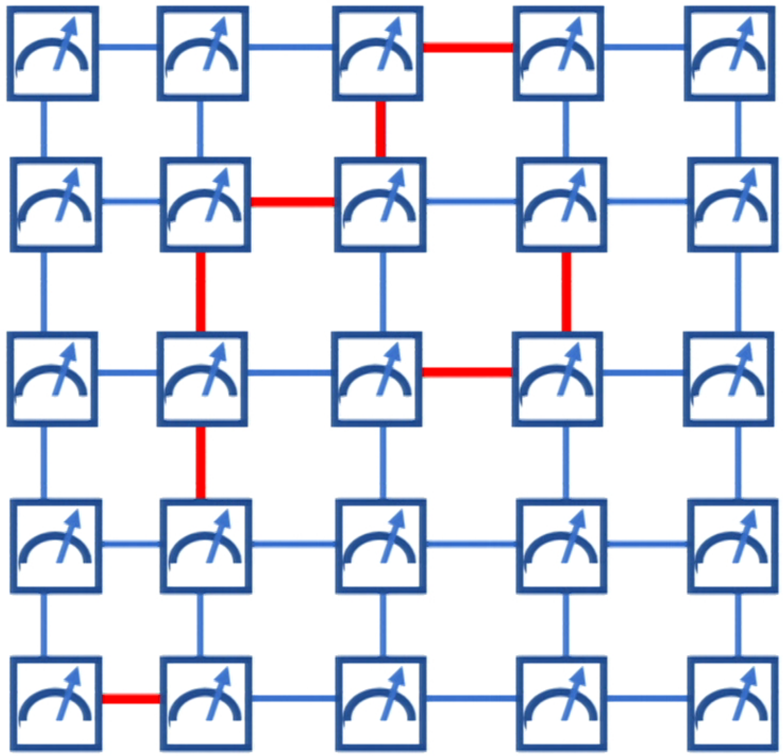
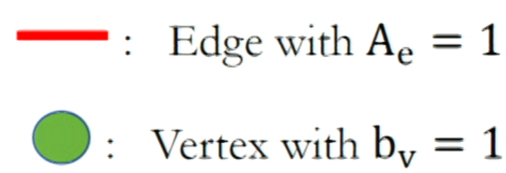




The 2D HLF problem is solved by a constant-depth quantum circuit with gates acting locally in 2D.

Next we show that it cannot be solved by a constant-depth classical circuit…

Theorem: The following holds for all sufficiently large N. Let $\mathcal{C}_N$ be a classical probabilistic circuit composed of gates of fan-in $\leq K$ which solves size-N instances of the 2D HLF Problem with probability greater than $7/8$. Then

$$\text{depth}(\mathcal{C}_N) \geq \frac{\log(N)}{8\log(K)}$$

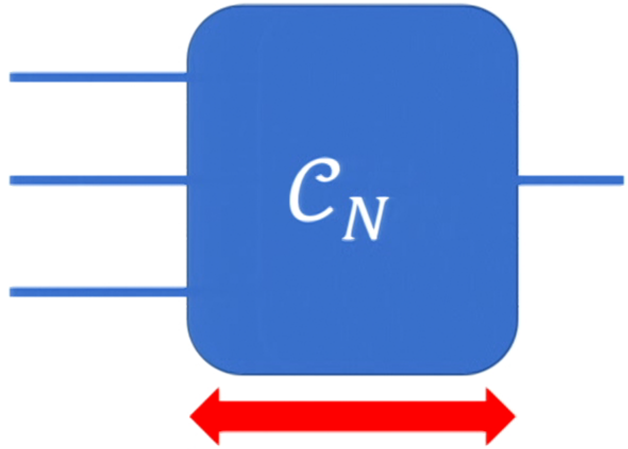
Input
(instance on
N × N grid)

$A \in \{0,1\}^{|E|}$

$b \in \{0,1\}^{|V|}$

Random bits
(drawn from any
joint distribution)

$r \in \{0,1\}^{\ell}$



Output

$z \in \{0,1\}^{|V|}$

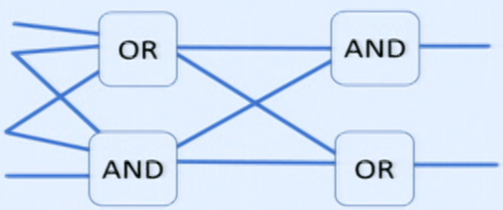
**Solution with
probability > 7/8**

**Circuit must have
depth $\mathbf{\Omega(\log(N))}$**

Proof Ideas

Locality in shallow classical circuits

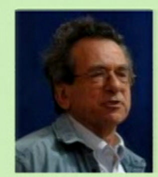
Each output bit can only depend on $O(1)$ input bits.



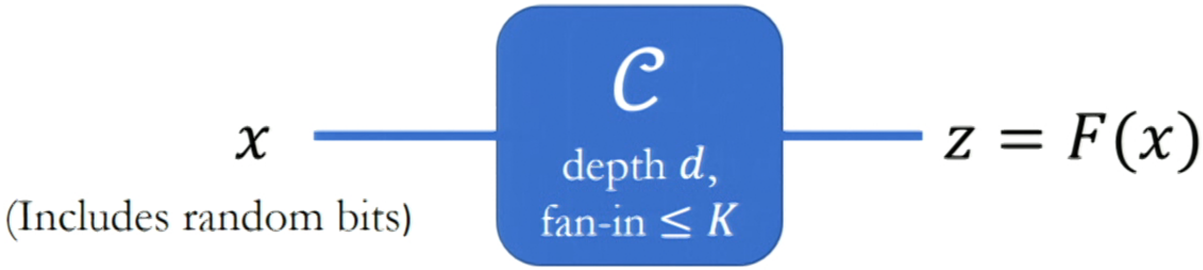
Vs.

Quantum nonlocality

Measurement statistics of entangled quantum states cannot be reproduced by local hidden variable models



Locality in classical circuits



Input bit x_j is **correlated** with output bit z_k iff flipping the jth input bit can flip the kth output bit. The **lightcone** $L(z_k)$ is the set of input bits that are correlated with z_k.

$$|L(z_k)| \leq K^d$$

"Constant-depth locality"

We'll see that the 2D Hidden Linear Function problem cannot be solved by "constant-depth local" circuits. First consider simpler forms of locality…

Quantum nonlocality beats completely local circuits

[Greenburger et al. 1990][Mermin 1990]

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$ satisfies:

$$P|GHZ\rangle = |GHZ\rangle$$

$$P \in \{XXX, -XYY, -YXY, -YYX\}$$

Choose bits b_1, b_2, b_3 and then measure each qubit of $|\text{GHZ}\rangle$ in either the X basis (if $b_j = 0$) or the Y basis (if $b_j = 1$). Outcomes $z_j \in \{-1, +1\}$ satisfy:

$$i^{b_1+b_2+b_3} z_1 z_2 z_3 = 1 \quad \text{whenever} \quad b_1 \oplus b_2 \oplus b_3 = 0$$

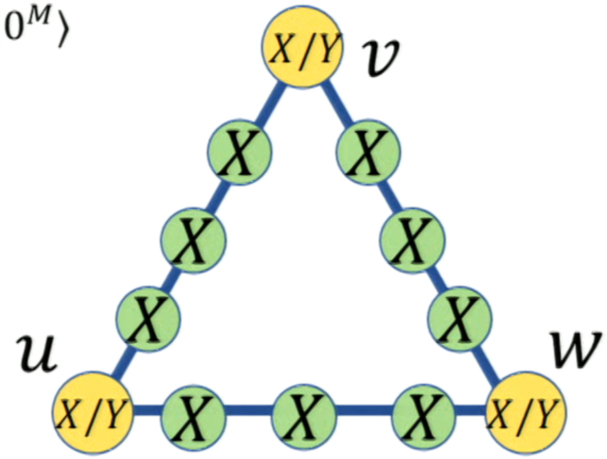
"GHZ relation"

The GHZ relation cannot be satisfied by a completely local classical probabilistic circuit where each output bit z_j is correlated with at most one of the input bits b_k.

Quantum nonlocality beats geometrically local circuits

[Barrett et al. 2007]

Graph state on an M-cycle (M even): $|\Phi_M\rangle = \left(\prod_{j=1}^{M} CZ_{j,j+1}\right) H^{\otimes M}|0^M\rangle$



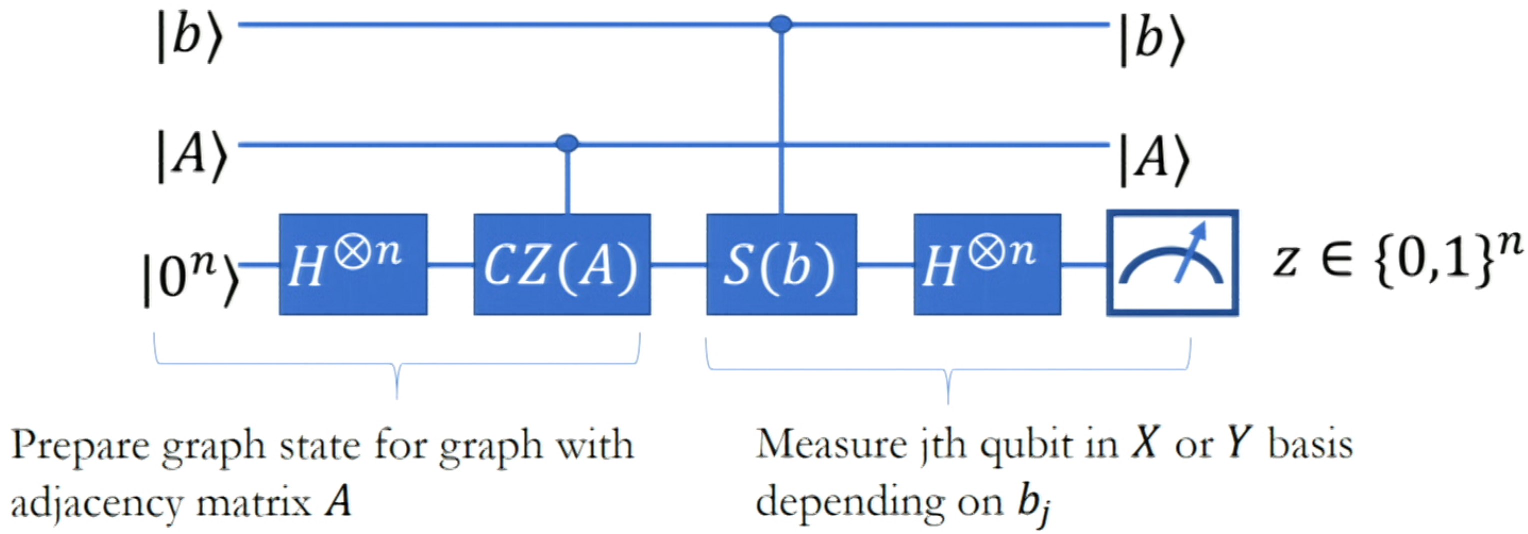
Choose 3 qubits u, v, w on the even sublattice. Measure u, v, w in X or Y basis and all other qubits in X basis.

Input		Output
$b_u, b_v, b_w \in \{0,1\}$	→	$z \in \{0,1\}^M$
Measurement bases		Measurement outcomes

Fact: Input/output satisfy a **"cycle relation"** $R(b_u, b_v, b_w, z) = 1$ similar to the GHZ relation.

Lemma: Suppose a classical circuit satisfies the cycle relation with probability $> 7/8$. Then some output bit z_k is correlated with a **distant** input bit b_u, b_v or b_w.
(this means it is not the nearest vertex of the triangle)

…How is this related to the 2D Hidden Linear Function Problem?



Choosing A to describe the adjacency matrix of a cycle and choosing b appropriately we infer (from Barrett et al.) a cycle relation satisfied by input/output.

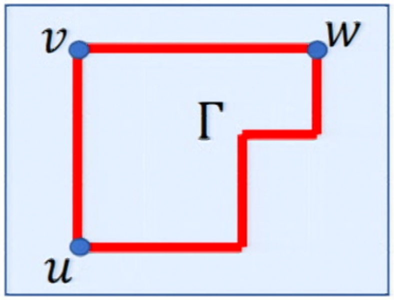
A classical circuit which solves the 2D HLF problem must also satisfy all such cycle relations….

Quantum nonlocality beats "constant-depth local" circuits

We use constant-depth locality (every output bit has constant-sized lightcone) and a probabilistic argument to prove the following:

Lemma: Suppose a classical circuit has depth less than $\frac{\log(N)}{8\log(K)}$.

Then we can find 3 vertices u, v, w on the even sublattice of the $N \times N$ grid and a cycle Γ which passes through them, such that **input bits $\boldsymbol{b_u, b_v, b_w}$ are not correlated with any distant output bits on $\boldsymbol{\Gamma}$.**



→ The circuit does not w.h.p satisfy the cycle relation for Γ

→ It does not w.h.p solve instances of 2D HLF problem where A is the adjacency matrix of Γ.

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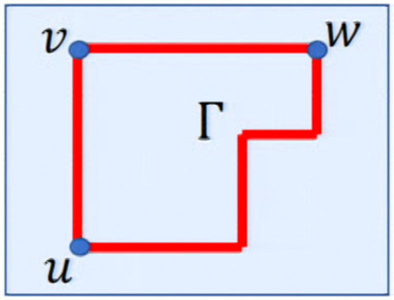

→ The circuit does not w.h.p satisfy the cycle relation for Γ → It does not w.h.p solve instances of 2D HLF problem where A is the adjacency matrix of Γ.

This provides our lower bound on the depth of any classical circuit which solves the 2D HLF problem with probability greater than 7/8.

Open questions

Stronger classical circuits? Can the 2D HLF be solved by AC^0 circuits? (constant depth unbounded fan-in)

Recursive HLF problems? The recursive version of Bernstein-Vazirani gives a superpolynomial speedup in query complexity.

Sampling problems? Can constant-depth quantum circuits sample from a distribution that can't be sampled by classical constant depth circuits? A recent characterization of distributions sampled by NC^0 circuits might be useful [Viola 2014].